Advanced topics in differential geometry
Summary of the content

The class will be devoted to rigidity and flexibility in negative curvature.

The following topics will be covered.

(1) Holonomy and the theorem of Cartan, Ambrose, Hicks.
(2) Symmetric spaces, their rank and their holonomy.
(3) Volume entropy and growth of the fundamental group.
(4) The Mostow rigidity theorem in the rank one case and minimal volume.

Time permitting and depending on the audience, either recent constructions of examples of manifolds with arbitrarily pinched negative curvature or a short introduction to harmonic maps with some geometric applications will also be discussed.

The first part belongs to classical differential geometry. So-called special holonomy has obtained a lot of attention recently, and some basic questions and results (without proof) will be explained.

The second part will only cover some basic differential geometric features of symmetric spaces as Riemannian manifolds with parallel curvature tensor, focussing mainly on holonomy and the role of Killing fields. The rank one examples will be discussed in more detail. The classification of symmetric spaces will only be stated, without proof.

Volume entropy measures the growth of the volume of balls in the universal covering of a closed manifold and is closely related to the growth of the fundamental group. This together with some basic additional geometric features of negatively curved manifolds will be introduced.

The Mostow rigidity theorem states that any finite volume rank one (=negatively curved) locally symmetric manifolds of dimension at least 3 is essentially determined by its fundamental group: Any two such manifolds with the same fundamental group are homothetic, that is, isometric up to scaling the metric with a constant. There are also versions of this theorem in higher rank, giving stronger results (Margulis' superrigidity).

The proofs in the rank one and higher rank case build on different principles. There are three proofs available in the rank one case. Mostow's original proof using quasi-conformal maps, a proof of Gromov and Thurston, restricted to the constant curvature case, using simplicial volume, and a differential geometric proof by Besson, Courtois and Gallot using barycenter maps. The latter proof also yields the minimal volume theorem for hyperbolic manifolds and will be discussed.

Some recent applications of the methods to rigidity and finiteness results for manifolds with volume and topological constraints, for example in the context of integral curvature bounds, are given (e.g. results of Petersen, Wei and collaborators).

Time permitting, the last part is devoted to constructions of manifolds with curvature arbitrarily close to -1 which do not admit hyperbolic metrics. The first examples are due to Gromov and Thurston, a very recent and much more general construction is due to Ontaneda. I'll try to explain how these examples fit into the finiteness and rigidity theorems covered earlier.

Alternatively, harmonic maps will be introduced and some recent applications of such maps to rigidity questions will be discussed.

Prerequisits: The lecture builds on the class *Advanced Geometry I* in the winter semester 20/21.

Below is an incomplete list of references which will be extended later on.

References

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[BCG95] G. Besson, C. Courtois, S. Gallot, *Entropies et rigidites des espaces localement symétriques de courbure strictement négative*, Geom. Funct. Anal. 5 (1995), 731–799.

[CE75] J. Cheeger, D. Ebin, *Comparison theorems in negative curvature*, North Holland 1975.

[E96] P. Eberlein, *Geometry of nonpositively curved manifolds*, Chicago Lectures in Math. 1996.

[GT87] M. Gromov, W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. 89 (1987), 1–12.

[H78] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press 1978.

[M73] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton Univ. Press 1973.

[O20] P. Ontaneda, *Riemannian hyperbolization*, Publ. IHES, 1-72, (2020).

[S96] T. Sakai, *Riemannian geometry*, AMS 1996.