HOMEWORK #8 IN ALGEBRAIC STRUCTURES 2

Problem 1. Prove Lemma 8.1.

Problem 2. Prove Lemma 8.3 (a)–(e).

Problem 3. Prove Lemma 8.3'.

Problem 4. Let $K \subset L_1, L_2 \subset \bar{K}$ be two finite extensions and let $L := L_1L_2 \subset \bar{K}$ be the subfield generated by L_1 and L_2. L is called the *compositum* of L_1, L_2.

(a) Show that if L_1/K, L_2/K are normal extensions, so is L/K.
(b) Show that if L_1/K, L_2/K are Galois, so is L/K, and $\mathrm{Gal}(\bar{K}/L) = \mathrm{Gal}(\bar{K}/L_1) \cap \mathrm{Gal}(\bar{K}/L_2)$.
(c) Show that under the assumptions of (b), the mapping $\mathrm{Gal}(L/K) \to \mathrm{Gal}(L_1/K) \times \mathrm{Gal}(L_2/K)$ defined by taking $\sigma \in \mathrm{Gal}(L/K)$ to $(\sigma_{|L_1}, \sigma_{|L_2})$ is a monomorphism of groups.

Problem 5. Let K be a field such that $\mathrm{ch}(K) \neq 2$, $a_1, \dots, a_n \in K$.

(a) Show that the extension $K(\sqrt{a_1}, \dots, \sqrt{a_n})/K$, $a_i \in K$ is a Galois extension and the Galois group $\mathrm{Gal}(K(\sqrt{a_1}, \dots, \sqrt{a_n})/K)$ is direct product of k copies of $\mathbb{Z}/2\mathbb{Z}$ where $k \leq n$.
(b) Under which conditions do we have $[K(\sqrt{a_1}, \dots, \sqrt{a_n}) : K] = 2^n$?

Problem 6. Let K be a field of characteristic zero, $n > 1$ be an integer such that the equation $t^n - 1$ has n solutions in K.

(a) Show that the extension $K(\sqrt[n]{a_1}, \sqrt[n]{a_2})/K$, $a_i \in K$ is a Galois extension and the Galois group $G := \mathrm{Gal}(K(\sqrt[n]{a_1}, \sqrt[n]{a_2})/K)$ is isomorphic to the direct product
$$G = \mathbb{Z}/r_1\mathbb{Z} \times \mathbb{Z}/r_2\mathbb{Z}$$
where r_i are divisors of n,
(b) Under which conditions do we have $[K(\sqrt[n]{a_1}, \sqrt[n]{a_2}) : K] = n^2$?