The Adams spectral sequence
S4D2 Graduate Seminar on Topology

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WiSe 24/25

Schedule

The seminar meets at 10:00 ct on Tuesday in Room 1.007. The first meeting for talk distribution will exceptionally be at 10:00 on Thursday July 18 in Room 0.011

Date		Topic	Speaker
18/07	0	Talk distribution	Organisers
15/10	1	Steenrod algebra and its dual	
22/10	2	Comodules	
29/10	3	Construction of the ASSeq	
5/11	4	More on the ASSeq	
12/11	5	ASSeq for ko and ku	
19/11	6	Hopf Invariant one	
26/11	7	Adams vanishing line	
3/12	8	May spectral sequence	
10/12	9	Adams–Novikov spectral sequence	
17/12	10	Chromatic spectral sequence	
07/01	11	Miller square	
14/01	12	Synthetic spectra	

Schedule of talks

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Syllabus

Talk 1: The Steenrod algebra and its dual

Recall the definition of the Steenrod algebra as the algebra of operations in $\mathbf{F}_2$-cohomology and its explicit presentation in terms of Steenrod squares. Now dualise this to define the dual Steenrod algebra, describe a presentation for the dual Steenrod algebra in terms of Milnor generators, and discuss the formulas for the Hopf algebra structure. Discuss the comodule structure on the homology of a spectrum. Define the subalgebras $\mathscr{A}(n)$ and $\mathscr{E}(n)$ as well as their duals. If time permits, discuss odd primes as well.
References: [Rog23], you may also want to check out the original reference [Mil58]. Section 7.5 of [Rog12] discusses these subalgebras as well.

Talk 2: Comodules

Discuss the general theory of comodules over a (flat) Hopf algebroid. Define Ext and Cotor over a Hopf algebroid, and introduce the cobar resolution that computes these. Discuss the Change-of-Rings theorem and the Cartan–Eilenberg spectral sequence.
References: Appendix A1 of [Rav23].

Talk 3: Construction of the Adams spectral sequence

Set up the Adams spectral sequence based on a homotopy ring spectrum E and discuss the properties that E needs to satisfy for this to admit a description in terms of Ext on comodules. Introduce Adams grading for spectral sequences and discuss the filtration theorem. In the case $E = \mathbf{F}_2$, discuss the cohomological version as well.
References: Sections 1 and 2 of Chapter 2 in [Rav23], Section 4 in [Rog12], Chapter 1 in [McC01].

Talk 4: More on the Adams spectral sequence

Discuss convergence of the Adams spectral sequence: this spectral sequence converges to the homotopy groups of the $\mathbf{F}_p$-nilpotent completion of a spectrum. Define nilpotent completion and relate it to the p-completion of a spectrum in certain cases. Discuss multiplicaitvity of the Adams spectral sequence.
References: The original reference for localisations and completions is [Bou75], with a streamlined discussion in [Lur]. For convergence and multiplicativity consult 4.5-4.6 and Section 5 in [Rog12].

Talk 5: Adams spectral sequence for ko and ku

Discuss the Adams spectral sequence (at the prime two) for ko and ku. The key part here is computing their cohomologies and seeing that these are particularly nice comodules.
References: Section 6.4 in [Rog12], there is also a more terse homological discussion in Section 3 of Chapter 3 of [Rav23], but beware of notational differences! For more details on Stong's theorem, see [Rog00].

Talk 6: Hopf Invariant one

Discuss the $\mathbf{F}_2$-based Adams spectral sequence for the sphere, and take this opportunity to state what we know about the first three filtration stages (Theorem 3.4.1 in [Rav23]). Of interest to us are the classes h_i in filtration one. Show that the first four of these classes are permanent cycles, while the higher ones die. Relate this to the Hopf invariant one problem and the classification of spheres admitting an h-space structure. Discuss the proof using Steenrod operations in the Adams spectral sequence.
References: Section 4.4 of [Rog12] proves the straightforward part, have a look at the original reference [Ada60]. For the hard part, look at Sections 5 and 6 of [Vic13]

Talk 7: Adams vanishing line

Discuss the Adams vanishing line at the prime two, relating this to the approximation lemma. Discuss the Adams periodicity theorem as well, sketching the proof if time permits.
References: Section 4 of Chapter 3 of [Rav23]. More details can be found in Section 6.3 of [Rog12].

Talk 8: May spectral sequence

Construct the May spectral sequence computing the Adams E_2-page for the sphere at the prime two. Mention the more general algebraic phenomenon at hand. Use basic computations with this spectral sequence to compute the first thirteen stable stems at the prime two.
References: Section 2 of Chapter 3 in [Rav23], and Section 9.6 of [McC01].

Talk 9: Adams–Novikov spectral sequence

Construct the Adams–Novikov spectral sequence (it's best to fix a prime and work with BP) and describe what we know about its E_2-page. Introduce the algebraic Novikov spectral sequence, which is our primary tool for computing its E_2-page. Construct the Greek letter elements, and mention what we know about their corresponding elements in the stable stems. Discuss the "Thom reduction" map from the ANSS to the Adams spectral sequence, and how this can be used to extract information about the latter.
References: [Zah72], Chapter 4 Section 4 of [Rav23], [MRW77], and [Wil13].

Talk 10: Chromatic spectral sequence

Introduce the Chromatic spectral sequence, and use it to recover our Greek letter elements. Discus the relation to the Morava stabiliser algebras.
References: [MRW77], Chapter 5 of [Rav23]

Talk 11: Miller square

Discuss Miller's take on the Adams spectral sequence in terms of injective resolutions. This allows us to construct certain commutative diagrams of spectral sequences and compare d_2 differentials. Discuss some applications of this construction.
References: [Mil81].

Talk 12: Synthetic spectra

Define the ∞-category of synthetic spectra based on an Adams-type homology theory E. Discuss the special and generic fibres of this deformation, and use this to relate the τ-Bockstein spectral sequence and the E-Adams spectral sequence. If time permits, discuss how this gives rise to a refinement of the Miller square mentioned above.
References: [Pst23], [BHS19], [BX23], [BJM24].

References

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