Fundamental Notions in Algebra – Exercise No. 10

Definition: A ring R is called *semi-primitive* if for every element $a \neq 0$ of R there exists a simple R-module M such that $a \notin Ann(M)$.

Definition: We say that a ring R is a *subdirect product* of rings R_α, if there exists an embedding $\iota : R \to \prod R_\alpha$ such that the composition $\pi_\alpha \circ \iota : R \to R_\alpha$ is surjective for each α.

1. Let R be a ring. Show that the following conditions are equivalent:
   (a) R is semi-primitive;
   (b) R has a faithful semi-simple module;
   (c) R is a subdirect product of primitive rings
   (d) $J(R) = 0$.

2. (a) Show that a ring R is primitive if and only if it contains a left ideal I which does not contain a non-zero two-sided ideal.
   (b) Show that a commutative ring R is primitive if and only if it is a field.
   (c) Show that an artinian ring R is primitive if and only if it is simple.
   (d) Show that an artinian ring R is semi-primitive if and only if it is semi-simple.

3. (a) Show that if R is a primitive ring such that for each $r \in R$ there exists an integer $n(r) > 1$ such that $r^{n(r)} = r$, then R is a division ring.
   (b) In the assumptions of (a) show that R is a field. [**Hint:** Show first that R is an algebra over a finite field $\mathbb{F}_p$ for some prime p. Assume that $R \neq Z(R)$ and choose any $r \in R - Z(R)$. Show that in this case there exists $s \in R$ such that $srs^{-1} = r^p$ and that r and s generate a finite division algebra.]
   (c) Show that if R is a ring such that for each $r \in R$ there exists an integer $n(r) > 1$ such that $r^{n(r)} = r$, then R is semi-primitive.
   (d) Show that if R is a ring such that for each $r \in R$ there exists an integer $n(r) > 1$ such that $r^{n(r)} = r$, then R is commutative.

4. Let k be a field, and A be a k-algebra. Show that the exists a primitive k-algebra R and a surjection R onto A.

   Hint: Let V be a direct sum of infinitely many copies of A. Let R be the subring of $\mathrm{End}_k(V)$ generated by A (acting diagonally) and the set of linear transformations of V of finite rank. Show that
   (a) the ring R is primitive;
   (b) there exists a surjection of R onto A.