Throughout, we work in ZF with the axiom of dependent choice, to allow conclusions under the axiom of determinacy. Let $\mathcal{B}$ be one of the following Boolean algebras of subsets of Polish spaces (in section 3 we give more precise requirements):

- The Boolean algebra generated by the F_σ-sets.
- The Borel sets.
- Assuming projective determinacy, the projective sets.
- Assuming the axiom of determinacy, all sets.

Note that as it is independent of ZFC if coanalytic sets have the perfect set property, there are formulas in the theory with $\mathcal{B}$ generated by the analytic sets whose validity is independent of ZFC.

Theorem 1. *(a) The monadic theory of order of $\mathbb{R}$ with quantification restricted to $\mathcal{B}$ coincides with the one where $\mathcal{B}$ is given by Boolean combinations of F_σ-sets.*

(b) The theory of (a) is decidable.

This immediately implies the corresponding result with $\mathbb{R}$ replaced by $2^{\mathbb{N}}$, or $\mathbb{N}^{\mathbb{N}}$, or $\mathcal{C} = 2^{\mathbb{N}} \setminus \{0^\infty, 1^\infty\}$.

The proof is self-contained: Until and including section 3, we essentially only repeat Shelah's results [She75].

1 Basic results on monadic theories

For convenience, throughout we allow empty parts in a partition.

For now, let $\mathcal{B}$ be a Boolean algebra of subsets of a linear order X, and consider it as a monadic second-order structure in the following way: We consider X as first-order structure in the language $(\subseteq, \cap, \cup, \complement, \emptyset, \leq)$, with $A \leq B$ if $A = \{a\}, B = \{b\}$ are singletons with $a \leq b$. We moreover add unary relation symbols $\mathcal{M}_1, \ldots, \mathcal{M}_u$, and constant symbols $P_1, \ldots, P_m$.

Let $n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}^n, P = (P_j)_j \in \mathcal{B}^m$; we will also write $n = |k|$. We define $Th_k(X, P)$ by recursion: For $n = 0$, it is the set of valid quantifier-free monadic sentences, modulo logical equivalence in the empty theory. For $n > 0$, it is $\left\{Th_{k_1,\ldots,k_{n-1}}(X, PQ) \mid Q \in \mathcal{B}^{k_n}\right\}$. The set FTh_k of formally possible types is defined analogously by omitting satisfiability in the case $n = 0$ (and thus the dependence on X, P_j).

Note that $Th_k(X, P) = Th_k(Y, Q)$ if and only if (X, P) and (Y, Q) satisfy the same formulas in prenex normal form with quantifier block of length k, that is, with $n-1$ alternations and blocks of length $k_n, \ldots, k_1$. Thus, if we call a partial type k-complete whenever it only consists of such formulas and is maximal among those types, then elements of $Th_k(X)$ correspond to $(k_1, \ldots, k_{n-1})$-complete types in k_n variables. (Since our Boolean algebras generated by formulas with fixed quantifier prefix are finite, all filters are principal and thus types are in bijection with formulas.) Hence, in the following we will use the terms type and theory interchangeably for these Th_k.

Furthermore, Th_k is hereditarily finite as there are only finitely many quantifier-free formulas up to equivalence. Thus the computability of $k \mapsto Th_k(X)$ is equivalent to decidability of the monadic second-order theory of X. Clearly, $k \mapsto FTh_k$ is always computable.

Given $P \in \mathcal{B}^m$ and $Y \subseteq X$, we will also write $Th_k(Y, P)$ for $Th_k(Y, \{p \cap Y \mid p \in P\})$. Next let $I, (X_i)_{i \in I}$ linear orders, and for any i, $\mathcal{B}_i$ be a Boolean algebra of subsets of X_i, with unary relations $\mathcal{M}_1^i, \ldots, \mathcal{M}_u^i$ on subsets. We also apply the above construction to I with $\mathcal{B} = 2^I, u = 0$. We define $\sum_{i \in I} X_i$ as follows: Its underlying set is $\coprod_i X_i$, with the lexicographic order. A Boolean algebra $\mathcal{B}$ of subsets is given by $A \in \mathcal{B}$ if all $A \cap X_i \in \mathcal{B}_i$, and similarly for $\mathcal{M}_j$.

Theorem 2. *Let $X = \sum_i X_i$ with $P \in \mathcal{B}^m$, set $t_i = Th_k(X_i, P)$, and for $t \in FTh_k$ let $Q_t = \{i \mid t_i = t\}$. There is ℓ_k with $|\ell_k| = |k|$ such that $Th_{\ell_k}(I, (Q_t)_{t \in FTh_k}) \mapsto Th_k(X, P)$ is computable, uniformly in k.*

More formally, there are a computable map $k \mapsto \ell_k$ and a computable map $FTh_{\ell_k, \#FTh_k} \to FTh_{k,m}$, such that for all X, P, I as above, $Th_{\ell_k}(I, (Q_t)_t)$ is mapped to $Th_k(X, P)$. In particular, the left hand side uniquely determines the right hand side, and if the left hand side is computable, so is the right hand side.

Proof. We describe an algorithm recursive in $|k|$.

For handling $|k| = 0$, it suffices to consider validity of atomic sentences in X, so let s, s' be terms:

- $X \models s = s'$ if and only if for all $t \in FTh_k$ with $(s \neq s') \in t$, we have $I \models Q_t = \emptyset$.
- $X \models s \leq s$ (that is, s is interpreted by a singleton) if and only if there is a unique $t \in FTh_k$ with $(s \leq s) \in t$ and $I \models Q_t \leq Q_t$, and for all other $t \in FTh_k$ we have $I \models Q_t = \emptyset$.
- $X \models s \leq s'$ if and only if the previous condition holds for s, s' with some $t_s, t_{s'}$, and we have either $I \models Q_{t_s} < Q_{t'_s}$ or both $(s \leq s') \in t_s = t_{s'}$.
- $X \models \mathcal{M}_j(s)$ if and only if for all $t \in FTh_k$ with $(\mathcal{M}_j(s)) \notin t$, we have $I \models Q_t = \emptyset$.

Now suppose $|k| = n > 0$. We need to compute the possible

$$Th_{k_1,\ldots,k_{n-1}}(X, PR) = \sum_i Th_{k_1,\ldots,k_{n-1}}(X_i, PR).$$

Each such sum can be computed recursively if we know all possibilities for $Q_t = \{i \mid Th_{k_1,\ldots,k_{n-1}}(X_i, PR) = t\}$. The condition is that they form a partition of I refining $Q'_{t'} = \{i \mid Th_k(X_i, P) = t'\}$ such that that whenever $Q_t \subseteq Q'_{t'}$, then $t \in t'$. □

Corollary 3. *If X, Y have a decidable monadic second-order theory, so has $X + Y$, uniformly in X, Y.*

Proof. Set $I = \{1, 2\}$, which has a decidable theory by finiteness. □

If X is one of the orders $\{1, \ldots, n\}, \mathbb{N}, \mathbb{Z}$, we always consider the full monadic theory (i.e., $\mathcal{B} = 2^X, u = 0$).

Lemma 4. *(a) The monadic second-order theory of $\{1, \ldots, n\}$ is decidable uniformly in n.*

(b) The set of monadic-second order formulas which hold in some $\{1, \ldots, n\}$ is decidable.

Proof. (a): First, the theories of $\emptyset, \{1\}$ are decidable by finiteness. Now the theory of $\{1, \ldots, n\}$ can be computed uniformly by iterating corollary 3.

(b): It suffices to consider Th_k for fixed k, which is finite. Thus taking the iterated sums of (a) yields the same value for some $n < m$. Then each formula in Th_k true for some $\{1, \ldots, \ell\}$ is also true for such a set with $\ell < m$. □

Theorem 5 (S1S)**.** *The monadic theory of $\mathbb{N}$ is decidable.*

Proof. We compute $Th_k(\mathbb{N})$ by recursion on $|k|$. For $|k| = 0$ we only have to compute a finite set. Now suppose $|k| = n > 0$ and set $k' = (k_1, \ldots, k_{n-1})$.

Given $P \in (2^{\mathbb{N}})^{k_n}$, color pairs of $i < j$ by $t_{ij} = Th_{k'}(\{i, \ldots, j-1\}, P)$. By Ramsey's theorem, there are t and an infinite set of $n_0 < n_1 < \ldots$ such that $t_{n_i n_{i+1}} = t$ for all i. Set $s = Th_{k'}(\{0, \ldots, n_0 - 1\}, P)$. Thus $Th_{k'}(\mathbb{N}, P) = s + \sum_{i \in \mathbb{N}} t$. Since any type is of this form for some s, t types of finite linear orders (and all such sums may occur), by theorem 2 it remains to compute (for the index set) $Th_{k'}(\mathbb{N})$ (here no parameters are necessary as the type of the summands is constant), and possible types of finite orders. Finite orders are handled by lemma 4, and $Th_{k'}(\mathbb{N})$ by the recursion. □

Corollary 6. *The monadic theories of the order dual $\mathbb{N}^{\mathrm{op}}$ and $\mathbb{Z}$ are decidable.*

2 A Ramsey theorem

Lemma 7. *Let X a topological space, C finite, $A \subseteq X$ somewhere dense and $f\colon A \to C$. Then some fiber of f is somewhere dense.*

Proof. This follows as finite unions of nowhere dense sets are nowhere dense. $\square$

Let X a dense linear order without endpoints, with the order topology. Let C be a finite semigroup, and $f\colon \{(x,y) \in X \times X \mid x < y\} \to C$ such that $f(x,y) + f(y,z) = f(x,z)$ for all $x, y, z \in X$ (the existence of such a semigroup structure is equivalent to $f(x,z) = f(x',z')$ whenever $f(x,y) = f(x',y'), f(y,z) = f(y',z')$ for some y, y', possibly adding a point to C to make it total). Call $A \subseteq X$ homogeneous if f is constant on $A \times A$.

Theorem 8. *There is a somewhere dense homogeneous subset.*

Proof. Define $F, X \times 2^X \to 2^C, (a,J) \mapsto \bigcap_{b>a} f(a,\bullet)(J \cap (a,b))$.

We define a decreasing sequence of somewhere dense J_i by recursion. Let $J_0 = X$. By lemma 7 for any i there is $D_i \subseteq C$ such that $J_{i+1} = F(\bullet, J_i)^{-1}(\{D_i\})$ is somewhere dense.

By finiteness choose $D \subseteq C$ such that $\{i \mid D_i = D\}$ is infinite, say of the form $\{k_0, k_1, \dots\}$ with $k_i < k_{i+1}$, and set $J^i = J_{k_i+1}$.

Then D is a subsemigroup: Given $d_1, d_2 \in D$, choose any $x \in J^1$ and let $b > x$ arbitrary. Since $F(x, J_{k_1}) = D$, there is $y \in J_{k_1} \subseteq J^0$ with $y < b$ and $f(x,y) = d_1$. Since $F(y, J_{k_0}) = D$, there is $z \in J_{k_0}$ with $z < b$ and $f(y,z) = d_2$. Thus $d_1 + d_2 = f(x,y) + f(y,z) = f(x,z) \in f(x,\bullet)(J_{k_0} \cap (x,b))$, and as b was arbitary, $d_1 + d_2 \in F(x, J_{k_0}) = D$.

Choose $a < b$ such that $a \in J^{\#C}$ and $J^{\#C}$ is dense in (a,b). By finiteness and replacing b we may assume $D = F(a, J_{k_0}) = f(a,\bullet)(J_{k_0} \cap (a,b))$. By lemma 7 there is $d \in D$ such that $(f(a,\bullet))|^{-1}_{(a,b)\cap J^{\#C}}(\{d\})$ is somewhere dense, say in $(a',b') \subseteq (a,b)$.

Since D is a finite semigroup, there are $p, q \in \mathbb{Z}_{>0}$ with $p \cdot d = (p+q) \cdot d$ and $p + q \leq \#D + 1 \leq \#C + 1$. We claim that

$$J = \left\{x \in (a',b') \cap J^{\#C+1-q} \mid f(a,x) = q \cdot d\right\}$$

is homogeneous of color $q \cdot d$ and dense in (a',b').

For homogeneity let $x, y \in J$. Define recursively $x_i \in J^i$, where $x_p = x \in J \subseteq J^{\#C+1-q} \subseteq J^p$. Given x_{i+1}, since $F(x_{i+1}, J^i) \supseteq F(x_{i+1}, J_{k_{i+1}}) = D \ni d$, there is $x_i \in J^i \cap (x_{i+1}, y)$ with $f(x_{i+1}, x_i) = d$. Now

$$f(x,y) = f(x_p,y) = f(x_p,x_{p-1}) + \cdots + f(x_1,x_0) + f(x_0,y) = p \cdot d + f(x_0,y)$$

and so

$$f(x,y) = (p+q) \cdot d + f(x_0,y) = q \cdot d + (p \cdot d + f(x_0,y)) = q \cdot d + f(x,y).$$

Hence as $x, y \in J$,

$$f(x,y) = q \cdot d + f(x,y) = f(a,x) + f(x,y) = f(a,y) = q \cdot d.$$

For showing it dense let $a' < a'' < b'' < b'$. Define recursively $x_i \in J^i \cap (a'', b'')$. Here x_{q-1} may be chosen such that $f(a, x_{q-1}) = d$ as $f(a,\bullet)^{-1}(\{d\})$ is dense in (a',b'). Given x_{i+1}, since $F(x_{i+1}, J^i) \supseteq F(x_{i+1}, J_{k_{i+1}}) = D \ni d$, there is $x_i \in J^i \cap (x_{i+1}, b'')$ with $f(x_{i+1}, x_i) = d$. Hence

$$f(a,x_0) = f(a,x_{q-1}) + f(x_{q-1},x_{q-2}) + \cdots + f(x_1,x_0) = q \cdot d$$

and so $x_0 \in J \cap (a'', b'')$. $\square$

3 Dense uniform orders

Now we specialize the results of section 1 to X, X_i such that the order topology is Polish, $u = 2, \mathcal{M}_1 = \mathrm{Cnt}, \mathcal{M}_2 = \mathrm{Mgr}$. In the following, we will only use sums with all but countably many summands singletons and the remaining ones intervals. For them, a set lies in $\mathcal{M}_i$ if it does in all summands, whence theorem 2 continues to hold.

We impose the following requirements on $\mathcal{B}$, which in particular imply that sums of subspaces carry the same Boolean algebra as in section 1:

- Each F_σ-set lies in $\mathcal{B}$.
- Each element of $\mathcal{B}$ has the Baire property and the perfect set property.
- If C is G_δ, thus Polish, then pulling back $\mathcal{B}$ along the inclusion map of C yields $\mathcal{B}$.
- $\mathcal{B}$ is generated as a Boolean algebra by a set which is stable under countable unions and contains all intervals.

Next, we allow Th_k for $|k| = 0$ to contain first-order formulas up to some fixed number of quantifiers. For this stronger version of Th_k, theorem 2 still holds: The induction start is proven by using an analogue of theorem 2 where only the first-order theories of X, X_i (but the monadic theory of I) are considered. This fixed number of quantifiers is in particular chosen large enough to express basic topological properties like being perfect or having empty interior.

Proposition 9. *Let $\mathcal{C} = 2^{\mathbb{N}} \setminus \{0^\infty, 1^\infty\}$. If we can compute $Th_k(\mathbb{R})$ for all k with $|k| = n$, then the same holds for $Th_k(\mathcal{C}), Th_k(2^{\mathbb{N}})$.*

Proof. Each perfect, meager, unbounded above and below subset of $\mathbb{R}$ is order-isomorphic to $\mathcal{C}$. Thus adding either an existential or a universal quantifier over such sets at the front of the given formula in $\mathcal{C}$ allows to reduce it to a formula in $\mathbb{R}$. Finally, $2^{\mathbb{N}} \cong 1 + \mathcal{C} + 1$. $\square$

In the following we will often instead of Th_k use a variant pTh_k for partitions, which is equivalent to compute for fixed $|k|$: For $|k| = 0$, set $pTh_k(X, P) = Th_k(X, P)$, but P is required to be a partition of X. For $|k| > 0$, we do require the added Q to form a refinement of the given partition.

We assume $k_{n+1} > 2k_n$ for some of our statements.

We call a partition P of $X = \mathbb{R}$ or $X = \mathcal{C}$ k-uniform if for any open interval (a, b) such that a, b have no predecessor or successor in the order, we have $pTh_k((a, b), P) = pTh_k(X, P)$.

Now let P be a k-uniform partition of $\mathbb{R}$ for $|k| = n > 0$ and set $k' = (k_1, \ldots, k_{n-1})$. We define $pUTh^1_k(\mathbb{R}, P)$ as the set of $pTh_{k'}(\mathbb{R}, Q)$ for Q a k-uniform refinement of P of length k_n, and $pUTh^2_k(\mathbb{R}, P)$ as the set of $pTh_{k'}(\mathcal{C}, Q)$ for some embedding $\mathcal{C} \subseteq \mathbb{R}$, and Q a refinement of P of length $K = k_n + k_n^2 \# FTh_k$ that is $(k_1, \ldots, k_{n-1}, K)$-uniform in $\mathcal{C}$, such that for all gaps $a < b$ in $\mathcal{C}$ (that is, b is the successor of a), $pTh(\{a\}, Q) = pTh(\{b\}, Q)$, and with the first k_n variables used to partition the non-gaps and the remaining ones to partition the gaps.

Define $E_k(\mathbb{R}, P)$ to be the smallest subset of FTh_k satisfying the following conditions:

- $pUTh^1_k(\mathbb{R}, P) \subseteq E_k(\mathbb{R}, P)$.
- Let $t_1, t_2 \in E_k(\mathbb{R}, P)$ and $t \in pTh(\{a\}, Q)$. Then $t_1 + t + t_2 \in E_k(\mathbb{R}, P)$.
- Let $t_1 \in E_k(\mathbb{R}, P)$ and $t \in pTh(\{a\}, Q)$. Then $\sum_{i \in \mathbb{Z}}(t_1 + t) \in E_k(\mathbb{R}, P)$.
- For $t \in pUTh^2_k(\mathbb{R}, P)$, consider the partition of $\mathbb{R}$ obtained by replacing any gap (a, b) by a closed interval whose k'-theory is determined by $pTh(\{a\}, Q) = pTh(\{b\}, Q)$, whenever this lies in $E_k(\mathbb{R}, P)$ (else omit this t). Note that its theory $g(t)$ is computable from t (using theorem 2) and require $g(t) \in E_k(\mathbb{R}, P)$.

Note that $E_k(\mathbb{R}, P)$ is computable from $pUTh^1_k(\mathbb{R}, P), pUTh^2_k(\mathbb{R}, P)$.

Lemma 10. *(a) Let $I \subseteq \mathbb{R}$ convex without endpoints such that for $x \in I$ there are $a < x < b$ such that for all $a < a' < b' < b$ we have $pTh_k((a', b'), P) \in E_k(\mathbb{R}, P)$. Then $pTh_k(I, P) \in E_k(\mathbb{R}, P)$.*

(b) Suppose there are p, q and a dense $D \subseteq \mathbb{R}$ such that for $a, b \in D$ with $a < b$ we have $p = pTh_k((a, b), P)$ and $q = pTh(\{a\}, P)$. Then P is k-uniform.

(c) Any P is k-uniform on some interval.

Proof. (a): By assumption there is a $\mathbb{Z}$-indexed sequence $(a_i)_i$ with $a_{-n} \to \inf I$, $a_n \to \sup I$ and $pTh_k((a_n, a_{n+1}), P) \in E_k(\mathbb{R}, P)$. By taking repeated binary sums, for $m < n$ we have $pTh_k((a_m, a_n), P) \in E_k(\mathbb{R}, P)$. By Ramsey's theorem there are a color $(t_1, t) \in E_k(\mathbb{R}, P) \times pTh(\{a\}, P)$ and a subsequence with $t_1 = pTh_k\big((a_{n_i}, a_{n_{i+1}}\big), P)$ and $t = pTh(\{a_{n_i}\}, P)$ for all i. Hence $pTh_k(I, P) = \sum_{i \in \mathbb{Z}} (t_1 + t) \in E_k(\mathbb{R}, P)$.

(b): Again write any convex set I without endpoints as a sum of intervals (a_n, a_{n+1}) with endpoints in D. Then $pTh_k(I, P) = \sum_{i \in \mathbb{Z}} (p + q)$ does not depend on I.

(c): By theorem 8 there is an interval satisfying the assumptions of (b). □

Theorem 11. *We have $E_k(\mathbb{R}, P) = pTh_k(\mathbb{R}, P)$.*

Proof. Clearly $pUTh_k^1(\mathbb{R}, P) \subseteq pTh_k(\mathbb{R}, P)$, and $pTh_k(\mathbb{R}, P)$ is stable under the types of sums given above, using uniformity to create copies on intervals and adding these intervals. And by construction $g(t) \in pTh_k(\mathbb{R}, P)$.

For the converse inclusion let $pTh_{k_1, \ldots, k_{n-1}}(\mathbb{R}, Q) \in pTh_k(\mathbb{R}, P)$. Let C be the set of $x \in \mathbb{R}$ such that for all $a < x < b$ there are $a < a' < b' < b$ with $pTh_k((a', b'), Q) \notin E_k(\mathbb{R}, P)$.

Clearly C is closed.

And C has no isolated points: Suppose $C \cap (a, b) = \{x\}$. It suffices to show for all $a < a' < x < b' < b$ that $pTh_k((a', b'), Q) \in E_k(\mathbb{R}, P)$. By lemma 10 (a) we have $pTh_k((a', x), Q) \in E_k(\mathbb{R}, P)$ and $pTh_k((x, b'), Q) \in E_k(\mathbb{R}, P)$, whence $pTh_k((a', b'), Q) = pTh_k((a', x), Q) + pTh(\{x\}, Q) + pTh_k((x, b'), Q) \in E_k(\mathbb{R}, P)$.

Furthermore, C has empty interior: If I is any interval, by lemma 10 (c), Q is k-uniform on some interval $J \subseteq I$, and its theory on any subinterval lies in $pUTh_k^1(\mathbb{R}, P) \subseteq E_k(\mathbb{R}, P)$. Thus $J \cap C = \emptyset$ and so $I \nsubseteq C$.

Hence C is either empty or homeomorphic to $\mathcal{C}$ after possibly adding endpoints.

If $C = \emptyset$, then lemma 10 (a) applies with $I = \mathbb{R}$.

Else by lemma 10 (c), applied to C with gaps replaced by singletons, Q is k-uniform on some open interval of C, and the theory of any open subinterval of its convex hull in $\mathbb{R}$ lies in $E_k(\mathbb{R}, P)$, using $g(t)$. This contradicts the definition of C. □

4 Preliminary results

Moreover, here are some certainly well-known basic results about the topology of $\mathbb{R}$:

Lemma 12. *Any open subset of a Cantor set is a countable disjoint union of clopen Cantor sets.*

Proof. It is a countable union of clopens U_i, and replacing U_i by $U_i \setminus (U_0 \cup \cdots \cup U_{i-1})$ this union may be chosen disjoint. Finally, any nonempty clopen by compactness is a finite union of basic clopens and thus a Cantor set. □

Corollary 13. *Any countable union of Cantor sets may be refined to a countable disjoint union of Cantor sets.*

Lemma 14. *Any meager F_σ-set of $\mathbb{R}$ is a countable disjoint union of Cantor sets and points.*

Proof. By local compactness it is a countable union of compact sets K_i. Since K_i is also meager, hence has empty interior, it is a compact totally disconnected metric space, hence the union of at most one Cantor set C_i and a countable set D_i. Finally apply corollary 13. □

Corollary 15. *Any meager subset of $\mathbb{R}$ is contained in a countable disjoint union of Cantor sets.*

Proof. Since any point is contained in a Cantor set, by lemma 14 it is contained in a countable union of Cantor sets. Now use corollary 13. □

Since we added predicates for countable and meager sets in section 3, it may be good to know (although not logically necessary) that these notions are definable in the theory, thus only serve for eliminating quantifiers. And the proof for meager sets is a toy version of theorems 17 (b) and 17 (c).

First, it is easy to define standard topological notions as open, closed, dense, connected or if the number of (isolated) points of a set equals a fixed finite number. Thus, using the perfect set property, we can define countability.

Definability of meagerness follows from the following result, where a subset of a Polish space without isolated points is called very dense if its intersection with every nonempty open is uncountable.

Lemma 16. *For a Borel $A \subseteq \mathbb{R}$ the following are equivalent:*

(i) A is meager.

(ii) For each nonempty open U there is a very dense Borel $B \subseteq U$ such that there is no Cantor set $C \subseteq A \cup B$ with both A, B dense in C.

Proof. (i) $\implies$ (ii): A is contained in a countable union of Cantor sets A_i. For each basic open U_i, the uncountable $U_i \setminus A$ contains a Cantor set B_i, and we set $B = \bigcup_i B_i$. If $C \subseteq A \cup B = \bigcup_i A_i \cup B_i$, some A_i or B_i is nonmeager, hence by closedness has nonempty interior. Thus as A, B are disjoint, B or A is not dense.

$\neg$(i) $\implies$ $\neg$(ii): Choose U such that A is comeager in U and let $B \subseteq U$ very dense. Shrinking A, B we may assume that A is G_δ and B is a countable union of Cantor sets B_i. The complement of A is contained in countably many pairwise disjoint Cantor sets C_i.

Enumerate the basic opens U_i and set $j_{-1} = 0$. Since $U_i \setminus \bigcup_{p<i} C_{j_p}$ has nonempty interior, it meets B in an uncountable set. Thus $A \cap B \cap U_i$ is uncountable or there are $j_i > j_{i-1}$ and k_i with $C_{j_i} \cap B_{k_i} \cap U_i$ uncountable. Then $A \cup \bigcup_i B_{k_i} \cap C_{j_i}$ is G_δ, hence Polish, with both A, B very dense, and thus contains the desired Cantor set. □

5 The new part

Theorem 17. *Let P be k-uniform P with $|k| = n$, and suppose $pTh_k(\mathbb{R})$ is given.*

(a) If $n > 0$, then $pTh_k(\mathbb{R}, P)$ can be computed from the comeager part P_0 and $pUTh^2_k(\mathbb{R}, P)$.

(b) Suppose P_0 is the comeager part, and $\complement P_0$ is contained in a countable disjoint union of Cantor sets C_i. Then $pTh_k(\mathbb{R}, P)$ can be computed from $\{pTh_k(C_i, P) \mid i\}$.

(c) For any set T of satisfiable $pTh_k(2^{\mathbb{N}}, P)$, there is a partition of $\mathbb{R}$ with comeager part P_0 that is of the form of (b), called uniform sum and denoted $\sum^{P_0} T$ or $\sum^{P_0}_{pTh_k(C_i,P)\in T} C_i$.

Proof. We use recursion on n and set $k' = (k_1, \ldots, k_{n-1})$.

(b): $n = 0$ is easy: P_0 is the unique comeager part and a part is countable or empty if and only if it is so in all summands. The first-order theory reduces to completeness and decidability of the theory of dense linear orders without endpoints and with a partition into n dense sets, uniformly in n.

For $n > 0$, by theorem 11 it suffices to compute $pUTh^1_k, pUTh^2_k$. Thus consider some $pTh_{k'}(X, Q)$ for $X = \mathbb{R}$ or $X \cong \mathcal{C}$.

First suppose $P_0 \cap X$ is comeager in X. Then some part $Q_0 \subseteq P_0$ is still comeager, and by adding parts with trivial partition P_0 for $P_0 \setminus Q_0$ to the decomposition (whose types can be computed using $pTh_k(\mathbb{R})$), we may assume $P_0 = Q_0$. Now replace each C_i by a nonempty set $\left\{C^1_i, \ldots, C^{k_i}_i\right\}$ of possible refinements of (C_i, P) to a partition Q. Then $pTh_{k''}(\mathbb{R}, Q)$ can be computed recursively for appropriate k'' with $|k''| = n-1$, whence the set of possible $pTh_{k'}(X, Q)$ can by the proof of proposition 9. And all elements of $pUTh^1_k$ or with $P_0 \cap X$ comeager are of this form.

Else by uniformity and the Baire property, $P_0 \cap X$ is meager in X. Thus the union of all C_i and hence some C_i are nonmeager in X. Since $C_i \cap X$ is closed in X, it even has nonempty interior and thus by uniformity of X, X has the type of an open interval of C_i, thus of a copy of $\mathcal{C}$ in C_i, and the possible ones can be computed from $pTh_k(C_i)$.

(c): If T is empty, this is trivial. Else taking finite sums we may assume $T = \{pTh_k(C, P)\}$.

We define an increasing sequence D_i of Cantor sets by recursion, with $D_0 = C$. Given D_i, add a copy of C in each complementary interval and call the resulting Cantor set D_{i+1}, ensuring that D_{i+1} meets each subinterval of $[-i, i]$ of length at least $\frac{1}{i}$. At the end, combine the given partition on $\bigcup_i D_i$ with the trivial partition into P_0 on its complement to obtain a partition of $\mathbb{R}$.

It remains to show this partition k-uniform. By lemma 10 (c) it is k-uniform on some interval. The claim follows as any interval contains a subinterval homeomorphic to the partition.

(a): By theorem 11 it remains to compute $pUTh^1_k(\mathbb{R}, P)$. By (b) and corollary 15 it suffices to compute the possible $pTh_{k'}(C, Q)$ for C a Cantor set and Q a not necessarily uniform refinement of P: Then the possible uniform refinements of $(\mathbb{R}, P)$ have types given by uniform sums of (c) with the summands a subset S of the possible $pTh_{k'}(C, Q)$ such that for any possible (C, P) there is a refinement in S.

For this we modify the proof of theorem 11: Define $E_k(2^{\mathbb{N}}, P)$ to be the smallest subset of FTh_k satisfying the conditions $pUTh^2_k(\mathbb{R}, P) \subseteq E_k(\mathbb{R}, P)$, the same conditions for sums, and the same condition for $g(t)$, but where we fill the gaps with Cantor sets instead of real intervals. Then $E_k(2^{\mathbb{N}}, P)$ is the set of these $pTh_{k'}(C, Q)$ by the same proof, defining for example C as the set of $x \in 2^{\mathbb{N}}$ such that the same condition holds. □

We define for a k-uniform partition P of $\mathbb{R}$ or $\mathcal{C}$, or more precisely for its type $t = pTh_k(\mathbb{R}, P)$, the rank $\operatorname{rk} t = \operatorname{rk} P \in \mathbb{N} \cup \{\infty\}$ as the largest number satisfying:

- A trivial partition, that is, a partition with only one uncountable part, has rank 0.
- If P is a partition of $\mathcal{C}$, then its rank is the rank of the partition of $\mathbb{R}$ obtained by replacing gaps by singletons.
- If $\operatorname{rk} C_i \leq n$ for all i, then $\operatorname{rk} \sum_i^{P_0} C_i \leq n + 1$.

For $t = pTh_{k_1,\dots,k_n}(\mathbb{R}, P)$ and $m \leq n$ write $t|m = pTh_{k_1,\dots,k_m}(\mathbb{R}, P)$.

If P is k-uniform, we call t minimal, if t is trivial, or if t is a $\subseteq$-least element of the uniform t' with $t'|n-1 = t|n-1$. We call the type of $(\mathcal{C}, P)$ minimal if the type of $\mathbb{R}$ obtained by replacing gaps by singletons is minimal.

Lemma 18. *Uniform sums of minimal types are minimal.*

Proof. This follows from the description of the n-types of sums in the proof of theorem 17 (b), and the fact that any realization contains the $n-1$-types of all summands in any interval by uniformity. To handle trivial summands, use the perfect set property. □

Theorem 19. *(a) The rank of any uniform partition is finite.*

(b) If $t = pTh_k(\mathbb{R}, P)$ with $n = |k|$ is k-uniform, there is a minimal $M(t)$ with $M(t)|n = t$ and $\operatorname{rk} M(t) = \operatorname{rk} t$.

Proof. We use simultaneous induction on $n = |k|$.

(a) for $n = 0$: We show that every nontrivial partition has rank 1. Indeed, it is determined by knowing which parts are empty, countable, or meager. Thus it is a uniform sum, with the given comeager part, a summand for each uncountable meager part, and a trivial summand for the countable parts.

(a) implies (b) for the same n: We use induction on $\operatorname{rk} t$, and trivial partitions are easy. Else by (a) write $t = \sum_i^{P_0} C_i$ with C_i uniform and $\operatorname{rk} C_i < \operatorname{rk} t$. Set $M(t) = \sum_i^{P_0} M(C_i)$, which is valid by the induction hypothesis. This is uniform with $\operatorname{rk} M(t) \leq 1 + \max_i \operatorname{rk} M(C_i) = 1 + \max_i \operatorname{rk} C_i \leq \operatorname{rk} t$. Minimality follows from lemma 18.

(b) for n implies (a) for $n+1$: Let T be an $n+1$-type of a uniform partition P with comeager part P_0. By (b) for any $t \in pUTh^2_k(\mathbb{R}, P)$, $M(t)$ exists. Thus it suffices to show $T = \sum^{P_0}_{t \in pUTh^2_k(\mathbb{R},P)} M(t)$. By theorem 17 (a) it suffices to show that they have the same $pUTh^2_k$, and clearly the one of T is contained in the one of the sum. Equality follows from lemma 18. $\square$

Lemma 20. *Given $pTh_k(\mathbb{R})$, the k-theory of a trivial partition is computable.*

Proof. By theorem 17 (a) it suffices to compute $pUTh^2_k$. Its elements are trivial with an arbitrary subset of the given countable parts. $\square$

Corollary 21. *The monadic second-order theory of $(\mathbb{R}, <)$ with quantification restricted to $\mathcal{B}$ is decidable.*

Proof. We compute $Th_k(\mathbb{R})$ by recursion on $|k|$. The case $|k| = 0$ is clear. For $|k| > 0$ by theorem 11 it suffices to compute $pUTh^1_k(\mathbb{R}), pUTh^2_k(\mathbb{R})$.

By proposition 9 we can reduce the second one to the first one after increasing the last component of k, and so it suffices to compute all possible types of uniform partitions. By theorem 19 we can compute them by recursion on the rank, using theorem 17 (b) for positive ranks, and lemma 20 and the recursion on $|k|$ for rank 0. $\square$

References

[She75] Saharon Shelah. The monadic theory of order. *Annals of Mathematics*, 102(3):379–419, 1975.