Fundamental Notions in Algebra – Exercise No. 2

1. Let A and B be R-algebras. Prove that the tensor product $A \otimes_R B$ has a structure of an R-algebra with a multiplication $(a \otimes b) \cdot (c \otimes d) = (ac \otimes bd)$.

2. Let M and N be two modules over a commutative ring R.

   (a) Show that there exists a homomorphism of R-algebras
   $$f : \mathrm{End}_R(M) \otimes_R \mathrm{End}_R(N) \to \mathrm{End}_R(M \otimes_R N)$$
   such that $f(A \otimes B)(m \otimes n) := A(m) \otimes B(n)$.

   (b) Show that homomorphism f from (a) is an isomorphism, if M and N are free and finitely generated.

   (c) Consider dual module $M^* := Hom_R(M, R)$. Show that there exists a homomorphism $g : M^* \otimes_R N \to Hom_R(M, N)$ of R-modules such that $g(f \otimes n)(m) := f(m)n$.

   (d) Show that homomorphism g from (c) is an isomorphism, if M is free and finitely generated.

   (e) Are the assumptions in (b) and (d) necessary?

3. (a) Show that for any two sets I and J there is a natural isomorphism of R-modules $R^I \otimes_R R^J \cong R^{I \times J}$.

   (b) Let R be a commutative ring, and let G and H be groups. Show that there are natural isomorphisms of R-algebras
   $$R[G \times H] \cong R[G] \otimes_R R[H] \cong (R[G])[H].$$

   (c) Let $\mathbb{H}$ be the algebra of real quaternions. Show that the $\mathbb{R}$-algebras $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$ is isomorphic to $\mathrm{Mat}_4(\mathbb{R})$.
   Hint: Construct an algebra homomorphism $f : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{End}_{\mathbb{R}}(\mathbb{H})$ such that $f(x \otimes y)(z) := x \cdot z \cdot \bar{y}$.

4. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of R-modules. Show that the following are equivalent.

   (a) The sequence splits.

   (b) The submodule $f(A)$ is a direct summand in B, (that is, there exists a submodule $D \subset B$ such that $D \cap f(A) = 0$ and $B = D + f(A)$, thus $B := D \oplus f(A)$).

   (c) There exists a homomorphism $r : B \to A$ such that $r \circ f = id_A$.

   (d) There exists a homomorphism $s : C \to B$ such that $g \circ s = id_C$.

5. (a) Show that an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ always splits if C is free.

   (b) Find an example of a non-split short exact sequence.

   (c) Show that for a ring R the following are equivalent:

      i. Every short exact sequence of R-modules splits

      ii. For every R-module M, every R-submodule N is a direct summand.