Decoupling and Polynomial Methods in Analysis

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1 L^p regularity of averages over curves and bounds for associated maximal operators, Part II

after M. Pramanik and A. Seeger [3]
A summary written by David Beltran

Abstract

The maximal function generated by dilations of curves with nonvanishing curvature and torsion, for which the prototypical example is the helix, is bounded on L^p for $p > 4$. Via a Sobolev embedding argument, the result amounts to a *local smoothing* estimate for the associated averaging operator. A key ingredient of the proof is the sharp ℓ^p-decoupling estimate for cones.

1.1 Results and strategy

Let I be a compact interval, χ be a smooth function supported in the interior of I and $\gamma : I \to \mathbb{R}^3$ be a smooth curve. Define a measure μ_t supported on a dilate of the curve by

$$\langle \mu_t, g \rangle := \int g(t\gamma(s))\chi(s)ds,$$

and set

$$\mathcal{A}_t f(x) := f * \mu_t(x) \quad \text{and} \quad \mathcal{M}f(x) := \sup_{t>0} |\mathcal{A}_t f(x)|.$$

One of the main results of the paper under review is the boundedness of the maximal function over dilates of curves with nonvanishing curvature and torsion. The prototypical example for these curves is the helix $\gamma(t) = (\cos t, \sin t, t)$.

Theorem 1. *Suppose $\gamma \in C^5(I)$ has nonvanishing curvature and torsion. Then $\|\mathcal{M}f\|_p \lesssim \|f\|_p$ for all $p > 4$.*

The strategy to prove this boundedness comes in three steps. First, the use of a Sobolev embedding to replace the L^∞ in the t-variable by an L^p-norm, with the loss of carrying $s > 1/p$ derivatives in the t-variable. Then, the use of a *local smoothing* estimate, that is, an estimate which incorporates a gain in the regularity with respect to the fixed-time estimate when integrating locally in time. Finally, the use of a variant of the ℓ^p-decoupling estimate for cones in order to prove the required local smoothing estimate.

One may put the above strategy in action after a Littlewood–Paley reduction. This reduces the problem to the frequency projections $\mathcal{P}_k f$ for $k > 0$, which have Fourier support in $\{|\xi| \sim 2^k\}$. Namely, one should prove that for $p > 4$, the estimate for a single localised Fourier projection decays exponentially in k, so that one may sum in $k > 0$ and obtain Theorem 1. It should also be noted that by a discretisation and a scaling argument, one may assume $1 \leq t \leq 2$ in the definition of the maximal function.

The use of the Sobolev embedding converts the maximal estimate into a full $L^p_{x,t}$ estimate for $\mathcal{A}_t f(x)$, but brings a factor $2^{k\left(\frac{1}{p}+\varepsilon\right)}$. This cannot be compensated by simply integrating over $1 \leq t \leq 2$ the sharp fixed time Sobolev estimate

$$\|\mathcal{A}_t \mathcal{P}_k f\|_{L^p(\mathbb{R}^n)} \leq C_{p,t} 2^{-k/p} \|f\|_p$$

for $p > 4$. The resulting maximal estimate would carry a factor $2^{k\varepsilon}$, which is not summable in $k > 0$. However, one expects to obtain a gain when integrating locally in time, that is to obtain

$$\left(\int_1^2 \|\mathcal{A}_t \mathcal{P}_k f\|^p_{L^p(\mathbb{R}^n)} dt \right)^{1/p} \leq C_{p,t} 2^{-k/p - k\varepsilon(p)} \|f\|_p \tag{1}$$

for some $\varepsilon(p) > 0$ and $p > 4$. These types of estimates are commonly referred to as *local smoothing estimates* due to the gain in regularity obtained after integrating locally in time. This effect was first observed and conjectured by Sogge [4] in the context of solutions to the wave equation; see Section 1.2 for further details.

In particular, in the case of curves with nonvanishing curvature and torsion one may obtain the following.

Theorem 2. *For any $\varepsilon > 0$, one has $\varepsilon(p) = \frac{1}{3p} - \varepsilon$ in (1) for all $p > 6$.*

This is a very good local smoothing estimate for our purposes, as it provides a gain of $1/(3p)$ derivatives when only an $\varepsilon > 0$ gain was needed. However it has the constraint $p > 6$.

Despite this apparent constraint, one may still use it to interpolate with a not so good but available estimate in the L^2-case, which is a consequence of Van der Corput's lemma.

Lemma 3.

$$\Big(\int_1^2 \|\mathcal{A}_t\mathcal{P}_k f\|_2^2 dt\Big)^{1/2} \leq C2^{-k/3}\|f\|_2$$

Following the notation of (1), this estimate would correspond to $\varepsilon(2) = -\frac{1}{6}$. The result of interpolating Theorem 2 and Lemma 3 is that $\varepsilon(p) > 0$ for all $p > 4$. Thus one obtains

$$\|\sup_{1<t<2} |\mathcal{A}_t\mathcal{P}_k f|\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-k\varepsilon}\|f\|_p,$$

which allows to sum in k and then conclude the proof of Theorem 1. The hard part relies in the proof of the local smoothing estimate in Theorem 2.

Decoupling inequalities were first introduced by Wolff [5] in order to prove such local smoothing estimates in the original setting of the wave equation. In that case, the passage from a decouping inequality to a local smoothing estimate is relatively straightforward; we illustrate this in the next section. However, in the case of averages over the helix, that passage is much harder and technical, although ℓ^p-decoupling still plays a crucial role. The details of that passage are left for the presentation, which involve many decompositions of the multiplier associated to the operator $\mathcal{A}_t \circ \mathcal{P}_k$ until one is in a good shape to apply a decoupling estimate (in this case at very local scales).

Remark 4. *Theorem 1 may be extended for curves of finite type, with exponents p depending on the type of the curve. We refer to [3] for further details.*

1.2 Decoupling and local smoothing

We breafly sketch the connection between local smoothing and decoupling estimates in the classical (original) context of the solution to the wave equation, and also their connection to spherical averages.[1] It is instructive to recall this classical case, as it may be seen as a departure point towards understanding the maximal function along the helix in $\mathbb{R}^3$.

Let Γ_k denote the portion of the forward light cone Γ at scale 2^k,

$$\Gamma_k := \{(\xi, |\xi|) \in \mathbb{R}^3 : 2^k \leq |\xi| \leq 2^{k+1}\}$$

[1]For the purposes of this review and for consistency with the previous section, we only consider the case of $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, although the discussion works as well in higher dimensions.

and $\mathcal{N}_1(\Gamma_k)$ denote the 1-neighbourhood of Γ_k. Let Θ_k denote a decomposition of $\mathcal{N}_1(\Gamma_k)$ into plates θ of dimension $2^{k/2} \times 1 \times 2^k$. Let $\{\chi_\theta\}_{\theta \in \Theta_k}$ denote a partition of unity adapted to the plates θ and let f_θ be defined by $\widehat{f_\theta} = \chi_\theta \widehat{f}$.

After contributions by many authors, Bourgain and Demeter [2] proved the sharp ℓ^2 decoupling conjecture for cones, which in its ℓ^p form and in $\mathbb{R}^3$ reads the following.

Theorem 5 ([2])**.** *Let $f \in \mathscr{S}(\mathbb{R}^3)$ be such that* $\operatorname{supp} \hat{f} \subseteq \mathcal{N}_1(\Gamma_k)$*. Then for any $\varepsilon > 0$*

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim 2^{k(\frac{1}{2}-\frac{2}{p})+k\varepsilon} \Big(\sum_{\theta \in \Theta_k} \|f_\theta\|^p_{L^p(\mathbb{R}^3)} \Big)^{1/p}$$

for $6 \le p \le \infty$.

The *local smoothing conjecture* was posed by Sogge [4] in the context of the wave equation. Let $e^{it\sqrt{-\Delta}}$ denote the wave propagator in $\mathbb{R}^2$, that is,

$$e^{it\sqrt{-\Delta}} f(x) := \int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{it|\xi|} \widehat{f}(\xi) d\xi.$$

It was conjectured that [2]

$$\|e^{it\sqrt{-\Delta}} \mathcal{P}_k f\|_{L^p(\mathbb{R}^n \times [1,2])} \lesssim 2^{-k[-(\frac{1}{2}-\frac{1}{p})+\varepsilon(p)]} \|f\|_p \tag{2}$$

should hold for $\epsilon(p) = \frac{1}{p}$ and $p > 4$. This conjecture is still open, but the range $p > 6$ is now settled via decoupling arguments. Note that (2) with any $\varepsilon(p) > 0$ amounts to a gain in regularity when integrating locally in time with respect to the fixed time estimate, which is known to hold for $\varepsilon(p) = 0$ and $1 < p < \infty$.

In this case, the use of ℓ^p decoupling in Theorem 5 to obtain the local smoothing conjecture for $p > 6$ is relatively straightforward. First, one may replace

$$\|e^{it\sqrt{-\Delta}} \mathcal{P}_k f\|_{L^p(\mathbb{R}^2 \times [1,2])} \le \|\chi(t) e^{it\sqrt{-\Delta}} \mathcal{P}_k f\|_{L^p(\mathbb{R}^2 \times \mathbb{R})},$$

where $\chi(t) \ge 1$ on $[1,2]$ and $\operatorname{supp}(\widehat{\chi}) \subset [-1,1]$. Then the space-time Fourier transform of $T^k f(x,t) := \chi(t) e^{it\sqrt{-\Delta}} \mathcal{P}_k f(x)$ is supported in $\mathcal{N}_1(\Gamma_{2^k})$. Applying Theorem 5, it is clear that (2) holds with $\varepsilon(p) = \frac{1}{p} - \varepsilon$ for $p > 6$ and any $\varepsilon > 0$, as we can put pieces together with the following much more elementary "operator–recoupling" estimate.

[2] The conjecture is not phrased in terms of Littlewood–Paley frequency projections, but it is instructive to present it this way in views of the applications described in Section 1.1.

Proposition 6. *For* $2 \leq p \leq \infty$*,*

$$\Big(\sum_{\theta \in \Theta_k} \|T_\theta^k f\|_{L^p(\mathbb{R}^{2+1})}^p\Big)^{1/p} \lesssim \|f\|_{L^p(\mathbb{R}^2)}.$$

The proof of this estimate follows from interpolating a trivial L^2 estimate consequence of Plancherel's theorem, and a L^∞ estimate, consequence of Young's inequality after a careful analysis of the L^1 norm of the associated kernel to T_θ^k.

Connection to circular averages

The local smoothing estimates for the wave equation may be interpreted as local smoothing estimates for the circular averages, which may be used to prove the boundedness of the circular maximal function in $\mathbb{R}^2$, a celebrated result of Bourgain [1], who originally proved it with a different argument.

The connection of the circular averages with the wave propagator follow from the Fourier inversion formula, after noting that the Fourier transform of the measure of the circle is $\widehat{d\sigma}(\xi) = \sum_{\pm} a_\pm(\xi) e^{i|\xi|}$, where $a_\pm \in S^{-1/2}$. Then, one may realise $f * d\sigma_t$ as the sum of the Fourier integral operators

$$T_\pm f(x,t) := \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it|\xi|} a_\pm(t\xi) \widehat{f}(\xi) d\xi,$$

and the connection with the wave propagator is now apparent.

Remark 7. *Theorem 5 was only known for a partial range of* p *at the time the work in [3] was done; let* p_W *denote the best exponent for which was known. Thus, the numerology of Theorem 1 depended upon that value* p_W*, which was still far from the conjectured value* $p_W = 6$*; in particular, it was proved for* $p > (p_W + 2)/2$*. After Bourgain and Demeter [2] established the sharp* $p_W = 6$*, one has* $p > 4$ *in Theorem 1.*

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2 A short proof of the multilinear Kakeya inequality

after L. Guth [5]
A summary written by Constantin Bilz

Abstract

We discuss Guth's proof of the Bennett–Carbery–Tao multilinear Kakeya inequality that uses multiscale analysis and the Loomis-Whitney inequality. An overview of previous proofs is provided.

2.1 Introduction

In [1], Bennett, Carbery and Tao introduced a multilinear Kakeya conjecture, which can be formulated as follows using the notation of [5].

Let $l_{j,a}$ be affine lines in $\mathbb{R}^n$, where $j = 1, \dots, n$, and where $a = 1, \dots, N_j$. Denote by $T_{j,a}$ the characteristic function of the 1-neighborhood of $l_{j,a}$.

Conjecture 1 (Multilinear Kakeya, cf. [1, Conjecture 1.8 and Remark 1.11])**.** *There is a constant $\delta > 0$ depending only on n, such that the following holds: Suppose that $\frac{n}{n-1} \leq q \leq \infty$. If each line $l_{j,a}$ makes an angle of at most δ with the x_j-axis, then*

$$\left\| \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right) \right\|_{L^{q/n}(\mathbb{R}^n)} \lesssim_q \prod_{j=1}^{n} N_j. \tag{1}$$

Note that the conjecture is equivalent via log-convexity of L^p norms to the endpoint case $q = \frac{n}{n-1}$, because the case $q = \infty$ is trivial.

Bennett, Carbery and Tao showed that the above conjecture is essentially equivalent to a multilinearized version of the Fourier restriction conjecture. They proved Conjecture 1 up to the endpoint case $q = \frac{n}{n-1}$, for which they gave an estimate with an additional factor. From this they derived a near-optimal multilinear restriction estimate. Their method of proof for the multilinear Kakeya conjecture involved replacing the characteristic functions $T_{j,a}$ by Gaussians and using properties of heat flow along with multiscale analysis.

Guth [4] gave a proof of the full endpoint case that was motivated by Dvir's polynomial method [3]. This proof makes use of techniques from algebraic topology, including the Lusternik-Schnirelmann vanishing lemma about cup products. Carbery and Valdimarsson [2] subsequently simplified this approach: The only tool from algebraic topology that they use is the Borsuk–Ulam theorem, which permits purely analytic proofs. There will be lectures on both [3] and [2].

The aim of this summary is to discuss another more recent and independent proof due to Guth [5] of a slightly weaker version of the endpoint case. Namely, we will show the following

Theorem 2 ([5, Theorem 1])**.** *Let $l_{j,a}$ and $T_{j,a}$ be as above. Suppose that each $l_{j,a}$ makes an angle of at most $(10n)^{-1}$ with the x_j-axis. Then for any cube Q_S of side length $S \geq 1$ and any $\epsilon > 0$, the following inequality holds:*

$$\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim_\epsilon S^\epsilon \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}. \tag{2}$$

The proof is similar to [1] in that it also uses multiscale analysis. Furthermore, Theorem 2 still implies the aforementioned Bennett–Carbery–Tao restriction estimate.

In Subsection 2.2, we discuss Guth's proof of Theorem 2. Some generalizations of the theorem are given in Subsection 2.3.

2.2 Guth's proof of the multilinear Kakeya inequality

2.2.1 Reduction to the case of small angles depending on ϵ

As a first step, it was observed in [1] that it suffices to prove Theorem 2 when the angle $(10n)^{-1}$ is replaced by a small angle δ that may depend on ϵ:

Theorem 3 ([5, Theorem 2])**.** *For every $\epsilon > 0$, there is some $\delta > 0$ such that the following holds. Let $l_{j,a}$ and $T_{j,a}$ be as before. Suppose that each $l_{j,a}$ makes an angle of at most δ with the x_j-axis. Then for any cube Q_S of side length $S \geq 1$ and any $\epsilon > 0$, the following inequality holds:*

$$\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim_\epsilon S^\epsilon \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}.$$

It can be shown that Theorem 3 implies Theorem 2 by using the following strategy: Given $\epsilon > 0$, we choose $\delta > 0$ as in Theorem 3. By the hypotheses of Theorem 2, every $l_{j,a}$ intersects the spherical cap S_j of radius $(10n)^{-1}$ around the j-th standard unit vector e_j. It is possible to subdivide S_j into at most $C_n \delta^{1-n} \lesssim_\epsilon 1$ subsets $S_{j,\beta}$, each of which is contained in a spherical cap of radius $\delta/10$. Then, the left-hand side of (2) is controlled by

$$\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim_\epsilon \max_{\beta_1,\ldots,\beta_n} \int_{Q_S} \prod_{j=1}^{n} \left(\sum_{l_{j,a} \cap S_{j,\beta_j} \neq \emptyset} T_{j,a} \right)^{\frac{1}{n-1}}. \tag{3}$$

One can check that, after a suitable linear transformation whose norm and determinant are close to 1, Theorem 3 applies to the right-hand side of equation (3).

In order to prove Theorem 3, Guth conducts an induction on scales argument (see 2.2.3 below). The induction step involves replacing neighborhoods of small segments of $l_{j,a}$ by axis parallel tubes. Hence, he first considers the axis parallel case of Theorem 3 (see 2.2.2 below).

2.2.2 The axis parallel case

Let $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection that forgets the j-th coordinate. The case of axis parallel lines $l_{j,a} = \pi_j^{-1}(y_{j,a})$ with $y_{j,a} \in \mathbb{R}^{n-1}$ in Theorem 2 follows immediately from the classical Loomis–Whitney inequality that goes back to [6]:

Theorem 4 (Loomis–Whitney)**.** *The following inequality holds for all measurable functions $f_j : \mathbb{R}^{n-1} \to \mathbb{R}$.*

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j(\pi_j(x))^{\frac{1}{n-1}} \leq \prod_{j=1}^{n} \|f_j\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}} \tag{4}$$

We now choose $f_j = \sum_a \chi_{B(y_{j,a},1)}$ so that $\sum_a T_{j,a}(x) = f_j(\pi_j(x))$ and further $\|f_j\|_{L^1(\mathbb{R}^{n-1})} = \omega_{n-1} N_j \lesssim_\emptyset N_j$. The axis parallel case follows:

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \overset{(4)}{\leq} \prod_{j=1}^{n} \|f_j\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}} \lesssim \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}.$$

2.2.3 Multiscale analysis

We perform an induction on scales S. Denote by $T_{j,a,W}$ the characteristic function of the W-neighborhood of $l_{j,a}$ and write

$$f_{j,W} = \sum_{a=1}^{N_j} T_{j,a,W}.$$

The induction step will be given by the following

Lemma 5 ([5, Lemma 4]). *Let $\delta > 0$ and suppose that $l_{j,a}$ makes an angle of at most δ with the x_j-axis. If Q_S is a cube of side length $S \geq \delta^{-1}W$, then*

$$\int_{Q_S} \prod_{j=1}^{n} f_{j,W}^{\frac{1}{n-1}} \leq C_n \delta^n \int_{Q_S} \prod_{j=1}^{n} f_{j,\delta^{-1}W}^{\frac{1}{n-1}}. \tag{5}$$

The idea is as follows: We subdivide Q_S into small subcubes Q of side length around $(10n\delta)^{-1}W$. Then it suffices to show (5) for Q instead of Q_S. But inside Q each tube $T_{j,a}$ looks fairly axis parallel, in particular its intersection with Q is contained in axis parallel tubes of slightly larger radius. An application of the Loomis–Whitney inequality finishes the proof.

If the side length in Theorem 3 is $S = \delta^{-M}$, we can now apply Lemma 5 multiple times to get

$$\begin{aligned} \int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} &= \int_{Q_S} \prod_{j=1}^{n} f_{j,1}^{\frac{1}{n-1}} \leq C_n^M S^{-n} \int_{Q_S} \prod_{j=1}^{n} f_{j,\delta^{-M}}^{\frac{1}{n-1}} \\ &\leq C_n^M \prod_{j=1}^{n} N_j^{\frac{1}{n-1}} = S^{\frac{\log C_n}{\log \delta^{-1}}} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}. \end{aligned}$$

When we choose $\delta > 0$ small enough, this shows Theorem 3 in the case $S = \delta^{-M}$. The general case follows by a covering argument.

2.3 Generalizations

There are some mild generalizations of Theorem 2 given in [5]. First, we may introduce weights:

Corollary 6 ([5, Corollary 5]). *Let $l_{j,a}$, $T_{j,a}$, Q_S, and $\epsilon > 0$ be as in Theorem 2. Suppose that $w_{j,a} \geq 0$ are numbers. Then the following inequality holds:*

$$\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} w_{j,a} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim_\epsilon S^\epsilon \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} w_{j,a} \right)^{\frac{1}{n-1}}.$$

The proof is straightforward: Integer weights are handled by including lines multiple times, rational weights reduce to the integer case by scaling, and the corollary also holds for real weights by density.

Secondly, the assumption in Theorem 2 that $l_{j,a}$ should make an angle of at most $(10n)^{-1}$ with the x_j-axis can be relaxed with essentially the same proof:

Corollary 7 ([5, Corollary 6]). *Suppose that $l_{j,a}$ are affine lines in $\mathbb{R}^n$ and that the direction of $l_{j,a}$ is $v_{j,a} \in S^{n-1}$. Suppose that for any $1 \leq a_j \leq N_j$,*

$$|v_{1,a_1} \wedge \ldots \wedge v_{n,a_n}| \geq \nu.$$

Let $T_{j,a}$, Q_S, and $\epsilon > 0$ be as in Theorem 2. Then the following holds:

$$\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim_\epsilon \mathrm{Poly}(\nu^{-1}) S^\epsilon \prod_{j=1}^{n} N_j^{\frac{1}{n-1}},$$

where $\mathrm{Poly}(\nu^{-1})$ is a polynomial in ν^{-1} depending only on n.

Lastly, one can see that the argument given in Subsection 2.2 generalizes to Lipschitz curves in place of affine lines. Namely, let $g_{j,a} : \mathbb{R} \to \mathbb{R}^{n-1}$ be a Lipschitz function of Lipschitz constant at most δ. Let $\gamma_{j,a}$ be the graph of $g_{j,a}$ which is defined by $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) = g_{j,a}(x_j)$. Denote by $T_{j,a}$ the characteristic function of the 1-neighborhood of $\gamma_{j,a}$. Then the following generalization of Theorem 3 holds.

Corollary 8 ([5, Theorem 7]). *For every $\epsilon > 0$, there is a $\delta > 0$ so that the following holds. Let $T_{j,a}$ be the characteristic function of the 1-neighborhood of the Lipschitz curve $\gamma_{j,a}$ as described above. Let Q_S be any cube of side length $S \geq 1$. Then we have*

$$\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim_\epsilon S^\epsilon \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}.$$

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3 Behaviour of the Schrödinger evolution for initial data near $H^{\frac{1}{4}}$

after L. Carleson [1] and after B. Dahlberg, and C. Kenig [2]
A summary written by Gianmarco Brocchi

Abstract

We study convergence of solutions of the Schrödinger equation on $\mathbb{R}$ as $t \to 0$. For initial data in the Sobolev space $H^s(\mathbb{R})$, Carleson showed that we have almost everywhere convergence when $s \geq \frac{1}{4}$. Dahlberg and Kenig proved that this result is also sharp.

3.1 Introduction

We consider the initial value problem for the Schrödinger equation in $\mathbb{R}$:

$$\begin{cases} i\partial_t \Psi(x,t) + \Delta\Psi(x,t) = 0 \\ \Psi(x,0) = f(x) \end{cases}$$

The solution to this problem is given by

$$e^{it\Delta} f(x) = \int_{\mathbb{R}} e^{ix\xi + it\xi^2} \hat{f}(\xi) \frac{d\xi}{2\pi}.$$

The operator $e^{it\Delta}$ is bounded on L^2, so it is continuous; in particular $\lim_{t\to 0} e^{it\Delta} f = f$ in L^2, or equivalently

$$\lim_{t\to 0} \| e^{it\Delta} f - f \|_{L^2} = 0.$$

But what can we say about the pointwise limit of $e^{it\Delta} f(x)$ as $t \to 0$? For which class of initial data does it hold that

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x) \quad \text{for almost every } x \in \mathbb{R}?$$

In the 1980's Lennart Carleson gave an answer when the initial data f is compactly supported and α-Hölder continuous with $\alpha > \frac{1}{4}$. Here we state and prove this result for f belonging to the Sobolev space $H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$.

Theorem 1 (Carleson)**.** *If $f \in H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$ then*

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x) \quad \text{for almost every } x \in \mathbb{R}.$$

The key of the proof is the bound of the maximal Schrödinger operator for some $p > 1$

$$\left\| \sup_{t>0} |e^{it\Delta} f| \right\|_{L^p} \leq C\|f\|_{H^s(\mathbb{R})}.$$

One year later, Dahlberg and Kenig proved that the above result is *sharp*. They proved the following

Theorem 2 (Dahlberg & Kenig)**.** *Let $s \in \left[0, \frac{1}{4}\right)$. There exists a function $f \in H^s(\mathbb{R})$ and a set E with positive measure such that, for every $x \in E$*

$$\limsup_{t\to 0} |e^{it\Delta} f(x)| = +\infty.$$

3.2 Positive result

In order to prove Theorem 1, we will use an a priori estimate for the maximal operator $\sup_{t>0} |e^{it\Delta} f|$.

Proposition 3 (A priori estimate)**.** *Let $f \in \mathcal{S}(\mathbb{R})$ Schwartz function. Then there exists a constant $C > 0$ such that*

$$\left\| \sup_{t>0} |e^{it\Delta} f| \right\|_{L^4(\mathbb{R})} \leq C\|f\|_{H^{\frac{1}{4}}(\mathbb{R})}. \tag{1}$$

Proof. First we aim to prove a *local* estimate, namely

$$\left\| \sup_{t>0} |e^{it\Delta} f| \right\|_{L^4([-R,R])} \leq C\|f\|_{H^{\frac{1}{4}}([-R,R])}$$

where the constant C is independent of R. The estimate (1) will follow by taking the limit as $R \to \infty$. We split the proof in steps.

Step 1 We would like to get rid of the supremum. Fix $x \in \mathbb{R}$. There exists a time $t(x) > 0$ such that

$$|e^{it(x)\Delta} f(x)| \geq \frac{1}{2} \sup_{t>0} |e^{it\Delta} f(x)|.$$

Step 2 Then we use *duality*. There exists a function $w \in L^{\frac{4}{3}} \cong (L^4)'$, with $\|w\|_{\frac{4}{3}} = 1$, with $\mathrm{supp}(w) \subset [-R, R]$, such that

$$\|e^{it\Delta} f\|_{L^4([-R,R])} = \int_{\mathbb{R}} e^{it(x)\Delta} f(x) w(x) dx.$$

Step 3 Expand the integral, use Fubini[3] and Cauchy-Schwarz.

$$\begin{aligned}
\int_{\mathbb{R}} e^{it(x)\Delta} f(x) w(x) &= \iint_{\mathbb{R}^2} \hat{f}(\xi) e^{2\pi i (x\xi - 2\pi t(x)\xi^2)} d\xi \, w(x) \, dx \\
&= \int_{\mathbb{R}} \hat{f}(\xi) |\xi|^{\frac{1}{4}} \int_{\mathbb{R}} e^{2\pi i (x\xi - 2\pi t(x)\xi^2)} \frac{w(x)}{|\xi|^{\frac{1}{4}}} dx \, d\xi \\
&\leq \left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\xi|^{\frac{1}{2}} d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{2\pi i (x\xi - 2\pi t(x)\xi^2)} \frac{w(x)}{|\xi|^{\frac{1}{4}}} dx \right|^2 d\xi \right)^{\frac{1}{2}} = \mathrm{I} \cdot \mathrm{II}.
\end{aligned}$$

Step 4 We bound the two factors separately.

$$\mathrm{I} \leq \left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^{\frac{1}{4}} d\xi \right)^{\frac{1}{2}} = \|f\|_{H^{\frac{1}{4}}(\mathbb{R})}.$$

For II, a careful estimate of the oscillatory integral inside leads to

$$\mathrm{II}^2 \leq C \int_{\mathbb{R}^2} \frac{w(x)w(y)}{|x-y|^{\frac{1}{2}}} dx dy.$$

Use Hölder and Hardy-Littlewood-Sobolev inequalities to conclude

$$\mathrm{II}^2 \leq C \|w\|_{L^{\frac{4}{3}}} \left\| \int_{\mathbb{R}} \frac{w(y)}{|x-y|^{\frac{1}{2}}} dy \right\|_{L^4} \leq C \|w\|^2_{L^{\frac{4}{3}}(\mathbb{R})}.$$

To sum up:

$$\left\| \sup_{t>0} |e^{it\Delta} f| \right\|_{L^4([-R,R])} \leq 2 \left\| e^{it(\cdot)\Delta} f \right\|_{L^4([-R,R])} \leq C \|w\|_{L^{\frac{4}{3}}(\mathbb{R})} \|f\|_{H^{\frac{1}{4}}(\mathbb{R})}.$$

[3] The function $w \in L^{\frac{4}{3}}([-R, R]) \subset L^1([-R, R])$. In particular w is integrable and we can use Fubini.

By taking the limit as $R \to \infty$, we conclude. $\square$

Idea of the proof of Theorem 1. By density of Schwartz functions $\mathcal{S}(\mathbb{R})$ in the Sobolev space $H^{\frac{1}{4}}(\mathbb{R})$, the bound (1) holds true for functions in $H^{\frac{1}{4}}(\mathbb{R})$, and also in any $H^s(\mathbb{R})$ for $s \geq \frac{1}{4}$, since they are all contained in $H^{\frac{1}{4}}$.

Thus the maximal function $\sup_{t>0}|e^{it\Delta}f|$ is bounded from $H^s(\mathbb{R})$ to $L^4(\mathbb{R})$ for $s \geq \frac{1}{4}$. This bound implies pointwise almost everywhere convergence for the family of operators $\{e^{it\Delta}\}_{t\in[0,1]}$, in particular we have

$$\lim_{t\to t_0} e^{it\Delta}f(x) = e^{it_0\Delta}f(x) \qquad \text{for almost every } x \in \mathbb{R},$$

and when $t_0 = 0$, when we get back $f(x)$. $\square$

3.3 Negative result

In his work, Carleson already proved that the convergence to $f \in H^s(\mathbb{R})$ might fail for $s < \frac{1}{8}$. For the proof of the Theorem 2 Björn Dahlberg and Carlos Kenig exploited a theorem by Nikišin, published the same year in [3]. We recall first some notations from [4].

Let (X, μ) and (Y, ν) two σ-finite measure spaces. Let $L^0(Y, \nu)$ the space of a.e. finite real-values measurable functions on Y endowed with the metric of the convergence in measure.

We say that $T\colon L^p(X, \mu) \to L^0(Y, \nu)$ is *linearizable*[4] if for each $f_0 \in L^p(X)$ there exist a *linear* operator H_{f_0} such that

1. $|H_{f_0}f_0| = |Tf_0|$ ν- a.e. and
2. $|H_{f_0}f| \leq |Tf|$ ν- a.e. for all $f \in L^p(X)$.

Remark 4. *For an operator T being linearizable means that there is a family $\{H_{f_0}\}_{f_0\in L^p(X)}$ of linear operators such that T majorizes each one of them and coincides with in absolute value with H_{f_0} in f_0.*

Example 5. *Given a sequence of operator $\{T_n\}_n\colon L^p(X, \mu) \to L^0(Y, \nu)$. The truncated maximal operator of the family T_N^* is linearizable.*

We are ready to state the theorem.

[4] or *hyperlinear* in Nikišin's terminology

Theorem 6 (Nikišin)**.** *Let $1 \leq p < \infty$, and let $T\colon L^p(X,\mu) \to L^0(Y,\nu)$ linearizable and continuous in measure at 0. Then for every $\epsilon > 0$ there exists a set $E_\epsilon \subset Y$ with $|E_\epsilon| \geq |Y| - \epsilon$ such that*

$$|\{y \in E_\epsilon : Tf(y) > \lambda\}| \leq C_\epsilon \left(\frac{\|f\|_{L^p}}{\lambda}\right)^q,$$

for all $\lambda > 0$, $f \in L^p(X)$, and $q = \min\{p, 2\}$.

To show that pointwise convergence a.e. fails, it is enough to show that it fails on an finite interval $I \subset \mathbb{R}$. Aiming to a contradiction, assume that we have convergence a.e. for every $f \in H^s(\mathbb{R})$ with $s < \frac{1}{4}$, then

$$\limsup_{t\to 0} |e^{it\Delta} f(x)| < +\infty \qquad \text{for almost every } x \in I.$$

Consider an even function $f \in C_c^\infty(\mathbb{R})$ supported in $I = [-1,1]$.
For $0 < t < 1$ we rescale and modulate f

$$f_t(x) = f\left(\frac{x}{t}\right) e^{2ix/t^2},$$

such that its Sobolev norm is

$$\|f_t\|_{H^s}^2 \leq Ct^{1-4s}.$$

Then let $t(x) = t^2 x$ for $x > 0$. Moreover, we have that

$$|e^{it(x)\Delta} f_t| = \left| \frac{1}{\sqrt{x}} \int_{\mathbb{R}} f(y) e^{iy^2/x} dy \right| =: g(x).$$

Notice that g is a continuous function independent of t.
We can view $e^{it\Delta}$ as an operator acting on the Fourier side and mapping to measurable functions:

$$\begin{aligned} e^{it\Delta}\colon L^2(\mathbb{R}, \langle\xi\rangle^s\, d\xi) &\to L^0(I) \\ \hat{f} &\mapsto \mathcal{F}^{-1}(e^{it\xi^2}\hat{f}) \end{aligned}$$

By our previous assumption, this is a bounded operator from a (weighted) L^2 to measurable functions on an interval. We apply Theorem 6 with $p = 2$, $X = \mathbb{R}$ with the measure $\mu = (1 + |\xi|^2)^{s/2}\, d\xi$, so that $L^2(\mathbb{R}, \mu) = H^s(\mathbb{R})$, and $Tf = \sup_{0<t<1} |e^{it\Delta} f|$.

Then there exists a closed set $E \subset [-1,1]$ with positive Lebesgue measure[5], and $C > 0$, such that

$$\left| \{ y \in E \,:\, \sup_{0<t<1} |e^{it\Delta} f(y)| > \lambda \} \right| \leq C \left(\frac{\|\hat{f}\|_{L^2(\mathbb{R},\langle\xi\rangle^s)}}{\lambda} \right)^2 \qquad \text{for all } \lambda > 0. \quad (2)$$

The restriction $g \restriction E$ is continuous. Let $\lambda_0 := \min_{x\in E} g(x)$. Using (2) we have that

$$\begin{aligned} |E| &= |\{x \in E \,:\, g(x) > \lambda_0\}| = |\{x \in E \,:\, |e^{it(x)\Delta} f_t| > \lambda_0\}| \\ &\leq |\{x \in E \,:\, \sup_{t\in[0,1]} |e^{it(x)\Delta} f_t| > \lambda_0\}| \leq \frac{C}{\lambda_0^2} \|f_t\|^2_{H^s(\mathbb{R})} \lesssim t^{1-4s}. \end{aligned}$$

This is a contradiction as long $s < \frac{1}{4}$, since one has

$$0 < |E| \lesssim t^{1-4s} \to 0 \qquad \text{as } t \to 0.$$

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[5] actually, the set E is an arbitrarily large subset of $[-1,1]$

4 Efficient congruencing

A brief account of Wooley's method, written by Sam Chow

Abstract

The purpose of this talk is to provide a brief but substantial description of Wooley's efficient congruencing method. We focus on the cubic case of Vinogradov's mean value theorem, first solved by Wooley in 2014. The method generalises to arbitrary degree, as demonstrated in a very recent preprint by Wooley. The efficient congruencing and ℓ^2 decoupling approaches are one and the same, the former being p-adic and the latter being real. This exposition is based on Heath-Brown's simplified account of Wooley's proof of the cubic case.

4.1 Introduction

Efficient congruencing (2010–) was introduced by Wooley [3] to make substantial progress on Vinogradov's mean value theorem, with powerful applications to Waring's problem and other problems in analytic number theory. In this talk, we shall describe the method in the context of the cubic case of Vinogradov's mean value theorem [4]. The proof has since been generalised to arbitrary degree [5]. This generalisation is not straightforward, but the cubic case contains the key ideas. We follow a simplified account of Wooley's proof, as given by Heath-Brown [1]. For the history of the method and its arithmetic consequences, see Pierce's Bourbaki notes [2].

4.2 The cubic case of VMVT

Define the exponential sum

$$f(\boldsymbol{\alpha}) = \sum_{x \le X} e(\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3).$$

By orthogonality, the quantity

$$J(X) = \int_{\mathbb{T}^3} |f(\boldsymbol{\alpha})|^6 \mathrm{d}\boldsymbol{\alpha}$$

counts solutions $(x_1, x_2, x_3, y_1, y_2, y_3) \in \{1, 2, \ldots, X\}^6$ to the system

$$x_1^j + x_2^j + x_3^j = y_1^j + y_2^j + y_3^j \qquad (j = 1, 2, 3). \tag{1}$$

There are at least X^3 diagonal solutions, and our goal is to show that

$$J(X) \ll_\varepsilon X^{3+\varepsilon} \qquad (\varepsilon > 0). \tag{2}$$

4.2.1 Congruence conditions

Let $p \geq 5$ be a prime; it will depend on X, but is fixed throughout the argument. Defining the auxiliary exponential sum

$$f_a(\boldsymbol{\alpha}; \xi) = \sum_{\substack{x \leq X \\ x \equiv \xi \bmod p^a}} e(\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3),$$

observe that the quantity

$$I_m(X; \xi, \eta; a, b) = \int_{\mathbb{T}^3} |f_a(\boldsymbol{\alpha}; \xi)|^{2m} |f_b(\boldsymbol{\alpha}; \eta)|^{6-2m} \mathrm{d}\boldsymbol{\alpha} \qquad (m = 0, 1, 2)$$

counts solutions to (1) for which

$$x_i \equiv y_i \equiv \xi \bmod p^a \qquad (1 \leq i \leq m)$$

and

$$x_i \equiv y_i \equiv \eta \bmod p^b \qquad (m+1 \leq i \leq 3).$$

Note that

$$I_0(X; \xi, \eta; a, b) = \int_{\mathbb{T}^3} |f_b(\boldsymbol{\alpha}, \eta)|^6 \mathrm{d}\boldsymbol{\alpha}$$

is independent of ξ and a.

It is convenient to consider maxima over ξ and η, introducing the quantities

$$I_0(X; a, b) = \max_{\eta \bmod p^b} I_0(X; \xi, \eta; a, b)$$

and

$$I_m(X; a, b) = \max_{\xi \not\equiv \eta \bmod p} I_m(X; \xi, \eta; a, b) \qquad (m = 1, 2).$$

We can upper bound $J(X)$ in terms of $I_2(X; 1, 1)$.

Lemma 1. *If $p \leq X$ then*

$$J(X) \ll pJ(2X/p) + p^{12} I_2(X; 1, 1).$$

Proof. By considering whether or not all of the variables are congruent modulo p, we obtain, for some $\xi \not\equiv \eta \bmod p$, the bound

$$J(X) \le pJ(2X/p) + \binom{12}{2} p(p-1) \int_{\mathbb{T}^3} |f_1(\boldsymbol{\alpha};\xi) f_1(\boldsymbol{\alpha};\eta) f(\boldsymbol{\alpha})^{10}| \mathrm{d}\boldsymbol{\alpha};$$

for the first term we have changed variables by $x_i = px_i' - \xi$ and used translation–dilation invariance. Hölder gives

$$\begin{aligned}\int_{\mathbb{T}^3} &|f_1(\boldsymbol{\alpha};\xi) f_1(\boldsymbol{\alpha};\eta) f(\boldsymbol{\alpha})^{10}| \mathrm{d}\boldsymbol{\alpha} \\ &\le \left(\int_{\mathbb{T}^3} |f_1(\boldsymbol{\alpha};\xi)^4 f_1(\boldsymbol{\alpha};\eta)^8| \mathrm{d}\boldsymbol{\alpha} \right)^{1/12} \left(\int_{\mathbb{T}^3} |f_1(\boldsymbol{\alpha};\xi)^8 f_1(\boldsymbol{\alpha};\eta)^4| \mathrm{d}\boldsymbol{\alpha} \right)^{1/12} \\ &\qquad \left(\int_{\mathbb{T}^3} |f(\boldsymbol{\alpha})|^{12} \mathrm{d}\boldsymbol{\alpha} \right)^{5/6},\end{aligned}$$

and so

$$J(X) \ll pJ(2X/p) + p^2 I_2(X;1,1)^{1/6} J(X)^{5/6},$$

from which we deduce the asserted bound. □

We can compare I_2 to I_1 using Hölder's inequality, obtaining

$$I_2(X;a,b) \le I_2(X;b,a)^{1/3} I_1(X;a,b)^{2/3}.$$

4.2.2 Recursive estimates

The power of the method comes from starting with a system of congruences modulo powers of p and inferring congruences modulo higher powers of p.

Lemma 2. *If* $1 \le a \le 3b$ *then*

$$I_1(X;a,b) \le p^{3b-a} I_1(X;3b,b).$$

Proof. The quantity $I_1(X;\xi,\eta;a,b)$ counts solutions to (1) with

$$x_1 = \xi + p^a x_1', \qquad y_1 = \xi + p^a y_1'$$

and

$$x_i = \eta + p^b x_i', \qquad y_i = \eta + p^b y_i' \qquad (i = 2,3).$$

Putting $\nu = \xi - \eta$,

$$z_1 = \nu + p^a x_1', \qquad w_1 = \nu + p^a y_1'$$

and

$$z_i = p^b x_i', \qquad w_i = p^b y_i' \qquad (i = 2, 3),$$

we find by translation–dilation invariance that (1) holds with $(\mathbf{x}, \mathbf{y})$ replaced by $(\mathbf{z}, \mathbf{w})$. In particular, the cubic equation implies that

$$(\nu + p^a x_1')^3 \equiv (\nu + p^a y_1')^3 \bmod p^{3b}.$$

As $p \nmid \nu$ and $3 \nmid \varphi(p^{3b})$, this forces $\nu + p^a x_1' \equiv \nu + p^a y_1' \bmod p^{3b}$, and so $z_1 \equiv w_1 \bmod p^{3b-a}$. Now $x_1 \equiv y_1 \equiv \xi' \bmod p^{3b}$ for one of p^{3b-a} possible values of ξ', so

$$I_1(X; \xi, \eta; a, b) \leq p^{3b-a} I_1(X; 3b, b).$$

$\square$

We can similarly bound $I_2(X; a, b)$ recursively — this does present additional difficulties of a geometric nature — and ultimately show that if $1 \leq a \leq b$ and $p^b \leq X$ then

$$I_2(X; a, b) \leq 2bp^{-10a/3+14b/3} I_2(X; b, 2b-a)^{1/3} I_2(X; b, 3b)^{1/6} J(2X/p^b)^{1/2}. \quad (3)$$

4.2.3 Proof by contradiction

Define the real number $\Delta \geq 0$ by

$$\Delta = \inf\{\delta \in \mathbb{R} : J(X) \ll_\delta X^{6+\delta} \qquad (X \geq 1)\},$$

and assume for a contradiction that $\Delta > 0$. Using (3), one can show by induction on $n \in \mathbb{Z}_{\geq 0}$ that if

$$1 \leq a \leq b, \qquad p^{3^n b} \leq X$$

then

$$I_2(X; a, b) \ll_{\varepsilon,n,a,b} X^{6+\Delta+\varepsilon} p^{-2a-4b} p^{(3-n\Delta/6)(3b-a)}.$$

Applying this with $a = b = 1$, and with p in the range

$$\frac{1}{2} X^{1/3^n} \leq p \leq X^{1/3^n},$$

we obtain from Lemma 1 the inequality

$$J(X) \ll pJ(2X/p) + p^{12}I_2(X;1,1) \ll_{\varepsilon,n} p(X/p)^{6+\Delta+\varepsilon} + X^{6+\Delta+\varepsilon}p^{12-n\Delta/3}.$$

Choosing $n \geq 39/\Delta$, and X large in terms of n, gives

$$J(X) \ll_{\varepsilon,n} X^{6+\Delta+\varepsilon}p^{-1} \ll_{\varepsilon,n} X^{6+\Delta-3^{-n}+\varepsilon}.$$

This contradiction shows that $\Delta = 0$, which establishes (1).

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5 Polynomial method in number theory

after L. Guth [1]
A summary written by Dong Dong

Abstract

We show how polynomial method can be used to study Diophantine equation (more narrowly, rational approximation) problems. The idea of Thue's work is explained.

5.1 Introduction

In 1909, Thue [8] made a breakthrough in the study of a category of Diophantine equitations (which is afterwards called Thue equation). More precisely, he proved

Theorem 1. *Let $P \in \mathbb{Z}[x, y]$ be a homogeneous irreducible polynomial of degree at least 3. Then for any $A \in \mathbb{Z}$, $P(x, y) = A$ has only finitely many integer solutions.*

The key ingredient of the proof of Theorem 1 is the following theorem about rational approximation of algebraic numbers:

Theorem 2. *Let β be an algebraic number of degree $d \geq 3$. Then for any $s > \frac{d}{2} + 1$, there are only finitely many rationals $\frac{p}{q}$ that satisfy*

$$\left|\beta - \frac{p}{q}\right| \leq \frac{1}{q^s}$$

Let's first see how to derive Theorem 1 from Theorem 2. We will focus on the idea of the proof and thus will consider only a special $P(x, y) = y^d - 2x^d$ for some integer $d \geq 3$. The general case can be proved in a similar way. So our goal is to prove that there are only finitely many integer solutions of the equation $y^d - 2x^d = 1$. Obviously when $x = 0$, y could be 1 or -1. Now without loss of generality assume $x > 0$ (the case $x < 0$ can be handled in the same way), and therefore y is positive as well. Dividing both sides by x^d, we have

$$\left(\frac{y}{x}\right)^d - 2 = \frac{1}{x^d}.$$

Next, take out the factor $\frac{y}{x} - 2^{1/d}$ from the left-hand-side of the above equation, and we obtain that

$$\left|2^{1/d} - \left(\frac{y}{x}\right)\right| \lesssim_d \frac{1}{x^d}.$$

Now apply Theorem 2 for $\beta = 2^{1/d}$, which is algebraic of degree d. So there exists some $s < d$ (say $s = d - \frac{1}{1000}$) such that there are only finitely many rational numbers $\frac{p}{q}$ satisfying

$$\left|2^{1/d} - \frac{p}{q}\right| \leq \frac{1}{q^s}.$$

In other words, with finitely many exceptions,

$$\frac{1}{q^s} \leq \left|2^{1/d} - \frac{p}{q}\right|.$$

Therefore, for positive integers x, y satisfying $y^d - 2x^d = 1$,

$$\frac{1}{x^s} \leq \left|2^{1/d} - \left(\frac{y}{x}\right)\right| \lesssim_d \frac{1}{x^d},$$

which implies $|x| \lesssim_d 1$. Clearly for each x, there are at most d y's so that $y^d - 2x^d = 1$. This finishes the proof of (a special case of) Theorem 1, assuming the validity of Theorem 2.

We remark that before Thue's proof of Theorem 2, the best result was due to Liouville (1844), who showed the same conclusion holds for $s > d$. Liouville's Theorem is just insufficient to prove Theorem 1, where $s < d$ is needed. After Thue's theorem, a few mathematicians continued to improve the lower bound for s (e.g. Siegel (1921), Dyson (1947), Gelfond (1947)), and finally Roth (1955, [3]) closed the project, showing Theorem 2 holds for $s > 2$. Some good references on this topic are [2, 4, 5, 6, 7].

5.2 Outline of the proof of Theorem 2

5.2.1 Three steps of polynomial methods

Let's recall the three main steps of polynomial methods: first we find a polynomial with controlled degree (using parameter counting); next prove that the polynomial vanishes at many points (vanishing lemma); finally, obtain a contradiction by showing that the polynomial cannot vanish that much.

Now we give descriptions of the above three steps in the proof of Theorem 2. Suppose that β has two good rational approximations r_1 and r_2 with large denominators.

(1) Find a none-zero polynomial $P \in \mathbb{Z}[x_1, x_2]$ with controlled degree and coefficients that vanishes to high order at (β, β). (Use parameter counting, more precisely, Siegel's lemma in this setting).

(2) Because r_1 and r_2 are very good approximations of β, the polynomial P must also vanish to high order at (r_1, r_2). (use Taylor's theorem)

(3) The polynomial P vanishes too much at (r_1, r_2), and so it must be zero. (use a variant of Gauss's lemma)

The contradiction implies that β can have at most one good rational approximation with large denominator. In other words, the other approximations have bounded denominators and therefore there must be finitely many of them.

5.2.2 Lemmas needed

We summarize lemmas needed to prove Theorem 2. The proofs are long and thus omitted.

The first two steps in the polynomial methods are simultaneously achieved in the following lemma.

Lemma 3. *Let β be algebraic of degree $d \geq 3$. Suppose $s > \frac{d}{2} + 1$. There are positive constants $c(\beta, s) < 1$, $C(\beta, s)$ and $C(\beta)$ such that the following holds.*

Suppose that $r_1 = \frac{p_1}{q_1}$ and $r_2 = \frac{p_2}{q_2}$ are good rational approximations of β, i.e.

$$|\beta - r_i| \leq q_i^{-s}, \quad i = 1, 2.$$

We assume that $q_1^m \leq q_2 < q_1^{m+1}$ for some large m. Assume q_1 is sufficiently large. Then there exists a polynomial $P \in \mathbb{Z}[x_1, x_2]$ of the form $P(x_1, x_2) = P_1(x_1)x_2 + P_0(x_1)$ so that

(1) $\partial_1^j P(r_1, r_2) = 0$ for $0 \leq j < c(\beta, s)m$;

(2) $|P| \leq C(\beta, s)^m$;

(3) Deg $P \leq C(\beta)m$.

The next lemma is the key in the final step of the polynomial methods.

Lemma 4. *Let $P(x_1, x_2) = P_1(x_1)x_2 + P_0(x_1) \in \mathbb{Z}[x_1, x_2]$. Suppose that for some $l \geq 2$ and rational numbers $r_1 = \frac{p_1}{q_1}, r_2 = \frac{p_2}{q_2}$, $\partial_1^j P(r_1, r_2) = 0$ for $j = 0, \ldots, l-1$. Then*

$$|P| \geq \min\{(2DegP)^{-1} q_1^{\frac{l-1}{2}}, q_2\}.$$

5.2.3 Detailed proof

Having all the lemmas needed, we are ready to prove Theorem 1. It turns out that the main argument in the proof of Theorem 2, like many other examples of polynomial methods, is quite short.

Fix β algebraic of degree $d \geq 3$ and $s > \frac{d}{2} + 1$. Our goal is to show that the inequality

$$\left|\beta - \frac{p}{q}\right| \leq \frac{1}{q^s}$$

holds for only finitely many rational numbers $\frac{p}{q}$, or equivalently, for finitely many q's.

Assume otherwise. Then q can be arbitrarily large. Let $r_1 = \frac{p_1}{q_1}$ and $r_2 = \frac{p_2}{q_2}$ be two rational solutions of the above inequality, with the property that q_1 is large and $q_1^m \leq q_2 < q_1^{m+1}$ for some large m. By Lemma 3, there is a polynomial $P(x_1, x_2) \in \mathbb{Z}[x_1, x_2]$ of the form $P(x_1, x_2) = P_1(x_1)x_2 + P_0(x_1)$ so that $\partial_1^j P(r_1, r_2) = 0$ for $0 \leq j \leq l-1$, with $l = c(\beta, s)m$; $|P| \leq C(\beta, s)^m$; Deg $P \leq C(\beta)m$. A lower bound for $|P|$ can be obtained from Lemma 4:

$$|P| \gtrsim \min\{m^{-1} q_1^{\frac{l-1}{2}}, q_2\} = m^{-1} q_1^{c(\beta,s)m}.$$

Comparing this lower bound with the upper bound of $|P|$, we immediately get $q_1 \leq C(\beta, s)$, a contradiction.

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6 A restriction estimate using polynomial partitioning, part I

after L. Guth [1]
A summary written by Xiumin Du

Abstract

We give an improvement on the three-dimensional restriction problem using polynomial partitioning. In this first part, we state the main results and review some preliminaries.

6.1 Introduction

Let E be the extension operator for the truncated paraboloid in $\mathbb{R}^3$. More precisely, if f is a function $B^2(0,1) \to \mathbb{C}$, then for $x = (x', x_3) = (x_1, x_2, x_3) \in \mathbb{R}^3$,

$$Ef(x) := \int_{B^2(0,1)} f(\omega) e^{i(x' \cdot \omega + x_3 |\omega|^2)} \, d\omega \, .$$

Our main result is the following improvement on the restriction problem.

Theorem 1. *For all $p > 3.25$,*

$$\|Ef\|_{L^p(\mathbb{R}^3)} \leq C_p \|f\|_{L^\infty(B^2(0,1))}$$

holds for all $f \in L^\infty(B^2(0,1))$.

By ϵ-removal argument, it suffices to prove the following local estimate at the endpoint.

Theorem 2. *For any $\epsilon > 0$ and any radius R,*

$$\|Ef\|_{L^{3.25}(B^3_R)} \leq C_\epsilon R^\epsilon \|f\|_{L^\infty(B^2(0,1))}$$

holds for all $f \in L^\infty(B^2(0,1))$.

We introduce a concept of α-broadness, using which we can get a bilinear structure later on. Let $K = K(\epsilon)$ be a large constant. We divide $B^2(0,1)$ into $\sim K^2$ balls τ of radius $\sim K^{-1}$. Let f_τ denote the restriction of f to τ. For $\alpha \in (0,1)$, we say that x is *α-broad* for Ef if:

$$\max_{\tau} |Ef_\tau(x)| \leq \alpha |Ef(x)|.$$

We define $\mathrm{Br}_\alpha Ef(x)$ to be $|Ef(x)|$ if x is α-broad and zero otherwise. To prove Theorem 2, we bound $\int_{B_R} |Ef|^{3.25}$ by

$$\int_{B_R} (\mathrm{Br}_{K^{-\epsilon}} Ef)^{3.25} + K^{O(\epsilon)} \sum_{\tau} \int_{B_R} |Ef_\tau|^{3.25}.$$

The second term can be easily handled by parabolic rescaling and induction on the radius. Our strongest result is the following estimate about L^p norms of the broad part of Ef.

Theorem 3. *For any $\epsilon > 0$, there exists $K = K(\epsilon)$ so that for any radius R,*

$$\|\mathrm{Br}_{K^{-\epsilon}} Ef\|_{L^{3.25}(B_R^3)} \leq C_\epsilon R^\epsilon \|f\|_{L^2(B^2(0,1))}^{12/13} \|f\|_{L^\infty(B^2(0,1))}^{1/13}$$

holds for all $f \in L^\infty(B^2(0,1))$. Also, $\lim_{\epsilon \to 0} K(\epsilon) = +\infty$.

We remark that the exponent 3.25 is the sharp exponent in Theorem 3, given the right-hand side of the inequality. We will prove Theorem 3 using polynomial partitioning.

6.2 Preliminaries

In this section, we will first recall polynomial partitioning and wave packet decomposition, and then give an example showing that the exponent 3.25 is sharp.

6.2.1 Polynomial partitioning

First we have a variation of the classic ham sandwich theorem, which relies on the Borsuk-Ulam Theorem.

Theorem 4 (Borsuk-Ulam)**.** *If $F : S^N \to \mathbb{R}^N$ is a continuous function obeying the antipodal condition $F(-v) = -F(v)$, then there exists a $v \in S^N$ with $F(v) = 0$.*

The following polynomial ham sandwich theorem is an elegant application of the Borsuk-Ulam theorem.

Theorem 5. *If $W_1, \cdots, W_N$ are nonnegative L^1-functions on $\mathbb{R}^n$, then there exists a non-zero polynomial P of degree $\leq C_n N^{1/n}$ so that for each W_j,*

$$\int_{\{P>0\}} W_j = \int_{\{P<0\}} W_j \, .$$

By applying the polynomial ham sandwich theorem repeatedly, we obtain the following partitioning result.

Theorem 6. *Suppose that W is a nonnegative L^1 function on $\mathbb{R}^n$. Then for any degree D there exists a non-zero polynomial P of degree at most D so that $\mathbb{R}^n \backslash Z(P)$ is a union of $\sim D^n$ disjoint open sets O_i, and the integrals $\int_{O_i} W$ are all equal.*

For technical reasons, it is helpful in our arguments later to use non-singular polynomials. Using the density of non-singular polynomials, we have the following partitioning result involving non-singular polynomials, at a cost of weakening perfectly equal partitioning to approximately equal partitioning.

Theorem 7. *Suppose that W is a nonnegative L^1 function on $\mathbb{R}^n$. Then for any degree D there exists a non-zero polynomial P of degree at most D so that $\mathbb{R}^n \backslash Z(P)$ is a union of $\sim D^n$ disjoint open sets O_i, and the integrals $\int_{O_i} W$ agree up to a factor of 2. Moreover, the polynomial P is a product of non-singular polynomials.*

6.2.2 Wave packet decomposition

We study Ef by breaking it into wave packets. Given a large radius R, we tile $B^2(0,1)$ by $\sim R$ balls θ of radius $R^{-1/2}$ and tile B^2_R by $\sim R$ balls υ of radius $R^{1/2}$. Then for function f supported in $B^2(0,1)$, we break it into small pieces

$$f = \sum_{\theta,\upsilon} f_{\theta,\upsilon} \, ,$$

where each piece $f_{\theta,\upsilon}$ has support in θ and Fourier support essentially in υ. Therefore, we have orthogonality

$$\|f\|_2^2 \sim \sum_{\theta,\upsilon} \|f_{\theta,\upsilon}\|_2^2 \, .$$

Correspondingly, Ef can be decomposed as

$$Ef = \sum_{\theta,\upsilon} Ef_{\theta,\upsilon}\,,$$

where each piece $Ef_{\theta,\upsilon}$ has Fourier support in the R^{-1}-neighborhood of the parabolic cap over θ. We denote this neighborhood by θ^*, which is essentially a block of dimensions $\sim R^{-1/2} \times R^{-1/2} \times R^{-1}$. Moreover, $Ef_{\theta,\upsilon}$, when restricted to B_R^3, is essentially supported in a tube $T_{\theta,\upsilon}$ with dimensions $R^{1/2+\delta} \times R^{1/2+\delta} \times R$, and with direction $(-2c(\theta), 1)$ and center $(c(\upsilon), 0)$, where $c(\theta)$ and $c(\upsilon)$ denote the centers of θ and υ respectively, and $\delta = \epsilon^2$ is a small parameter. Morally, each piece $Ef_{\theta,\upsilon}$ is well approximated by the following model:

$$\text{for } x \in B_R, Ef_{\theta,\upsilon} \text{ is approximately } a_{\theta,\upsilon}\chi_{T_{\theta,\upsilon}} e^{i(c(\theta),|c(\theta)|^2)\cdot x}\,,$$

where $a_{\theta,\upsilon}$ is a complex number with $|a_{\theta,\upsilon}| \sim R^{-1/2}\|f_{\theta,\upsilon}\|_2$.

6.2.3 Example

We now give an example to show that the exponent 3.25 is sharp in Theorem 3, given the right-hand side in the inequality. This example is a planar example, in which Ef is essentially supported in a planar slab of dimensions $R^{1/2} \times R^{1/2} \times R$. There are $\sim R^{1/2}$ balls θ in a $\sim R^{-1/2}$-strip contained in $B^2(0,1)$, for which the directions $(-2c(\theta), 1)$ lie within an angle $\sim R^{-1/2}$ of the plane. For each θ, there are $\sim R^{1/2}$ tubes of the form $T_{\theta,\upsilon}$ that lie in the planar slab. We pick a number B between 1 and $R^{1/2}$, and for each θ, we randomly pick B tubes of the form $T_{\theta,\upsilon}$ that lie in our planar slab. We have now picked $\sim BR^{1/2}$ tubes. An average point of the planar slab lies in $\sim B$ of our tubes. Since the tubes were selected randomly, most points of the planar slab lie in $\sim B$ of our tubes.

For each of our chosen tubes $T_{\theta,\upsilon}$, we choose $f_{\theta,\upsilon}$ so that $|Ef_{\theta,\upsilon}(x)| \sim \chi^*_{T_{\theta,\upsilon}}$, which is a smooth approximation of the characteristic function on the tube, and $\|f_{\theta,\upsilon}\|_2 \sim R^{1/2}$ and $\|f_{\theta,\upsilon}\|_\infty \sim R$. Now we let f be a sum with random signs: $f = \sum_{\theta,\upsilon} \pm f_{\theta,\upsilon}$. Then $|Ef(x)| \sim B^{1/2}$ on most points in the planar slab, and $\|Ef\|_{L^p(B_R)} \gtrsim B^{1/2}R^{\frac{5}{2p}}$. Moreover, a typical point lies in B different tubes in random directions. If $B \geq K^{10\epsilon}$, then almost every point will be $K^{-\epsilon}$-broad. Therefore, we get

$$\|\mathrm{Br}_{K^{-\epsilon}} Ef\|_{L^p(B_R)} \gtrsim B^{1/2} R^{\frac{5}{2p}} .$$

On the other hand we have $\|f\|_2 \sim B^{1/2}R^{3/4}$ by orthogonality, and also $\|f\|_\infty \lesssim BR$.

We take $B \sim K^{10\epsilon}$, which is a constant independent of R. In this case, if $\|\mathrm{Br}_{K^{-\epsilon}} Ef\|_{L^p(B_R)} \leq C_\epsilon R^\epsilon \|f\|_2^{12/13}\|f\|_\infty^{1/13}$, then a direct computation shows that $p \geq 3.25$.

It might be possible to get a smaller exponent p by weighting $\|f\|_\infty$ more heavily. For instance, the following estimate is consistent with the planar example and appears plausible:

Conjecture 8.

$$\|\mathrm{Br}_{K^{-\epsilon}} Ef\|_{L^3(B_R)} \leq C_\epsilon R^\epsilon \|f\|_2^{2/3}\|f\|_\infty^{1/3} .$$

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7 A note on the Schrödinger maximal function

after J. Bourgain [1]
A summary written by Daniel Eceizabarrena

Abstract

In the context of the initial value problem of the free Schrödinger equation, it is shown that for the convergence property $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ to hold for almost every $x \in \mathbb{R}^n$ and for all $f \in H^s(\mathbb{R}^n)$ it is necessary that $s \geq \frac{n}{2(n+1)}$.

7.1 Introduction

The initial value problem for the free Schrödinger equation is

$$\begin{cases} u_t(x,t) = i\Delta u(x,t), & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & \text{in } \mathbb{R}^n, \end{cases}$$

which by using the Fourier transform in the form $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx$ is solved by

$$u(x,t) = e^{it\Delta} f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i (x\cdot\xi - 2\pi t|\xi|^2)}\, d\xi. \tag{1}$$

We seek a space H such that

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x) \tag{2}$$

holds for a.e. $x \in \mathbb{R}^n$ and for all $f \in H$.

This problem has been analysed since the 1980s, and it has become evident that the convenient spaces to work with are the Sobolev spaces H^s of fractional order, described as

$$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1+|\xi|^2)^s\, d\xi < \infty \right\}, \quad s \geq 0.$$

The objective is thus to find the right range for s so that (2) is satisfied. In [2], Carleson showed that if $n = 1$, it is enough to ask $s \geq 1/4$. Dahlberg and

Kenig found in [3] that if $s < 1/4$, there are functions for which the property fails, a result which was valid for any dimension. The problem was thus solved in $n = 1$, and moreover, it placed a conjecture asserting the threshold $s = 1/4$ was the correct one for any $n \in \mathbb{N}$.

There were many improvements in the following years, as the one by P. Sjölin [4] and L. Vega [5], who independently asserted that $s > 1/2$ was sufficient for convergence in any dimension. The conjecture remained alive until J. Bourgain proved in [6] that $s \geq 1/2 - 1/n$ was necessary, since when $n \geq 5$ this bound is greater that $1/4$.

Recently, the problem has been almost solved in $\mathbb{R}^2$ thanks to works by Bourgain, who proved the necessity of $s \geq 1/3$ in the result we are to present, and Du, Guth and Li in [7], who showed sufficiency of $s > 1/3$. The behaviour of the endpoint remains unknown.

The best general conditions we have so far are due to Bourgain, and state that $s > 1/2 - 1/4n$ is sufficient if $n \geq 3$ (in [6]) and that it is necessary that $s \geq \frac{n}{2(n+1)}$. It is this last result that we present here.

7.2 The Method

The standard method for proving convergence is to obtain a maximal estimate like

$$\|\sup_{t \in I} \left| e^{it\Delta} f(\cdot) \right| \|_{L^p(B)} \leq C \|f\|_{H^s(\mathbb{R}^n)}, \quad \forall f \in H^s(\mathbb{R}^n), \tag{3}$$

where the L^p space can be any with $p \in [1, \infty)$, I is some interval around zero and B is some set with positive measure, such as a ball or a cube. With this estimate in hand, the proof of the convergence can be performed in the same way as the standard proof for the Lebesgue Differentiation Theorem and is strongly based on Chebyshev's inequality.

Counterexamples are the most trivial way to prove the necessity of some condition. In our case, having fixed some $s < \frac{n}{2(n+1)}$, it would be enough to find some function $f \in H^s(\mathbb{R}^n)$ such that $\lim_{t \to 0} e^{it\Delta} f(x) \neq f(x)$. Nevertheless, sometimes this is not the easiest approach to the problem. On the other hand, it can be proved that convergence implies an estimate like (3) with $p = 2$ and $I = (0, 1)$. Therefore, instead of finding a counterexample for the convergence property, it is usual to show that the estimate (3) cannot hold for the orders s under consideration.

7.3 The Result

As mentioned before, the result presented in Bourgain's note [1] is the best necessary condition known for convergence.

Theorem 1. *If $s < \frac{n}{2(n+1)}$, there exists $f \in H^s(\mathbb{R}^n)$ such that*

$$\lim_{t \to 0} e^{it\Delta} f(x) \neq f(x)$$

in some set with positive measure.

The proof of this result is achieved by means of the maximal estimate method described in the previous section. Denoting an annulus as $A(R) = \{x \in \mathbb{R}^n \mid |x| \sim R\}$, what is proved is the following.

Theorem 2. *Let $n \geq 2$ and $s < \frac{n}{2(n+1)}$. Then, there exist sequences $\{R_k\}_{k \in \mathbb{N}}$ and $\{f_k\}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^2)$ such that $\lim_{k \to \infty} R_k = \infty$, $\operatorname{supp} \widehat{f_k} \subset A(R_k)$ and*

$$\lim_{k \to \infty} \frac{\|\sup_{0<t<1} \left|e^{it\Delta} f_k(\cdot)\right| \|_{L^1(B(0,1))}}{R_k^s \|f_k\|_2} = \infty.$$

This result shows the maximal estimate cannot hold. Indeed, observe that for big $k \in \mathbb{N}$, the support condition and Plancherel's theorem imply that

$$\|f_k\|_{H^s}^2 = \int_{\mathbb{R}^n} |\widehat{f_k}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi \sim R^{2s} \|f_k\|_{L^2}^2.$$

Also, since the L^1 norm is taken in a fixed ball, it is bounded by the L^2 norm. Therefore, the limit in Theorem 2 implies that there cannot exist a constant for an estimate like (3), and consequently that convergence cannot always hold.

Let us sketch the proof of Theorem 2. Let $\varphi \in \mathcal{S}(\mathbb{R})$ and $\Phi \in \mathcal{S}(\mathbb{R}^{n-1})$ having non-negative images be such that $\operatorname{supp} \widehat{\varphi} \subset [-1, 1]$, $\operatorname{supp} \widehat{\Phi} \subset B(0, 1)$ and $\varphi(0) = 1 = \Phi(0)$. Set $D = R^{\frac{n+2}{2(n+1)}}$ and denote $x \in \mathbb{R}^n$ as $x = (x_1, \ldots, x_n) := (x_1, x')$. Define

$$f(x) = e^{2\pi i R x_1} \varphi(R^{\frac{1}{2}} x_1) \Phi(x') \prod_{j=2}^{n} \left(\sum_{\frac{R}{2D} < l_j < \frac{R}{D}} e^{2\pi i D l_j x_j} \right) \tag{4}$$

where $l = (l_2, \ldots, l_n) \in \mathbb{Z}^{n-1}$. One can compute

$$\|f\|_2 \sim R^{-\frac{1}{4}} \left(\frac{R}{D}\right)^{\frac{n-1}{2}} \qquad \text{and} \qquad \operatorname{supp} \widehat{f} \subset A(R).$$

Writing (4) in the expression for the solution (1) and separating variables in the integral, we can write

$$\left|e^{it\Delta} f(x)\right| \sim \varphi(R^{\frac{1}{2}}(x_1 - 4\pi R t)) \left|\sum_l e^{2\pi i(Dl\cdot x' - 2\pi D^2 |l|^2 t)}\right|.$$

Choosing $t = \frac{x_1}{4\pi R} - \tau$ with τ small enough (say $|\tau| < cR^{-3/2}$ with $c \ll 1$), we can write $\varphi(R^{\frac{1}{2}}(x_1 - 4\pi R t)) \sim \varphi(0) = 1$ and we are left only with the sum, which after the substitution of t we can write as

$$\left|e^{it\Delta} f(x)\right| \sim \left|\sum_l e^{2\pi i(Dl\cdot x' - \frac{D^2}{2R}|l|^2 x_1 + 2\pi D^2 |l|^2 \tau)}\right| = \prod_{j=2}^{n} \left|\sum_{\frac{R}{2D} < l_j < \frac{R}{D}} e^{2\pi i(l_j y_j + l_j^2 (y_1 + s))}\right|. \tag{5}$$

In the last equality, we have set $y' = \{Dx'\}$ and $y_1 = \left\{-\frac{D^2}{2R} x_1\right\}$ (where $\{\cdot\}$ is the fractional part) and $s = 2\pi D^2 \tau$ satisfies $|s| < cD^2 R^{-3/2}$. Observe that we can now see $y = (y_1, y')$ as an element of the n-dimensional torus $\mathbb{T}^n = [0,1]^n$.

The idea is the following: choosing an appropriate subset of $\mathbb{T}^n$, we will transform each of the $n-1$ sums in (5) into a quadratic Gauss sum, which we will be able to estimate. More precisely, we will choose y that lies close to $\frac{a}{q}$, for $a \in (\mathbb{Z}/q\mathbb{Z})^n$ and for some prime $q \in \mathbb{N}$.

Following this, we define

$$\Omega = \bigcup_{q,a} \left\{ y = (y_1, y') \mid \left|y_1 - \frac{a_1}{q}\right| < cD^2 R^{-\frac{3}{2}} \text{ and } \left|y' - \frac{a'}{q}\right| < c\frac{D}{R} \right\} \subset \mathbb{T}^n,$$

where $q \in \mathbb{N}$ is prime, $q \sim R^{\frac{n-1}{2(n+1)}}$ and $a = (a_1, a') \in (\mathbb{Z}/q\mathbb{Z})^n$. The measure of this set is $|\Omega| \sim 1/\log R^{\frac{n-1}{2(n+1)}}$.

Consider $y \in \Omega$ and q and a as above. Also, let $s = \frac{a_1}{q} - y_1$ so that $|s| < cD^2 R^{-\frac{3}{2}}$ is satisfied. Then, the sum in (5) is

$$\sum_{\frac{R}{2D} < l_j < \frac{R}{D}} e^{2\pi i\left(l_j y_j + \frac{a_1 l_j^2}{q}\right)}.$$

Calling $y_j = \frac{a_j}{q} + b_j$ where $|b_j| < c\frac{D}{R}$, we have $e^{2\pi i l_j y_j} = e^{2\pi i \frac{a_j l_j}{q}} e^{2\pi i l_j b_j}$, where $|l_j b_j| \lesssim c\frac{R}{D}\frac{D}{R} = c \ll 1$. Since that phase is small, we may write

$$\left| \sum_{\frac{R}{2D} < l_j < \frac{R}{D}} e^{2\pi i (l_j y_j + \frac{a_1 l_j^2}{q})} \right| \sim \left| \sum_{\frac{R}{2D} < l_j < \frac{R}{D}} e^{2\pi i \frac{a_j l_j + a_1 l_j^2}{q}} \right|.$$

That exponential has period q in the variable l_j, so there are really $\frac{R}{2D}/q \sim R^{\frac{1}{2(n+1)}}$ copies of the sum from 0 to $q-1$, which is precisely a generalised Gauss sum. In other words, we have

$$\left| \sum_{\frac{R}{2D} < l_j < \frac{R}{D}} e^{2\pi i \frac{a_j l_j + a_1 l_j^2}{q}} \right| \sim R^{\frac{1}{2(n+1)}} \left| \sum_{l_j=0}^{q-1} e^{2\pi i \frac{a_j l_j + a_1 l_j^2}{q}} \right| \sim R^{\frac{1}{2(n+1)}} \sqrt{q} \sim R^{\frac{1}{4}}.$$

This means that, from (5) and for $y \in \Omega$,

$$\left| e^{it\Delta} f(x) \right| \sim R^{\frac{n-1}{4}}.$$

Now, it can be checked that if we call B the set of $x \in B(0,1)$ such that $y \in \Omega$, then $|B| \sim |\Omega|$. As a consequence,

$$\frac{\|\sup_{0<t<1} |e^{it\Delta} f(\cdot)\|_{L^1(B(0,1))}}{\|f\|_2} \gtrsim |B| R^{\frac{n-1}{4}} R^{\frac{1}{4}} \left(\frac{R}{D}\right)^{-\frac{n-1}{2}} = |B| R^{\frac{n}{2(n+1)}}.$$

Finally, if we take $s < \frac{n}{2(n+1)}$, we might write

$$\frac{\|\sup_{0<t<1} |e^{it\Delta} f(\cdot)\|_{L^1(B(0,1))}}{R^s \|f\|_2} \gtrsim \frac{R^{\frac{n}{2(n+1)} - s}}{\log R^{\frac{n-1}{2(n+1)}}} \to \infty$$

as $R \to \infty$. The proof is now complete.

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8 On the Erdős distinct distances problem in the plane. Part I.

after L. Guth and N.H. Katz [3]
A summary written by Marco Fraccaroli

Abstract

We want to prove that a set of N points in $\mathbb{R}^2$ has at least $c\frac{N}{\log N}$ distinct distances, thus obtaining the sharp exponent in a problem of Erdős. We describe the setup of Elekes and Sharir which, in the spirit of the Erlangen program, allows us to study the problem in the group of rigid motions of the plane. This converts the distinct distances problem to one of point-line incidences in $\mathbb{R}^3$, which will be solved by means of the polynomial ham sandwich theorem, flecnode polynomial and the geometry of ruled surfaces in Part II.

8.1 Introduction

In [2], Erdős posed the question about how few distinct distances are determined by N points in $\mathbb{R}^2$. He checked that if the points are arranged in a square grid, then the number of distinct distances is $\sim \frac{N}{\sqrt{\log N}}$[6], and he conjectured this lower bound for any arrangement of the points.

Guth and Katz in [3] proved the following result.

Theorem 1. *A set of N points in the plane determines $\gtrsim \frac{N}{\log N}$ distinct distances.*

They followed the approach introduced by Elekes and Sharir in [1], based on the idea to translate the distinct distances problem to one of incidence geometry in $\mathbb{R}^3$. In particular, Thm. 1 is a consequence of the following result.

Theorem 2. *Let $\mathfrak{L}$ be a set of N^2 lines in $\mathbb{R}^3$. Suppose that $\mathfrak{L}$ contains $\lesssim N$ lines in any plane or any regulus. Suppose that $2 \leq k \leq N$. Then the number of points that lie in at least k lines is $\lesssim N^3 k^{-2}$.*

[6]From now on, we use the notation $A \gtrsim B$ to mean that there is a universal constant $C > 0$ with $A \geq CB$. In an analogous way we define $A \lesssim B$. By $A \sim B$ we mean that $A \gtrsim B$ and $A \lesssim B$ hold simultaneously.

A *regulus* is the locus of lines meeting three given skew lines in $BBR^{?}?$. It is a quadratic surface which is doubly ruled, meaning that each point in the surface lies in two lines in the surface, and each line from one ruling intersects all the lines from the other ruling.

8.2 Elekes-Sharir framework

Let $P \subset \mathbb{R}^2$ be a set of N points. We denote by $d(P)$ the set of nonzero distances among points of P defined by

$$d(P) := \{d(p,q) \colon p,q \in P, p \neq q\}.$$

To obtain a lower bound on $|d(P)|$, we will prove an upper bound on the cardinality of the of quadruples $Q(P)$ defined by

$$Q(P) := \{(p_1,p_2,p_3,p_4) \in P^4 \colon d(p_1,p_2) = d(p_3,p_4) \neq 0\}.$$

If $d(P)$ is small, then $Q(P)$ needs to be large. By applying the Cauchy-Schwarz inequality, we quantify this statement with the following inequality.

Lemma 3. *For any set $P \subset \mathbb{R}^2$ with N points, it holds:*

$$|d(P)| \geq \frac{N^4 - 2N^3}{|Q(P)|}.$$

To prove Thm. 1, it suffices to show the following upper bound on $|Q(P)|$.

Proposition 4. *For any set $P \subset \mathbb{R}^2$ of N points, then*

$$|Q(P)| \lesssim N^3 \log N.$$

Elekes and Sharir studied $Q(P)$ from the point of view of the symmetries of $\mathbb{R}^2$. We denote by G the group of *positively oriented rigid motions of the plane*, i.e. the group of bijections from the plane to itself that preserve angles, distances and orientation. The first connection between $Q(P)$ and G come from the following simple proposition.

Proposition 5. *Let $(p_1,p_2,p_3,p_4) \in Q(P)$. Then there is a unique $g \in G$ so that $g(p_1) = p_3$ and $g(p_2) = p_4$.*

We denote by E the map defined by the proposition and we observe that E is not necessarily injective, as the number of quadruples in $E^{-1}(g)$, where $g \in G$, depends on the size of $P \cap gP$.

Lemma 6. *Let $g \in G$ such that $|P \cap gP| = k$. Then $|E^{-1}(g)| = 2\binom{k}{2}$.*

Let $G_{=k}(P)$ denote the set of $g \in G$ with $|P \cap gP| = k$. By the last lemma, we have

$$|Q(P)| = \sum_{k=2}^{N} 2\binom{k}{2}|G_{=k}(P)|.$$

If we denote by $G_k(P)$ the set of $g \in G$ so that $|P \cap gP| \geq k$, we have $|G_{=k}(P)| = |G_k(P)| - |G_{k+1}(P)|$, thus

$$|Q(P)| = \sum_{k=2}^{N}(2k-2)|G_k(P)| \tag{1}$$

We will bound the cardinality of $G_k(P)$ by showing the following result.

Proposition 7. *For any set $P \subset \mathbb{R}^2$ of N points, and any $1 \leq k \leq N$, then*

$$|G_k(P)| \lesssim N^3 k^{-2}.$$

Plugging the bound into equation (1) we get the wanted bound for $|Q(P)|$, and hence for $|d(P)|$.

The following step in the Elekes and Sharir framework is to relate the sets $G_k(P)$ to an incidence problem involving certain curves in G. For any points $p, q \in \mathbb{R}^2$, we define the set $S_{pq} \subset G$ given by

$$S_{pq} := \{g \in G \colon g(p) = q\}.$$

Each S_{pq} is a smooth one-dimensional curve in the three-dimensional Lie group G. The connection between $G_k(P)$ and the curves S_{pq} is provided by

Lemma 8. *A $g \in G$ belongs to $G_k(P)$ if and only if it lies in at least k of the curves $\{S_{pq} \colon p, q \in P\}$.*

By making a careful change of coordinates we can reduce this problem to an incidence problem for lines in $\mathbb{R}^3$.

Let $G^{\text{trans}} \subset G$ denote the subset of translations, and let G' denote $G \setminus G^{\text{trans}}$. We then divide $G_k(P)$ accordingly into $G'_k(P) \cup G_k^{\text{trans}}(P)$. For the translations we have the wanted bound fairly easily, namely

Lemma 9. *Let P be any set of N points in $\mathbb{R}^2$. Then $|E^{-1}(G^{\text{trans}})| \leq N^3$. Moreover, $|G_k^{\text{trans}}(P)| \lesssim N^3 k^{-2}$ for all $2 \leq k \leq N$.*

To bound $G'_k(P)$ we pick a nice set of coordinates $\rho\colon G' \to \mathbb{R}^3$ defined in the following way. Each element of G' has a unique fixed point (x, y) and an angle θ of rotation around the fixed point with $0 < \theta < 2\pi$. We define

$$\rho(x, y, \theta) = \left(x, y, \cot\frac{\theta}{2}\right).$$

Proposition 10. *Let $p = (p_x, p_y)$ and $q = (q_x, q_y)$ be points in $\mathbb{R}^2$. Then the set $\rho(S_{pq} \cap G')$ is a line in $\mathbb{R}^3$.*

We denote by L_{pq} the line $\rho(S_{pq} \cap G')$, which is parametrized by the equation

$$\left(\frac{p_x + q_x}{2}, \frac{p_y + q_y}{2}, 0\right) + t\left(\frac{q_y - p_y}{2}, \frac{p_x - q_x}{2}, 1\right).$$

If we denote by $\mathfrak{L}$ the set of lines $\{L_{pq}\colon p, q \in P\}$, it is easy to check that these are N^2 distinct lines. Moreover, if g lies in $G'_k(P)$, then $\rho(g)$ lies in at least k lines of $\mathfrak{L}$. We would like to prove that there are $\lesssim N^3k^{-2}$ points that lie in at least k lines of $\mathfrak{L}$. However, such an estimate does not hold for an arbitrary set of N^2 lines. There are two important examples to consider carefully:

- if the lines of $\mathfrak{L}$ lie in a plane, then one may expect $\sim N^4$ points that lie in at least two lines;
- if $\mathfrak{L}$ contains $N^2/2$ lines in each of the two rulings of a regulus, then we would have $\sim N^4$ points that lie in at least two lines.

Because of these examples we have to show that not too many lines of $\mathfrak{L}$ lie in a plane or a regulus. In fact,

Proposition 11. *No more than N lines of $\mathfrak{L}$ lie in a single plane. No more than $O(N)$ lines of $\mathfrak{L}$ lie in a single regulus.*

We have finally connected the distinct distances problem in Thm. 1 to the incidence geometry problem in Thm. 2. In particular, it will be proved by the following two results on incidence geometry, that will be shown in Part II.

Theorem 12. *Let $\mathfrak{L}$ be any set of N^2 lines in $\mathbb{R}^3$ for which no more than N lie in a common plane and no more than $O(N)$ lie in a common regulus. Then the number of points of intersection of two lines in $\mathfrak{L}$ is $O(N^3)$.*

Theorem 13. *Let $\mathfrak{L}$ be any set of N^2 lines in $\mathbb{R}^3$ for which no more than N lie in a common plane, and let k be a number $3 \leq k \leq N$. Let $\mathfrak{G}_k$ be the set of points where at least k lines meet. Then $|\mathfrak{G}_k| \lesssim N^3 k^{-2}$.*

The obtained estimates also show that sets with few distinct distances must have many partial symmetries. For example, if $G_3(P)$ is empty, then our results show that $|Q(P)| \lesssim N^3$ and $|d(P) \gtrsim N$. Also, any set with $|d(P) \lesssim N(\log N)^{-1/2}$ must have a partial symmetry with $k \geq \exp(c\sqrt{\log N})$ for a universal constant $c > 0$. Any set with $|d(P) \lesssim N(\log N)^{-1}$ must have a partial symmetry with $k \geq N^c$ for a universal constant $c > 0$.

8.3 Sharpness of the estimates for the square grid

In the Erdős' example of a square grid of N points, it can be shown that $|Q(P)| \gtrsim N^3 \log N$ and $|G_k(P)| \gtrsim N^3 k^{-2}$ for all $2 \leq k \leq N/2000$. Therefore the estimates we made in Prop. 4 and Prop. 7 are sharp up to constant factors. To prove this, we consider $S \geq 1$ an integer and we let P be the grid of points (x, y) where x and y are integers with norm $\leq 2S$. The number of points in P is $N = (4S + 1)^2$. Let $\mathfrak{L}$ be the set of lines in $\mathbb{R}^3$ associated to the set P. Then

Lemma 14. *If a, b, c, d are positive integers with norm $\leq S$, then the line from $(a, b, 0)$ to $(c, d, 1)$ is contained in $\mathfrak{L}$.*

Let $\mathfrak{L}_0 \subset \mathfrak{L}$ be the set of lines from $(a, b, 0)$ to $(c, d, 1)$ where a, b, c, d are positive integers with norm $\leq S$. Note that $|\mathfrak{L}_0| = S^4$. The incidences of $\mathfrak{L}_0$ are studied by the following result.

Proposition 15. *Let $\mathfrak{G}_k$ be the set of points in $\mathbb{R}^3$ that lie in at least k lines of $\mathfrak{L}_0$. For any k in the range $2 \leq k \leq S^2/400$, $|\mathfrak{G}_k| \gtrsim S^6 k^{-2}$.*

Since $S^2 \sim N$, we obtain the sharpness of our results.

This observation is telling us that to improve the lower bound by Guth and Katz relying on the Elekes-Sharir framework, one either has to make a drastic change in the framework, or to separately handle the family of "problematic" point configurations. In particular, the gap in the bound is a result of the application of the Cauchy-Schwarz inequality in Lemma 3.

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9 The joints problem in R^n and On the size of Kakeya sets in finite fields

after René Quilodrán [1],Zeev Dvir [2]
A summary written by Zihui He

Abstract

This presentation includes 2 parts. First we will show that given a collection of A lines in R^n, $n \geq 2$, the maximum number of their joints is $O(A^{n/(n-1)})$. Then we will show a Kakeya set is a subset of F^n, where F is a finite field of q elements, that contains a line in every direction.we will show that the size of every Kakeya set is at least $C_n \cdot q^n$, where C_n depends only on n.

9.1 The joints problem in R^n

For a given collection of lines L in R^n consider the set J of points of the form $\cap_{i=1}^n \ell_i$, where $\ell_i \in L$ for all $1 \leq i \leq n$ and the directions of the lines $\ell_1, \ldots, \ell_n$, are linearly independent. We will refer to J as the set of transverse intersections, or joints, of L.

Theorem 1. *Let L be a collection of lines in R^n, then the cardinality of the set of joints of L, J, satisfies $|J| \lesssim |L^{n/(n-1)}|$.*

We start by showing the following Lemma.

Lemma 2. *Let J' be a subset of J with the property that every line $\ell \in L$ with $\ell \cap J' \neq \emptyset$ contains at least m points of J', that is $|\ell \cap J'| \geq m$, for some given constant m. Then $|J'| \geq C_n m^n$, where C_n is a constant depending on n only.*

A similar bound as the one in Theorems 1 can be proven if we replace lines by algebraic curves. We start by considering a special case of algebraic curves. Let $\mathcal{C}$ be a set of smooth curves, each parametrized by polynomials, that is, if $\gamma \in \mathcal{C}$ we can parametrize it as $\gamma(t) = (P_1(t), \ldots, P_n(t))$ where each P_i is a polynomial in one variable of degree at most d, for a given constant d. We let J denote the set of joints determined by $\mathcal{C}$.

A minor modification of Lemma 2 gives the following.

Lemma 3. *Let $\mathcal{C}$ and J be as in the previous paragraph, and let J' be a subset of J with the property that $|\gamma \cap J'| \geq m$ for every curve $\gamma \in \mathcal{C}$ with $\gamma \cap J' \neq \emptyset$, for some given constant m. Then $|J| = \Omega(m^n/d^n)$.*

The conclusion follows as in the case of lines, and the bound on the number of joints is $|J| \leq C_n |\mathcal{C}|^{n/(n-1)} d^{n/(n-1)}$, where C_n is a constant depending on n only.

More generally, if we consider an irreducible, smooth algebraic curve γ of degree d and if $Q \in R[x_1, \ldots, x_n]$ has degree $< m/d$ and its zero locus intersects γ on at least m different points, then the curve is contained in the zero set of Q, that is $Q|_\gamma \equiv 0$, by an application of Bezout's Theorem. Hence the same conclusion as in Lemma 3 holds if we let $\mathcal{C}$ consist of irreducible, smooth algebraic curves of degree at most d. Therefore we have the following Theorem.

Theorem 4. *Let $\mathcal{C}$ be a collection of irreducible, smooth algebraic curves of degree at most d in R^n. Let J denote the set of joints determined by $\mathcal{C}$. Then the cardinality of J satisfies $|J| \leq C_n |\mathcal{C}|^{n/(n-1)} d^{n/(n-1)}$, for some constant C_n depending on n only.*

9.2 On the size of Kakeya sets in finite fields

Let F denote a finite field of q elements. A Kakeya set (also called a Besicovitch set) in F^n is a set $K \subset F^n$ such that K contains a line in every direction. More formally, K is a Kakeya set if for every $x \in F^n$ there exists a point $y \in F^n$ such that the line

$$L_{y,x} \triangleq \{y + a \cdot x | a \in F\}$$

is contained in K.

Theorem 5. *Let $K \subset F^n$ be a Kakeya set. Then*

$$|K| \geq C_n \cdot q^{n-1},$$

where C_n depends only on n.

We derive Theorem from a stronger theorem that gives a bound on the size of sets that contain only 'many' points on 'many' lines. Before stating the theorem we formally define these sets.

Definition 6 ((δ,γ)-Kakeya Set)**.** *A set $K \subset F^n$ is a (δ,γ)-*Kakeya Set *if there exists a set $\mathcal{L} \subset F^n$ of size at least $\delta \cdot q^n$ such that for every $x \in \mathcal{L}$ there is a line in direction x that intersects K in at least $\gamma \cdot q$ points.*

Theorem 7. *Let $K \subset F^n$ be a (δ,γ)-Kakeya Set. Then*

$$|K| \geq \binom{d+n-1}{n-1},$$

where

$$d = \lfloor q \cdot \min\{\delta,\gamma\} \rfloor - 2.$$

Improving the bound to $\approx q^n$

Theorem 8. *Let $K \subset F^n$ be a Kakeya set. Then*

$$|K| \geq C_n \cdot q^n,$$

where C_n depends only on n.

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10 A Better Restriction Estimate for the Two-Dimensional Paraboloid

after J. Bourgain and L. Guth [1]
A summary written by Dominique Kemp

Abstract

We use a multi-scale version of the Bourgain-Guth multilinear method in order to improve the barrier on the adjoint restriction operator in $\mathbb{R}^3$ from 10/3 to 33/10.

10.1 Introduction

We focus our argument for the sake of simplicity on the case of restriction for the (truncated) paraboloid. It is worth commenting, though, that theorem actually holds more generally for all smooth, compact hypersurfaces S with positive definite second fundamental form. Tao proved [2] via bilinear restriction that $\|\widehat{fd\sigma}\|_p \leq \|f\|_{L_q(d\sigma)}$ for all $p > \frac{2(n+2)}{n}$ and $\frac{n+1}{n-1}q' \leq p$, where q' is the exponent conjugate to q, n is the dimension of the ambient space, and σ is the surface measure on the paraboloid. Note that in dimension 3, $\frac{2(n+2)}{n} = \frac{10}{3}$. Using multilinear theory via simple arithmetic reasoning and iteration of scales, Bourgain and Guth improved this estimate in dimension 3 to $\frac{33}{10}$.

Let T denote the extension operator over the paraboloid $\mathbb{P}^2$, i.e. $Tf(x) = \int_{[-1,1]^2} e^{-2\pi i(x_1y_1+x_2y_2+x_3|y|^2)} f(y)dy$. We will have need of considering the application of this operator to restrictions of f to subcubes α of $Q = [-1,1]^2$. Denote as T_α the operator $e^{2\pi i x\cdot(c(\alpha),|c(\alpha)|^2)}T(f\chi_\alpha)$.

We now briefly mention the essential machinery used before describing the proof. The authors use the multilinear restriction theorem famously proved by Bennett, Carberry, and Tao [2], as well as the $L_{\frac{5}{3}}$ Kakeya estimate proven by T. Wolff. The former result has the striking significance of being true for all exponents that arise in the restriction conjecture.

In what follows, we "weaken" our considerations to proving our theorem locally, and then further simplify to proving the following result. The justification for the latter may be found in a book by Mattila [4].

Theorem 1. *For all $p \geq \frac{33}{10}$ and all $R >> 1$,*

$$\|Tf\|_{L_{p_0}(B_R)} \leq C_{p,\epsilon} R^{\epsilon} \|f\|_{\infty}.$$

10.2 The Initial Decomposition

Let $K_1 < K$ be parameters where K is a constant larger than 1. In the sequel, we shall find that $K_1 = K^{\epsilon}$ works for our purposes. We consider partitions of Q into cubes α of side length $\frac{1}{K}$, whose centers we denote by y_α and also into cubes τ of side length $\frac{1}{K_1}$. Observing that for each $x \in \mathbb{R}^3$ we have $Tf(x) = \sum_\alpha T_\alpha f(x)$, we consider how we might bound $Tf(x)$ by an expression that involves a multilinear term for which [BCT] might apply. We do this by case-by-case analysis.

Fix x, and set $A = Tf(x), A_\alpha = T_\alpha f(x)$. Let $A_* = \max_\alpha A_\alpha$, and let $S_{big} = \{\alpha : A_\alpha > K^{-2} A_*\}$. We now consider the following (mutually exclusive) three cases.

1. There exist $\alpha_1, \alpha_2, \alpha_3 \in S_{big}$ such that

$$d(y_{\alpha_1}, y_{\alpha_2}) \geq d(y_{\alpha_1}, y_{\alpha_3}) \geq d(y_3, l(y_{\alpha_1}, y_{\alpha_2})) > \frac{10^3}{K},$$

   where $l(y_{\alpha_1}, y_{\alpha_2})$ is the line connecting these two points.

2. If $d(y_\alpha, y_{\alpha_*}) > \frac{1}{K_1}$, then $\alpha \notin S_{big}$.

3. There exists $\alpha_{**} \in S_{big}$ such that $d(y_{\alpha_*}, y_{\alpha_{**}}) > \frac{1}{K_1}$.

The third case is more complicated than the other two. It actually splits into two mutually exclusive subcases where we assume either $A \leq C \max_\tau A_\tau$ or $|A_{\tilde{\tau}}|, |A_{\tau'}| > \frac{1}{10K_1^2}|A|$ for some $\frac{1}{K_1}$-cubes $\tilde{\tau}, \tau'$ that are at least distance $\frac{10^6}{K_1}$ apart. We also make crucial use of the fact that the first case does not hold in this situation.

The reader may check that we thus obtain the following inequality for $Tf(x)$:

$$|Tf(x)| \leq C(K^4) \max_{\alpha_1,\alpha_2,\alpha_3 \, \frac{1}{K}-\text{transverse}} |T_{\alpha_1} f(x) T_{\alpha_2} f(x) T_{\alpha_3} f(x)|^{\frac{1}{3}}$$

$$+C(K_1^2)\max_{(l,E',E'')} |\sum_{\alpha\in E'} T_\alpha f(x)|^{\frac{1}{2}}|\sum_{\alpha\in E''} T_\alpha f(x)|^{\frac{1}{2}}+\max_\alpha |T_\alpha f(x)|+\max_\tau |T_\tau f(x)|,$$

where (l, E', E'') denotes a line l and collections E', E'' of K cubes α each, which are contained in the $\frac{10^3 K_1}{K}$-neighborhood of l and also satisfy $d(\alpha', \alpha'') > \frac{10^6}{K_1}$ for all $\alpha' \in E', \alpha'' \in E''$.

It turns out to be better for iteration to replace the bilinear term (*) above by a square function. Using a Cordoba-type argument, we may deduce that

$$\int_{B(a,K)} (*)^4 \le C(K_1^8) \sum_{\substack{\alpha'\in E' \\ \alpha''\in E''}} \int_{B(a,K)} |T_{\alpha'} f(x)|^2 |T_{\alpha''} f(x)|^2 \lesssim K^3 C(K_1^8) \sum_{\alpha\in E} \sum |T_\alpha f(x)|^2$$

[7]where $B(a,K)$ is some cube of side length K contained in B_R, x is any point in $B(a,K)$, and E is a collection of $2K$ many cubes α. Thus, we may replace (*) with $\phi(x)(\sum_{\alpha\in E} |T_\alpha f(x)|^2)^{\frac{1}{2}}$, where ϕ is constant over cubes of side length 1 and $⨍_{B(a,K)} \phi^4 \lesssim C(K_1^8)$ for all cubes $B(a,K)$.

10.3 Parabolic Rescaling

At this point, we would like to apply the above decomposition to $T_\beta f(x)$, where β is any δ-subcube of Q with center c. We wish to make such an application as will respect the scale length associated with β. Consequently, we use change of variables to rewrite $|T_\beta f(x)|$ as $\delta^2 |T(f\circ g)(h(x))|$, where g is an affine transformation mapping Q to β and $h(x_1, x_2, x_3) = (\delta x_1 + 2\delta c_1 x_3, \delta x_2 + 2\delta c_2 x_3, \delta^2 x_3)$, and then we transform back. Applying the decomposition above to the latter term, we find:

$$|T_\beta f(x)| \le C(K^4)\max_{\beta_1,\beta_2,\beta_3\ \frac{\delta}{K}-\text{transverse}} |T_{\beta_1} f(x) T_{\beta_2} f(x) T_{\beta_3} f(x)|^{\frac{1}{3}}$$

$$+C(K_1^2)\phi_\beta(x)\max_{E_\delta} (\sum_{\beta'\in E_\delta} |T'_\beta f(x)|^2)^{\frac{1}{2}} + \max'_\beta |T'_\beta f(x)| + \max_{\tilde\beta} |T_{\tilde\beta} f(x)|,$$

where all cubes are contained in β; β_i, β' have lengths $\frac{\delta}{K}$ while the length of $\tilde\beta$ is $\frac{\delta}{K_1}$; and E_δ is a collection of $O(\frac{1}{\delta})$ cubes β'. ϕ_β is also slightly different

[7]Note that here use the version of the uncertainty principle which guarantees that $T_\alpha f$ is virtually constant on cubes of side length K.

from its above counterpart; in this scenario, ϕ_β is constant over boxes of dimensions $\frac{1}{\delta} \times \frac{1}{\delta} \times \frac{1}{\delta^2}$ and also satisfies $\fint_B \phi_\beta^4 \lesssim C(K_1^8)$ for any $\frac{K}{\delta} \times \frac{K}{\delta} \times \frac{K}{\delta^2}$-box B.

Let us denote this estimate by (†).

10.4 Induction on Scales

In the previous section, we obtained a bound on the restriction of T. We now seek to iterate this result in order to compute a useful estimate of the contribution to $|Tf(x)|$ of each given scale. The process is what one might expect. Namely, leave all multilinear terms alone and only further decompose the other terms of the form T_β using (†). We cease decomposing a term when it is reduced to a scale length that is less than $\sqrt{R}$. Furthermore, strictly for convenience, we set $K = 2$.

We next must determine the relevant behavior of those products of the ϕ_β that arise from our iterative process. Let $\beta, \beta_i \subset \beta$ be arbitrary fixed subcubes of length δ, $\frac{\delta}{K}$ respectively. We denote the boxes "dual" to a given cube β by $\mathring{\beta}$, i.e. $\mathring{\beta}$ is a $\frac{1}{\delta} \times \frac{1}{\delta} \times \frac{1}{\delta^2}$-box. Also, let $K\mathring{\beta}$ denote a $\frac{K}{\delta} \times \frac{K}{\delta} \times \frac{K}{\delta^2}$-box. We shall consider the average integral of $\phi_\beta \phi_{\beta_i}$ over a $K\mathring{\beta}_i$-box B. First, partition B into $\mathring{\beta}_i$-boxes B_α. Then,

$$\int_B \phi_\beta^4 \phi_{\beta_i}^4 \sim \sum_\alpha [\phi_{\beta_i}|_{B_\alpha}^4 \int_{B_\alpha} \phi_\beta^4.$$

We next observe that $\mathring{\beta}_i$ can be covered by $O(K)$ $K\mathring{\beta} -$ boxes B'. This is because $\mathring{\beta}_i$ has direction given by a normal for β_i, which in turn makes an angle of at most δ with any normal of β. It follows that

$$\fint_{B_\alpha} \phi_\beta^4 \leq \max_{B'} \fint_{B'} \phi_\beta^4 < C(K_1^8).$$

We conclude that

$$\int_B \phi_\beta^4 \phi_{\beta_i}^4 \leq C(K_1^8) \sum_\alpha \int_{B_\alpha} \phi_{\beta_i}^4 = C(K_1^8) \int_B \phi_\beta^4 < C(K_1^8)^2 |B|.$$

We conclude this section with our final pointwise upper bound for $|Tf(x)|$.

$$|Tf| \leq \sum_{k=1}^{O(\log R)} K_1^{2k} \max_{E_k} [\sum_{\beta \in E_k} (\phi_\beta |T_{\beta_1} f|^{\frac{1}{3}} |T_{\beta_2} f|^{\frac{1}{3}} |T_{\beta_3} f|^{\frac{1}{3}})^2]^{\frac{1}{2}} +$$

$$(\log R) \max_{E_{\sqrt{R}}} [\sum_{\beta \in E_{\sqrt{R}}} (\phi_\beta |T_\beta f|)^2]^{\frac{1}{2}},$$

where E_k is a collection of at most 2^{k+2} disjoint 2^{-k}-cubes β; $\beta_1, \beta_2, \beta_3 \subset \beta$ are (2^{-k-1})-transverse and of that size as well; and (*) $⨍_B \phi_\beta^4 < C(K_1)^{8\frac{\log 2^k}{\log 2}} < R^{\frac{\log C(K_1)}{\log K}} = R^\epsilon$, for all $K\mathring{\beta}$-boxes B. We call the above inequality ††.

10.5 Main Argument

We have now reached the essential point of this section of the 2011 paper by Bourgain and Guth. We would like to find suitable L_3 and $L_{\frac{10}{3}}$-bounds for each term in ††, and then interpolate. To this end, we use a rescaled version of [BCT], introduce Kakeya-type pointwise estimates in order to apply Wolff's $L_{\frac{5}{3}}$ estimate, and finally apply these to suitable decompositions of our functions. We briefly explain as follows.

We assume that $|f| \leq 1$. Set $\delta = 2^{-k}$. We have by $[BCT]$ and rescaling that $\int_{B_R} |T_{\beta_1}||T_{\beta_2}||T_{\beta_3}| \leq \delta^2 R^\epsilon$ for all $\beta_1, \beta_2, \beta_3 \subset \beta$ as above with $l(\beta) = \delta$.

The L_3-bound is found using a simple application of Holder's inequality. Pointwise, we find that
$\max_{E_\delta} [\sum_{\beta \in E_k} (\phi_\beta |T_{\beta_1} f|^{\frac{1}{3}} |T_{\beta_2} f|^{\frac{1}{3}} |T_{\beta_3} f|^{\frac{1}{3}})^2]^{\frac{1}{2}} \leq \delta^{\frac{-1}{6}} (\sum_{\beta \in E_k} \phi_\beta^3 |T_{\beta_1} f||T_{\beta_2} f||T_{\beta_3} f|)^{\frac{1}{3}}$,
where in the latter expression we sum over all δ-cubes in Q. Thus, first taking a suitable partition of B_R and using [BCT] as above together with (*), we find that the

$$\|\max_{E_\delta} [\sum_{\beta \in E_k} (\phi_\beta |T_{\beta_1} f|^{\frac{1}{3}} |T_{\beta_2} f|^{\frac{1}{3}} |T_{\beta_3} f|^{\frac{1}{3}})^2]^{\frac{1}{2}} \|_{L_{p_0}(B_R)} \leq R^\epsilon \delta^{\frac{-1}{6}}.$$

Now, for the $L_{\frac{10}{3}}$ estimate, which we want to exhibit dependence solely upon a positive power of δ. Set $p_0 = \frac{10}{3}$. In the sequel, λ, μ will be dyadic

parameters such that $0 < \lambda < 1$ and $\mu \geq 1$. Define
$\phi_{\beta,\mu} = \phi_\beta \chi_{[\phi_\beta \sim \mu]}, \phi_{\beta,1} = \phi_\beta \chi_{\phi_\beta \leq 1}$, and similarly $g_{\beta,\lambda} = g_\beta \chi_{[g_\beta \sim \lambda\delta^2]}$, where $g_\beta = (|T_{\beta_1} f||T_{\beta_2} f||T_{\beta_3} f|)^{\frac{1}{3}}$. Using Holder's inequality as above and then making an appropriate use of our decompositions, we find that $\|\max_{E_\delta} [\sum\limits_{\beta \in E_k} (\phi_{\beta,\mu} g_{\beta,\lambda})^2]^{\frac{1}{2}}\|_{L_{p_0}(B_R)} \leq R^\epsilon \lambda^{\frac{1}{10}} \mu^{\frac{-1}{5}}$.

Now, we seek an estimate where the signs on the powers of λ, μ are reversed. We first bound $\max_{E_\delta} [\sum\limits_{\beta \in E_k} (\phi_{\beta,\mu} g_{\beta,\lambda})^2]^{\frac{1}{2}}$ by $\mu(\sum\limits_{\beta \in E_k} (\phi_{\beta,\mu} g_{\beta,\lambda})^2)^{\frac{1}{2}}$, where the latter sum is over a δ-partition of Q.

In light of the Fourier support of $T_{\beta_i} f$, we have essentially:

$$|T_{\beta_i} f| \leq |T_{\beta_i} f| \star (\delta^4 \chi_{\mathring{\beta}}).$$

Thus,

$$g^2_{\beta,\lambda}(x) \lesssim \delta^4 \int (\omega^2 \chi_{[\omega \gtrsim \lambda \delta^2]})(z) \chi_{\mathring{\beta}}(x - z)\ dz,$$

where $\omega = \prod\limits_{i=1}^{3} [|T_{\beta_i} f| \star (\delta^4 \chi_{\mathring{\beta}})]^{\frac{1}{3}}$. Using [BCT], we have

$$\int_{B_R} \omega^2 \chi_{[\omega \gtrsim \lambda\delta^2]} \lesssim R^\epsilon \lambda^{-1}.$$

It follows that $g^2_{\beta,\lambda}(x)$ may be bounded by a sum $\sum_{k \in \Gamma} c_k \chi_{\mathring{\beta}+k}(x)$, where $\{T + k\}$ is a partition of $\mathbb{R}^3$ and $\sum_{k \in \Gamma} c_k \lesssim 1$. Using Wolff's Kakeya estimate, we then obtain:

$$\|(\sum_{\beta \in E_k} (\phi_{\beta,\mu} g_{\beta,\lambda})^2)^{\frac{1}{2}}\|_{L_{p_0}(B_R)} \lesssim R^\epsilon \lambda^{\frac{-1}{2}} \delta^{\frac{1}{10}}.$$

Therefore,

$$\|\max_{E_\delta} [\sum_{\beta \in E_k} (\phi_{\beta,\mu} g_{\beta,\lambda})^2]^{\frac{1}{2}}\|_{p_0} \leq R^\epsilon \min\ (\lambda^{\frac{1}{10}} \mu^{-1} 5, \lambda^{\frac{-1}{2}} \mu \delta^{\frac{1}{10}}).$$

Summing over λ, μ, we get:

$$\|\max_{E_\delta} [\sum_{\beta \in E_k} (\phi_\beta g_\beta)^2]^{\frac{1}{2}}\|_{L_{p_0}(B_R)} \leq R^\epsilon \delta^{\frac{1}{60}}.$$

The other term in (††) may be treated similarly.

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11 Bounds on oscillatory integral operators based on multilinear estimates, I

after J. Bourgain and L. Guth [1]
A summary written by Zane Li

Abstract

We summarize the Bourgain-Guth approach to proving restriction estimates in the particular case of the paraboloid in $\mathbb{R}^3$. We briefly mention how to tackle the higher dimensional cases.

11.1 Introduction

Let $S \subset \mathbb{R}^n$ be a smooth compact hypersurface and let $d\sigma$ be its surface measure. The restriction problem asks for which $q < \infty$, does one have the estimate

$$\|\widehat{f\,d\sigma}\|_{L^q(\mathbb{R}^n)} \lesssim_{q,n,S} \|f\|_{L^\infty(S,d\sigma)}. \tag{1}$$

The restriction conjecture, due to Stein states that this should occur for $q > \frac{2n}{n-1}$. Bourgain and Guth in [1] are able to prove new results for dimensions ≥ 5 (see the end for the precise range of q) when S also has positive definite second fundamental form. Their results in dimensions 3 and 4 corresponding to $q > 10/3$ and 3, respectively coincide with a previous bilinear L^2 approach due to Tao. However Bourgain and Guth's proof is different and allows for insertion of additional inputs such as Wolff's bound for the Kakeya maximal function. This allows them to improve the restriction exponent in dimension 3 to $q > 3.3$ (discussed in a later talk).

We outline the most basic Bourgain-Guth approach by proving (1) when $n = 3$, $q > 10/3$ and S is a compact subset of the paraboloid. The Tomas-Stein exponent in this case is $q = 4$ and so for the remainder of this note we will assume that $10/3 < q < 4$.

Let Ω be a fixed compact neighborhood of 0 in $\mathbb{R}^2$. For $x \in \mathbb{R}^3$, let

$$(Tf)(x) = \int_\Omega e^{i\phi(x,y)} f(y)\,dy = \int_S f(\xi)e^{ix\cdot\xi}\,d\sigma(\xi) = (\widehat{fd\sigma})(x)$$

where $S = \{(y_1, y_2, |y|^2) : y \in \Omega\}$ and $\phi(x, y) = x \cdot (y_1, y_2, |y|^2)$. Fix a q with $10/3 < q < 4$. For any ball B_R of radius R, let

$$Q_R := \max_{\|f\|_{L^\infty(S)}\leq 1} \|Tf\|_{L^q(B_R)}.$$

To show (1), it suffices to show that Q_R is bounded uniformly in R. However, from an epsilon removal lemma due to Tao, it suffices instead to show that $Q_R \lesssim R^\varepsilon$. We may assume that B_R is centered at the origin.

Let K and K_1 be such that $R^\varepsilon \gg K \gg K_1$ to be chosen later. Partition $\Omega = \bigcup_\alpha \Omega_\alpha = \bigcup_\tau \widetilde{\Omega}_\tau$ where Ω_α are balls of radius $1/K$ and $\widetilde{\Omega}_\tau$ are balls of radius $1/K_1$. The ball Ω_α lies below a $1/K$-cap U_α on the paraboloid. We will often identify α with the $1/K$-cap that lies above it.

11.2 Preliminary tools

We mention two crucial tools which we will use. The proof of parabolic rescaling is just a change of variables.

Lemma 1 (Parabolic Rescaling)**.** *If U is a cap of size ρ on S, then*

$$\| \int_U f(\xi) e^{ix\cdot\xi}\, d\sigma(\xi) \|_{L^q(B_R)} \lesssim \rho^{2-4/q} Q_{\rho R}.$$

Lemma 2 (Specific case of [2], Theorem 1.16)**.** *Given an $x \in S$, let x' be the normal vector of S at x. Suppose U_1, U_2, and U_3 are such that ω_1', ω_2', and ω_3' are non-coplanar for all $\omega_i \in U_i$ (in other words, there exists a uniform lower bound for $|\omega_1' \wedge \omega_2' \wedge \omega_3'|$ for all $\omega_i \in U_i$). Then for each $\varepsilon > 0$ and $q \geq 3$*

$$\| \prod_{j=1}^{3} \int_{U_i} f(\xi) e^{ix\cdot\xi}\, d\sigma(\xi) \|_{L^{q/3}(B(0,R))} \lesssim_{q,n,U_1,U_2,U_3} R^\varepsilon \prod_{i=1}^{3} \|f\|_{L^2(U_i)}^{q/3}.$$

11.3 Local restriction for $q > 10/3$

Let y_α be the center of Ω_α. Let

$$Tf = \sum_\alpha e^{i\phi(x,y_\alpha)} \left(\int_{\Omega_\alpha} e^{i(\phi(x,y)-\phi(x,y_\alpha))} f(y)\, dy \right) = \sum_\alpha e^{i\phi(x,y_\alpha)} T_\alpha f$$

and

$$c_\alpha(x) := |(T_\alpha f)(x)|.$$

For each x, denote by α_* (potentially depending on x) the cap such that $c_{\alpha_*}(x) = \max_\alpha c_\alpha(x)$. For $1/K_1$-caps τ, similarly define $c_\tau(x)$ and τ_*. For each fixed $x \in B_R$, we will say a $1/K$-cap α is dominant if

$$c_\alpha(x) > K^{-4} c_{\alpha_*}(x).$$

For each $x \in B_R$, there are three possible cases :

I (Non-coplanar interaction): There are α_1, α_2, and α_3 which are dominant and

$$|y_{\alpha_1} - y_{\alpha_2}| \geq |y_{\alpha_1} - y_{\alpha_3}| \geq d(y_{\alpha_3}, \ell(y_{\alpha_1}, y_{\alpha_2})) > \frac{1000}{K} \tag{2}$$

where here $\ell(y_{\alpha_1}, y_{\alpha_2})$ is the line connecting y_{α_1} and y_{α_2}.

OR the negation of I and so one of the following must happen:

II (Non-transverse interaction): If α is such that $|y_\alpha - y_{\alpha_*}| > 1/K_1$, then it is not a dominant cap.

III (Transverse coplanar interaction): There is an α_{**} which is dominant and $|y_{\alpha_*} - y_{\alpha_{**}}| > 1/K_1$. Let $\ell = \ell(y_{\alpha_*}, y_{\alpha_{**}})$. Since we are in the negation of I, if α is such that $d(y_\alpha, \ell) > 1000K_1/K$, then α is not dominant.

By the uncertainty principle, $c_\alpha(x)$ is essentially constant on $K \times K \times K^2$ boxes oriented in the direction of the normal vector on S lying above y_α. If we restrict $x \in B(a, K)$ for some a, then essentially the dominant caps depend only on the box $B(a, K)$ (rather than x).

Assumption: *In the estimates that follow, we will ignore the contribution from all non-dominant caps. Removing this assumption will only require trivial bounds and is why we have a K^{-4} in our definition of dominant.*

11.3.1 Non-coplanar contribution

Suppose $x \in B_R$ and for such an x, the dominant caps satisfy Case I above. Then

$$|Tf(x)| \leq \sum_\alpha |T_\alpha f(x)| \lesssim K^2 c_{\alpha_*} \lesssim K^6 (\prod_{i=1}^{3} c_{\alpha_i}(x))^{1/3}$$

where here we note that α_i depends on x. Then

$$\int_{x \in \mathrm{I}} |Tf|^q \lesssim K^{6q} \sum_{\alpha_1, \alpha_2, \alpha_3 \in (2)} \int_{B_R} \prod_{i=1}^{3} |\int_{U_{\alpha_i}} f(\xi) e^{ix\cdot\xi}\, d\sigma(\xi)|^{q/3}\, dx$$

where the sum removes the dependence of α_i on x. Since $q \geq 3$, Bennett-Carbery-Tao followed by $\|f\|_{L^\infty(S, d\sigma)} \leq 1$ gives that the above is $\lesssim K^{O(1)} R^\varepsilon$.

11.3.2 Non-transverse contribution

Suppose $x \in B_R$ and for such an x, the dominant caps satisfy Case II. This implies that all dominant caps are within $1/K_1$ of α_*. Ignoring the contribution for the non-dominant caps, we essentially have

$$|Tf(x)| \le |\sum_{\alpha:|y_\alpha - y_{\alpha*}| \le \frac{10}{K_1}} e^{i\phi(x,y_\alpha)} T_\alpha f(x)| \lesssim c_{\tau_*}(x)$$

since there are only $O(1)$ many τ in $\bigcup_{\alpha:|y_\alpha - y_{\alpha*}| \le 10/K_1} \Omega_\alpha$. Here once again we note that τ_* may depend on x. Parabolic rescaling then gives

$$\int_{x \in \mathrm{II}} |Tf|^q \lesssim \sum_\tau \int_{B_R} c_\tau(x)^q \, dx \lesssim K_1^{6-2q} Q^q_{R/K_1}.$$

11.3.3 Transverse coplanar case

Fix an $x \in B(a, K)$ for some a. For this x, suppose the dominant caps satisfy Case III. We can regard which caps are dominant, α_*, α_{**}, and $\ell = \ell(y_{\alpha_*}, y_{**})$ to be independent of x (but may depend on $B(a, K)$).

Assumption: *Suppose that for all $x \in B(a, K)$ satisfying Case III, there are two $1/K_1$-caps τ_1, τ_2 such that $d(\widetilde{\Omega}_{\tau_1}, \widetilde{\Omega}_{\tau_2}) > 10000/K_1$ and $|Tf(x)| \le 10K_1^2 \min(c_{\tau_1}(x), c_{\tau_2}(x))$. Furthermore assume that τ_i are independent of x. The only other possibility for a given $x \in B(a, K)$ is $|Tf(x)| \lesssim \max_\tau c_\tau(x)$, but this is a non-dominant contribution. The dependence of τ_i on x can be removed with a maximum, paying a price of $O(K_1^2)$.*

With this assumption, we then have

$$|Tf(x)| \lesssim K_1^2 \prod_{i=1}^{2} |\int_{\widetilde{\Omega}_{\tau_i}} e^{i\phi(x,y)} f(y) \, dy|^{1/2}.$$

Let $\mathcal{L}$ be the $1000K_1/K$ neighborhood of ℓ intersected with Ω and let $\mathcal{L}_i = \widetilde{\Omega}_{\tau_i} \cap \mathcal{L}$. Since $d(\widetilde{\Omega}_{\tau_1}, \widetilde{\Omega}_{\tau_2}) > 10000/K_1$, $d(\mathcal{L}_1, \mathcal{L}_2) > 1/K_1$. Since we are in Case III, ignoring contributions from $1/K$-caps far away from ℓ, we essentially have

$$|\int_{\widetilde{\Omega}_{\tau_i}} e^{i\phi(x,y)} f(y) \, dy| \lesssim |\sum_{\Omega_\alpha \subset \mathcal{L}_i} e^{i\phi(x,y_\alpha)} T_\alpha f(x)|.$$

Since $\mathcal{L}_i$ may be different for different $B(a,K)$,

$$\int_{x\in \mathrm{III}} |Tf|^q \lesssim \sum_a K_1^{2q} \int_{B(a,K)} \prod_{i=1}^{2} |\sum_{\Omega_\alpha \subset \mathcal{L}_i} e^{i\phi(x,y_\alpha)} T_\alpha f|^{q/2}\, dx \tag{3}$$

where $\sum_a$ denotes the sum over a partition of B_R into balls of radius K. As $q < 4$, from Holder's inequality, each integral in the sum above is

$$\leq K^{3(1-q/4)} \bigg(\int_{B(a,K)} \prod_{i=1}^{2} |\sum_{\Omega_\alpha \subset \mathcal{L}_i} e^{i\phi(x,y_\alpha)} T_\alpha f|^2\, dx \bigg)^{q/4}. \tag{4}$$

Since $\mathcal{L}_i$ is independent of x, the inside integral above is bounded by

$$\sum_{\substack{\Omega_{\alpha_1}, \Omega_{\alpha_2} \subset \mathcal{L}_1 \\ \Omega_{\alpha_1'}, \Omega_{\alpha_2'} \subset \mathcal{L}_2}} \bigg| \int_{B(a,K)} T_{\alpha_1} f \overline{T_{\alpha_2} f T_{\alpha_1'} f} T_{\alpha_2'} f e^{i[\phi(x,y_{\alpha_1}) - \phi(x,y_{\alpha_2}) - \cdots]}\, dx \bigg|. \tag{5}$$

From $\phi(x,y)$ and some computation, the only $(\alpha_1, \alpha_2, \alpha_1', \alpha_2')$ which contribute are those such that $|y_{\alpha_1} - y_{\alpha_2}|, |y_{\alpha_1'} - y_{\alpha_2'}| \lesssim K_1^2/K$. Therefore (5) is

$$\lesssim K_1^{O(1)} \sum_{\substack{\Omega_{\alpha_1} \subset \mathcal{L}_1 \\ \Omega_{\alpha_2} \subset \mathcal{L}_2}} \int_{B(a,K)} |(T_{\alpha_1} f)(x)|^2 |(T_{\alpha_2} f)(x)|^2\, dx.$$

Using this, the fact that $|T_{\alpha_i} f|$ is essentially constant on $B(a,K)$, and Holder, we have that (4) is bounded by

$$K_1^{O(1)} \int_{B(a,K)} \prod_{i=1}^{2} (\sum_{\Omega_{\alpha_i} \subset \mathcal{L}_i} |T_{\alpha_i} f|^2)^{q/4}\, dx \lesssim K_1^{O(1)} K^{q/2-1} \sum_\alpha \|T_\alpha f\|_{L^q(B(a,K))}^q.$$

Parabolic rescaling then gives that (3) is

$$\lesssim K_1^{O(1)} K^{q/2-1} \sum_\alpha \|T_\alpha f\|_{L^q(B_R)}^q \lesssim K_1^{O(1)} K^{5-3q/2} Q_{R/K}^q.$$

11.3.4 The endgame

Combining the results from all three cases gives that

$$Q_R \lesssim K^{O(1)} R^\varepsilon + K_1^{-2(1-\frac{3}{q})} Q_{R/K_1} + K_1^{O(1)} K^{\frac{5}{q}-\frac{3}{2}} Q_{R/K}.$$

For $q > 10/3$, $1 - 3/q > 0$ and $5/q - 3/2 < 0$ and so choosing first K_1 sufficiently small and then K appropriately shows that $Q_R \lesssim R^\varepsilon$ for $10/3 < q < 4$.

11.4 Higher dimensional remarks

The method of proving higher dimensional restriction estimates is similar. Once again it suffices to prove a local restriction estimate up to epsilon losses. Then for each $x \in B_R$, one studies where are the dominant caps. The dominant caps are either non-coplanar in which case we use Bennett-Carbery-Tao, or the normal vectors of the dominant caps lie within a $1/K$ neighborhood of an $(n-1)$-dimensional subspace. This decreases the dimension by 1. Partition the $1/K$ neighborhood into slightly larger caps of size $1/K'$ with $K' \ll K$ and ask for the dominant ones. Repeat this process and either we use Bennett-Carbery-Tao again or reduce the dimension by 1. Continuing this process, we obtain that $Q_R \lesssim R^{\varepsilon}$ in n-dimensions provided q satisfies

$$q > 2 \max_{2 \le k \le n} \min\left(\frac{k}{k-1}, \frac{2n-k+1}{2n-k-1}\right) = \begin{cases} \frac{2(4n+3)}{4n-3} & \text{if } n \equiv 0 \pmod 3 \\ \frac{2n+1}{n-1} & \text{if } n \equiv 1 \pmod 3 \\ \frac{4(n+1)}{2n-1} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

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12 On the Erdös distinct distance problem in the plane, Part II

after Larry Guth and Nets Hawk Katz [GK]
A summary written by José Madrid

Abstract

In [E] Paul Erdös posed the question: how few distinct distances are determined by N points in the plane. He conjectured that for any arrangement of N points, the number of distinct distances is $\gtrsim \frac{N}{\sqrt{\log N}}$. Recently Larry Guth and Nets Katz [GK] proved that the number of distinct distances is $\gtrsim \frac{N}{\log N}$, we will disscus their strategy and main ideas.

12.1 Results and Strategy

Erdös Conjecture. A set of N points in the plane determines $\gtrsim \frac{N}{\sqrt{\log N}}$ distinct distances.
Erdös observed that if N points are arranged in a square grid, then the number of distinct distances is $\sim \frac{N}{\sqrt{\log N}}$. The next teorem is the main result in [GK], it solved the Erdös conjecture (up to a small gap of $\sqrt{\log N}$).

Theorem 1 (Main Theorem)**.** *A set of N points in the plane determines $\gtrsim \frac{N}{\log N}$ distinct distances.*

Guth and Katz's seminal work [GK] was based on several novel ideas. Some of the fundamental were the following:

(i) A reduction from the distinct distances problem to a problem about line intersections in $\mathbb{R}^3$. This part is usually referred to as the Elekes-Sharir framework.

(ii) Polynomial partitioning [see Lemma 8].

(iii) Analytic geometry tools related to ruled surfaces, such as flecnode polynomials (It is fundamental to prove Theorem 4).

(iv) Analysis of *joints* (triple intesections).

12.2 Elekes-Sharir framework

Given a set P of N points, we define

$$d(P) := \{d(p,q)\}_{p,q \in P, p \neq q},$$

$$Q(P) := \{(p_1,p_2,p_3,p_4) \in P^4, d(p_1,p_2) = d(p_3,p_4) \neq 0\},$$

and

G denotes the group of positively oriented rigid motions of the plane.

By Hölder inequality we have

$$|d(p)| \geq \frac{N^4 - 2N^3}{|Q(P)|}. \tag{1}$$

Moreover, we can see $Q(P)$ as a subset of G by the map $E : Q(P) \to G$ given by $(p_1,p_2,p_3,p_4) \to g$, where g is the unique unique $g \in G$ such that $g(p_1) = p_3$ and $g(p_2) = p_4$. With an elementary computation we obtain

$$|Q(P)| = \sum_{k=2}^{N} (2k-2)|G_k(P)|. \tag{2}$$

Here $G_k(P) = \{g \in G, |P \cap gP| \geq k\}$. Thus by (1) and (2) to get the expected lower bound for $d(P)$ is enough to get a nice upper bound for any $G_k(P)$.

12.3 Geometrical Ideas – Algebraic Topology

We can denote by G^{trans} de transalations and by $G' = G \setminus G^{trans}$. Thus

$$G_k(P) = (G_k(P) \cap G^{trans}) \cup (G_k(P) \cap G') = G_k^{trans}(P) \cup G'_k(P).$$

With an elementary computation we get

$$|G_k^{trans}(P)| \lesssim N^3 k^{-2}. \tag{3}$$

The main point is to bound $G'_k(P)$. To do this we use incidence geometry, we start with the following observation: For any points $p, q \in \mathbb{R}^2$ we define the set

$$S_{p,q} := \{g \in G, g(p) = q\}.$$

Each $S_{p,q}$ is a smooth 1-dimensional curve in the 3-dimensional Lie group G. the curves $S_{p,q}$ are closely related to the sets $G_k(P)$.

Lemma 2. *A rigid motion g lies in $G_k(P)$ if and only if it lies in at least k of the curves $\{S_{p,q}\}_{p,q \in P}$.*

Every element in G' is determinated by a fixed point (x, y) and an angle θ about the fixed point with $0 < \theta < 2\pi$, considering the map $\rho : G' \to \mathbb{R}^3$ given by

$$\rho(x, y, \theta) = (x, y, \cot(\frac{\theta}{2})),$$

and using the Lemma 2 the problem bounding $G'_k(P)$ can be reduced in an incidence problem in $\mathbb{R}^3$ as a consequence of the following observation.

Proposition 3. *Given $p = (p_x, p_y)$ and $q = (q_x, q_y)$ points in $\mathbb{R}^2$. Then with ρ as above, the set $\rho(S_{p,q} \cap G')$ is a line in $\mathbb{R}^3$.*

An inequality like (3) for $G'_k(P)$ will be a consequence of the two next results on incidence geometry.

Theorem 4. *Let σ any set of N^2 lines in $\mathbb{R}^3$ for which no more than N lie in a common plane and no more than $O(N)$ lie in a common regulus. Then the number of points of intersections of two lines in σ is $O(N^3)$.*

Theorem 5. *Let σ any set of N^2 lines in $\mathbb{R}^3$ for which no more than N lie in a common plane, and let k be a number $3 \leq k \leq N$. Let Ω_k be the set of points where at least k lines meet. Then*

$$|\Omega_k| \lesssim N^3 k^{-2}.$$

We remmber that a Regulus is a doubly ruled surface (every point lies in two lines in the regulus), and each line from one ruling intersects all the lines from the other ruling. An algebraic surface (in $\mathbb{R}^3$) is ruled if it contains a line passing through every point. A ruled surface is called singly-ruled if a generic point in the surface lies in only one line in the surface (some points in a singly-ruled surface may lie in two lines.) Except for reguli and planes, every irreducible ruled surface (in $\mathbb{R}^3$) is singly-ruled.

The following is the main geometrical lemma to prove Theorem 4.

Lemma 6. *Let p be a polynomial od degree less than N so that $p = 0$ is ruled and so that p is plane-free and regulus-free. Let σ_1 be a set of lines contained in the surface $p = 0$ with $|\sigma_1| \lesssim N^2$. Let Q_1 be the set of points of intersections of lines in σ_1. Then*

$$|Q_1| \lesssim N^3.$$

Using this lemma by a contradiction argument we can obtain Theorem 4. By an inductive argument we see that to prove Theorem 5 it is enough to prove

Proposition 7. *Let $k \geq 3$. Let σ be a set of L lines in $\mathbb{R}^3$ with at most B lines in any plane. Let Ω be a set of S points in $\mathbb{R}^3$ so that each point intersects between k and $2k$ lines of σ. Also we assume that there are $\geq \frac{1}{100}L$ lines in σ which each contain $\geq \frac{1}{100}SkL^{-1}$ points of Ω. Then the following inequality holds:*

$$|S| \leq C[L^{3/2}K^{-2} + LBk^{-3} + Lk^{-1}].$$

The main element to prove this proposition is the following lemma which is a consequence of a discrete version of the Stone–Turkey generalized Ham Sandwich Theorem for finite sets of points in $\mathbb{R}^n$.

Lemma 8. *If σ is a set of S points in $\mathbb{R}^n$ and $J \geq 1$ is an integer, then there is a polynomial surface Z of degree $d \lesssim 2^{J/n}$ with the following peoperty. The complement $\mathbb{R}^n \setminus Z$ is the union of 2^J open cells O_i, and each cell contains $\lesssim 2^{-J}S$ points of σ.*

Remark 9. *Theorem 4 and Theorem 5 were essentially conjectured by Elekes and Sharir [ES], with a different coordinates system. In case $k = 3$, Theorem 5 was proven in [EKS]*

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13 An elementary proof of the Hasse-Weil theorem for hyperelliptic curves.

after S. Stepanov [5] and the notes of W. Schmidt [4]
A summary written by Dominique Maldague

Abstract

Stepanov gives an elementary proof of the Hasse-Weil theorem about the number of solutions of the hyperelliptic congruence $y^2 = f(x)(\mod p)$, where the polynomial $f(x)$ has odd degree. We follow the notes of Schmidt which summarize the results of Stepanov.

13.1 The problem: hyperelliptic equations.

We consider the number of points (x, y) that satisfy the equation $y^2 = f(x)$ where f is in the polynomial ring $\mathbb{F}[x]$ of a finite field $\mathbb{F}$. Since $\mathbb{F}$ is contained in larger fields K, the number of points depends on the size of the field we are working in.

Recall some of facts about finite fields. If $\mathbb{F}$ is a finite field, then it has a non-zero characteristic which must be a prime p. This means it contains an isomorphic copy of $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$. We can regard a field as a vector space over its base field with dimension d, meaning that $\mathbb{F}$ has p^d elements. To construct such a finite field, let $h(x)$ be a monic, irreducible polynomial in $\mathbb{Z}_p[x]$ of degree d. Then $\mathbb{F}_{p^d} \cong \mathbb{Z}_p/(h(x))$ as fields.

Let $n \geq 3$ be an odd number. We fix a field $\mathbb{F}_{p^r}$ with p^r elements where r is any natural number and $p > 9n^2$ is a prime. Let $q = p^r$, so $\mathbb{F}_{p^r} = \mathbb{F}_q$. Consider the equation

$$Y^2 = f(X) \tag{1}$$

where $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ is a polynomial with coefficients in $\mathbb{F}_q$ is an example of a hyperelliptic equation since $\deg f = n \geq 3$.

If we fix an algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$ (which, by definition, contains all points $(x, y) \in \overline{\mathbb{F}_q} \times \overline{\mathbb{F}_q}$ satisfying $y^2 = f(x)$), for each $s \geq 0$ there is a unique intermediate field $\mathbb{F}_{q^s}$ of size q^s so that $\mathbb{F}_q \subset \mathbb{F}_{q^s} \subset \overline{\mathbb{F}}$. For such intermediate extensions $\mathbb{F}_{q^s}$, we define the set

$$C(\mathbb{F}_{q^s}) = \{(x, y) \in \mathbb{F}_{q^s} \times \mathbb{F}_{q^s} : y^2 = f(x)\}.$$

The number of non-zero squares in $\mathbb{F}_{q^s}$ is $\frac{q^s-1}{2}$. For each $x \in \mathbb{F}_{q^s}$ so that $f(x)$ is a non-zero square, there are two solutions $(x, y), (x, -y) \in C(\mathbb{F}_{q^s})$. Thus we might expect $|C(\mathbb{F}_{q^s})|$ to be roughly q^s. Stepanov proves that this is the case asymptotically in the following theorem.

Theorem 1. *Let $s \geq 0$. Then*

$$|C(\mathbb{F}_{q^s}) - q^s| \leq (n-1)\sqrt{q^s}. \tag{2}$$

13.2 Strategy for counting: groups of solutions.

Let $\mathbb{F}_{q^s}^*$ denote the multiplicative group $\mathbb{F}_{q^s} \setminus \{0\}$ of order $q^s - 1$. Note that every element of $x \in \mathbb{F}_{q^s}^*$ satisfies $x^{q^s-1} = 1$, and so every element of $\mathbb{F}_{q^s}$ is a solution to $X^{q^s} - X \in \mathbb{F}_{q^s}[X]$. Since q is an odd number, we have the factorization

$$X^{q^s} - X = X(X^{\frac{q^s-1}{2}} - 1)(X^{\frac{q^s-1}{2}} + 1)$$

and the corresponding partition of $\mathbb{F}_{q^s}$:

$$\mathbb{F}_{q^s} = \{x \in \mathbb{F}_{q^s} : f(x) = 0\} \cup \{x \in \mathbb{F}_{q^s} : f(x)^{\frac{q^s-1}{2}} = 1\} \cup \{x \in \mathbb{F}_{q^s} : f(x)^{\frac{q^s-1}{2}} = -1\}.$$

Let $N_0 = |\{x \in \mathbb{F}_{q^s} : f(x) = 0\}|$, $N_1 = |\{x \in \mathbb{F}_{q^s} : f(x)^{\frac{q^s-1}{2}} = 1\}|$, and $N_{-1} = |\{x \in \mathbb{F}_{q^s} : f(x)^{\frac{q^s-1}{2}} = -1\}|$. Note that $q^s = N_0 + N_1 + N_{-1}$. The following lemma will allow us to group the solutions of $y^2 = f(x)$ according to this partition.

Lemma 2. *Let $x \in \mathbb{F}_{q^s}$. If $f(x) = 0$, then there is exactly one corresponding solution $(x, 0) \in \mathbb{F}_{q^s} \times \mathbb{F}_{q^s}$ satisfying $0 = f(x)$. If $f(x)^{\frac{q^s-1}{2}} = 1$, then there are exactly two solutions $(x, y), (x, -y) \in \mathbb{F}_{q^s} \times \mathbb{F}_{q^s}^*$ satisfying $y^2 = f(x)$. If $f(x)^{\frac{q^s-1}{2}} = -1$, then there exists no element $y \in \mathbb{F}_{q^s}$ satisfying $y^2 = f(x)$.*

As a consequence of the lemma, we have $|C(\mathbb{F}_{q^s})| = N_0 + 2N_1$. See the discussion following Lemma 2D in Chapter 1 of [4] for a proof of the lemma.

13.3 Tools of the polynomial method.

The idea of Stepanov is to construct an auxiliary polynomial of degree r having zeros of high multiplicity (say at least l) at each x-coordinate of points in $C(\mathbb{F}_{q^s})$. In doing so, we easily observe

$$|C(\mathbb{F}_{q^s})| \leq 2rl^{-1},$$

two being the largest multiplicity of the coordinate x in $C(\mathbb{F}_{q^s})$. This inequality turns out to be so strong that it gives the upper bound of the theorem. A trick involving the partitioning we did in the previous section then gives the lower bound in a similar way.

13.3.1 Vanishing lemma.

To produce polynomials vanishing to large order, we wish to use derivatives to characterize when this occurs. However, in characteristic $p > 0$, it is no longer true that if $D^k g(a) = 0$ for $0 \leq k < l$, then $(X - a)^l$ divides the polynomial $g(X)$. For example, if $g(X) = X^p$, all derivatives vanish at $x = 0$, yet $g(X)$ has a zero only of order p at $x = 0$. The solution to this is to consider other differential operators.

Definition 3. *Let K be any field. For any $k \geq 0$, the k-th hyperderivative (also Hasse derivative) is the linear operator $E^k : K[X] \to K[X]$ defined by*

$$E^l(X - c)^t = \binom{t}{l}(X - c)^{t-l}$$

for all $n \geq 0$, and extended to $K[X]$ by linearity. We also write $E = E^1$ (but beware that $E^k \neq E \circ E \circ \cdots \circ E$).

If D is the differentiation operator, then $D^l = l!E^l$. Hyperderivatives have the advantage described in the following lemma.

Lemma 4. *Let $f \in K[X]$ and $a \in K$. Suppose that $(E^k g)(a) = 0$ for all $k < l$. Then g has a zero of order $\geq l$ at a, i.e., is divisible by $(X - a)^l$.*

13.3.2 Parameter counting.

Let $f \in \mathbb{F}_{q^s}[X]$ be as above, so $n = \deg f$. Consider for $a \in \mathbb{F}_{q^s}$ the set $\mathcal{S}_a = \{x \in \mathbb{F}_{q^s} : f(x) = 0 \quad \text{or} \quad f(x)^{\frac{q-1}{2}} = a\}$.

Lemma 5. *Let $a \in \mathbb{F}_{q^s}$ and let l be an integer $n < l \leq q^s/8$. Then there exists a non-zero polynomial $r \in \mathbb{F}_{q^s}[X]$ of degree*

$$\deg(r) < \frac{1}{2}(q^s - 1)l + 2nl(l - 1) + nq^s$$

which has a zero of order at least l at all points $x \in \mathcal{S}_a$.

The proof proceeds via the method of indeterminate coefficients (i.e. parameter counting). We consider a polynomial of the special form

$$r(X) = f^l(X) \sum_{0 \leq j < J} (r_j(X) + s_j(X) f(X)^{\frac{q^s-1}{2}}) X^{jq^s}$$

for some polynomials $r_j, s_j \in \mathbb{F}_{q^s}[X]$, to be constructed, each of which has bounded degree and J to be chosen. By requiring that $E^k r(x) = 0$ for each $x \in \mathcal{S}_a$, we obtain a system of linear equations and then count the parameters to ensure we have a nontrivial solution.

13.4 Proof of the (preliminary) main theorem.

Recall that $p > 9n^2$ so $\sqrt{q^s}/2 > 3n/2$. Fix $l = [\sqrt{q^s}/2] + 1$, where $[\cdot]$ denotes the floor function, and $a = 1$ in the parameter counting lemma. Since the auxiliary polynomial r is non-zero and vanishes to order l for each $x \in \mathcal{S}_1$, we have

$$\begin{aligned} |\mathcal{S}_1| &\leq l^{-1} \deg(r) < \frac{1}{2}(q^s - 1) + 2n(l-1) + l^{-1} n q^s \\ &\leq \frac{q^s}{2} + 3n\sqrt{q^s}. \end{aligned}$$

Note that $|C(\mathbb{F}_{q^s})| = N_0 + 2N_1$, and $|\mathcal{S}_1| = N_0 + N_1$, so we obtain the upper bound

$$|C(\mathbb{F}_{q^s})| \leq 2|\mathcal{S}_1| < q^s + 6n\sqrt{q^s}.$$

If we take $a = -1$ and l as above, then we can use the parameter counting lemma again to conclude that

$$N_0 + N_{-1} = |\mathcal{S}_{-1}| < \frac{q^s}{2} + 6n\sqrt{q^s}$$

Since $|\mathcal{S}_{-1}| = N_0 + N_{-1}$ and $q^s = N_0 + N_1 + N_{-1}$, we can rearrange the above line to obtain the lower bound

$$N_0 + 2N_1 \geq 2N_1 \geq q^s - 6n\sqrt{q^s}.$$

Putting together the upper and lower bounds the weaker version of (1)

$$||C(\mathbb{F}_{q^s})| - q^s| \leq 6n\sqrt{q^s}, \tag{3}$$

which we will improve in the following section.

13.5 Relation to a zeta-function.

We homogenize the equation $Y^2 = f(X)$ as follows: for a third variable Z, consider solutions to

$$Z^{n-2}Y^2 = X^n + a_{n-1}X^{n-1}Z + \cdots + a_0 Z^n.$$

If we look for solutions $(x : y : z)$ in the projective space $P^3(\mathbb{F}_{q^s})$ of equivalence classes under scalar multiplication, for $z \neq 0$, we get a correspondence with the number $C(\mathbb{F}_{q^n})$ defined earlier. If $z = 0$, then we must have $x = 0$, so $y \neq 0$ (since projective space is an equivalence class on $\mathbb{F}_{q^s} \times \mathbb{F}_{q^s} \times \mathbb{F}_{q^s} \setminus \{(0,0,0)\}$). This is the reason we see the quantity $C(\mathbb{F}_{q^s})+1$ appear in the following definition.

Definition 6. *The zeta-function associated to our curve $y^2 = f(x)$ is*

$$Z(u) = \exp\left(\sum_{s=1}^{\infty} \frac{(|C(\mathbb{F}_{q^s})|+1)u^s}{s}\right)$$

One part of the Weil conjectures described in [1] says that $Z(u)$ is a rational function of the form

$$Z(u) = \frac{\prod_{i=1}^{n-1}(1-\alpha_i u)}{(1-u)(1-qu)},$$

so for $\alpha_i \in \mathbb{C}$. By taking a logarithmic derivative of both expressions of $Z(u)$ and matching coefficients, we conclude that

$$|C(\mathbb{F}_{q^s})| = q^s - \sum_{i=1}^{n-1} \alpha_i^s. \tag{4}$$

Rewriting (3) using this identity, we have for each $s \geq 0$ that

$$||C(\mathbb{F}_{q^s})| - q^s| = \left|\sum_{i=1}^{n-1} \alpha_i^s\right| \leq 6n\sqrt{q^s}.$$

It follows from an elementary lemma about complex numbers (see p. 138 of [3]) that $|\alpha_i| \leq \sqrt{q}$ for each i so that the above inequality automatically improces to (2).

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14 Bounds on oscillatory integral operators based on multilinear estimates (part III)

Jean Bourgain and Larry Guth [1]
A summary written by Darío Mena

Abstract

We apply the Benett-Carbery-Tao multilinear restriction estimate in order to bound restriction operators and more general oscillatory integral operators. In section 5, the estimate is proved for a phase function with variable coefficient.

14.1 The Variable Coefficient Case

Consider the operator

$$(T_\lambda f)(x) = \int e^{i\lambda\psi(x,y)} f(y)\,dy, \quad \psi(x,y) = x\cdot(y, \langle Ay, y\rangle + O(|y|^3)) + O(|x|^2|y|^2), \tag{1}$$

where A is a non-degenerate quadratic form, x is in a neighborhood of $0 \in \mathbb{R}^d$ and y in a neighborhood of $0 \in \mathbb{R}^{d-1}$. The main result of this section is the following

Theorem 1. *Consider the operator (1). Then, if the index p satisfies*

$$\begin{cases} p > \dfrac{2(4d+3)}{4d-3} & \text{if } d \equiv 0 \mod 3, \\ p > \dfrac{2d+1}{d-1} & \text{if } d \equiv 1 \mod 3, \\ p > \dfrac{4(d+1)}{2d-1} & \text{if } d \equiv 2 \mod 3, \end{cases}$$

we have the inequality

$$\|T_\lambda f\|_{L^p_{loc}} \le C_p \lambda^{-d/p} \|f\|_\infty. \tag{2}$$

The idea is to modify the arguments from sections 2 and 3. By rescaling, we can replace $\lambda\psi$, by ϕ defined by

$$\phi(x,y) = x\cdot(y, \langle Ay, y\rangle + O(|y|^3)) + \lambda\phi_\nu(\tfrac{x}{\lambda}, y), \quad \phi_\nu \text{ at least quadratic.} \tag{3}$$

To obtain the operator $(Tf)(x) = \int e^{i\phi(x,y)} f(y)\, dy$, for $|x| < o(\lambda)$.

Let $Z(x,y) = \partial_{y_1}(\nabla_x \phi) \wedge \cdots \wedge \partial_{y_{d-1}}(\nabla_x \phi)$, and consider $U_1, \ldots, U_k$ small caps satisfying the transversality condition

$$|Z_1(x, y^{(1)}) \wedge \cdots \wedge Z_1(x, y^{(k)})| > c, \quad \text{for all } x \text{ and } y^{(i)} \in U_i \,.$$

Applying the variable coefficient case result in [1], for the operators

$$T_i f(x) = \int_{U_i} e^{i\phi_i(x,y)} f(y)\, dy, \quad (1 \le i \le k),$$

we have for $q = \frac{2k}{k-1}$ and x restricted to $|x| < o(|\lambda|)$,

$$\left\| \left(\prod_{i=1}^{k} |T_i f_i| \right)^{1/k} \right\| \le C_\varepsilon \lambda^\varepsilon \left(\prod_{i=1}^{k} \|f_i\|_2 \right)^{1/k} . \tag{4}$$

In this particular instance of the result in [1], the factor λ^ε can be removed by increasing q to $q > \frac{2k}{k-1}$. So, under the same transversality conditions, we get

$$\left\| \left(\prod_{i=1}^{k} |T_i f_i| \right)^{1/k} \right\| \le C_\varepsilon \left(\prod_{i=1}^{k} \|f_i\|_2 \right)^{1/k}, \qquad \text{for } q > \tfrac{2k}{k-1}. \tag{5}$$

Adjusting the parabolic rescaling of sections 2 and 3, we need to consider a more general operator $(Tf)(x) = \int e^{i\phi(x,y)} f(y)\, dy$, with $x \in Q = \{|x_i| \le R_1 \text{ for } 1 \le i \le d-1, |x_d| < R\}$, with $R \le R_1$, and the phase function is given by

$$\phi(x,y) = x \cdot (y, \langle Ay, y\rangle + O(|y|^3)) + R\phi_\nu \left(\frac{x_1}{R_1}, \cdots, \frac{x_{d-1}}{R_1}, \frac{x_d}{R}; y \right). \tag{6}$$

To show (5) for this particular operator, proceed by subdividing Q into R-cubes Q_s and the y-domain Ω into cubes Ω_α of size approximately $1/R$, centered at y_α. For a fix Q_s, if $\bar{x}$ denotes its center, taking

$$g_i(y) = e^{i\phi(x,y)} \int_{\Omega_\alpha} f_i(y) e^{i[\phi(\bar{x},y) - \phi(\bar{x},y_\alpha)]},$$

and $\eta(z,y) = \phi(\bar{x}+z, y) - \phi(\bar{x}, y)$, we can write

$$|T_i f_i|(\bar{x}+z) \approx R^{d-1} \left| \int e^{i\eta(z,y)} g_i(y)\, dy \right|,$$

where the phase function η is such that you can apply 5 directly. Using this, the L^2-norms of the g_i, and other standard estimates, when we sum over s we get

$$\int_Q \left[\prod_1^k |T_i f_i|\right]^{q/k} < CR^{\frac{q(d-1)}{2}-d} \int_Q \prod_1^k \left[\sum_\alpha \left|\int_{\Omega_\alpha} f_i(y) e^{i\phi(x,y)}\, dy\right|^2\right]^{q/2k}.$$

By the previous steps, standard orthogonality and Hölder estimates, it is easy to see that the previous quantity is bounded by $C(\prod_1^k \|f_i\|_2^q)^{1/k}$, and so, we obtain the multilinear estimate (5) for the setting of (6).

At a local scale, the phase function ϕ in (6) can be linearized to reduce the problem to the restriction setting: For $x = a + z \in B(a, \rho)$, we can write $\phi(x, y) = \phi(a, y) + \psi(z, y) + \Omega(z, y)$, where Ω is easy to control, and ψ is given by

$$\psi(z, y) = z \cdot (y, \langle A' y, y\rangle + O(|y|^3)).$$

With this, and using the multilinear estimate, the analysis from sections 2 and 3 can be followed to obtain the proof of Theorem 1.

For d odd, the condition $p \geq \frac{2(d+1)}{d-1}$ may be the optimal range of validity for inequality (2). For d even, we have the following result

Theorem 2. *Consider the operator (1). For d even, we have*

$$\|T_\lambda f\|_{L^p_{loc}} \leq C_p \lambda^{-d/p} \|f\|_\infty, \qquad \text{for} \quad p > \frac{2(d+2)}{d}.$$

For the proof, we look again at the more general operator given by the phase function (6). Taking the value $k = \frac{d}{2} + 1$, gives the condition $q > \frac{2(d+2)}{d}$ in (5). The proof follows the procedures of sections 2 and 3. We restrict x to the ball $B(a, K)$, for K large, and subdivide the domain for y into balls Ω_α of size $\frac{1}{K}$. There are two cases, if we look at the operators

$$(T_\alpha f)(x) = \int_{\Omega_\alpha} e^{i\phi(x,y)} f(y)\, dy,$$

1) On B_K, $|Tf| < C(K)|T_{\alpha_i} f|$, for some $\alpha_1, \dots, \alpha_k$ such that $\Omega_{\alpha_1}, \dots, \Omega_{\alpha_k}$ satisfy the transversality condition. In this case, we can use the k-linear bound to control the collected contribution.

2) The failure of the first case, in which case $\#A \lesssim K^{k-2}$. By some crude orthogonality estimates and rescaling, we can obtain the desired result.

14.2 Some examples

This section introduces two examples in dimension $d = 3$, one of which gives the optimality of the index 10/3 in Theorem 1.

First, consider the hyperbolic example, given by the phase function

$$\psi(x, y) = -x_1y_1 - x_2y_2 + 2x_3y_1y_2 + x_3^2y_2^2.$$

We can restrict y to non-collinear disks $U_1, U_2, U_3 \subset \mathbb{R}^2$ to satisfy the transversality condition. By explicit computation of the Kakeya type sets associated to the phase function, we see that there is 2D-compression, since the the respective tubes are contained in the surface $x_2 = x_1x_3$.

Using the function $f(y) = e^{iy_1^2}$, the 2D-compression and stationary phase estimates imply

$$\int e^{i\lambda\psi(x,y)} f(y)\, dy \approx \int_{\text{loc}} e^{\lambda[(y_1+x_3y_2)^2 - x_1(y_1+x_3y_2)]}\, dy \sim \lambda^{-1/2}.$$

And this is less than $\lambda^{-3/q}$ for $q \geq 4$.

If we consider now the elliptic case, given by the phase function

$$\phi(x, y) = -x_1y_1 - x_2y_2 + \frac{1}{2}x_3^2y_1y_2 + \frac{1}{2}(x_3 + x_3^2)y_2^2.$$

In this case, we have the same 2D-compression, but the same construction is not possible. We use the function f given by

$$f(y) = \sum_{s<\sqrt{\lambda}} \sigma_s \mathbb{1}_{\left[\frac{s}{\sqrt{\lambda}}, \frac{s+c}{\sqrt{\lambda}}\right]}(y_2) e^{i\lambda \frac{s}{\sqrt{\lambda}} y_1},$$

where σ_s is a choice of signs and c is a small positive constant. By a choice of signs, and some algebraic manipulation, we can see that

$$\left\| \int e^{i\lambda\phi(x,y)} f(y)\, dy \right\|_q \gtrsim \left(\frac{1}{\lambda}\right)^{\frac{3}{4}+\frac{1}{2q}}.$$

And this is controlled by $\lambda^{-3/q}$, for $q \geq 10/3$.

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15 Decouplings for surfaces in $\mathbb{R}^4$

after J. Bourgain and C. Demeter [?]
A summary written by Martina Neuman

Abstract

We prove a sharp decoupling for nondegenerate surfaces in $\mathbb{R}^4$. This puts the progress in [2] on the Lindelöf hypothesis into a more general perspective.

15.1 Introduction

Consider a compact C^3 surface in $\mathbb{R}^4$:

$$\Psi(t,s) = (\psi_1(t,s),\ \dots\ ,\psi_4(t,s)), \qquad \psi_i : [0,1]^2 \to \mathbb{R}$$

which is a ssumed to satisfy the nondegeneracy condition

$$rank[\Psi_t(t,s), \Psi_s(t,s), \Psi_{tt}(t,s), \Psi_{ss}(t,s), \Psi_{ts}(t,s)] = 4,$$

for each t, s. Then:

Theorem 1. *For each $p \geq 2, g : [0,1]^2 \to \mathbb{C}$ and each ball $B_N \subset \mathbb{R}^4$ with radius N*

$$\begin{aligned} &\| \int_{[0,1]^2} g(t,s) e(x \cdot \Psi(t,s))\, dtds \|_{L^p(w_{B_N})} \\ &\qquad \leq D(N,p) (\sum_{\substack{\Delta \subset [0,1]^2 \\ l(\Delta) = N^{-1/2}}} \| \int_{\Delta} g(t,s) e(x \cdot \Psi(t,s))\, dtds \|^p_{L^p(w_{B_N})})^{1/p} \qquad (1) \end{aligned}$$

where the sum on the right hand side is over a partition of $[0,1]^2$ into squares Δ with side length $l(\Delta) = N^{-1/2}$, and

$$D(N,p) \lesssim_\epsilon N^{1/2-1/p+\epsilon},\ 2 \leq p \leq 6$$

$$D(N,p) \lesssim_\epsilon N^{1-4/p+\epsilon},\ p \geq 6.$$

Here, $e(z) = e^{2\pi i z}$, and for each ball B centered at c with radius R, w_B will denote the weight $w_B(x) = \frac{1}{(1+\frac{|x-c|}{R})^{100}}$. The theorem is essentially sharp as a standard computation with $g = 1_{[0,1]^2}$ reveals that

$$D(N,p) \gtrsim N^{1/2-1/p},\ 2 \le p \le 6$$

$$D(N,p) \gtrsim N^{1-4/p},\ p \ge 6.$$

The result of this type follows from interpolation. Note that $D(N,2) \lesssim N^0$ from the fact that $(\int_\Delta g(t,s)e(x\cdot\Psi(t,s))\,dtds)|_B$ are almost orthogonal, and for $p \ge 1$, we have the trivial upper bound $D(N,p) \lesssim N^{1-1/p}$ from the Cauchy-Schwarz inequality. Hence we will show that $D(N,6) \lesssim_\epsilon N^{1/3+\epsilon}$.

15.2 Preliminaries

15.2.1 Reduction to quadratic surfaces

The nondegeneracy condition means that the surface Ψ is locally nonflat in the following sense. For each $(t_0,s_0) \in [0,1]^2$ there is no unit vector $\gamma \in \mathbb{R}^4$ such that $|\langle\gamma, \Psi(t_0+\Delta t, s_0+\Delta s) - \Psi(t_0,s_0)\rangle| = O(|(\Delta t,\Delta s)|^3)$. This allows us to locally present the surface, with respect to an appropriate system of coordinates as

$$(t, s, A_1t^2 + 2A_2ts + A_3s^2, A_4t^2 + 2A_5ts + A_6s^2) + O(|(t,s)|^3)$$

where

$$rank \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \end{pmatrix} = 2. \tag{2}$$

We show that the result for a generic nondegenerate surface will follow if it holds for quadratic surfaces of the above type.

Let $\mathcal{N}_N(\Psi)$ denote the N^{-1} neighborhood of the surface $\Psi([0,1]^2)$ and $\mathcal{P}_N$ be an associated cover of $\mathcal{N}_N$ with N^{-1} neighborhoods τ of $\Psi(\Delta)$. Let f be Fourier supported in $\mathcal{N}_N(\Psi)$ and f_τ an appropriate smooth Fourier restriction of f to τ so that

$$f = \sum_{\tau\in\mathcal{P}_N} f_\tau.$$

Then we denote $K_{p,\Psi}(N)$ to be the best constant such that

$$\|f\|_{L^p(\mathbb{R}^4)} \le K_{p,\Psi}(N)(\sum_{\tau\in\mathcal{P}_N} \|f_\tau\|^p_{L^p(\mathbb{R}^4)})^{1/p}.$$

Proving $D(N,6) \lesssim_\epsilon N^{1/3+\epsilon}$ is equivalent to proving $K_{6,\Psi}(N) \lesssim_\epsilon N^{1/3+\epsilon}$. Now each τ, after a rescaling, can be mapped into $\mathcal{N}_{O(N^{1/3})}(\Psi_A)$ for some

$\Psi_A(t,s) = (t, s, A_1t^2 + 2A_2ts + A_3s^2, A_4t^2 + 2A_5ts + A_6s^2) + O(|(t,s)|^3)$ satisfying (2). This allows us to obtain the following factorization

$$K_{p,\Psi}(N) \le K_{p,\Psi}(N^{2/3}) \sup_{A\in\mathcal{A}} K_{p,\Psi_A}(N^{1/3}) \tag{3}$$

with

$$\mathcal{A} = \{A : |A_i| \le C_\Psi, \max\left\{\left|det\begin{pmatrix} A_1 & A_2 \\ A_4 & A_5\end{pmatrix}\right|, \left|det\begin{pmatrix} A_1 & A_3 \\ A_4 & A_6\end{pmatrix}\right|, \left|det\begin{pmatrix} A_3 & A_2 \\ A_6 & A_5\end{pmatrix}\right|\right\} \ge C_\Psi^{-1}\}.$$

The constant C_Ψ depends only on Ψ. Hence if we can show

$$\sup_{A\in\mathcal{A}} K_{p,\Psi_A}(N^{1/3}) \lesssim_{\epsilon, C_\Psi} N^{1/3+\epsilon}$$

then the desired result will follow from iterating (3).

Remark: This quadratic reduction showcases why the scale $l(\Delta) = N^{-1/2}$ is the right scale for nondegenerate surfaces in any dimension n, instead of the bigger scale $N^{-1/2n}$.

15.2.2 Transversality and a bilinear theorem

Denote $Eg(x) = \int g(t,s)e(x\cdot\Psi(t,s))\,dtds$. Fix $A \in \mathcal{A}$. Define $c_{1,A} = det\begin{pmatrix} A_1 & A_2 \\ A_4 & A_5\end{pmatrix}$, $c_{2,A} = det\begin{pmatrix} A_1 & A_3 \\ A_4 & A_6\end{pmatrix}$, $c_{3,A} = det\begin{pmatrix} A_6 & A_5 \\ A_3 & A_2\end{pmatrix}$.

Definition Let $\nu \le 1$. We say that two sets $S_1, S_2 \subset [0,1]^2$ are ν-transverse if

$$c_{1,A}(t_1-t_2)^2 + c_{2,A}(t_1-t_2)(s_1-s_2) + c_{3,A}(s_1-s_2)^2 \ge \nu$$

for each $(t_i, s_i) \in S_i$.
To take care of contributions from transverse and non-tranverse squares we have the following three results:

Theorem 2. *Let $R_1, R_2 \subset [0,1]^2$ be ν-transverse squares. Then for each*

$4 \leq p \leq \infty$ *and* $g_i : R_i \to \mathbb{C}$ *we have:*

$$\|(\prod_{i=1}^{2} \sum_{l(\Delta)=N^{-1/2}} |E_\Delta g_i|^2)^{1/4}\|_{L^p(w_{B_N})}$$

$$\lesssim_\nu N^{-4/p}(\prod_{i=1}^{2} \sum_{l(\Delta)=N^{-1/2}} \|E_\Delta g_i\|^2_{L^{p/2}(w_{B_N})})^{1/4}.$$

Proposition 3. *For* $K = 2^m \geq 1$*, consider the collection* Col_K *of the* K^2 *dyadic squares in* $[0,1]^2$ *with side length* K^{-1}*. For each* $R \in Col_L$*, there are* $O(K)$ *squares* $R' \in Col_K$ *which are* K^{-2}*-transverse to* R*.*

Proposition 4. *Let* $R_1, \ \dots \ , R_K$ *be pairwise disjoint squares in* $[0,1]2$ *with side length* K^{-1}*. Then for each* $2 \leq p \leq \infty$

$$\|\sum_i E_{R_i} g\|_{L^p(w_{B_K})} \lesssim_p K^{1-2/p}(\sum_i \|E_{R_i} g\|^p_{L^p(w_{B_K})})^{1/p}.$$

Remark: Since R_i are not required to be transverse, this last proposition allows us to deal with L^p contributions from non-transverse squares. We can't do better than this trivial decoupling for the non-transverse case.

15.3 Linear versus bilinear decoupling

We introduct a bilinear version of $D(N,p)$. Given $\nu \leq 1$, let $D_{multi}(N,p,\nu)$ be the smallest constant such that the bilinear decoupling

$$\||E_{R_1} g_1 E_{R_2} g_2|^{1/2}\|_{L^p(w_{B_N})} \leq D_{multi}(N,p,\nu)(\prod_{i=1}^{2} \sum_{l(\Delta)=N^{-1/2}} \|E_\Delta g_i\|^p_{L^p(w_{B_N})})^{1/2p}$$

holds true for all ν-tranverse squares $R_1, R_2 \subset [0,1]^2$ with arbitrary side lengths, all $g_i : R_i \to \mathbb{C}$ and all balls $B_n \subset \mathbb{R}^4$ with radius N.
Hölder's inequality shows that $D_{multi}(N,p,\nu) \leq D(N,p)$. The reverse inequality is also essentially true:

Theorem 5. *For each* $\nu \leq 1/10$ *and* $p \geq 2$ *there exists* $C_\nu > 0$ *and* $\epsilon(\nu,p)$ *with* $\lim_{\nu \to 0} \epsilon(\nu,p) = 0$ *such that for each* $N \geq 1$

$$D(N,p) \leq C_\nu N^{\epsilon(\nu,p)} \sup_{1 \leq M \leq N} (\frac{M}{N})^{1/p-1/2} D_{multi}(M,p,\nu). \tag{4}$$

The key step is to achieve the following inequality

$$\|E_{[0,1]^2} g\|^p_{L^p(w_{B_N})}$$
$$\leq C_p\Big(K^{p-2} \sum_{l(R)=1/K} \|E_R g\|^p_{L^p(w_{B_N})} + K^{4p} D_{multi}(N, p, K^{-2})^p \sum_{l(\Delta)=N^{-1/2}} \|E_\Delta g\|^p_{L^p(w_{B_N})}\Big).$$

It's in the reduction to scales $l(R) = K^{-2}$ that we need the estimates in **Thereom 2**, **Proposition 3** and **Proposition 4**.

15.4 Final part

For $p \geq 4$ define κ_p such that $\frac{2}{p} = \frac{1-\kappa_p}{2} + \frac{\kappa_p}{p}$. By using Hölder's inequality, Cauchy-Schwarz inequality and the almost orthogonality specific to L^2 when passing from scale $N^{-1/4}$ to $N^{-1/2}$, we obtain:

Proposition 6. *Let R_1, R_2 be ν-transvers squares in $[0,1]^2$ with arbitrary side lengths. We have that for each radius $R \geq N$, $p \geq 4$ and $g_i : R_i \to \mathbb{C}$*

$$\|(\prod_{i=1}^{2} \sum_{l(\tau)=N^{-1/4}} |E_\tau g_i|^2)^{1/4}\|_{L^p(w_{B_R})}$$
$$\lesssim_{\nu,p} N^{\kappa_p/2(1/2-1/p)} \|(\prod_{i=1}^{2} \sum_{l(\Delta)=N^{-1/2}} |E_\Delta g_i|^2)^{1/4}\|^{1-\kappa_p}_{L^p(w_{B_R})}$$
$$\times \big(\prod_{i=1}^{2} \sum_{l(\tau)=N^{-1/4}} \|E_\tau g_i\|^p_{L^p(w_{B_R})}\big)^{\kappa_p/2p}. \quad (5)$$

Cauchy-Schwarz inequality also gives the following trivial estimate

Lemma 7. *Consider two rectangles $R_1, R_2 \subset [0,1]^2$ with arbitrary side lengths. Assume g_i is supported on R_i. Then for $1 \leq p \leq \infty$ and $s \geq 2$*

$$\|(\prod_{i=1}^{2} |E_{R_i} g_i|)^{1/2}\|_{L^p(w_{B_R})} \leq N^{2^{-s}} \|(\prod_{i=1}^{2} \sum_{l(\tau_s)=N^{-2^{-s}}} |E_{\tau_s} g_i|^2)^{1/4}\|_{L^p(w_{B_R})}. \quad (6)$$

Parabolic scaling provides that for each square $R \subset [0,1]^2$ with side length $N^{-\rho}, \rho \leq 1/2$

$$\|E_R g\|_{L^p(w_{B_N})} \leq D(N^{1-2\rho}, p)\Big(\sum_{\substack{\Delta \subset R \\ l(\Delta)=N^{-1/2}}} \|E_\Delta g\|^p_{L^p(w_{B_N})}\Big)^{1/p}. \tag{7}$$

As a result of Minkowski's, Hölder's inequality and (6) - continued with iterating (5) $s-1$ times and invoke (7) at each step, we obtain:

$$\begin{aligned} D_{multi}(N,p,\nu) \leq C^{s-}_{p,\nu} N^{2^{-s}} N^{\kappa_p 2^{-s}(1-\frac{2}{p})\frac{1-2(1-\kappa_p))^{s-1}}{2\kappa_p - 1}} \\ \times D(N^{1-2^{-s+1}}, p)^{\kappa_p} D(N^{1-2^{-s+2}}, p)^{\kappa_p(1-\kappa_p)^{s-2}} N^{O_p((1-\kappa_p)^s)}. \end{aligned} \tag{8}$$

Let γ_p be the unique positive number such that

$$\lim_{N\to\infty} \frac{D(N,p)}{N^{\gamma_p+\epsilon}} = 0, \text{ for each } \epsilon > 0,$$

$$\limsup_{N\to\infty} \frac{D(N,p)}{N^{\gamma_p-\epsilon}} = \infty, \text{ for each } \epsilon > 0.$$

Substitute $D(N,p) \lesssim_\epsilon N^{\gamma_p+\epsilon}$ in (8) we obtain, for each $\nu, \epsilon > 0, s \geq 2$

$$\limsup_{N\to\infty} \frac{D_{multi}(N,p,\nu)}{N^{\gamma_{p,\epsilon,s}}} < \infty \tag{9}$$

where $\gamma_{p,\epsilon,s}$ is the exponent one obtains after simplifying exponents of N in (8) after the substitution. By choosing convenient s, ϵ and ν, we obtain from (9)

$$\gamma_p \leq \frac{p-6}{2p-8} + \frac{1}{2} - \frac{1}{p}. \tag{10}$$

Let $p \to 6$ in (10) to get $\gamma_6 \leq 1/3$. Hence $\gamma_6 = 1/3$.

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16 A Sharp Schrödinger Maximal Estimate in $\mathbb{R}^2$

after Xuimin Du, Larry Guth, and Xiaochun Li [3]
A summary written by Kevin O'Neill

Abstract

We give a summary of [3], which provides a proof of almost everywhere convergence for solutions to the Schrödinger equation in $\mathbb{R}^2$ for initial data in H^s, $s > 1/3$.

16.1 Introduction

Let

$$e^{it\Delta} f(x) = (2\pi)^{-n} \int e^{i(x\cdot\xi + t|\xi|^2)} \hat{f}(\xi) d\xi \tag{1}$$

denote the solution to the free Schrödinger equation $iu_t - \Delta u = 0$ on $\mathbb{R}^n \times \mathbb{R}$ with initial data $u(x, 0) = f(x)$. The main result of [3] is the following.

Theorem 1. *For every $f \in H^s(\mathbb{R}^2)$ with $s > 1/3$, $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ almost everywhere.*

This statement is sharp in s up to the endpoint. Theorem 1 is proven from a maximal estimate (as is standard), which in turn is derived from the following more complicated result.

Theorem 2. *For $p > 3$, for any $\epsilon > 0$, there exists a constant $C_{p,\epsilon}$ such that for any $q > 1/\epsilon^4$,*

$$||e^{it\Delta} f||_{L^p_x L^q_t(B(0,R)\times[0,R])} \leq C_{p,\epsilon} M^{-\epsilon^2} R^{\epsilon} ||f||_2 \tag{2}$$

holds for all $R \geq 1$, and $\xi_0 \in B^2(0, 1)$, any $M \geq 1$ and all f with supp$\hat{f} \subset B^2(\xi_0, M^{-1})$.

To obtain Theorem 1 from Theorem 2, begin by setting $M = 1$. (We note here that the reason M is included as a parameter in the theorem is so we may perform induction on the frequency radius in addition to the physical radius.) Then take $q \to \infty$ to establish a maximal estimate which applies to functions whose Fourier supports are contained in a ball of radius 1. To extend the estimate to arbitrary $f \in H^s$, perform a Littlewood-Paley decomposition on f and apply it to each piece. Expanding the maximal estimate to include functions with Fourier support in balls of radius R requires parabolic rescaling, and in this process we obtain a factor of $R^{1/3}$. However, the regularity of $f \in H^s$ $(s > 1/3)$ is enough for the summation of estimates on the pieces of f to converge to the desired estimate for f itself.

16.2 Wave Packets and Polynomial Partitioning

The proof of Theorem 2 uses the method of polynomial partitioning, which we summarize here. (In fact, this is why we use L_t^q norms for arbitrarily large q rather than L_t^∞ norms.)

We begin by decomposing the given function f into wave packets. Let θ denote an $R^{-1/2}$-cube in frequency space and ν denote an $R^{1/2}$-cube in physical space. Then, under appropriate choice of a collection of θ's and ν's, we may write

$$f = \sum_{\theta,\nu} f_{\theta,\nu} \tag{3}$$

where $f_{\theta,\nu}$ is essentially supported on θ in frequency space and ν in physical space. More importantly, $e^{it\Delta} f_{\theta,\nu}$ is essentially supported on a tube $T_{\theta,\nu}$ which has length R and radius $R^{1/2+\delta}$, where $\delta = \epsilon^2$ is a small positive parameter. Futhermore, $T_{\theta,\nu}$ is in the direction $(-2c(\theta), 1)$ and intersects $\{t = 0\}$ at an $R^{1/2+\delta}$-ball centered at $c(\nu)$, where $c(X)$ denotes the center of X.

Let $Z(g)$ denote the zero set of a polynomial g. Now, by a basic theorem in polynomial partitioning, there exists a nonzero polynomial P of degree at most D such that $(\mathbb{R}^2 \times \mathbb{R}) \setminus Z(P)$ is a union of $\sim D^3$ disjoint open sets O_i such that

$$||e^{it\Delta} f(x)||^p_{L_x^p L_t^q(B(0,R)\times[0,R])} \leq cD^3 ||\chi_{O_i} e^{it\Delta} f(x)||^p_{L_x^p L_t^q(B(0,R)\times[0,R])}. \tag{4}$$

We would like to use the Fundamental Theorem of Algebra to deduce that each $T_{\theta,\nu}$ intersects at most D cells O_i, but this is not necessarily true due to the width of the tubes. Thus, we define the wall $W = N_{R^{1/2+\delta}} Z(P) \cap B(0,R) \times [0,R])$ and let $O_i' = [O_i \cap B(0,R) \times [0,R])] \setminus W$. From here, we may split into two cases: when each of the O_i' contain most of the mass of $e^{it\Delta} f$ on O_i and when they don't.

When they do, we may bound $||e^{it\Delta} f(x)||^p_{L_x^p L_t^q(B(0,R)\times[0,R])}$ as desired by choosing a particular O_i whose intersecting tubes form a small portion of

$||f||_2$ and applying induction on the radius, which closes by taking large D when $p > 3$ due to the presence of a D^{3-p} factor.

When they don't, we in turn bound the mass of $e^{it\Delta}f$ from tubes which intersect the wall. These split into tubes which intersect the wall transversely and those which intersect tangentially. More specifically, we first divide $B(0,R) \times [0,R])$ into balls B_j of radius $R^{1-\delta}$ and consider intersections of tubes in each B_j. The sum of wave packets which intersect $B_j \cap W$ tangentially will be denoted $f_{j,tang}$ (and likewise for transverse terms).

In contrast to other papers which use similar methods, here it is sufficient to bound the transverse terms and a bilinear tangent term instead of a linear one. By induction on the frequency radius, we may assume that f is spread out in frequency. Thus, the bilinear term

$$\mathrm{Bil}(e^{it\Delta}f_{j,tang}(x)) := \max_{dist(\tau_1,\tau_2)\geq 1/(KM)} |e^{it\Delta}f_{t_1,j,tang}(x)|^{1/2}|e^{it\Delta}f_{t_2,j,tang}(x)|^{1/2} \tag{5}$$

is sufficient in replacement of the linear one. Here, $f = \sum_\tau f_\tau$ is a frequency decomposition of f with respect to balls τ of radius $1/(KM)$.

Bounding the transverse terms becomes a simple application of induction on the physical radius to the balls B_j, whose number are bounded. The majority of the work of the paper is dealing with the bilinear tangent term. This seems fitting when one considers the counterexamples to almost everywhere convergence in H^s for $s < 1/3$ in [1] which involve tubes contained in a neighborhood of a variety. It will suffice to prove the following bound:

Proposition 3. *For $p > 3$, the following maximal estimate of the bilinear tangent term holds, uniformly in M:*

$$\left(\int_{B(0,R)} \sup_{t:(x,t)\in W\cap B_j} |Bil(e^{it\Delta}f_{j,tang}(x)|^p dx\right)^{1/p} \leq C_\epsilon R^{\epsilon/2}||f||_2. \tag{6}$$

16.3 The Bilinear Tangent Term

We note here that $f_{j,tang}$ is defined with respect to a ball B_j of scale $R^{1-\delta}$ and the tangency condition resulting from a wave packet decomposition at scale R. This "mismatch" is addressed with help from the following definition.

We say that $T_{\theta,\nu}$ is $ER^{-1/2}$ tangent to a variety Z if $T_{\theta,\nu} \subset N_{ER^{1/2}}Z \cap B(0,R) \times [0,R])$ and $\mathrm{Angle}((-2c(\theta),1), T_z[Z(P)]) \leq ER^{-1/2}$ for any nonsingular point $z \in N_{2ER^{1/2}}(T_{\theta,\nu}) \cap (B(0,R) \times [0,R])$. (Note that setting $E =$

R^δ yields the usual definition of tangency.) We say that f is concentrated in wave packets from $T_Z(E) = \{(\theta,\nu) | T_{\theta,\nu}$ is $ER^{-1/2}$ tangent to $Z\}$ if

$$\sum_{(\theta,\nu)\notin T_Z(E)} ||f_{\theta,\nu}||_2 \leq \text{RapDec}(R)||f||_2. \tag{7}$$

We may now state the theorem which addresses the bilinear tangent term.

Theorem 4. *For functions f_1 and f_2 with separated Fourier supports in $B^2(0,1)$, separated by ~ 1, suppose that f_1 and f_2 are concentrated in wave packets from $T_Z(E)$. Suppose that $Q_1, ..., Q_N$ are lattice $R^{1/2}$-cubes in $B^3(R)$ so that for each i,*

$$||e^{it\Delta} f_i||_{L^6(Q_j)} \text{ is essentially constant in } j. \tag{8}$$

Let $Y = \cup_{j=1}^N Q_j$. Then, for all $\epsilon > 0$,

$$\left|\left||e^{it\Delta} f_1 e^{it\Delta} f_2|^{1/2}\right|\right|_{L^6(Y)} \leq C_\epsilon R^{\epsilon - 1/6} E^{O(1)} N^{-1/6} ||f_1||_{L^2}^{1/2} ||f_2||_{L^2}^{1/2}. \tag{9}$$

The exponent $p = 6$ is used because it is the optimal exponent for ℓ^2 decoupling on the parabola from [2], a result used in the proof of the above. [3] also includes a linear version of the above theorem which may also be viewed as an improved Strichartz estimate for the case $||e^{it\Delta} f||_6$ is spread out. However, the bilinear version stated above will be the one used to prove Proposition 3 and conclude the proof of Theorem 2.

Before getting to the proof of Theorem 4, let us first address how it proves Proposition 3. First, Theorem 4 is expanded to include functions with Fourier support in a ball of radius $1/M$ and separated by $1/(KM)$ by the processes of parabolic rescaling and dyadic pigeonholing of L^6 norms. Next, we replace the L^6 norm with an $L^3_x L^\infty_t$ norm by proving the estimate $H|U|^{1/3} \leq C_\epsilon R^\epsilon E^{O(1)} ||f_1||_2^{1/2} ||f_2||_2^{1/2}$, where U is the set in which $H \sim \sup_t |e^{it\Delta} f_1 e^{it\Delta} f_2|^{1/2}$. This is done by covering U with the projections of dual rectangles on which certain reverse Hölder-type inequalities hold. Lastly, it is shown that $f_{j,tang}$ is concentrated in wave packets from $T_Z(E)$ with respect to the wave packet decomposition at scale $R^{1-\delta}$ so the necessary hypotheses are satisfied.

The proof of Theorem 4 uses parabolic rescaling, dyadic pigeonholing and ℓ^2 decoupling for the parabola. We begin by writing

$$f = \sum_\square f_\square \tag{10}$$

where the $f_\square$ are localized on $R^{3/4}$-balls in physical space and $R^{-1/4}$-balls in frequency space. Through parabolic rescaling and induction on R, we may apply the linear estimate to each $f_\square$. To do so properly (in a way which satisfies the hypotheses of the linear estimate), we fix a particular scale for the number of $||e^{it\Delta} f_\square||_{L^6(S)}$ and the number of boxes S in the strip at this scale. This is done via dyadic pigeonholing, leading to $\log R$ terms which are absorbed into the R^ϵ. Furthermore, we dyadically pigeonhole by the scale of $||f_\square||_2$ and the number of tubes S containing each box Q_j.

In summing eveything together, we use the ℓ^2 decoupling result on the parabola. This is applicable because the set of tubes $T_{\theta,\nu}$ which are tangent to Z in a box Q_j have frequencies restricted to what is, in essence, a parabola.

While this is enough to establish the linear estimate, Theorem 4 follows from the same process except for a clever counting of boxes which results from the transverse intersection of strips due to the separation of Fourier supports.

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17 L^p regularity of averages over curves and bounds for associated maximal operators, Part I

after A. Seeger and M. Pramanik [1]
A summary written by Itamar Oliveira

Abstract

The averaging operator associated to curves with non-vanishing curvature and torsion maps $L^p(\mathbb{R}^3)$ to $W^{\frac{1}{p},p}(\mathbb{R}^3)$ for $p > 38$ ([1]). The proof uses a variant of Wolff's ℓ^p-decoupling theorem for the cone ([2]). Using Bourgain and Demeter's ℓ^2-decoupling theorem in [3], the same argument proves the original result for $p > 4$. The proof is based on several fine decompositions of the associated multiplier on the frequency side.

17.1 Introduction and main ideas

Let $\gamma : I \to \mathbb{R}^3$ be a smooth curve, where I is a compact interval, and χ is a smooth function supported in the interior of I. We define a measure μ_t supported on a dilate of the curve by

$$\langle \mu_t, f\rangle := \int f(t\gamma(s))\chi(s)ds,$$

and we set $\mathcal{A}_t f(x) := f * \mu_t(x)$. About this operator, it is shown:

Theorem 1. *Suppose that $\gamma \in C^5(I)$ has non-vanishing curvature and torsion. Then $\mathcal{A} := \mathcal{A}_1$ maps L^p to $W^{\frac{1}{p},p}(\mathbb{R}^3)$ for $p > 4$.*

Theorem 1 is an improvement of the original result in [1], where the statement is proven for $p > 38$. The authors use a variant of a celebrated result of Wolff [2]. Wolff's original statement reads as follows: consider $f \in \mathcal{S}'(\mathbb{R}^3)$ with Fourier transform supported in a neighborhood of width $\delta \ll 1$ of the light cone $\xi_3^2 = \xi_1^2 + \xi_2^2$ at $\xi_3 \approx 1$. Let $\{\Psi_\nu\}$ be a collection of smooth functions which are supported in $1 \times \delta^{1/2} \times \delta$-plates that fit the light cone and have appropriate size and differentiability properties. Then for $p > 74$ and all $\varepsilon > 0$ there exists $C_{\varepsilon,p} > 0$ such that

$$\left\| \sum_{\nu} \hat{\Psi}_{\nu} * f \right\|_{p} \leq C_{\varepsilon,p} \delta^{-\frac{1}{2}+\frac{2}{p}-\varepsilon} \left(\sum_{\nu} \|\hat{\Psi}_{\nu} * f\|_{p}^{p} \right)^{\frac{1}{p}}. \tag{1}$$

This is an ℓ^p-decoupling estimate for the cone multiplier. The variant of Wolff's inequality in [2] is generalized to cones generated by curves $g(\alpha) = (g_1(\alpha), g_2(\alpha))$ that satisfy some differentiability conditions. Roughly speaking, (1) is the particular case for $g(\alpha) = (\cos 2\pi\alpha, \sin 2\pi\alpha)$ and $-1/2 \leq \alpha \leq 1/2$.

Bourgain and Demeter proved an ℓ^2-decoupling estimate in [3] that implies the one above in $\mathbb{R}^3$ for $p > 6$:

$$\|f\|_p = \left\| \sum_{\nu} \hat{\Psi}_{\nu} * f \right\|_{p} \leq C_{\varepsilon,p} \delta^{-\frac{1}{4}+\frac{3}{2p}-\epsilon} \left(\sum_{\nu} \|\hat{\Psi}_{\nu} * f\|_{p}^{2} \right)^{\frac{1}{2}}. \tag{2}$$

Theorem may be obtained 1 by using (2) in the argument of [1].

The proof depends on understanding the multiplier $\hat{\mu}(\xi)$. A Littlewood-Paley decomposition allows us to break this multiplier in a sum of multipliers m_k concentrated at different dyadic scales. By using van der Corput's lemma, we will be able to restrict our analysis to a narrow tubular neighborhood of the binormal cone $\mathcal{B} = \{rB(s) : r > 0, s \in I\}$ (here $B(s)$ is the binormal vector to γ at s).

On the other hand, each m_k is associated to a symbol $a_k(s, \xi)$ that depends on a spatial parameter. The crucial summable bound $\|m_k\|_{M_p} \leq 2^{-k/p}$ is obtained only after a careful analysis of this symbol, and makes use of (2).

17.2 Decomposition of the multiplier

Recall that $\widehat{\mathcal{A}f}(\xi) := \hat{f}(\xi)\hat{\mu}(\xi)$, where $\mu := \mu_1$. As usual, let ψ and ϕ_0 be smooth and compactly supported such that

$$1 = \sum_{k \geq 0} \psi(2^{-k}\xi) + \phi_0(\xi)$$

and set $a_k(s,\xi) = \chi(s)\psi(2^{-k}\xi)$. Thus,

$$\begin{aligned}\hat{\mu}(\xi) = \int_I e^{-2\pi i\langle\gamma(s),\xi\rangle}\chi(s)ds &= \sum_{k\geq 0}\int_I e^{-2\pi i\langle\gamma(s),\xi\rangle}\chi(s)\psi(2^{-k}\xi)ds \\ &= \sum_{k\geq 0}\int_I e^{-2\pi i\langle\gamma(s),\xi\rangle}a_k(s,\xi)ds \\ &= \sum_{k\geq 0} m_k(\xi),\end{aligned}$$

Consider for $k > 0$ the Fourier multipliers

$$m_k(\xi) = \int e^{-it\langle\gamma(s),\xi\rangle}a(s, 2^{-k}\xi)ds,$$

where a vanishes outside the annulus $\{\xi : 1/2 < |\xi| < 1\}$ and satisfies the estimates

$$|\partial_s^j\partial_\xi^\alpha a(s,\xi)| \leq C_2, \quad \alpha \leq 2, 0 \leq j \leq 3;$$

We now invoke the following fact regarding Littlewood-Paley projections and Sobolev spaces.

Lemma 2. *let T_s, $s \geq 0$, be given by*

$$\widehat{T_s(f)}(\xi) = \left(\sum_{k>0} 2^{ks}m_k(\xi)\right)\widehat{f}(\xi),$$

If T_s maps L^p to L^p then T_0 maps L^p to $W^{s,p}$ for $p > 1$.

The goal now becomes proving that $\sum_{k>0} 2^{k/p}m_k$ maps L^p to L^p for $p > 4$. After a localization, one verifies that the level curve of $\mathcal{B}$ at height $\xi_3 = 1$ satisfies the conditions of g in the variant of (1) that we mentioned before.

First of all, it suffices to understand the localization of $\sum_{k>0} 2^{k/p}m_k$ to a narrow tubular neighborhood of the binormal cone $\mathcal{B} = \{rB(s) : r > 0, s \in I\}$. Indeed, if θ is smooth away from the origin, homogeneous of degree zero and

$$|\langle\gamma''(s), \xi/|\xi|\rangle| \geq c > 0, \forall\xi \in \operatorname{supp}(\theta) \cap \operatorname{supp}(a_k),$$

then van der Corput's Lemma gives us the desired bound for $\theta(\xi)\sum_{k>0} 2^{k/p}m_k(\xi)$.

Lemma 3 (van der Corput). *Suppose ϕ is real-valued and smooth in (a,b), and that $|\phi^k(x)| \geq 1$ for all $x \in (a,b)$. Then*

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \leq c_k\lambda^{-1/k}$$

holds when $k=2$ or $k=1$ and $\phi'(x)$ is monotonic. The bound c_k does not depend on ϕ and λ.

Therefore we assume that ξ is in the support of $a_k(s,\cdot)$ and, being in a neighborhood of $\mathcal{B}$, it can be expressed as

$$\xi = rB(\sigma) + uT(\sigma) := \Xi(r,u,\sigma)$$

with $(r(\xi), u(\xi), \sigma(\xi))$ depending smoothly on ξ.

Note that the binormal cone $\mathcal{B}$ has one nonvanishing principal curvature. That is crucial for the subsequent argument.

17.2.1 Decomposition of the dyadic multipliers

The bounds $\|m_k\|_{M_p} \lesssim 2^{-k/p}$ for $p > 4$ will come only after a delicate decomposition of a_k. We write a_k as

$$a_k(s,\xi) = \tilde{a}_k(s,\xi) + \sum_{l\leq k/3} a_{k,l}(s,\xi) + \sum_{l\leq k/3} b_{k,l}(s,\xi),$$

where $a_{k,l}(s,\cdot)$ is supported in $\mathrm{dist}(\xi,\mathcal{B}) \approx 2^{-2l}$ and $|s-\sigma(\xi)| \lesssim 2^{-l}$, $\tilde{a}_k(s,\cdot)$ is supported in a $C2^{-2k/3}$ neighborhood of the binormal cone with $|s-\sigma(\xi)| \lesssim 2^{-k/3}$ and $b_{k,l}(s,\cdot)$ is supported in a $C2^{-2l}$ neighborhood of the binormal cone but now $|s-\sigma(\xi)| \approx 2^{-l}$. For a general symbol a set

$$m_k[a](\xi) = \int a(s, 2^{-k}\xi)e^{-i\langle\gamma(s),\xi\rangle}ds.$$

We prove the following bounds:

Proposition 4. *For $6 < p < \infty$,*

$$\|m_k[\tilde{a}_k]\|_{M_p} \leq C_\varepsilon 2^{-4k/3p+k\varepsilon},$$

$$\|m_k[a_{k,l}]\|_{M_p} \leq C_\varepsilon 2^{-k/p}2^{-l/p+l\varepsilon},$$

$$\|m_k[b_{k,l}]\|_{M_p} \leq C_\varepsilon 2^{-2k/p}2^{2l/p+l\varepsilon}.$$

On the other hand, to prove Proposition 4 one has to decompose $\tilde{a}_k$, $a_{k,l}$ and $b_{k,l}$ once more. More precisely, let $\zeta \in C_0^\infty$ be supported in $(-1,1)$ so that $\sum_{\nu\in\mathbb{Z}} \zeta(\cdot - \nu) \equiv 1$. We set

$$a_{k,l,\nu}(s,\xi) = \zeta(2^l s - \nu) a_{k,l}(s,\xi),$$

$$b_{k,l,\nu}(s,\xi) = \zeta(2^l s - \nu) a_{k,l}(s,\xi),$$
$$\tilde{a}_{k,\nu}(s,\xi) = \zeta(2^{k/3} s - \nu) \tilde{a}_k(s,\xi).$$

Estimates on the supports of $a_{k,l,\nu}$ and $b_{k,l,\nu}$ and on the growth of their directional derivatives along $T(2^{-l}\nu), N(2^{-l}\nu), B(2^{-l}\nu)$ together with the variant of (1) from [1], allow us to prove L^2 and L^∞ bounds on the associated multiplier operators. Proposition 4 then follows by interpolation.

Finally, another interpolation argument gives:

Corollary 5. *For $p > 4$ there is $\varepsilon_0 = \varepsilon_0(p) > 0$ so that*

$$\|m_k[a_{k,l}]\|_{M_p} \leq C_p 2^{-k/p} 2^{-\varepsilon_0 l/p},$$

$$\|m_k[\tilde{a}_k]\|_{M_p} \leq C_p 2^{-k(1+\varepsilon_0)/p}.$$

Moreover,

$$\sum_{k\geq 3l} 2^{k/l} \|m_k[b_{k,l}]\|_{M_p} \leq C_p,$$

$$\sum_{k} 2^{k/p} \|m_k[\tilde{a}_k]\|_{M_p} \leq C_p.$$

A bound for $\sum_{k\geq 3l} 2^{k/l} \|m_k[a_{k,l}]\|_{M_p}$ can be obtained with the aid of a vector-valued version of the Fefferman-Stein inequality for the #-function. Putting this estimate with the ones in Corollary 5 together, one can finish the proof of Theorem 1.

Remark 6. *The authors actually prove Theorem 1 for* curves of finite type. *For the extension of the result to this kind of curve we refer to [1].*

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18 The proof of the ℓ^2 decoupling conjecture

after J. Bourgain and C. Demeter [3]
A summary written by Yumeng Ou

Abstract

We prove the ℓ^2 Decoupling Conjecture for compact hypersurfaces with positive definite second fundamental form and also for the cone, which implies a wide range of applications in additive combinatorics, incidence geometry, number theory and PDE.

18.1 Introduction

Consider the truncated (elliptic) paraboloid in $\mathbb{R}^n$

$$\mathbb{P}^{n-1} := \{(\xi_1, \dots, \xi_{n-1}, \xi_1^2 + \cdots + \xi_{n-1}^2) \in \mathbb{R}^n : 0 \le \xi_i \le 1\}.$$

For each cube $\tau \subset [0,1]^{n-1}$ and $g : \tau \to \mathbb{C}$ define the extension operator

$$E_\tau g(x) := \int_\tau g(\xi_1, \dots, \xi_{n-1}) e^{2\pi i(\xi_1 x_1 + \cdots + \xi_{n-1}x_{n-1} + (\xi_1^2 + \cdots + \xi_{n-1}^2)x_n)}\, d\xi,$$

and write $Eg = E_{[0,1]^n} g$.

Theorem 1. *Let $Dec(\delta, p, n)$ be the smallest constant such that*

$$\|Eg\|_{L^p(B)} \le Dec(\delta,p,n) \left(\sum_{\tau:\, \delta^{1/2}\text{-}cube} \|E_\tau g\|^2_{L^p(w_B)} \right)^{1/2} =: Dec(\delta,p,n)\|Eg\|_{L^{p,\delta}(w_B)}$$

holds for every cube $B = B_{\delta^{-1}} \subset \mathbb{R}^n$ (centered at c_B) with side length δ^{-1} and $g : [0,1]^{n-1} \to \mathbb{C}$, where the weight $w_B(x) := (1 + R^{-1}|x - c_B|)^{-100n}$. Then for $2 \le p \le \frac{2(n+1)}{n-1}$ we have the sharp (up to $\delta^{-\epsilon}$ losses) upper bound

$$Dec(\delta,p,n) \lesssim_{\epsilon,p,n} \delta^{-\epsilon}. \tag{1}$$

In the range $2 \le p \le 2n/(n-1)$, this is first obtained by Bourgain in [1], and can be viewed as a weaker (via Minkowski inequality) substitute for the *square function conjecture*

$$\|Eg\|_{L^p(B)} \lesssim \delta^{-\epsilon} \left\| \left(\sum_{\tau:\, \delta^{1/2}\text{-cube}} |E_\tau g|^2 \right)^{1/2} \right\|_{L^p(w_B)}, \quad \forall 2 \le p \le \frac{2n}{n-1},$$

which is wide open in $n \geq 3$. By rescaling and Taylor's formula, Theorem 1 extends to all compact C^2 hypersurfaces in $\mathbb{R}^n$ with positive definite second fundamental form. Moreover, a surprisingly short application of Theorem 1 extends it to a sharp decoupling for the (truncated) cone

$$C^{n-1} = \{(\xi_1, \dots, \xi_{n-1}, \sqrt{\xi_1^2 + \cdots + \xi_{n-1}^2}) \in \mathbb{R}^n : 1 \leq \sqrt{\xi_1^2 + \cdots + \xi_{n-1}^2} \leq 2\}.$$

It has some striking consequences such as progress on the local smoothing conjecture for the wave equation.

18.2 Proof strategy

In [3], the authors formulate a mutlilinear ℓ^2 decoupling theory and use a method adapted from [6] to show that it is essentially equivalent to linear decoupling, i.e. Theorem 1. More precisely, let $\pi : \mathbb{P}^{n-1} \to [0,1]^{n-1}$ be the projection map. We say cubes $\tau_1, \dots, \tau_n \subset [0,1]^{n-1}$ are *ν-transversal*, if the volume of the parallelepiped spanned by unit normals $n(P_j)$ is greater than ν, for each choice of $P_j \in \mathbb{P}^{n-1}$ with $\pi(P_j) \in \tau_j$.

Theorem 2 (Multilinear decoupling)**.** *Denote by $MD(\delta, p, \nu, n)$ the smallest constant such that*

$$\left\| \prod_{j=1}^{n} |E_{\tau_j} g|^{1/n} \right\|_{L^p(B)} \leq MD(\delta, p, \nu, n) \prod_{j=1}^{n} \|E_{\tau_j} g\|_{L^{p,\delta}(w_B)}^{1/n}$$

for all ball $B = B_{\delta^{-1}} \subset \mathbb{R}^n$, $g : [0,1]^{n-1} \to \mathbb{C}$, and ν-transversal cubes $\tau_j \subset [0,1]^{n-1}$. Then for $2 \leq p \leq 2(n+1)/(n-1)$ and $\epsilon > 0$, $MD(\delta, p, \nu, n) \lesssim \delta^{-\epsilon}$.

By Höder's inequality, it is easily seen that linear decoupling implies multilinear decoupling, i.e. $MD(\delta, p, \nu, n) \lesssim D(\delta, p, n)$ for each ν. What is quite surprising is that the reverse is also essentially true, which makes decoupling more accessible compared to restriction or Kakeya. More precisely, the authors proved in [3] that

Theorem 3. *Suppose that in dimension $n-1$, the decoupling constant $Dec(\delta, p, n-1) \lesssim \delta^{-\epsilon}$ for any $\epsilon > 0$. Then for any $\epsilon > 0$,*

$$Dec(\delta, p, n) \lesssim_\nu \delta^{-\epsilon} MD(\delta, p, \nu, n).$$

This result is proved following the same strategy of [6], where multilinear restriction is shown to imply improvement in linear restriction. The idea is that one splits the function into a *broad* and a *narrow* part, depending on whether most of the contribution is made by wave packets that are lying near a lower-dimensional plane. Then the broad part can be bounded by multilinear estimates and the narrow part is dealt with by induction on scales. Because of the self-similarity of $\mathbb{P}^{n-1}$, there is a natural passage linking the operator E_τ to $E_{[0,1]^{n-1}}$ using parabolic rescaling, i.e. stretching each square-like cap on the paraboloid $\mathbb{P}^{n-1}$ to the whole $\mathbb{P}^{n-1}$ via an affine transformation.

Proposition 4. *Let $0 < \delta \leq \sigma < 1$ and $p \geq 2$. For each cube $\tau \subset [0,1]^{n-1}$ with $\ell(\tau) = \sigma^{1/2}$ and each cube $B \subset \mathbb{R}^n$ with $\ell(B) \geq \delta^{-1}$ one has*

$$\|E_\tau g\|_{L^p(B)} \lesssim Dec(\frac{\delta}{\sigma}, p, n) \left(\sum_{\theta:\, \delta^{1/2}\text{-cube in } \tau} \|E_\theta g\|^2_{L^p(w_B)} \right)^{1/2}.$$

Using the multilinear restriction theorem of Bennett-Carbery-Tao ([2]), one can prove Theorem 2 (and even the multilinear square function estimate) in the smaller range $2 \leq p \leq 2n/(n-1)$. This implies a quick proof of Theorem 1 for $2 \leq p \leq 2n/(n-1)$, which is first observed by Bourgain in [1]. In order to cover the rest of the range, the authors in [3] set up an iteration scheme estimating how the decoupling constants change when one decouples into finer and finer scales (Proposition 10.4 of [4]). The iteration estimate interpolates between L^2 and L^p estimates. In particular, the very simple but efficient L^2 decoupling (Proposition 5 below) exploits the L^2 orthogonality and allows one to decouple to the smallest possible scale (the inverse of the radius of the ball).

Proposition 5. *Let τ be a cube with $\ell(\tau) \geq \delta$. Then for each ball $B \subset \mathbb{R}^n$ with radius δ^{-1} one has*

$$\|E_\tau g\|_{L^2(B)} \lesssim \left(\sum_{\theta:\, \delta^{1/2}\text{-cube in } \tau} \|E_\theta g\|^2_{L^2(w_B)} \right)^{1/2}.$$

This method allows them to simultaneously prove multilinear and linear decoupling using induction on dimension. Briefly speaking, one assumes that the linear decoupling constant satisfies $Dec(\delta, p, n) \sim \delta^{-\eta_p}$. First, one applies parabolic rescaling (Proposition 4), the iteration estimate (Proposition 10.4 of [4]), and a trivial decoupling (Cauchy-Schwarz) to derive an upper bound on the multilinear constant $MD(\delta, p, \nu, n)$ involving $\delta^{-\eta_p}$. Then, one plays this again Theorem 3, which gives a lower bound for $MD(\delta, p, \nu, n)$ in terms of $\delta^{-\eta_p}$, to show that η_p has to be zero.

18.3 Applications of decoupling

Theorem 1 immediately implies a wide range of striking applications in many areas. For example, it can be used to establish Strichartz estimates for the Schrödinger equation on the torus. More precisely, for $\phi \in L^2(\mathbb{T}^{n-1})$, define

$$\begin{aligned} &e^{it\Delta}\phi(x_1, \dots, x_{n-1}, t) := \\ &\sum_{(\xi_1,\dots,\xi_{n-1})\in\mathbb{Z}^{n-1}} \hat{\phi}(\xi_1, \dots, \xi_{n-1}) e^{i(x_1\xi_1+\cdots+x_{n-1}\xi_{n-1}+t(\xi_1^2\theta_1+\cdots+\xi_{n-1}^2\theta_{n-1}))} \end{aligned}$$

on the irrational torus $\prod_{i=1}^{n-1} \mathbb{R} \setminus (\theta_i\mathbb{Z})$ with $1/2 < \theta_i < 2$.

Theorem 6 (Strichartz estimates for irrational tori)**.** *Let $\phi \in L^2(\mathbb{T}^{n-1})$ with supp$(\hat{\phi}) \subset [-N, N]^{n-1}$. Then for each $\epsilon > 0$, $p \geq 2(n+1)/(n-1)$ and each interval $I \subset \mathbb{R}$ with $|I| \gtrsim 1$ we have*

$$\|e^{it\Delta}\phi\|_{L^p(\mathbb{T}^{n-1}\times I)} \lesssim_\epsilon N^{\frac{n-1}{2}-\frac{n+1}{p}+\epsilon}|I|^{1/p}\|\phi\|_2,$$

and the implicit constant is independent of I, N and θ_i.

Moreover, Theorem 1 can be used to count solutions of Diophantine inequalities. One such example is:

Theorem 7. *For fixed $k \geq 2$ and C the system*

$$\begin{cases} |n_1^k + n_2^k + n_3^k - n_4^k - n_5^k - n_6^k| \leq CN^{k-2} \\ n_1 + n_2 + n_3 = n_4 + n_5 + n_6 \end{cases}$$

has $O(N^{3+\epsilon})$ solutions with $n_i \sim N$.

There are many other number theoretical consequences of the decoupling theory that are investigated elsewhere, such as the proof of the Vinogradov's mean value theorem in [5].

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19 A restriction estimate using polynomial partitioning, part II

after L. Guth [1]
A summary written by João Pedro Ramos

Abstract

We will go deeper into the details and finally effectively prove our restriction estimate, passing through the core harmonic analysis techniques, as well as some algebraic properties that help us in the process

19.1 Introduction

First of all, as our aim is to prove Theorem 3 in the previous summary, and therefore we will prove a (slightly) more general result, whose proof yields, by a simple especification argument, our desired conclusion. Here, it is worthwhile to notice that our functions f_τ are going to be restrictions of f to caps τ that *might* overlap. We take our caps τ to be graphs of balls $B^2(\omega_\tau, r)$ under our parametrising function h. We further assume the centers ω_τ to be K^{-1} separated, and therefore the *multiplicity* of a certain covering by saying that the radii of the caps lie between K^{-1} and $\mu^{1/2}K^{-1}$. With this condition, it is then obvious that each point lies in at most $O(\mu)$ caps. Finally, we simplify the notation a little bit, expanding a function as a wave packet decomposition $f_T = f_{\theta,v}$.

We then state our more general Theorem:

Theorem 1. *For any given ϵ greater than zero, there are K, L and a small δ_{trans} depending only on ϵ so that the following holds:*
Suppose that S is the graph of a "good" h, with control over L derivatives. Suppose also that we cover S by caps as above with multiplicity μ at most, and that we take $\alpha \geq K^{-\epsilon}$. If, for any cap τ,

$$\fint_{B(w,R^{-1/2})\cap S} |f_\tau|^2 \leq 1,$$

then

$$\int_{B_R} Br_\alpha Ef^{3.25} \leq C_\epsilon R^\epsilon \left(\sum_\tau \int_S |f_\tau|^2 \right)^{3/2+\epsilon} R^{\delta_{trans}\log(K^\epsilon \alpha\mu)}.$$

Moreover, $\lim_{\epsilon\to 0} K(\epsilon) = +\infty$.

Recovering Theorem 3 is just a matter of adjusting the parameters in the above theorem. The parameters have to be, however, cared about: we will need our δ_{trans} to be way smaller than ??$epsilon$, therefore we take $\delta_{trans} = \epsilon^6$. We also take $K(\epsilon) = e^{\epsilon^{-10}}$ big enough. As the tubes to be used in our wave packet decomposition have thickness $R^{1/2+\delta}$, we take $\delta = \epsilon^2$. Finally, we have to choose a degree for the partitioning, which will be $D = R^{\delta_{deg}} = R^{\epsilon^4}$.The key here is that our parameters satisfy

$$\delta_{trans} \lll \delta_{deg} \lll \delta \lll \epsilon.$$

The choice of K is justified by the fact that one needs in the proof that $R^{\delta_{deg} \log(10^{-6} K^{\epsilon})} \geq R^{1000}$.

19.2 Polynomial partitioning

Our degree, as mentioned before, is taken to be $D = R^{\epsilon^4}$. Our partition will be respective to the function $\chi_{B_R} \mathrm{Br}_\alpha E f^{3.25}$. We know there is a non-zero polynomial P of degree at most D so that its zero set divides the three dimensional space into $\sim D^3$ cells O_i, so that

$$\int_{O_i \cap B_R} \mathrm{Br}_\alpha E f^{3.25} \sim D^{-3} \int_{B_R} \mathrm{Br}_\alpha E f^{3.25}.$$

Moreover, we can also assume P is a product of non-syngular polynomials. Further, we define also the cell wall $W = N_{R^{1/2+\delta}} Z(P)$, and the shrunk cells $O_i' = (O_i \cap B_R) \backslash W$. We then define also $\mathbb{T}_i \subset \mathbb{T}$ as the set of tubes that intersect the cell O_i'. Finally, take $f_{\tau,i} = \sum_{T \in \mathbb{T}_i} f_{\tau,T}$, and $f_i = \sum_\tau f_{\tau,i}$.
It is straightforward to get the following Lemma from the definitions:

Lemma 2. *Each tube belongs to at most $D+1$ sets $\mathbb{T}_i$.*

From this point on, we have to subdivide our analyis: the integral over the cells O_i' is going to be estimated using an inductive argument, whereas the integral over the cell wall has to be controlled by other means, which have to do with the position of the tubes T with respect to our zero set $Z(P)$. More especifically, we divide our original ball B_R into $\sim R^{3\delta}$ balls B_j of radius $R^{1-\delta}$ each. If an intersection $B_j \cap W$ is non-empty, we divide tubes into the following sets:

Definition 3. *$\mathbb{T}_{j,tang}$ is the set of all tubes T satisfying:*

- $T \cap W \cap B_j \neq \emptyset$.
- *If z is a non-singular point of $Z(P)$ in $2B_j \cap 10T$, then*

$$\angle(v(T), T_z Z) \leq R^{-1/2+2\delta}.$$

Definition 4. *$\mathbb{T}_{j,trans}$ is the set of all tubes T satisfying:*

- $T \cap W \cap B_j \neq \emptyset$.
- *There is a z non-singular in $Z(P)$ and lying in $2B_j \cap 10T$, so that*

$$\angle(v(T), T_z Z) > R^{-1/2+2\delta}.$$

By the way we constructed our cell wall, tubes and balls, with help of the fact that P is a product of non-singular polynomials – which implies that the set of non-singular points of $Z(P)$ is dense –, we see that, indeed, a tube must lie in one of those sets defined above.

Finally, before we can proceed to the inductive step of the proof, we have to state two geometric Lemmas that are going to help us estimate the contribution coming from the cell wall in each of the tangent and transversal cases:

Lemma 5. *Each tube belongs to at most $R^{O(\delta_{deg})}$ different sets $\mathbb{T}_{j,trans}$.*

Lemma 6. *For each fixed j, the number of different θ so that $\mathbb{T}_{j,tang} \cap \mathbb{T}(\theta) \neq \emptyset$ is at most $R^{1/2+O(\delta)}$.*

the proof of those estimates is going to be explored more during the talk, and for now we will focus on the core proof of the theorem. We just defone aditionally the functions $f_{\tau,j,tang} = \sum_{T \in \mathbb{T}_{j,tang}} f_{\tau,T}$ and $f_{j,tang} = \sum_{\tau} f_{\tau,j,tang}$, and $f_{\tau,j,trans}$ and $f_{j,trans}$ analogously.

19.3 Inductive step

First, we break the contribution in the integral $\int_{B_R} \mathrm{Br}_\alpha E f^{3.25}$ into cellular, transverse and tangential parts, and use an inductive argument to close up the proof. We start with an estimate that comes directly from the wave packet estimates.

Lemma 7. *If $x \in O_i'$ and R is large enough, then*

$$Br_\alpha Ef(x) \leq 2Br_{2\alpha}Ef_i(x) + R^{-900}\sum_\tau \|f_\tau\|_2$$

For the cell wall contributions, we have a more delicate situation. In fact, the function Ef is approximately equal to $Ef_{j,trans} + Ef_{j,tang}$ over the set $W \cap B_j$, but we will have to use aditional terms to bound our functions well.

Let I be a subset of the $\sim K^2$ into which our surface S is subdivided. Let us define then $f_{I,j,trans} = \sum_{\tau \in I} f_{\tau,j,trans}$. This will allow us to deal with the transverse term using an induction, whereas we can estimate the tangent term directly. The following estimate is, essentially, what allows us to do so:

Lemma 8. *Let $x \in W \cap B_j$ and $\alpha\mu \leq 10^{-5}$. Then:*

$$Br_\alpha Ef(x) \leq 2\left(\sum_I Br_{2\alpha}Ef_{I,j,trans} + K^{100}Bil(Ef_{j,tang}) + R^{-900}\sum_\tau \|f_\tau\|_2\right),$$

where we define the bilinear version of $Ef_{j,tang}$ *as*

$$Bil(Ef_{j,tang}) := \sum_{dist(\tau_1,\tau_2)\geq K^{-1}} |Ef_{\tau_1,j,tang}|^{1/2}|Ef_{\tau_2,j,tang}|^{1/2}.$$

Given the last two Lemmas, we are just an estimate away from being able to prove the Theorem. This estimate is a consequence of standard bilinear arguments, and its proof, although the place where the condition $p = 3.25$ comes in the heaviest, is not going to be our focus.

Proposition 9.

$$\int_{B_j} Bil(Ef_{j,tang})^{3.25} \lesssim R^{O(\delta)}\left(\sum_\tau \int |f_\tau|^2\right)^{3/2}.$$

Now the proof of the theorem follows, roughly, the following program: One wants to prove the theorem by induction on R and/or the size of $\sum_\tau |f_\tau|^2$. In order to do that, one does polynomial partitioning with the specified degree D. The contribution from the cells is easily controlled by the inductive assumptions and Lemma 7. On the other hand, to handle the terms coming from the cell wall, one has to make use of the Lemmas we have about transversal/tangential tubes in the end of the last subsection, as well as Lemma 8 and Proposition 9. The details are, of course, too extensive to be put here, and shall me developped more carefully in the talk.

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20 The optimal restriction estimate for a class of surfaces with curvature

after Ioan Bejenaru [1]
A summary written by Olli Saari

Abstract

We discuss optimal trilinear restriction estimate in $\mathbb{R}^{n+1}$ for double conic surfaces. Curvature is shown to improve the admissible range for L^p estimates over the generic case. A counterexample shows that $p > \frac{2(n+4)}{3(n+2)}$ is the universal threshold for the trilinear adjoint restriction estimate.

20.1 Introduction

Let $n \geq 1$ and let $S \subset \mathbb{R}^{n+1}$ be an n-dimensional submanifold parametrized through smooth $\Sigma : U \to \mathbb{R}^{n+1}$ with $U \subset \mathbb{R}^n$ open, bounded and connected. We define the associated extension operator acting on smooth f as

$$\mathcal{E}f(x) = \int_U f(x) e^{ix\cdot\Sigma(\xi)}\, d\xi.$$

It is obvious that $\|\mathcal{E}f\|_{L^\infty(\mathbb{R}^{n+1})} \leq C\|f\|_{L^1(\mathbb{R}^n)}$ and choosing S a piece of a hyperplane shows that no L^p with $p \neq \infty$ can be inserted to the left hand side. If S has everywhere non-vanishing Gaussian curvature, a classical result of Tomas–Stein tells that

$$\|\mathcal{E}f\|_{L^p(\mathbb{R}^{n+1})} \leq C\|f\|_{L^2(\mathbb{R}^n)}$$

for $p \geq \frac{2(n+2)}{n}$. The curvature imposes some decay on $\mathcal{E}f$, which improves the estimate compared to the flat case.

Given $k \in \{2, \ldots, n+1\}$ hypersurfaces S_i parametrized through $\Sigma_i : U_i \to \mathbb{R}^{n+1}$ with $i \in \{1, \ldots, k\}$, we define the corresponding extension operators $\mathcal{E}_i$ as above. A result of Bennett–Carbery–Tao [3] shows that

$$\|1_{B(0,R)} \prod_{i=1}^{k} \mathcal{E}_i f_i\|_{L^p(\mathbb{R}^{n+1})} \leq CR^\epsilon \prod_{i=1}^{k} \|f_i\|_{L^2(\mathbb{R}^n)}$$

for any $p \geq \frac{2}{k-1}$, $\epsilon > 0$ and R large provided that the surfaces S_i are transversal meaning that

$$vol(v_1, \ldots, v_k) > c > 0$$

for a fixed c and all choices of normal vectors $v_i \in NS_i$. Here vol denotes the k-dimensional volume of the parallelepiped spanned by the vectors. The transversality assumption encodes the normal vectors being linearly independent in a uniform way. Whether or not ϵ can be taken to be zero is an open problem.

Without any curvature assumptions, the lower bound $p \geq \frac{2}{k-1}$ is sharp (with $k = n+1$ and S_i transversal hyperplanes this reduces to the classical Loomis–Whitney inequality). When some curvature is present, it can be improved to

$$p > \frac{2(n+1+k)}{k(n+k-1)},$$

(at least for $k = 3$), as the main result of the paper under review shows, and this threshold is universal in the sense that no amount of curvature can improve it. The positive result is shown for double conic surfaces, which have two vanishing principal curvatures, and the negative result is proved by giving a counter-example with constant Gaussian curvature.

20.2 Geometry

Compared to the linear theory where the number of non-vanishing principal curvatures determines the optimal exponent in the extension estimate, the multilinear theory is more complicated since the interaction of many surfaces determines which curvatures matter and which ones do not.

Given a normal field N of a hypersurface S, we define the shape operator acting on the tangent space $T_\xi S$ at ξ as

$$S_{N(\xi)} v = -\frac{d}{dt} N(\gamma(t)) \bigg|_{t=0}$$

where γ is any smooth curve with $\gamma(0) = \xi$ and $\dot{\gamma}(0) = v$. It gives the derivative of the normal field to the direction of the tangent vector v.

Definition 1. *We say that a hypersurface S is double conic if*

- *S admits a foliation $S = \bigcup_\alpha S^\alpha$ where each of the disjoint leaves S^α is a two-dimensional submanifold.*

- *Each S^α is flat in the sense that $S_{N(\xi)}v = 0$ for all $v \in T_\xi S^\alpha$.*

Three double conic hypersurfaces $S_i, i = 1, 2, 3$ are admissible if there is $\nu > 0$ such that for any triple $(\xi_1, \xi_2, \xi_3) \in (S_1, S_2, S_3)$ and all orthonormal bases $v_4, \ldots, v_{n+1}$ of $T_{\xi_l} S_l \setminus T_{\xi_l} S_l^\alpha$ with any $l \in \{1, 2, 3\}$ the following holds:

$$vol(N_{\xi_1}, N_{\xi_2}, N_{\xi_3}, S_{N_l(\xi_l)}v_4, \ldots, S_{N_l(\xi_l)}v_{n+1}) \geq \nu. \tag{1}$$

The last condition contains both transversality of the hypersurfaces as well as a curvature assumption.

Theorem 2. *Let $S_i, i = 1, 2, 3$ be admissible double conic hypersurfaces. Then*

$$\|\mathcal{E}_1 f_1 \mathcal{E}_2 f_2 \mathcal{E}_3 f_3\|_{L^p(\mathbb{R}^{n+1})} \leq C(p) \prod_{i=1}^{3} \|f_i\|_{L^2(U_i)}, \quad \forall f_i \in L^2(U_i)$$

whenever $p > \frac{2(n+4)}{3(n+2)} = p(3)$.

20.3 Some ideas used in the proof

20.3.1 The free wave

We assume that S is given as a graph $\{(\xi, \varphi(\xi)) : \xi \in U\}$ of smooth φ. We call

$$\phi(x, t) = \mathcal{E}f(x, t) = \int_{\mathbb{R}^n} f(x) e^{i(\xi \cdot x + t\varphi(\xi))}\, d\xi$$

with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ a free wave. We define its mass at time t to be $M(\phi(t)) = \|\phi(t)\|^2_{L^2(\mathbb{R}^n)}$. A simple computation with n-dimensional Fourier transform shows that the mass is constant in t (which can indeed be thought as a time parameter).

20.3.2 Induction on scales

Fix $p \in (p(3), 1)$ and let $R > C_0$ with C_0 be a large dimensional constant. We let $A_p(R)$ be the best constant for which

$$\|\phi_1 \phi_2 \phi_3\|_{L^p(Q_R)} \leq A_p(R) \prod_{i=1}^{3} M(\phi_i)^{1/2}$$

holds for all cubes Q_R with side-length R and triples of free waves (ϕ_1, ϕ_2, ϕ_3) satisfying a margin requirement (the Fourier support well inside a reference set). Define $\overline{A}_p(R) = \sup_{r\in(C_0,R]} A_p(r)$ to obtain an increasing quantity. Then it suffices to prove

Proposition 3. *Let $\epsilon \in (0,1)$. If $R > 2^{2C_0}$ and $R^{-1/4+\epsilon} < c/C_0 < 1/C_0^2$, then there is $C(\epsilon)$ with*

$$A_p(R) \leq (1+cC)\left((1+cC)^p \overline{A}_p(R/2)^p + \left(C(\epsilon)c^{-C}R^{\frac{n+4}{2}\left(\frac{1}{p}-\frac{3(n+2)}{2(n+4)}\right)+\epsilon}\right)^p\right)^{1/p}$$

The inequality above is enough to infer that $A_p(R)$ is bounded uniformly in R.

20.3.3 The table construction

An averaging lemma allows to estimate

$$\|\phi_1\phi_2\phi_3\|_{L^p(Q_R)} \leq (1+cC)\|\phi_1\phi_2\phi_3\|_{L^p(I^{c,j}(Q))}$$

where $Q \subset 4Q_R$ has side-length at most $2R$ and $I^{c,j} = \bigcup_{i=1}^{2^{jn}}(1-c)Q_i$ where Q_i are cubes of equal size that partition Q. Based on a wave packet decomposition, one constructs the *wave table* $\Phi_1 = (\Phi_1^{(k)})_{k=1}^{2^{jn}}$ from the data (ϕ_1, ϕ_2, Q) and defines its mass as $M(\Phi_1) = \sum_k M(\Phi^{(k)})$. The following properties follow from the construction:

- $\phi_1 = \sum_{k=1}^{2^{jn}} \Phi_1^{(k)}$.
- The margin requirement holds for all $\Phi_1^{(k)}$.
- $M(\Phi_1) \leq (1+cC)M(\phi_1)$.
- For $l \neq l'$, it holds

$$\|\Phi_1^{(l)}\phi_2\phi_3\|_{L^1((1-c)Q_{l'})} \lesssim_\epsilon c^{-C}R^{-\frac{n-2}{4}+\epsilon}\prod_{i=1}^{3} M(\phi_i)^{1/2}.$$

This decomposition is for ϕ_1 and it encodes its interaction with ϕ_2. There are analogous decompositions for other waves and interactions, and they are used in the proof.

Using the decomposition, one may write

$$\|\phi_1\phi_2\phi_3\|^p_{L^p(I^{c,j}(Q))} \leq \sum_{l,k} \|\Phi_1^{(k)}\phi_2\phi_3\|^p_{L^p(Q_l)}.$$

The off-diagonal terms are estimated interpolating the L^1 bound (from the decomposition) and an $L^{2/3}$ bound (Hölder plus $\|\phi_1\|_{L^2(Q_l)} \leq R^{1/2}M(\phi_1)^{1/2}$), yielding the the power of R appearing in Proposition 3. The decomposition argument is iterated on the second and third factors in the diagonal terms. This results in terms in smaller scale that are estimated using the bound with constant $\overline{A}_p(R/2)$ that is allowed to appear in the final estimate.

Together with the induction on scales argument, the last item in the decomposition Φ is the key ingredient in the proof. Its proof is based on a combination of a localized version of multilinear restriction estimate [2] and a geometric argument involving the transversality and curvature assumption of the theorem.

20.4 Optimality

The example showing the optimality of the estimate is obtained by choosing suitable caps from the unit sphere. This example has constant Gaussian curvature.

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21 Decoupling on moment curve

after J. Bourgain, C. Demeter and L. Guth [1]
A summary written by Hong Wang

Abstract

We discuss a few ideas on the proof of decoupling theorem on moment curve. The decoupling theorem leads to a proof of the main conjecture in Vinogradov's Mean Value Theorem.

21.1 Introduction

Given $f : [0,1] \to \mathbb{C}$ and an interval $J \subseteq [0.1]$, we define the *extension* operator E_J in $\mathbb{R}^n$ as follows

$$E_J f(x) = \int_J f(t)e(tx_1 + t^2x_2 + \cdots + t^n x_n)dt,$$

with $x = (x_1, \dots, x_n)$ and $e(z) = e^{2\pi i z}, z \in \mathbb{R}$.

Theorem 1. *Let $n \geq 2$ and $0 < \delta \leq 1$. For each ball $B \subset \mathbb{R}^n$ with radius δ^{-n} and each $f : [0,1] \to \mathbb{C}$ we have*

$$\|E_{[0,1]}f\|_{L^{n(n+1)}(w_B)} \lesssim_\epsilon \delta^{-\epsilon}(\sum_{J\subseteq[0,1],|J|=\delta} \|E_J f\|^2_{L^{n(n+1)}(w_B)})^{1/2}.$$

The implicit constant is independent of δ, B, f.

One important application is the Vinogradov's Mean Value Theorem.

Theorem 2. *For each integers $s \geq 1$ and $n, N \geq 2$ denote by $J_{s,n}(N)$ the number of integral solutions for the following system*

$$X_1^i + \cdots + X_s^i = X_{s+1}^i + \cdots + X_{2s}^i, 1 \leq i \leq n,$$

With $1 \leq X_1, \dots, X_{2s} \leq N$.

$$J_{s,n}(N) \lesssim_\epsilon N^{s+\epsilon} + N^{2s-\frac{n(n+1)}{2}+\epsilon}.$$

We show the proof of Theorem 2 in the case of $s = \frac{n(n+1)}{2}$ (which is also the most interesting case) using Theorem 1.

Proof. Let $f = \sum_{\xi \in \frac{1}{N}\mathbb{Z}} \delta(\cdot - \xi)$.

$$E_{[0.1]}f = \sum_{\xi \in \frac{1}{N}\mathbb{Z}, 0<\xi\leq 1} e(\xi x_1 + \xi^2 x_2 + \cdots + \xi^n x_n).$$

We apply Theorem 1 with $\delta = N^{-1}$. The left-hand side of the decoupling inequality (raised to the $n(n+1)$–power) is

$$\begin{aligned}\|E_{[0,1]}f\|_{L^{2s}(w_B)}^{2s} &\gtrsim \|E_{[0,1]}f\|_{L^{2s}(B)}^{2s} \\ &= \int_B \sum_{0<\xi_1,\cdots,\xi_{2s}\leq 1} e(\sum_{j=1}^{n}(\xi_1^j + \cdots + \xi_s^j - \xi_{s+1}^j - \cdots - \xi_{2s}^j)x_j)dx \\ &= J_{s,n}(N)|B|\end{aligned}$$

On the right-hand side, each interval J of length $\frac{1}{N}$ contains only one point $\xi \in \frac{1}{N}\mathbb{Z}$. We have a simple expression for $E_J f = e(\xi x_1 + \cdots + \xi^n x_n)$ and

$$\|E_J f\|_{L^{2s}(w_B)} \lesssim |B|^{1/2s}.$$

□

One can find another proof of Vinogradov's Mean Value Theorem using efficient congruencing method developed by Trevor D. Wooley [2].

21.2 Tools

In this subsection, we discuss two general tools in studying decoupling type problem.

21.2.1 Locally constant

Let $K \subset \mathbb{R}^n$ be a bounded open convex set with center of mass ω_K. We define the dual convex body K^* by

$$K^* := \{x \in \mathbb{R}^n \text{ so that } |x \cdot (\omega_K - \omega)| \leq 1 \text{ for all } \omega \in K\}.$$

Lemma 3. *There is a rapidly decaying μ so that the following holds. If $supp\widehat{g} \subset K$ (a bounded open convex set), then $|g| \lesssim |g| * \mu_{K^*}$.*

Moreover, for any $x \in \mathbb{R}^n$,

$$\max_{x+K^*} |g| \lesssim \min_{x+K^*} |g| * \mu_{K^*}.$$

Lemma 3 says that we can visualize $|g|$ as roughly a constant on any translation of K^* provided that supp$\widehat{g} \subset K$.

21.2.2 L^2 orthogonality

Lemma 4. *For each ball $B \subset \mathbb{R}^n$ with radius δ^{-1}, and each function g with supp$\widehat{g} \subset B_1$, a ball of radius 1,*

$$\|g\|_{L^2(w_B)}^2 \lesssim \sum_{\theta \subset B_1} \|g_\theta\|_{L^2(w_B)}^2,$$

where θ is a small ball of radius δ and $\widehat{g}_\theta = \widehat{g}\chi_\theta$.

The proof is by Plancherel's inequality and using cutoff function with support in B_δ. One can view Lemma 4 as an L^2 decoupling inequality in Theorem 1. Although the L^2 decoupling inequality might seem easy, it is more efficient: we can decouple into a much smaller interval fixing the radius of B. In the "decoupling factory", Lemma 4 is a machine that can only cut L^2–norm, other techniques, for example induction on scale, Hölder's inequality and Minkowski's inequality, ball inflation, produce "L^2 material" for Lemma 4 to cut and then transfer the decoupled pieces into different L^p–norms.

21.3 Property of the moment curve

21.3.1 Translation-Dilation Invariance

We can view $E_{[0,1]}f$ as the fourier transform of a function supported on the unit moment curve $\mathcal{C} = \{(t, t^2, \ldots, t^n), 0 \leq t \leq 1\}$. The moment curve $(t, t^2, \ldots, t^n)$ is translation-dilation invariant. In other words, given a small (connected) piece of $\mathcal{C}$, one can transform it into a unit size curve $\mathcal{C}$ via translation and dilation. In other words, decoupling $[0,1]$ into Js of length δ is the same as decoupling an I of length τ into $\widetilde{J}$s of length $\delta\tau$. This property of moment curve implies that one can use the information on smaller scale to study larger scale. Looking at the problem in many scales plays an important role on the study of decoupling. For instance, one can reduce the study of $\|E_{[0,1]}f\|_{L^{n(n+1)}(w_B)}$ to the study of multi-linear inequality

$$\|(\Pi_{i=1}^M E_{I_i} f)^{1/M}\|_{L^{n(n+1)}(w_B)} \lesssim_\epsilon \delta^{-\epsilon} \Pi_{i=1}^M \Big(\sum_{J \subseteq I_i, |J| = \delta} \|E_J f\|_{L^{n(n+1)}(w_B)}^2\Big)^{1/2},$$

for some large constant M with I_i pairwise distance at least $O_M(1)$.

21.3.2 Ball inflation

In "decoupling factory", the following theorem, named *ball inflation*, is the bridge that connects different scales.

Theorem 5. *Fix $1 \leq k \leq (n-1)$ and $p \geq 2n$. Let B be an arbitrary ball in $\mathbb{R}^n$ with radius $\rho^{-(k+1)}$, and let $\mathcal{B}$ be a finitely overlapping cover of B with balls Δ of radius ρ^{-k}. Then for each $f : [0,1] \to \mathbb{C}$ we have*

$$\frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}}\left[\Pi_{i=1}^{M_n}\Big(\sum_{J_i\subset I_i,|J_i|=\rho}\|E_{J_i}f\|^2_{L^{\frac{pk}{n}}_{avg}(w_\Delta)}\Big)^{1/2}\right]^{p/M_n}$$

$$\lesssim_{\epsilon,K}\rho^{-\epsilon}\left[\Pi_{i=1}^{M_n}\Big(\sum_{J_i\subset I_i,|J_i|=\rho}\|E_{J_i}f\|^2_{L^{\frac{pk}{n}}_{avg}(w_B)}\Big)^{1/2}\right]^{p/M_n}$$

On the left-hand side, we integral $E_{J_i}f$ in a smaller ball Δ. Inside Δ, we are like a nearsighted person who can only see the ρ^k–neighborhood of the moment curve $\mathcal{C}$. In order to further decouple $\mathcal{C}$, we need to gather information of E_{J_i} in a larger ball B. The above Ball Inflation Theorem tells us that we can do so under $L^{\frac{pk}{n}}$–norm without essential loss. The next step is to use Hölder's inequality to produce some L^2–norm and then apply Lemma 4.

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22 Algebraic methods in discrete analogs of the Kakeya problem

after L. Guth and N. Katz [3]
A summary written by Michał Warchalski

Abstract

In this talk we shall give a proof of the joints problem in three dimensions, a discrete analogue of the Kakeya conjecture. We use the polynomial method, following the paper [3] by Guth and Katz.

22.1 Joints problem in $\mathbb{R}^3$

We consider a discrete analogue of the Kakeya conjecture in dimension $d = 3$. $\mathbb{R}^d$ with $d \geq 2$ will be considered in the talk by Zihui He.

Suppose we are given L lines in $\mathbb{R}^3$.

Definition 1. *A* joint *is a point where d lines in L intersect, not all in a common hyperplane. We denote the set of joints formed by L with J.*

The joints problem is to answer the question: what is the upper bound on $|J|$ in terms of $|L|$? Taking a $N^{1/2} \times N^{1/2} \times N^{1/2}$ cube and considering the lines in coordinate directions intersecting the cube which intersect the lattice, it is easy to notice that this system of lines forms $\sim N^{3/2}$ joints. It was conjectured that the number of joints is $O(|L|^{3/2})$. The problem was resolved by Guth and Katz in [3]:

Theorem 2. *The number of joints $|J|$ formed by a system of lines L is $O(|L|^{3/2})$.*

During the talk we shall give a proof of Theorem 2. Before we collect the necessary ingredients, let us note the connection between the joints problem and the Kakeya conjecture.

22.2 Relation to the maximal Kakeya conjecture

Just like in the previous subsection we state all results and conjectures in dimension $d = 3$.

Let δ-tube be a tube in $\mathbb{R}^3$ of length 1 and cross section of radius δ. The maximal Kakeya conjecture in $\mathbb{R}^3$ can be stated as the following inequality.

Conjecture 3. *Let $\mathbb{T} = \{T_i\colon i \in I\}$ be any collection of δ-tubes in $\mathbb{R}^3$, whose orientation are δ-separated in S^2. Then*

$$\|(\sum_{T\in\mathbb{T}} \chi_T)^3\|_{L^{1/2}} \leq C_\epsilon \delta^{-\epsilon}(\delta^2|\mathbb{T}|)^3 \tag{1}$$

Bennett, Carbery and Tao posed a related question for a multilinear version of the above inequality in [1]: let $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3$ be families of δ-tubes in $\mathbb{R}^3$ having the property that any $T \in \mathbb{T}_j$ is pointing in a direction within a small fixed neighbourhood of the j-th standard basis vector. Then

$$\|\prod_{j=1}^{3}(\sum_{T\in\mathbb{T}_j} \chi_T)\|_{L^{1/2}} \leq C\prod_{j=1}^{3}\delta^2|\mathbb{T}_j| \tag{2}$$

In [1] the above inequality with loss of factor $\delta^{-\epsilon}$ was proven. Later the conjectured inequality (2) was shown by Guth in [2].

Connection between (2) and the joints problem can be seen roughly as follows: consider a collection of lines L and replace the collections of tubes $\mathbb{T}_j$ by L for every $j = 1, 2, 3$. The left hand side of (2) which can be seen as the square of the volume of points belonging to at least one tube in each direction is replaced with $|J|^2$, where J is the set of joints. Forgetting the δ factors we view (2) as $|J|^2 \leq C|L|^3$ which is precisely the bound in the joints problem.

22.3 Sketch of the proof

The strategy of [3] to prove Theorem 2 is to use the so-called polynomial method. The proof follows the steps:

1. Suppose to the contrary that the number of joints $|J| \geq KN^{3/2}$ for a big K.

2. Find a polynomial of low degree ($\leq CN^{1/2}K^{-1/3}$) which vanishes on most joints.

3. Factorize the polynomial and find an irreducible polynomial which vanishes on a big subset of J and on a big subset of L.

4. The gradient of the irreducible polynomial vanishes on a big subset of L as well.

5. Bezout's theorem leads to a contradiction because it implies that the polynomial and its gradient should have a common factor.

We elaborate now a little bit on the above steps, collecting several necessary ingredients of the proof.

22.3.1 Some algebra

We state the following fact that we wish to apply later to p and ∇p, where p is the constructed irreducible polynomial vanishing on many joints; more specifically it is the key fact in the proof of Proposition 8.

Proposition 4 (Bezout's theorem). *Let $f, g \in \mathbb{R}[x_1, x_2, x_3]$ and suppose that f and g have positive degrees l and m respectively. Suppose that there are more than lm lines on which f and g simultaneously vanish identically. Then f and g have a common factor.*

Next we give a proof of the following simple, however important, proposition. Using this fact one can find a polynomial of small degree $\sim N^{1/2}$ when given a set of points of size $\sim N^{3/2}$; that is roughly what happens in **step 2** of the method.

Proposition 5. *Let X be a set of N points in $\mathbb{R}^3$. Then there is a nontrivial polynomial in $\mathbb{R}[x_1, x_2, x_3]$ which vanishes at every point of X of degree less than $CN^{1/3}$.*

Proof. On one hand, note that a polynomial of degree d and three variables has roughly d^3 coefficients. On the other hand requiring a polynomial to vanish on a set of points of size N gives a system of N equations for the coefficients. However, underdetermined system always have a nontrivial solution. $\square$

22.3.2 Some geometry

Let p be irreducible polynomial on $\mathbb{R}^3$ of degree $d > 0$, consider the zero set of this polynomial

$$S = \{(x, y, z) \colon p(x, y, z) = 0\} \tag{3}$$

Definition 6. *We call a point $a \in S$ critical if $\nabla p(a) = 0$ and regular if $\nabla p(a) \neq 0$.*

Definition 7. *We call a line $l \subset S$ critical if $\nabla p \equiv 0$ on l.*

The following proposition implies that there cannot be too many lines on which both p and ∇p vanish identically, the fact that is used in **step 5** of the procedure.

Proposition 8. *The set S contains no more than $d(d-1)$ critical lines.*

Proof. Suppose to the contrary that there are $> d(d-1)$ critical lines. But this leads to a contradiction applying Proposition 4. □

22.3.3 A pigeonhole principle for bipartite graphs

We shall use the language of graph theory to deal with the lines and the joints formed by them. We shall create a bipartite graph between the set of lines L and the set of joints J. Here we give a variant of the pigeonhole principle we shall need in the proof.

Proposition 9. *Let (X, Y, E) be a bipartite graph with E the edges, X and Y the two sets of vertices. Suppose that $|E| > \rho|Y|$. Let $Y' \subset Y$ be the subset of vertices having degree at least μ and let $E' \subset E$ be the set of edges between Y' and X. Then*

$$|E'| > (\rho - \mu)|Y| \tag{4}$$

Proof. The vertices of $Y \setminus Y'$ are incident to at most $\mu|Y|$ edges. Let $|E''|$ be the set of these edges. We have

$$|E'| = |E| - |E''| \geq \rho|Y| - \mu|Y| = (\rho - \mu)|Y|. \tag{5}$$

□

The point of the proposition is that taking the set of all lines that form many (each $\geq \delta N^{1/2} K$) joints "covers" most of J (J', s.t. $|J'| \geq (1-\delta)|J|$). Since we assume that $|J|$ greatly dominates $|L|$, a polynomial vanishing on J' must vanish on L as well. Such arguments are used in **step 2** and **step 3** of the procedure.

22.4 One more discrete analogue

Similar methods to the ones presented in the previous subsection lead to a proof of Bourgain's conjecture, another discrete analogue of the Kakeya conjecture. The following theorem is the second main result of [3].

Theorem 10. *Let L be a set of N^2 lines in $\mathbb{R}^3$ and let P be a set of points in $\mathbb{R}^3$. Suppose no more than N lines of L lie in the same plane and suppose that each line of L contains at least N points of P. Then $|P| = \Omega(N^3)$.*

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23 Interlacing families I: bipartite Ramanujan graphs of all degrees

after Adam Marcus, Daniel Spielman, Nikhil Srivastava [1]
A summary written by Julian Weigt

23.1 Ramanujan graphs

For a graph G with vertices $1, 2, \ldots, n$ the *adjacency matrix* A_G is an $n \times n$-matrix where

$$A_G(i,j) = \begin{cases} 1 & i \sim j \\ 0 & \text{else} \end{cases}$$

A graph is called *d-regular* if every vertex has degree d. If G is d-regular graph, the eigenvalue of maximal absolute value of A_G is d. We call eigenvalues with absolute value d *trivial*. People are interested in the spectral gap of A_G, i.e. how small the nontrivial eigenvalues are in absolute value. $2\sqrt{d-1}$ is a lower bound in the sense that for every $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that all d-regular graphs with at least n vertices have an adjacency matrix with a nontrivial eigenvalue with absolute value at least $2\sqrt{d-1} - \varepsilon$ [3]. A graph is called *Ramanujan* if the nontrivial eigenvalues are bounded by $2\sqrt{d-1}$ in absolute value. For example this is the case for d-regular complete and complete bipartite graphs. The following theorem 1 is the main result of the paper.

Theorem 1. *For every degree d and $n \in \mathbb{N}$ there is a d-regular bipartite Ramanujan graphs with at least n vertices.*

23.2 2-lifts

In the bipartite case, the eigenvalues of the adjacency matrix are symmetric around 0. Hence there it suffices to bound the nontrivial eigenvalues by $2\sqrt{d-1}$.

We prove Theorem 1 by inductively constructing a sequence of bipartite Ramanujan graphs where in each step the number of vertices doubles. For a given Ramanujan graph $G = (V, E)$, the next one will be a *2-lift* of G. This is a graph on two copies of V. For a $v \in V$ denote the two corresponding vertices in the 2-lift by v_0, v_1. For every edge $(u, v) \in E$ exactly one of the following pairs of edges have to be be introduced to the 2-lift:

$$\begin{aligned} \text{either} \quad & (u_0, v_0), (u_1, v_1) \\ \text{or} \quad & (u_0, v_1), (u_1, v_0) \end{aligned}$$

A *signing* is a mapping $s : E \to \{-1, 1\}$. A 2-lift corresponds to the signing s which assigns $s(u, v) = 1$ if the first pair of edges belongs to the 2-lift and $s(u, v) = -1$ if the second pair does. For the adjacency matrix A of G we define A_s to be the matrix where the 1s in A that correspond to edges e with $s(e) = -1$ are replaced by -1s.

Proposition 2 ([2, Lemma 3.1])**.** *The eigenvalues of the adjacency matrix of the 2-lift are the union of the eigenvalues of A and A_s including multiplicities.*

Define

$$f_s(x) = \det(xI - A_s). \tag{1}$$

Now in order to prove Theorem 1 it suffices to find a signing s such that the second greatest root of f_s is at most $2\sqrt{d-1}$. This will be done using interlacing families of polynomials.

23.3 Interlacing families

Definition 3. *A polynomial $g = \prod_{k=1}^{n-1}(x - a_k)$ is an* interlacing *of $f = \prod_{k=1}^{n}(x - b_k)$ if for every $k = 1, \ldots, n-1$*

$$b_k \leq a_k \leq b_{k+1}.$$

Lemma 4 ([1, Lemma 4.2])**.** *Define*

$$q_\emptyset = \sum_{i=1}^{k} q_i.$$

Then if $q_1, \ldots, q_k$ have a common interlacing then $q_\emptyset$ is real rooted and there is an i such that the largest root of q_i is at most the largest root of $q_\emptyset$.

Definition 5. *Let $T_1, \ldots, T_m$ be sets and for every $(t_1, \ldots, t_m) \in T_1 \times \ldots \times T_m$ let $q_{t_1,\ldots,t_m}$ be a polynomial. For every $k = 0, \ldots, m$ define*

$$q_{t_1,\ldots,t_k} = \sum_{(t_{k+1},\ldots,t_m) \in T_{k+1} \times \ldots \times T_m} q_{t_1,\ldots,t_k,t_{k+1},\ldots,t_m}.$$

Then if for every $k = 0, \ldots, m-1$ and $(t_1, \ldots, t_k) \in T_1 \times \ldots \times T_k$ the polynomials $\{q_{t_1,\ldots,t_k,r} \mid r \in T_{k+1}\}$ have a common interlacing then the polynomials are called an interlacing family*.*

Theorem 6 ([1, Theorem 4.4])**.** *For an interlacing family there is a $(t_1, \ldots, t_m) \in T_1 \times \ldots \times T_m$ such that the largest root of $q_{t_1,\ldots,t_m}$ is at most the largest root of $q_\emptyset$.*

Theorem 7 ([1, Theorem 3.6],[4])**.** *Recall* (1)*. Then the roots of*

$$\sum_{s:E\to\{-1,1\}} f_s$$

are real and bounded by $2\sqrt{d-1}$.

In order to finish the induction step for Theorem 1 it suffices to show that $\{f_s \mid s : E \to \{-1,1\}\}$ is an interlacing family.

Lemma 8 ([1, Lemma 4.5])**.** *The polynomials $q_1, \ldots, q_k$ have a common interlacing iff every convex combination of them is real rooted.*

23.4 Real stable polynomials

We first show that the family $\{f_s \mid s\}$ belongs to a more general class of families which are all interlacing. Let $v_1, \ldots, v_m$ be independent random vectors such that v_i takes the values $w_{i,1}, \ldots, w_{i,l_i} \in \mathbb{C}^n$ with probabilities $p_{i,1}, \ldots, p_{i,l_i}$. We consider the family given by the polynomials

$$q_{j_1,\ldots,j_m} = [\prod_{i=1}^{m} p_{i,j_i}] \det(xI - \sum_{i=1}^{m} w_{i,j_i} w_{i,j_i}^*).$$

Now there is probabilistic interpretation of the last section's summing notation:

$$\begin{aligned} q_{j_1,\ldots,j_k} &= [\prod_{i=1}^{k} p_{i,j_i}] \sum_{j_{k+1},\ldots,j_m} [\prod_{i=k+1}^{m} p_{i,j_i}] \det(xI - \sum_{i=1}^{m} w_{i,j_i} w_{i,j_i}^*) \\ &= [\prod_{i=1}^{k} p_{i,j_i}] \mathbb{E}_{v_{k+1},\ldots,v_m} \det(xI - \sum_{i=1}^{k} w_{i,j_i} w_{i,j_i}^* - \sum_{i=k+1}^{m} v_i v_i^*) \qquad (2) \end{aligned}$$

Choose for each edge (u, v) a random vector $v_{(u,v)}$ which takes the two values

$$w_{(u,v),1} = (0, \ldots, 0, \underset{u}{1}, 0, \ldots, 0, \underset{v}{1}, 0, \ldots, 0)^*,$$
$$w_{(u,v),-1} = (0, \ldots, 0, \underset{u}{1}, 0, \ldots, 0, \underset{v}{-1}, 0, \ldots, 0)^*.$$

with equal probability $\frac{1}{2}$. Then we recover our case

$$2^{-m} f_s(x) = q_s(x + d).$$

Hence the following theorem 9 implies that $\{f_s \mid s\}$ is an interlacing family, finishing the proof of Theorem 1. Note, that the global shift of d does not effect the interlacing property.

Theorem 9 ([5, Theorem 4.5])**.** *For independent random vectors $v_1, \ldots, v_m$ the polynomials $\{q_{j_1,\ldots,j_m} \mid j_1, \ldots, j_m\}$ form an interlacing family.*

By Lemma 8 it suffices to prove that for all $k = 0, \ldots, m-1$ if $\lambda_1, \ldots, \lambda_{l_{k+1}} \geq 0$ and $j_1, \ldots, j_k$ then the polynomial

$$\sum_{r=1}^{l_{k+1}} \lambda_r q_{j_1,\ldots,j_k,r}$$

is real rooted. For that, define new random vectors $u_1, \ldots, u_m$ with

$$\begin{array}{lcr} i = 1, \ldots, k : & & P(u_i = w_{i,j_i}) = 1, \\ i = k+1 : & r = 1, \ldots, l_i & P(u_i = w_{i,r}) = \lambda_r, \\ i = k+2, \ldots, m : & & u_i = v_i. \end{array}$$

By equation (2) we get

$$\sum_{r=1}^{l_{k+1}} \lambda_r q_{j_1,\ldots,j_k,r} = [\prod_{i=1}^{k} p_{i,j_i}] \mathbb{E}_{u_1,\ldots,u_m} \det(xI - \sum_{i=1}^{m} u_i u_i^*).$$

Hence it suffices to prove the following theorem 10.

Theorem 10 ([5, Corollary 4.4])**.** *For independent random vectors $v_1, \ldots, v_m$*

$$\mathbb{E}_v \det(xI - \sum_{i=1}^{m} v_i v_i^*) \tag{3}$$

is real rooted.

For a univariate polynomial with real coefficients, such as (3), real rootedness is a special case of real stability:

Definition 11. *A (multivariate) polynomial $p(z_1, \ldots, z_m)$ with real coefficients is called* real stable *if for all $z_1, \ldots, z_m$ with positive imaginary part $p(z_1, \ldots, z_m) \neq 0$.*

Now we rewrite (3) in order to prove real stability.

Theorem 12 ([5, Theorem 4.1])**.**

$$\mathbb{E}_v \det(xI - \sum_{i=1}^{m} v_i v_i^*) = \left[\prod_{i=1}^{m}(1 - \partial_i)\right] \det(xI + \sum_{i=1}^{m} z_i \mathbb{E}_v v_i v_i^*) \mid_{z_1,\ldots,z_m=0} .$$

Theorem 10 then follows from Theorem 12 and the following propositions 13, 14 and 15.

Proposition 13 ([6, Proposition 2.4])**.** *If $A_1, \ldots, A_m$ are positive semidefinite Hermitian matrices then*

$$\det(\sum_{i=1}^{m} z_i A_i)$$

is real stable.

Proposition 14. *If $p(z_1, \ldots, z_m)$ is real stable then so is*

$$(1 - \partial_1)p(z_1, \ldots, z_m).$$

Proof. Let $z_2, \ldots, z_m$ have positive imaginary part. Then the roots $c_1, \ldots, c_l$ of the polynomial in z_1 $q(z_1) = p(z_1, z_2, \ldots, z_m)$ have nonpositive imaginary part. Hence for $\Im z_1 > 0$ we have that $\frac{\partial_1 q(z_1)}{q(z_1)} \sim \sum_{i=1}^{l} \frac{1}{z_1 - c}$ consists of summands with negative imaginary part and hence is not equal to 1. $\square$

Proposition 15 ([7, Lemma 2.4d])**.** *If $p(z_1, \ldots, z_m)$ is real stable and $a \in \mathbb{R}$ then*

$$q(z_2, \ldots, z_m) = p(a, z_2, \ldots, z_m)$$

is real stable.

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24 Decoupling, exponential sums and the Riemann zeta function

after Jean Bourgain [2]
A summary written by Błażej Wróbel

Abstract

Using decoupling techniques the paper gives an L^{12} norm estimate of a certain exponential sum. This is then applied to obtain pointwise bounds for another exponential sum which in turn imply improved estimates for $|\zeta(\frac{1}{2}+it)|$ as $t \to \infty$.

24.1 Introduction

The Lindelöf hypothesis states that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| = O(t^{\varepsilon}), \quad t \to \infty,$$

for arbitrary $\varepsilon > 0$. This is significantly weaker than the Riemann hypothesis. However, despite many efforts it remains wide open. One of the purpose of Bourgain's paper is to make a progress towards the proof of the Lindelöf hypothesis.

In the paper an important step to achieve this is an L^{12} norm estimate of an exponential sum. In what follows we set $e(x) = e^{2\pi i x}$ and, for $r \geq 1$ we define

$$A_r\left(\frac{1}{N^2}, \frac{1}{N}\right) :=$$
$$\int_0^1 \int_0^1 \int_{-1}^1 \int_{-1}^1 \Big|\sum_{n=1}^{N} e(nx_1 + n^2 x_2 + N^{1/2} n^{3/2} x_3 + N^{1/2} n^{1/2} x_4)\Big|^{12} dx_1\, dx_2\, dx_2\, dx_4$$

Throughout the summary by $J \ll K$ we mean that $J = O(K)$.

Theorem 1 (Theorem 2 in [2])**.** *For each $N \in \mathbb{N}$ and $\varepsilon > 0$ we have*

$$A_6\left(\frac{1}{N^2}, \frac{1}{N}\right) \ll N^{6+\varepsilon}. \tag{1}$$

Theorem 1 implies a corollary that is critical in obtaining bounds for exponential sums,. This in turn will lead to an estimate for $|\zeta(\frac{1}{2}+it)|$. In what follows we set

$$A_6(N,\delta,\Delta) = \int_0^1\int_0^1\int_{-1}^1\int_{-1}^1\left|\sum_{n\le N} e\left(nx_1+n^2x_2+\frac{1}{\delta}\left(\frac{n}{N}\right)^{3/2}x_3+\frac{1}{\Delta}\left(\frac{n}{N}\right)^{1/2}x_4\right)\right|^{12}dx.$$

Corollary 2 (Corollary 3 in [2])**.** *Let* $\frac{1}{N^2}\le\delta\le 1$, *and* $\frac{1}{N}\le\Delta\le 1$. *Then*

$$A_6(N,\delta,\Delta)\ll\delta\Delta N^{9+\varepsilon}. \tag{2}$$

From Corollary 2 the author deduces a bound for exponential sums that is crucial in bounding $|\zeta(\frac{1}{2}+it)|$. The implication from Corollary 2 to a pointwise bound for exponential sums follows an observation of Huxley [6]. Namely, obtaining good bounds on A_6 leads to further improvements in the Bombieri-Iwaniec "large sieve" method [1] (as developed by Huxley [4]). This is discussed in detail in [2, Section 3]. In the end one is reduced to a pointwise estimate for exponential sums which we describe below.

Let F be a smooth function on $[\frac{1}{2},1]$ satisfying, for some constant $c\in(0,1]$, the condition

$$\min\{|F''(x)|,|F'''(x)|,|F^{(iv)}(x)|\}>c. \tag{3}$$

For sufficiently large T and $M\ge 1$ we put $f(u)=TF(u/M)$ where $\frac{M}{2}\le u\le M$ and define

$$S=\sum_{m\approx M} e(f(m)).$$

The required bound for exponential sums is given in the theorem below.

Theorem 3 (Theorem 4 in [2])**.** *Set* $\alpha=\log M/\log T$. *Then with the above notation one has*

$$|S|\ll M^{1/2}T^{\varepsilon+13/84},\ \textit{whenever}\ \frac{1}{2}\ge\alpha\ge\frac{17}{42}. \tag{4}$$

For the applications to bounding $|\zeta(\frac{1}{2}+it)|$ one needs also (4) for the particular function $F(x)=\log x$ in the cases $\frac{13}{42}\le\alpha<\frac{17}{42}$. These follow from previously known estimates (via [5, Theorem 3] and [**?**, Section 5 20]). In summary, we have the following.

Corollary 4. *Let $F(x) = \log x$ and set $\alpha = \log M/\log T$. Then with the above notation one has*

$$|S| \ll M^{1/2}T^{\varepsilon+13/84}, \text{ whenever } \frac{1}{2} \geq \alpha \geq \frac{13}{42}. \tag{5}$$

From Corollary 4 with $F(x) = \log x$ Bourgain reaches an improvement over the known bounds for $|\zeta(\frac{1}{2}+it)|$ as $t \to \infty$. More precisely, Corollary 4 together with [6, Theorem 21.2.2] with $\sigma, \lambda \approx \frac{1}{2}$, and $\mu \approx 13/84$ implies the following.

Theorem 5 (Theorem 5 in [2])**.** *For any $\varepsilon > 0$ it holds*

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| = O(t^{13/84+\varepsilon}), \quad t \to \infty. \tag{6}$$

Theorem 5 improves a result of Huxley [7], who proved the estimate (6) with 13/84 replaced by 32/205.

24.2 A decoupling inequality for curves

We devote the reminder of the summary to a key ingredient in the proof of Theorem 1 (hence, also in deducing Corollary 2). This ingredient is a certain decoupling inequality for curves. Let $\Phi = (\phi_1, \ldots, \phi_d)\colon [0,1] \to \Gamma \subseteq \mathbb{R}^d$ be a smooth parametrization of a non-degenerate curve in $\mathbb{R}^d$. More precisely we assume that the Wronskian determinant

$$\det[\phi_j^{(s)}(t_s)_{1\leq j,s\leq d}] \neq 0 \text{ for all } t_1, \ldots, t_d \in [0,1].$$

Let d be even and denote

$$\|f\|_{L^p_\#(\Omega)} = \Big(\fint_\Omega |f|^p\,dx\Big)^{1/p} = \Big(\frac{1}{|\Omega|}\int_\Omega |f|^p\,dx\Big)^{1/p}.$$

By B_N we denote the Euclidean ball centered at the origin and of radius N.

Theorem 6 (Theorem 1 in [2])**.** *Let Γ be as above and let $I_1, \ldots, I_{\frac{d}{2}} \subset [0,1]$ be subintervals that are $O(1)$-separated. Let N be large and let $\{I_\tau\}$ be a partition of $[0,1]$ in $N^{-\frac{1}{2}}$-intervals. Then for arbitrary coefficient functions*

$a_j = a_j(t)$

$$\Big\| \prod_{j=1}^{d/2} \Big| \int_{I_j} a_j(t) e\big(x.\Phi(t)\big) dt \Big|^{2/d} \Big\|_{L^{3d}_{\#}(B_N)} \ll$$
$$N^{\frac{1}{6}+\varepsilon} \prod_{j=1}^{d/2} \Big[\sum_{\tau; I_\tau \subset I_j} \Big\| \int_{I_\tau} a_j(t) e\big(x.\Phi(t)\big) dt \Big\|^6_{L^6_{\#}(B_N)} \Big]^{\frac{1}{3d}} \tag{7}$$

holds, with $\varepsilon > 0$ arbitrary.

Remark 7. *Strictly speaking we should replace $L^6_{\#}(B_N)$ in (7) by some weighted $L^6_{\#}(w_N)$, where w_N is a smoothed version of 1_{B_N}.*

Sketch of the proof of Theorem 6. Let $b(N) > 0$ be the best constant such that the inequality,

$$\Big\| \prod_{j=1}^{d/2} \Big| \int_{I_j} a_j(t) e\big(x.\Phi(t)\big) dt \Big|^{2/d} \Big\|_{L^{3d}_{\#}(B_N)} \le$$
$$b(N) N^{\frac{1}{6}} \prod_{j=1}^{d/2} \Big[\sum_{I_\tau \subset I_j} \Big\| \int_{I_\tau} a_j(t) e\big(x.\Phi(t)\big) dt \Big\|^6_{L^6_{\#}(B_N)} \Big]^{\frac{1}{3d}} \tag{8}$$

holds, with arbitrary $\{a_j\}$. The aim is to establish a bootstrap inequality. It is shown in [2, eq. (1.14)], that $b(N) \le N^{1/6}$, so that $b(N)$ is finite.

With $K < N$ to specify later, partition B_N in K-cubes $\Delta = \Delta_K$. We may bound for each Δ

$$⨍_{\Delta} \prod_{j=1}^{d/2} \Big| \int_{I_j} a_j(t) e\big(x.\Phi(t)\big) dt \Big|^6 dx \le$$
$$b(K)^{3d} K^{\frac{d}{2}} \prod_{j=1}^{d/2} \Big[\sum_{I_\sigma \subset I_j} \Big\| \int_{I_\sigma} a_j(t) e\big(x.\Phi(t)\big) dt \Big\|^6_{L^6_{\#}(\Delta)} \Big], \tag{9}$$

where $\{I_\sigma\}$ is a partition in $K^{-\frac{1}{2}}$-intervals. Summation over $\Delta \subset B_N$ implies

$$\fint_{B_N} \prod_{j=1}^{d/2} \Big| \int_{I_j} a_j(t) e\big(x.\Phi(t)\big) dt \Big|^6 dx \leq$$

$$b(K)^{3d} K^{\frac{d}{2}} \sum_{I_{\sigma_1} \subset I_1, \dots, I_{\sigma_{d/2}} \subset I_{d/2}} \fint_{B_K^{d/2}} \Big\{ \fint_{B_N} \prod_{j=1}^{d/2} \Big| \int_{I_{\sigma_j}} a_j(t) e\big((x+z_j).\Phi(t)\big) dt \Big|^6 dx \Big\} \prod_j dz_j. \tag{10}$$

Fix an interval $I_{\sigma_j} = [t_j, t_j + K^{-\frac{1}{2}}] \subset I_j$. Then, taking

$$K = N^{3/4}, \tag{11}$$

so that $N = o(K^{3/2})$, we can write for $t = t_j + s \in I_{\sigma_j}$

$$(x+z_j).\Phi(t) = (x+z_j).\Phi(t_j) + (x+z_j).\Phi'(t_j)s + \frac{1}{2}(x+z_j).\Phi''(t_j)s^2 + o(1). \tag{12}$$

The $o(1)$ term in (12) produces a negligible error term. Hence, the inner integral in (10) may be replaced by

$$\fint_{B_N} \prod_{j=1}^{d/2} \Big| \int_0^{K^{-\frac{1}{2}}} a_j(t_j+s) e\Big((x+z_j).\Phi'(t_j)s + \frac{1}{2}(x+z_j).\Phi''(t_j)s^2\Big) ds \Big|^6 dx \tag{13}$$

Since $t_1 < t_2 < \cdots < t_{d/2}$ are $O(1)$-separated it, can be shown that the map taking $x \in \mathbb{R}^d$ to $\big(x.\Phi'(t_1), \frac{1}{2}x.\Phi''(t_1), \dots, x.\Phi'(t_{d/2}), \frac{1}{2}x.\Phi''(t_{d/2})\big) \in \mathbb{R}^d$ is a linear homeomorphism. The image measure of the normalized measure on B_N may be bounded by the normalized measure on B_{CN}, up to a factor and thus (13) may be controlled by

$$\prod_{j=1}^{d/2} \fint_{|u|,|v|<CN} \Big| \int_0^{K^{-\frac{1}{2}}} a_j(t_j+s)\, e(us+vs^2) ds \Big|^6 du dv. \tag{14}$$

This is the main point in the argument.

Namely, in the inner integral in (14) the exponent $e(us+vs^2)$ is integrated over a parabola in $d=2$. A decoupling inequality in this case follows from an earlier result of Bourgain and Demeter [3]. Using this observation Bourgain is able to control the left hand side of (10) by

$$b(K)^{3d}N^{\frac{d}{2}+\varepsilon}\prod_{j=1}^{d/2}\Big[\sum_{I_\tau\subset I_j}\Big\|\int_{I_\tau}a_j(t)\,e\big(x.\Phi(t)\big)dt\Big\|_{L^6_{\#}(B_N)}^6\Big]. \tag{15}$$

Since $b(N)$ is the best constant in (8) we conclude that $b(N)N^{\frac{1}{6}+\varepsilon}\leq b(K)N^{\frac{1}{6}}$. Hence, recalling (11) we arrive at $b(N)\leq b(N^{3/4})N^{\varepsilon}$. Finally, iterating this bound we obtain $b(N)\ll N^{4\varepsilon}$. This completes the sketch of the proof of Theorem 6.

□

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25 The multilinear restriction estimate: A short proof and a refinement

after Ioan Bejenaru [1]
A summary written by Zhen Zeng

Abstract

We provide an alternative and self-contained proof of a near-optimal version of the multilinear restriction conjecture in [2] and a refined estimate for the lower levels of multilinearity.

25.1 Introduction

The multilinear restriction estimate has long been a fundamental problem in harmonic analysis. In [2], Bennett, Carbery and Tao obtained an almost optimal estimate of this problem. In [1], the author presents an alternative proof of their results.

For $n \leq 1$, let $U \subset \mathbb{R}^n$ be an open, bounded neighborhood of the origin and let $\Sigma : U \to \mathbb{R}^{n+1}$ be a smooth parametrization of a n-dimensional submanifold of $\mathbb{R}^n$. We define the operator

$$\mathcal{E}f(x) = \int_U e^{ix\cdot\Sigma(\xi)} f(\xi) d\xi$$

For $1 \leq i \leq n+1$, let $\Sigma_i : U_i \to \mathbb{R}^{n+1}$ be the parametrization, satisfying the smooth condition:

$$||\partial^\alpha \Sigma_i||_{L^\infty(U_i)} \lesssim_\alpha 1$$

It also satisfies the transversality condition: there exists $\nu > 0$ such that

$$|det(N_1(\zeta_1), ...N_{n+1}(\zeta_{n+1}))| \geq \nu$$

Under these assumptions, the multilinear restriction conjecture states as the following:

$$||\prod_{i=1}^{n+1} \mathcal{E}_i f_i||_{L^{\frac{2}{n}}(\mathbb{R}???^{n+1})} \leq C \prod_{i=1}^{n+1} ||f_i||_{L^2}(U_i)$$

The result in [1] is an almost optimal version of the above conjecture.

Theorem 1. *Under the above assumptions, for any $\epsilon > 0$, there is $C(\epsilon)$ such that the following holds true*

$$||\prod_{i=1}^{n+1} \mathcal{E}_i f_i||_{L^{\frac{2}{n}}(B(0,R))} \leq C(\epsilon) R^{\epsilon} \prod_{i=1}^{n+1} ||f_i||_{L^2(U_i)}, \quad \forall f_i \in L^2(U_i), i = 1, ...n+1, \tag{1}$$

where $B(0, R) \subset \mathbb{R}^{n+1}$ is the ball of radius R centered at the origin.

In [2], the authors also obtain a similar result for lower levels of multilinearity. In [1], the author also provide a refinement for this case. We will state it in the last section.

The idea of the proof: we use localization both on the physical and frequency space to get an estimate on the smaller scales, then use discrete Loomis-Whitney inequality to pass to the larger scales.

25.2 Notation

Given N_i, $i = 1, ...n+1$ transversal unit vectors in $\mathbb{R}^{n+1}$, let $\mathcal{H}_i \subset \mathbb{R}^{n+1}$ be the hyperplanes passing through the origin to which N_i are normals. Define $\pi_{N_i} : \mathbb{R}^{n+1} \to \mathcal{H}_i$ to be the projection onto $\mathcal{H}_i$. The vectors N_i, i=1,...n+1 form a basis and thus gives a coordinate. Let $\mathcal{L} := \{z_1 N_1 + ... + z_{n+1} N_{n+1} : (z_1, ...z_{n+1}) \in \mathbb{Z}^{n+1}\}$ be the lattice in $\mathbb{R}^{n+1}$ generated by the unit vectors $N_1, ...N_{n+1}$. In each $\mathcal{H}_i$ we construct the induced lattice $\mathcal{L}(\mathcal{H}_i) = \pi_{N_i}(\mathcal{L})$. This is a lattice since the projection is taken along a direction of the original lattice $\mathcal{L}$.
Given $r > 0$ we define $\mathcal{C}(r)$ be the set of parallelepipeds of size r in $\mathbb{R}^{n+1}$ relative to the lattice $\mathcal{L}$; a parallelepiped in $\mathcal{C}(r)$ has the following form $q(j) := [r(j_1 - \frac{1}{2}), r(j_1 + \frac{1}{2})] \times ... \times [r(j_{n+1} - \frac{1}{2}), r(j_{n+1} + \frac{1}{2})]$ where $j = (j_1, ...j_{n+1}) \in \mathbb{Z}^{n+1}$ and $c(q) := rj$ is its center. Let $\mathcal{CH}_i(r) = \pi_{N_i}\mathcal{C}(r)$ be the set of parallelepipeds of size r in the hyperplane $\mathcal{H}_i$. Given two parallelepipeds $q, q' \in \mathcal{C}(r)$ or $\mathcal{CH}_i(r)$ we define $d(q, q')$ to be the distance between them as subsets of the underlying space $\mathbb{R}^{n+1}$ or $\mathcal{H}_i$.

Assume $\mathcal{H}_1 \subset \mathbb{R}^{n+1}$ is a hypersurface passing through the origin with normal N_1. We denote $\mathcal{F}_1 : \mathcal{H}_1 \to \mathcal{H}_1$ the standard Fourier transform. We denote variables in $\mathbb{R}^{n+1}$ as (x_1, x'), where x_1 are the coordinate along N_1 and x' are the coordinate along $\mathcal{H}_1$. Similarly define (ξ_1, ξ').

25.3 Tools

Let $\chi_0^n : \mathbb{R}^n \to [0, +\infty)$ be a Schwartz function, with $||\chi_0^n||_{L^1} = 1$ and its Fourier transform supported on the unit ball. For fixed $i \in 1, ...n+1$, $r > 0$, define $\mathcal{T}_i : \mathcal{H}_i \to \mathcal{H}_i$ to be the linear operator takes $\mathcal{L}(\mathcal{H}_i)$ to the standard lattice $\mathbb{Z}^n$ in $\mathcal{H}_i$. For each $q \in \mathcal{CH}_i$, define $\chi_q : \mathcal{H}_i \to \mathbb{R}$

$$\chi_q(x) = \chi_0^n(\mathcal{T}_i(\frac{x - c(q)}{r}))$$

$\mathcal{F}_i\chi_q$ is supported in a ball of radius $\lesssim r^{-1}$. By the Poisson summation formula, we have

$$\sum_{q \in \mathcal{CH}_i(r)} \chi_q = 1 \tag{2}$$

We also have

$$\sum_{q \in \mathcal{CH}_i(r)} ||\langle \frac{x - c(q)}{r} \rangle^N \chi_q g||_{L^2}^2 \lesssim_N ||g||_{L^2}^2 \tag{3}$$

To prove this, we just need to show $\sum_{q \in \mathcal{CH}_i(r)} ||\langle \frac{x-c(q)}{r} \rangle^N \chi_q(x)||_{L^\infty} \lesssim_N C$. Which is equivalent to show $\sum_{k \in \mathbb{Z}^n} ||(x-k)^N \chi_n^0(\mathcal{T}_i(x-k))||_{L^\infty} \lesssim C$. Notice χ_n^0 is a Schwartz function, $\chi_n^0(\mathcal{T}_i(x-k))(x-q)^N \leq C_N(x-k)^{-2}$, then we can get the desired inequality.

After properly decomposing the space, we need the following tools to sum them up.

Considering $L^p(S)$, $S \in \mathbb{R}^n$ which is a bounded domain. We recall the estimate for function $f_\alpha \in L^p(S)$ for $0 < p \leq 1$:

$$||\sum_\alpha f_\alpha||_{L^p}^p \leq \sum_\alpha ||f_\alpha||_{L^p}^p \tag{4}$$

Without loss of generality we can write down the parametrization explicitly. Assume $U_1 \subset \mathcal{H}_1$ is open and bounded. For $f : U_1 \to \mathbb{C}$, $f \in L^2(U_1)$ we define the operator $\mathcal{E}_1 : L^2(U_1) \to L^\infty(\mathbb{R}^{n+1})$ by

$$\mathcal{E}_1 f(x) = \int_{U_1} e^{i(x'\xi' + x_1\varphi(\xi'))} f(\xi') d\xi'$$

Analogous to Fourier transform, for all fixed $x_0 \in \mathbb{R}^{n+1}$ this operator has the following:

$$(x' - x'_0 - x_1 \nabla\varphi_1(\frac{D'}{i}))^N \mathcal{E}_1 f = \mathcal{E}_1(\mathcal{F}_1((x' - x'_0)^N \mathcal{F}_1^{-1} f)), \quad \forall N \in \mathbb{N} \qquad (5)$$

Where the differential operator $\nabla\varphi_1(\frac{D'}{i})$ is the operator with symbol $\nabla\varphi_1(\xi')$.

This can be shown by writing $\mathcal{E}_1 f = \mathcal{F}_1^{-1}(e^{ix_1\varphi_1(\xi')} f(\xi'))(x')$. We will use this equality to apply the induction hypothesis.

When we finish dealing with the estimate on the hypersurfaces, we can use the following discrete version of Loomis-Whitney inequality to get estimate on the original space.

$$||\prod_{i=1}^{n+1} g_i(\pi_{N_i}(z))||_{l^{\frac{2}{n}}(\mathcal{L})} \lesssim \prod_{i=1}^{n+1} ||g_i||_{l^2(\mathcal{L}(\mathcal{H}_i))}$$

25.4 Proof of theorem 1

First Deduction: It suffices to show theorem 1 holds for any parallelepiped of size R. And we will do induction on the size of the parallelepiped to find the best constant for the estimate.

Second Deduction: It suffices to prove theorem 1 for each surface $\Sigma_i(U_i)$ is a "small" piece. Given $0 < \delta << 1$ we split each domain U_i into smaller pieces of diameter $\leq \delta$. This will also split the surface $\Sigma_i(U_i)$ into pieces. If we can show the estimate holds in these "small" pieces then we can sum these pieces to get the estimate on U_i. This will result in a factor of $\delta^{C(n,k)}$. At the end, we will take δ respect to ϵ, so this factor will be absorbed into $C(\epsilon)$.

Since we deal with small pieces, without loss of generality, we can assume f_i is compactly supported in a slightly bigger set. When doing induction, we will need to relax the support of f_i. To quantify this, we introduce the concept of "margin". For a function f: $\mathcal{H}_i \to \mathbb{C}$, we define the margin:

$$margin^i(f) = dist(supp(f), B_i(0, 2\delta)^c), \quad i = 1, ...n + 1$$

The concept of margin is not of central importance. It is just a convenient tool for us to do induction.

Definition 2. *Without loss of generality, we can assume $R \geq \delta^{\frac{1}{2}}$. Define $A(R)$ to be the best constant for which the estimate*

$$||\prod_{i=1}^{n+1} \mathcal{E}_i f_i||_{L^{\frac{2}{n}}(Q)} \leq A(R) \prod_{i=1}^{n+1} ||f_i||_{L^2}$$

holds true for all parallelepipeds $Q \in \mathcal{R}$, with f_i obeying the margin requirement.

$$margin^i(f_i) \geq \delta - R^{-\frac{1}{2}}$$

Taking $i = 1$ as an example, we first write $\mathcal{E}_1 f_1 = \sum_{q' \in \mathcal{CH}_1(R)} \mathcal{E}_1 \mathcal{F}_1(\chi_{q'} \mathcal{F}_1^{-1} f_1)$.

We will show the following inequality

$$||\prod_{i=1}^{n+1} \mathcal{E}_i f_i||_{L^{\frac{2}{n}}(q)} \lesssim A(R) \prod_{i=1}^{n+1} (\sum_{q' \in \mathcal{CH}_i(R)} \langle \frac{d(\pi_{N_i} q, q')}{R} \rangle^{-(2N-n^2)} ||\langle \frac{x - c(q')}{R} \rangle^N \chi_{q'} \mathcal{F}_i^{-1} f_i||_{L^2}^2)^{\frac{1}{2}} \tag{6}$$

From this inequality, by applying the discrete Loomis-Whitney inequality, we will get the conclusion. Since

Define $g_i : \mathcal{L}(\mathcal{H}_i) \to \mathbb{R}$ by

$$g_i(\mathbf{j}) = (\sum_{q' \in \mathcal{CH}_i(R)} \langle \frac{d(q(\mathbf{j}), q')}{R} \rangle^{-(N-2n^2)} ||\langle \frac{x' - c(q')}{R} \rangle^N \chi_{q'} \mathcal{F}_i^{-1} f_i||_{L^2}^2)^{\frac{1}{2}}$$

Notice $\langle \frac{d(q(\mathbf{j}),q')}{R} \rangle$ takes integer values. For N(n) large enough, $g_i \in l^2(\mathbb{Z}^n)$

$$||g_i||_{l^2(\mathcal{L}(\mathcal{H}_i))} \lesssim ||f_i||_{L^2}$$

By Loomis-Whitney inequality, we have

$$||\prod_{i=1}^{n+1} \mathcal{E}_i f_i||_{L^{\frac{2}{n}}(Q)} \leq A(R) \prod_{i=1}^{n+1} ||f_i||_{L^2}$$

Thus we obtain

$$A(\delta^{-1} R) \leq C A(R)$$

This gives $A(\delta^{-n} R) \leq C^N A(R)$. For $R \in [\delta^{-N}, \delta^{-N-1}]$, this implies

$$A(R) \leq C^N C(\delta) \leq R^\epsilon C(\delta)$$

We can choose $\delta = C^{-\frac{1}{\epsilon}}$ such that $C^N \leq \delta^{-N\epsilon}$. This will give us the desired result.

So it suffices to show (6). We will show $i=1$, $N=1$ as an example.

$$||(x' - c(q') - x_1 \nabla\varphi_1(\xi_0'))\mathcal{E}_1\mathcal{F}_1(\chi_q'\mathcal{F}_1^{-1}f_1)\prod_{i=2}^{n+1}\mathcal{E}_i f_i||_{L^{\frac{2}{n}}(q)}$$

$$\leq ||(x' - c(q') - x_1 \nabla\varphi_1(\xi'))\mathcal{E}_1\mathcal{F}_1(\chi_q'\mathcal{F}_1^{-1}f_1)\prod_{i=2}^{n+1}\mathcal{E}_i f_i||_{L^{\frac{2}{n}}(q)}$$

$$+||x_1(\nabla\varphi_1(\xi_0') - \nabla\varphi_1(\xi'))\mathcal{E}_1\mathcal{F}_1(\chi_q'\mathcal{F}_1^{-1}f_1)\prod_{i=2}^{n+1}\mathcal{E}_i f_i||_{L^{\frac{2}{n}}(q)}$$

By (7) and the size of Q is $\delta^{-1}R$,

$$\leq ||\mathcal{E}_1\mathcal{F}_1(x' - c(q'))\chi_q'\mathcal{F}_1^{-1}f_1\prod_{i=2}^{n+1}\mathcal{E}_i f_i||_{L^{\frac{2}{n}}(q)}$$

$$+\delta^{-1}R||(\nabla\varphi_1(\xi_0') - \nabla\varphi_1(\xi'))\mathcal{E}_1\mathcal{F}_1(\chi_q'\mathcal{F}_1^{-1}f_1)\prod_{i=2}^{n+1}\mathcal{E}_i f_i||_{L^{\frac{2}{n}}(q)}$$

By the induction hypothesis and the second deduction, we have

$$\leq RA(R)||\langle\frac{x' - c(q')}{R}\rangle\chi_{q'}\mathcal{F}_1^{-1}f_1||_{L^2}\prod_{i=2}^{n+1}||f_i||_{L^2}$$

Notice in the above argument, the margin of f is relaxed, but still satisfies the condition to do induction.

We will write $\mathcal{E}_i f_i = \sum_{q'\in\mathcal{CH}_i(R)}\mathcal{E}_i\mathcal{F}_i(\chi_{q'}\mathcal{F}_i^{-1}f_i)$ for all $1 \leq i \leq n+1$ and then apply the above argument to each term.

Observe that

$$||\langle\frac{d(\pi_{N_1}q, q')}{R}\rangle^{-\frac{n^2}{2}}||_{l_{q'}^{\frac{2}{n-1}}} \lesssim 1$$

By (2), (4) and the following estimate for sequences

$$||a_i \cdot b_i||_{l_i^{\frac{2}{n}}} \lesssim ||a_i||_{l_i^2}||b_i||_{l_i^{\frac{2}{n-1}}}$$

We can obtain (6).

25.5 Lower levels of multilinearity

For the case with k surface where $2 \leq k \leq n+1$, the smooth conditions keeps the same, while the transversality condition is replaced by

$$vol(N_1(\zeta_1), ...N_k(\zeta_k)) \geq \nu$$

for all choices of $\zeta_i \in \Sigma_i(U_i)$. In this setting, the result is:

Theorem 3. *Under the above assumptions, for any $\epsilon > 0$, there is $C(\epsilon)$ such that the following holds true*

$$||\prod_{i=1}^{k} \mathcal{E}_i f_i||_{L^{\frac{2}{k-1}}(B(0,R))} \leq C(\epsilon) R^{\epsilon} \prod_{i=1}^{n+1} ||f_i||_{L^2(U_i)}, \quad \forall f_i \in L^2(U_i), i = 1, ...k, \tag{7}$$

If we have Σ_1 has small support in some directions, then there is a refined result for the above result.

Additional condition of Σ_1: Assume that $\Sigma_1 \subset B(\mathcal{H}, \mu)$, where $B(\mathcal{H}, \mu)$ is the neighborhood of size μ of the k-dimensional affine subspace $\mathcal{H}$.

To be compatible with the transversality condition, we also need:

If $N_i, i = k+1, ...n+1$ is a basis of the normal space $\mathcal{H}^{\perp}$ to $\mathcal{H}$, then $N_1(\zeta_1), ...N_k(\zeta_k), N_{k+1}, ...N_{n+1}$ satisfies the

$$|det(N_1(\zeta_1), ...N_{n+1}(\zeta_{n+1}))| \geq \nu$$

for any choice of $\zeta_i \in \Sigma_i$.

Theorem 4. *Assume $\Sigma_i, i = 1, ...k$ satisfy the smooth and the transversality condition with Σ_1 satisfies the additional condition as above. Then for any $\epsilon > 0$, there is $C(\epsilon)$ such that the following holds true*

$$||\prod_{i=1}^{k} \mathcal{E}_i f_i||_{L^{\frac{2}{k-1}}(B(0,R))} \leq C(\epsilon) \mu^{\frac{n+1-k}{2}} R^{\epsilon} \prod_{i=1}^{n+1} ||f_i||_{L^2(U_i)}, \quad \forall f_i \in L^2(U_i), i = 1, ...k, \tag{8}$$

The proof of the above two theorems is very similar to theorem 1.

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26 Solving the Kadison-Singer problem using interlacing families

after A. Marcus, D. Spielman and N. Srivastava [2]
A summary written by Ruixiang Zhang

Abstract

We use the method of interlacing families developed by Marcus-Spielman-Srivastava to give a solution to the Kadison-Singer problem

26.1 Introduction

In [1] and [2], Marcus, Spielman and Srivastava develop an approach to problems involving a lot of related characteristic polynomials of matrices using the so called "interlacing family" concept. We focus on [2] where this approach was used to give a (surprisingly) complete solution to the Kadison-Singer problem. We would not focus much on what was already in [1] as this appears in the report of Julian Weigt.

The main theorem in [2], which affirmatively answers the Kadison-Singer problem, states:

Theorem 1. *Every pure state on the (abelian) von Neumann algebra $\mathbb{D}$ of bounded diagonal operators on l_2 have a unique extension to a pure state on $B(l_2)$, the von-Neumann algebra of all bounded operators on l_2.*

Theorem 1 is known to be a corollary to either Theorem 2 or Theorem 3 below.

Theorem 2. *[Weaver's KS_2 conjecture] There exist universal constants $\eta \geq 2$ and $\theta > 0$ such that: If for any $w_1, \dots, w_m \in \mathbb{C}^d$ satisfying $\|w_i\| \leq 1$ and $\sum_{i=1}^m |< u, w_i >|^2 = \eta, \forall u \in \mathbb{C}^d, \|u\| = 1$, then there exists a partition S_1, S_2 of $\{1, 2, \dots, m\}$ such that:*

$$\sum_{i \in S_j} |< u, w_i >|^2 \leq \eta - \theta, \forall u \in \mathbb{C}^d, \|u\| = 1, \forall j \in \{1, 2\}. \tag{1}$$

Theorem 3. *[Anderson's paving conjecture] For every $\varepsilon > 0$, there is a positive integer r such that for every $n \times n$ Hermitian matrix T with zero diagonal, there are diagonal projections $P_1, \dots, P_r$ with $\sum_{i=1}^r P_i = I$ such that*

$$\|P_i T P_i\| \leq \varepsilon \|T\|, \forall i = 1, 2, \dots, r. \tag{2}$$

Theorems 2 and 3 were both proved in [2] and are both corollaries of the following core theorem:

Theorem 4. *[Main Theorem in [2]] If $\varepsilon > 0$ and $v_1, v_2, \ldots, v_m \in \mathbb{C}^d$ are independent random vectors with finite support such that*

$$\sum_{i=1}^{m} \mathbb{E} v_i v_i^* = I_d \tag{3}$$

and

$$\mathbb{E}\|v_i\|^2 \leq \varepsilon, \forall i, \tag{4}$$

then

$$\mathbb{P}\left[\| \sum_{i=1}^{m} v_i v_i^* \| \leq (1+\sqrt{\epsilon})^2 \right] > 0. \tag{5}$$

Theorem 4 is proved using interlacing families. Roughly speaking, a set of monic polynomials of degree d *have a common interlacing* if all of them are with real coefficients and real roots, and that there exist $(d-1)$ real numbers $\alpha_1 \leq \cdots \leq \alpha_{d-1}$ such that the roots $\beta_1 \leq \cdots \leq \beta_d$ of any given polynomial satisfy $\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_{d-1} \leq \beta_d$. A finite number of monic polynomials have a common interlacing if and only if any of their convex linear combination is real-rooted.

We can put a set of monic polynomials with real coefficients into some groups and choose a nonzero convex linear combination of the polynomials in each group to get a new set of (smaller number of) polynomials. We call this a *convex simplification.* If after a finite number of (successive) convex simplifications there is only one polynomial left and we find ourselves always taking a convex linear combination among some polynomials with a common interlacing in this whole process, we call the original set of polynomials an *interlacing family* and the final polynomial its *expectation.* It is elementary that in any interlacing family there has to be a polynomial whose largest (real) root is no more than the largest root of the expectation polynomial of the family.

26.2 interlacing property

It was proved in [2] that the characteristic polynomials of all possible $\sum_{i=1}^m v_i v_i^*$ in Theorem 4 form an interlacing family in a canonical way. A conclusion of this flavor also shows up in [1]. An essential tool in the proof and in other parts of the paper is the use of real stable polynomials. A nonzero multivariate polynomial $p(z_1, \ldots, z_m)$ is real stable if and only if its coefficients are real and $p(z_1, \ldots, z_m) \neq 0$ whenever $\Im z_i > 0$. The zero polynomial is real stable by definition. We can check real stability generalizes real-rootedness in the category of nonzero univariate polynomials real coefficient. Specialization into polynomial in less variables (with fixed real numbers substituting other variables) and the operator $(1 - \partial z_i)$ preserve real-stability. This and the fact that $\det(\sum_{i=1}^m z_i A_i)$ is real stable when all A_i are semi-definite Hermitian matrices are enough to prove the interlacing family property because of the formula (in the setting of Theorem 4):

$$\mathbb{E}\chi\left[\sum_{i=1}^m v_i v_i^*\right](x) = \left(\prod_{i=1}^m (1 - \partial_{z_i})\right) \det\left(xI + \sum_{i=1}^m z_i A_i\right)\Big|_{z_1 = \cdots = z_m = 0} \quad (6)$$

where $A_i = \mathbb{E} v_i v_i^*$.

(6) can be proved by induction. See Section 4 of [2].

It remains to bound the largest root of $\mathbb{E}\chi\left[\sum_{i=1}^m v_i v_i^*\right](x)$ from above. This is a new essential part of [2] compared to [1] (where the bound needed there was obtained by Heilmann and Lieb [5], or Godsil [3][4]). It is done by studying the so-called *barrier function*.

26.3 The barrier function argument

We have one more condition $\sum_{i=1}^m \mathbb{E} v_i v_i^* = I_d$ to use. In this case it can be further shown that both sides of (6) is equal to

$$\left(\prod_{i=1}^m (1 - \partial_{z_i})\right) \det\left(\sum_{i=1}^m z_i A_i\right)\Big|_{z_1 = \cdots = z_m = x}. \quad (7)$$

We would like to have a good upper bound of this polynomial to conclude the proof. To do this, the effect of $(1 - \partial_{z_i})$ on the roots of real stable polynomials was studied in [2] (note that before applying $\prod_{i=1}^m (1 - \partial_{z_i})$, the polynomial $\det\left(\sum_{i=1}^m x A_i\right) = x^d$ has no positive roots). An important tool is the (multivariate) barrier function.

Given a real stable polynomial $p \in \mathbb{R}[z_1, \ldots, z_m]$, we say $z \in \mathbb{R}^m$ is above the roots of p (denoted by $z \in \mathsf{Ab}_p$) if and only if $p(z + t) > 0, \forall t =$

$(t_1, \dots, t_m) \in \mathbb{R}^m, t_i \geq 0$. It was shown in [2] that some $z > 0$ (to be computed) and the polynomial

$$Q = \left(\prod_{i=1}^{m} (1 - \partial_{z_i}) \right) \det \left(\sum_{i=1}^{m} z_i A_i \right) \tag{8}$$

satisfy $(z, \dots, z) \in \mathsf{Ab}_Q$. Next we explain the proof and find a good such z along the way.

Back to a general real stable polynomial p. For any $z \in \mathsf{Ab}_p$, we define the i-th barrier function of p (or the barrier function of p in direction i) at z to be

$$\Phi_p^i(z) = \partial_{z_i} \log p(z) = \frac{\partial_{z_i} p(z)}{p(z)}. \tag{9}$$

It is shown in [2] that each barrier function is nonincreasing and convex in every coordinate. An important tool is the full characterization of bivariate real stable polynomials. The form (9) of the barrier function suggests that it can be used to connect p and $(1 - \partial_{z_i})p$. Indeed, an immediate corollary is that for any real stable p, as long as $\Phi_p^i(z) < 1$, we have $z \in \mathsf{Ab}_{(1-\partial_{z_i})p}$.

As for the way to make sure $\Phi_p^i(z) < 1$ along the way of taking $(1 - \partial_{z_i})$ repeatedly before we obtain Q, we have the following lemma from [2]:

Lemma 5. *For p real stable, if $z \in \mathsf{Ab}_p$ and $\Phi_p^j(z) \leq 1 - \frac{1}{\delta}$ for some j and $\delta > 0$, then for all i,*

$$\Phi_{(1-\partial_{z_j})p}^i(z + \delta e_j) \leq \Phi_p^i(z). \tag{10}$$

Lemma 5 tells us that as long as all the barrier functions are bounded nontrivially away from 1 at some point in Ab_p, we can move the point up a bit and make sure the barrier functions of $(1 - \partial_{z_i})p$ at the new point are not larger. Repeat this and do some computation, we find $t + \frac{1}{1 - \frac{\varepsilon}{t}}$ is an upper bound of the roots of $Q(x, x, \dots, x)$ $(t > \varepsilon)$. Optimizing in t and we deduce Theorem 4.

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27 Improved bounds in Weaver and Feichtinger conjectures

after M. Bownik, P. G. Casazza, A. W. Marcus, D. Speegle [1]
A summary written by Yi Zhang

Abstract

The constant in the KS_2 conjecture given by [2] is improved by [1], which can be applied to prove optimal asymptotic bounds on the Feichtinger conjecture.

27.1 Introduction

The main result in [2] is the following.

Theorem 1. *Let $\epsilon > 0$ and $v_1, \ldots, v_n$ are independent random vectors in $\mathbb{C}^d$ with each v_i taking a finite number of values. Assume the normalization*

$$\mathrm{E} \sum_{i=1}^{m} v_i v_i^* = \mathbf{I}$$

and the smallness condition, i.e. $\mathrm{E}\|v_i\|^2 \leq \epsilon$ for any $1 \leq i \leq m$. Then we have

$$\left\| \sum_{i=1}^{m} v_i v_i^* \right\| \leq (1 + \sqrt{\epsilon})^2 \tag{1}$$

with positive probability.

In the paper [1] the upper bound in (1) is improved under special assumptions.

Theorem 2. *Let $0 < \epsilon < 1/2$ and $v_1, \ldots, v_n$ are independent random vectors in $\mathbb{C}^d$ with each v_i taking two values. Assume the normalization*

$$\mathrm{E} \sum_{i=1}^{m} v_i v_i^* = \mathbf{I}$$

and $\mathrm{E}\|v_i\|^2 \leq \epsilon$ for any $1 \leq i \leq m$. Then we have

$$\left\| \sum_{i=1}^{m} v_i v_i^* \right\| \leq 1 + 2\sqrt{\epsilon}\sqrt{1 - \epsilon} \tag{2}$$

with positive probability.

This theorem leads to improved bounds in the Weaver conjecture and Feichtinger conjecture with almost the same proof as in [2].

27.2 Preliminaries

In this subsection let us recall some terminology used in [2]. A polynomial is called *real stable* if all of its coefficients are real and it does not have roots in

$$\{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon \mathrm{Im}(z_i) > 0, 1 \le i \le m\}$$

(and in

$$\{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon \mathrm{Im}(z_i) < 0, 1 \le i \le m\}$$

by conjugation). An important observation is that, for Hermitian positive semi-definite matrices $A_1, \ldots, A_m$, the polynomial $p(z, z_1, \ldots, z_m) = \det(z\mathbf{I}+z_1A_1+\cdots+z_mA_m)$ is real stable; indeed if $z, z_1, \ldots, z_m$ have positive imaginary part, then the skew-adjoint part of $z\mathbf{I}+z_1A_1+\cdots+z_mA_m$ is strictly positive definite and then the quadratic form $\mathrm{Im}\langle(z\mathbf{I}+z_1A_1+\cdots+z_mA_m)v, v\rangle$ is non-singular.

Real stable polynomials are closed under restriction to real numbers and under the differential operators $1 - \partial_j$. Moreover, for any two real stable polynomials $p(z)$, $q(z)$ of the same degree, if for each $0 \le t \le 1$ the convex combination $(1-t)p + tq$ is real stable, then the largest root of $(1-t)p+tq$ lies between those that of p and q. (Indeed they have a common interlacing polynomial of one lower degree).

We say that a point $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ is above the roots of a polynomial p if it is non-zero in

$$\{(z_1, \ldots, z_m) \colon z_i \ge x_i, 1 \le i \le m\}.$$

The following lemma is a key step in the proof of [2]

Lemma 3. *Let $A_1, \ldots, A_m$ be Hermitian positive semi-definite $d \times d$ matrices satisfying*

$$\sum_{i=1}^{m} A_i = \mathbf{I}.$$

Suppose that $\mathrm{tr}A_i \le \epsilon$ *for* $1 \le i \le m$ *and some* $\epsilon > 0$. *Write*

$$p(z_1, \ldots, z_m) = \det\left(\sum_{i=1}^{m} z_iA_i\right).$$

Then $((1+\sqrt{\epsilon})^2, \ldots, (1+\sqrt{\epsilon})^2)$ *is above the roots of* $(\Pi_{i=1}^m(1-\partial_i))p$.

Notice that $(\Pi_{i=1}^m(1-\partial_i))p$ is exactly $\det(z-A)$ if A_i, $1 \le i \le m$ are deterministic rank one $d \times d$ matrices and A is the sum of them. This observation with the lemma above (and the properties of stable polynomials) leads to Theorem 1.

Lemma 3 is improved in [1, Theorem 1.5] in a special case as follows.

Lemma 4. *Suppose* $A_1, \ldots, A_m$ *are Hermitian positive semi-definite* $d \times d$ *matrices of rank at most* 2,

$$\sum_{i=1}^m A_i = \mathbf{I},$$

and for $1 \le i \le m$*, we have* $\mathrm{tr}A_i \le \epsilon$ *for some* $\epsilon > 0$*. Then*

$$(1+2\sqrt{\epsilon}\sqrt{1-\epsilon}, \ldots, 1+2\sqrt{\epsilon}\sqrt{1-\epsilon})$$

is above the roots of $(\Pi_{i=1}^m(1-\partial_i))p$.

Likewise, this lemma leads to Theorem 2. The proof of Lemma 4 is based on an argument via induction, and the improvement comes from a more careful estimate under the assumption that $\det(\sum_i x_i A_i)$ is (at most) quadratic in each of its variables. We sketch the proof of Lemma 4 in the next subsection.

27.3 The idea of the proof of Lemma 4

First of all, let us observe that $(t, \ldots, t)$, $t > 0$ is above the roots of

$$Q_0 = \det\left(\sum_i x_i A_i\right);$$

indeed $\sum_i x_i A_i - t$ is positive semi-definite whenever $x_i > t$, and hence $Q_0(t, \ldots, t)$ is non-singular. This is the starting point of the proof.

In order to show Lemma 4 via induction, we wish to pass from Q_i to $Q_{i+1} := (1-\partial_{i+1})Q_i$. Towards this, one needs to estimate the logarithm j-th derivatives of Q_i

$$\Phi_{Q_i}^j := \frac{\partial_j Q_i}{Q_i}$$

(defined away from the zeros of Q_i) at points w_i, where $w_0 = (t, \dots, t)$ for some $t = t(\epsilon) > 0$ and $w_{i+1} = w_i + \delta e_{i+1}$ with $\delta = \delta(\epsilon) > 0$.

In the proof of Lemma 3, it is shown that if x is above the roots of Q_i, and for some $1 \leq j \leq m$ the bound

$$\Phi^j_{Q_i}(x) \leq 1 - \delta^{-1}$$

holds for some $\delta > 0$, then w_{i+1} is above the roots of Q_{i+1} with

$$\Phi^k_{Q_{i+1}}(x + \delta e_j) \leq \Phi^k_{Q_i}(x)$$

for every $1 \leq k \leq m$. This allows us to conclude $x + (\delta, \dots, \delta)$ is above the roots of $(\Pi_{i=1}^m (1 - \partial_i))p$.

However this is not the case for Lemma 4. Instead, one shows the following lemma.

Lemma 5 ([1, Lemma 4.2, Lemma 4.3])**.** *If w_i is above the roots of Q_i, and for $i+1 \leq j \leq m$ we have*

$$\Phi^j_{Q_i}(w_i) \leq \frac{\epsilon}{t}$$

with $t = (1 - 2\epsilon)\sqrt{\frac{\epsilon}{1-\epsilon}}$, then w_{i+1} is above the roots of Q_{i+1} with the monotonicity

$$\Phi^k_{Q_{i+1}}(w_{i+1}) \leq \Phi^k_{Q_i}(w_i)$$

holds for every $i + 2 \leq k \leq m$.

Throughout the whole proof it is heavily employed the fact that Q_i is (at most) quadratic in each of its variables; see [1, Lemma 3.7, Lemma 3.8]. Also in [1, Lemma 3.6] they study the monotonicity of the logarithm derivative $\Phi^j_{Q_i}$ with respect to i by a representative of real stable polynomials via the determinant of summation of real symmetric matrices [1, Corollary 3.4].

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28 The endpoint multilinear Kakeya theorem via the Borsuk–Ulam theorem

after A. Carbery and S. I. Valdimarsson [CV]
A summary written by Pavel Zorin-Kranich

Here I state all main steps from [CV] without proofs. A version with streamlined proofs is available at
`www.math.uni-bonn.de/people/pzorin/multilinear-kakeya.pdf`

28.1 Main result

The following result has been originally proved in [G]. We present the simpler proof from [CV].

Theorem 1. *Let $\mathcal{T}_1, \dots, \mathcal{T}_n$ be families of approximately axis parallel 1-tubes in $\mathbb{R}^n$, that is, each $T \in \mathcal{T}_j$ is the 1-neighborhood of a doubly infinite line with direction $e(T)$ that lies in a small (depending only on n) fixed neighborhood of the unit vector e_j. Then*

$$\int_{\mathbb{R}^n} \left(\sum_{T_1 \in \mathcal{T}_1} \chi_{T_1}(x) \cdots \sum_{T_n \in \mathcal{T}_n} \chi_{T_n}(x) \right)^{1/(n-1)} dx \lesssim (\#\mathcal{T}_1 \cdots \#\mathcal{T}_n)^{1/(n-1)} .$$

Here and later implicit constants are only allowed to depend on n.

The exponent $1/(n-1)$ is optimal because it recovers the Loomis–Whitney inequality in the case that all tubes are precisely axis parallel. The non-endpoint version has been previously proved in [BCT] with a short proof in [G2]. It is presented in a different talk and seems to suffice for all application in this summer school.

In [CV] a more general result involving d families of tubes, $2 \le d \le n$, and tubes not approximately aligned with coordinate axes is proved. For notational simplicity we omit these extensions. Much more general results, in which tubes are replaces by neighborhoods of varieties of arbitrary codimension, are proved in [Z].

28.2 Reduction to domination by tensor products

Let $\mathcal{Q}$ denote the lattice of unit cubes in $\mathbb{R}^n$. We may assume that the sets $\mathcal{T}_j$ are finite. It is then not hard to reduce Theorem 1 to the following statement.

Proposition 2. *For every function $M : \mathcal{Q} \to [0,\infty)$ satisfying $\sum_Q M(Q)^n = 1$, there exist functions $S_j : \mathcal{Q} \times \mathcal{T}_j \to [0,\infty)$ such that for all $T_j \in \mathcal{T}_j$ with $T_j \cap Q \neq \emptyset$,*

$$M(Q)^n \lesssim S_1(Q,T_1) \dots S_n(Q,T_n) \tag{1}$$

and, for all j and all $T_j \in \mathcal{T}_j$

$$\sum_{Q \,:\, T_j \cap Q \neq \emptyset} S_j(Q,T_j) \lesssim 1. \tag{2}$$

28.3 Directional surface area and visibility

The *directional area* of a hypersurface $Z \subset \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$ (termed *directed volume* by Guth) is defined by

$$\mathrm{surf}_v(Z) = \int_Z |v \cdot n(x)| \, d\mathcal{H}_{n-1}(x),$$

where $n(x)$ denotes the unit normal to Z at x (which is assumed to be defined for $\mathcal{H}_{n-1}$-a.e. $x \in Z$). This is a non-negative subadditive function of v, so it defines a seminorm on $\mathbb{R}^n$. We will denote seminorms on $\mathbb{R}^n$ by the letter $\mathfrak{s}$ (which stands both for "seminorm" and for "surface area").

For a polynomial p denote by $Z_p = \{x : p(x) = 0\}$ its zero set. If $p \not\equiv 0$, then for a.e. line in any given direction Z_p contains at most $\deg p$ points on that line. An immediate consequence is Guth's cylinder estimate:

Lemma 3. *If T is a 1-tube in $\mathbb{R}^n$ and p is a non-zero polynomial, then*

$$\mathrm{surf}_{e(T)}(Z_p \cap T) \lesssim \deg p.$$

To any seminorm $\mathfrak{s}$ on $\mathbb{R}^n$ we associate the centrally-symmetric convex body[8]

$$K_{\mathfrak{s}} := \mathbb{B} \cap \mathbb{B}_{\mathfrak{s}}, \tag{3}$$

where $\mathbb{B}$ is the Euclidean unit ball of $\mathbb{R}^n$ and $\mathbb{B}_{\mathfrak{s}}$ is the unit ball of $\mathfrak{s}$.

It is clear that $K_{\mathfrak{s}}$ is symmetric and convex. We then define the *visibility*[9] of $\mathfrak{s}$ as

$$\mathrm{vis}(\mathfrak{s}) := (\mathrm{vol}\, K_{\mathfrak{s}})^{-1/n}.$$

Note that since $K_{\mathfrak{s}} \subseteq \mathbb{B}$ we always have $\mathrm{vis}(\mathfrak{s}) \geq C$.

[8]Since we are using 1-tubes we have already broken scaling symmetry, so we start taking advantage of the fact that scale 1 plays a distinguished role.

[9]Guth's definition of visibility is the n'th power of the one given here.

Lemma 4. *Suppose that for all vectors $v \in \mathbb{R}^n$ we have $\|v\| \lesssim \mathfrak{s}(v)$. If $v_1, \ldots, v_n \in \mathbb{R}^n$ are approximately aligned with coordinate axes, then*

$$(\mathrm{vol}\, K_{\mathfrak{s}})^{-1} \lesssim \mathfrak{s}(v_1) \cdots \mathfrak{s}(v_n).$$

Lemma 5. *Suppose $C\mathbb{B}_{\mathfrak{s}} \not\subseteq \mathbb{B}$. Then there exists a unit vector $e \in \mathbb{S}^{n-1}$ such that*

$$\mathfrak{s}(e)^{n-1} \gtrsim (\mathrm{vol}\, K_{\mathfrak{s}})^{-1}.$$

In order to apply (a version of) the Borsuk–Ulam theorem we will need a continuous (in p) version of the directional surface area. Let $\mathcal{P}_k$ be the vector space of polynomials of degree at most k in n variables with real coefficients, then $\dim \mathcal{P}_k = \binom{k+n}{n} \sim k^n$. Since the class of polynomials with the desired properties for Theorem 8 is invariant under multiplication by non-zero scalars, it is natural to consider the unit sphere $\mathcal{P}_k^*$ of $\mathcal{P}_k$. Topologically $\mathcal{P}_k^* \cong \mathbb{S}^N$ with $N = N(k) \sim k^n$; we will use these notations interchangeably, the former when we are thinking of individual polynomials, the latter when continuity and topological considerations are foremost. The continuity property needed is most simply achieved by replacing $\mathrm{surf}_v(Z)$ by the mollified version

$$\mathfrak{s}_{p,U}(v) := \fint_{p' \in B(p,\epsilon) \subset \mathbb{S}^N} \mathrm{surf}_v(Z_{p'} \cap U),$$

where $\epsilon > 0$ will be chosen later (depending on M in Theorem 8) to be sufficiently small so that these norms behave in certain ways similarly to the unmollified versions. From this we define

$$K_{p,U} := K_{\mathfrak{s}_{p,U}} \tag{4}$$

and

$$\mathrm{vis}_{p,U} := (\mathrm{vol}\, K_{p,U})^{-1/n}. \tag{5}$$

Averaging Lemma 3 we obtain

Lemma 6. *If T is a $O(1)$-tube in $\mathbb{R}^n$ and p a non-zero polynomial, then*

$$\mathfrak{s}_{p,T}(e(T)) \lesssim \deg p$$

with implicit constant independent of T, p, and ϵ.

Since a cube is contained in a $O(1)$-tube in any direction we obtain

Corollary 7. *Let $Q \in \mathcal{Q}$ and p a non-zero polynomial. Then*

$$\mathfrak{s}_{p,Q}(v) \lesssim \|v\| \deg p.$$

28.4 The polynomial construction

Proposition 2 follows relatively quickly from the previous considerations about seminorms and the following result.

Theorem 8. *Given a nonnegative finitely supported function $M : \mathcal{Q} \to \mathbb{R}$, there exists a polynomial p on $\mathbb{R}^n$ with*

$$\deg p \lesssim \Big(\sum_Q M(Q)^n\Big)^{1/n}$$

and such that for some $\epsilon > 0$ and all $Q \in \mathcal{Q}$

$$\mathrm{vis}_{p,Q} \geq M(Q).$$

28.4.1 Borsuk–Ulam theorem

Let

$$S(Q) = \{p \in \mathcal{P}_k^* \ : \ \mathrm{vis}_{p,Q} \lesssim M(Q)\}.$$

We want to apply the Borsuk–Ulam theorem in the following form

Lemma 9. *Suppose that $A_i \subseteq \mathbb{S}^N$ for $1 \leq i \leq J$, and suppose that for each i, $A_i \cap (\overline{-A_i}) = \emptyset$. If $J \leq N$, then the $2J$ sets A_i and $-A_i$ do not cover $\mathbb{S}^N$.*

Hence we have to partition $S(Q)$ into a controlled number of sets that satisfy the hypothesis of Lemma 9.

28.4.2 Colours

We discretize the set of all norms on $\mathbb{R}^n$. Let $\mathcal{K}$ denote the class of centrally symmetric convex bodies in $\mathbb{R}^n$ with the metric

$$d(K, L) = \log \inf\{\alpha \geq 1 \ : \ \alpha^{-1}K \subseteq L \subseteq \alpha K\}.$$

Let $\mathcal{E} \subset \mathcal{K}$ denote the class of centred ellipsoids in $\mathbb{R}^n$. A centered ellipsoid is the image of the unit ball $\mathbb{B}$ under an invertible linear map. Thus $\mathcal{E} \cong GL(n)/O(n)$.

By the John ellipsoid theorem the set of ellipsoids $\mathcal{E}$ is $(\log n)/4$-dense in $\mathcal{K}$. The metric d is $GL(n)$-invariant and $d(A(\mathbb{B}), \mathbb{B}) = \log \max(\|A\|, \|A^{-1}\|)$ for all $A \in GL(n)$. It follows that the metric space $(\mathcal{E}, d)$ is homogeneous and locally compact.

Let $\mathcal{E}_0 \subset \mathcal{E}$ be a maximal $(\log n)/4$-separated subset. Then $\mathcal{E}_0$ is $(\log n)/4$-dense in $\mathcal{K}$. Using local compactness and homogeneity we can construct a partition

$$\mathcal{E}_0 = \uplus_{\theta \in \Theta} \mathcal{E}_0^\theta$$

into finitely many pairwise disjoint $(2 \log n)$-separated sets. We summarize the properties of this partition.

Lemma 10. *Every $K \in \mathcal{K}$ is $(\log n)/2$-close to some member of $\mathcal{E}_0$. For every $\theta \in \Theta$ every $K \in \mathcal{K}$ is $(\log n)$-close to at most one member of $\mathcal{E}_0^\theta$.*

We partition

$$S(Q) = \bigcup_{1 \lesssim 2^r \lesssim M(Q)} S^{(r)}(Q), \quad S^{(r)}(Q) = \{p \in \mathcal{P}_k^* \ : \ \mathrm{vis}_{p,Q} \sim 2^r\}$$

and $S^{(r)}(Q)$ into the sets

$$S^{(r),\theta}(Q) = \{p \in S^{(r)}(Q) \ : \ K_{p,Q} \text{ is } (\log n)/2\text{-close to a member of } \mathcal{E}_0^\theta\}.$$

28.4.3 Translates

Let $0 < \eta \ll 1$ be a parameter depending only on n to be chosen later. We now fix $Q \in \mathcal{Q}$, $r \geq 0$, and $\theta \in \Theta$.

For each $E \in \mathcal{E}_0$ of volume $\sim 2^{-rn}$ we can fit $\sim \eta^{-n} 2^{rn}$ disjoint translates of ηE inside Q. We label these ellipsoids with E_α, where α is an index in a set $\mathcal{A}_r$ of cardinality $\sim \eta^{-n} 2^{rn}$.

Let $E(p) = E(p, \theta)$ be the unique ellipsoid in $\mathcal{E}_0^\theta$ that is $(\log n)/2$-close to $K_{p,Q}$ if such elliposid exists.

Lemma 11. *There exist $\eta = \eta(n) > 0$ and $\epsilon = \epsilon(n, M) > 0$ such that the following holds. Let $p \in \mathcal{P}_k^*$ with $\mathrm{vis}_{p,Q} \sim 2^r \lesssim M(Q)$. Then the polynomial p does not bisect all $|\mathcal{A}_r|$ disjoint translates of $\eta E(p)$ in Q.*

The proof of this lemma uses the fact that if a surface (approximately) bisects a ball, then it has to have a certain (non-directed) area inside this ball. This fact can be deduced from the isoperimetric inequality. This fact is applied to the translates of ηE in an affine-invariant way.

Hence we can partition $S^{(r),\theta}(Q)$ into the sets

$$S_\alpha^{(r),\theta}(Q) := \{p \in S^{(r),\theta}(Q) : p \text{ does not bisect the } \alpha\text{'th translate of } \eta E(p) \text{ in } Q\}.$$

28.4.4 Antipodes

Finally we partition $S_\alpha^{(r),\theta} = S_\alpha^{(r),\theta+} \cup S_\alpha^{(r),\theta-}$, where the polynomial p goes into $S_\alpha^{(r),\theta+}$ iff

$$\mathrm{vol}(\{p > 0\} \cap E(p)_\alpha) > \mathrm{vol}(\{p < 0\} \cap E(p)_\alpha).$$

Then $S_\alpha^{(r),\theta-} = -S_\alpha^{(r),\theta+}$ and $S_\alpha^{(r),\theta+} \cap \overline{S_\alpha^{(r),\theta-}} = \emptyset$. For the second property we need the separation property of $\mathcal{E}_0^\theta$ and the continuity property of $\mathfrak{s}_{p,Q}$ to conclude that if $p_m \to p$, then $E(p_m)_\alpha$ eventually equals $E(p)_\alpha$.

Then we just have to count the sets $S_\alpha^{(r),\theta+}$, of which we turn out to have constructed at most $C \sum_Q M(Q)^n$.

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