HOMEWORK #2 IN ALGEBRAIC STRUCTURES 2

Problem 2.1. a) Show that for any field K that either ch $K = 0$ or it is a prime number,

b) let K be a field of characteristic 2. Show that for any $a, b \in K$ we have $(a+b)^2 = a^2 + b^2$,

c) let K be a field of characteristic $p > 0$. Show that for any $a, b \in K$ we have $(a+b)^p = a^p + b^p$.

Problem 2.2. Let L be an extension of K and $\{\alpha_1, ..., \alpha_n\}$ a set of elements in L. Define inductively a sequence of subfields $F_i \subset L, 0 \leq i \leq n$ by

$$F_0 = K, F_i = F_{i-1}(\alpha_i)$$

Show that $K(\alpha_1, ..., \alpha_n) = F_n$.

To formulate the next problem we will use the notations from the Definition 2.3. Let A be an integral commutative ring.

Problem 2.3. Show that

a) if $(a, s) \equiv (a', s'), (b, u) \equiv (b', u')$ then $(a, s)(b, u) \equiv (a', s')(b', u')$ and $(a, s) + (b, u) \equiv (a', s') + (b', u')$,

as follows from a) we have well defined operations $(\alpha, \beta) \to \alpha\beta$ and $(\alpha, \beta) \to \alpha + \beta, \alpha, \beta \in K(A)$ on $K(A)$. We define elements $0, 1 \in$ as the equivalence classes of pairs $(0, 1)$ and $(1, 1)$ correspondingly.

b) show that the set $K(A)$ with the operations $(\alpha, \beta) \to \alpha\beta$ and $(\alpha, \beta) \to \alpha + \beta, \alpha, \beta \in K(A)$ is a field and the map $a \to (a, 1)$ defines a monomorphism from A to $K(A)$. We will always consider A as a subring of $K(A)$.

Let K is a field. Define inductively a sequence of fields $L_i, 0 \leq i$ by $L_0 := K, L_i := L_{i-1}(t_i)$ (that is, L_i is the field of rational functions in one variable over L_{i-1}). By the construction, the ring $K[t_1, ..., t_n]$ is a subring of L_n.

c) construct an isomorphism of the fields L_n and $K(t_1, ..., t_n)$ which induces the identity map on $K[t_1, ..., t_n]$ which we consider as a subring of both fields L_n and $K(t_1, ..., t_n)$.

Problem 2.4. a)Find the greatest common divisor of $q(t) = t^7 - t^4 + t^3 - 1$ and $r(t) = t^3 - 1$

b) show that the polynomial $p(t) = t^2 + 1$ over the field F_{19} is irreducible and show that the quotient ring $F_{19}[t]/(t^2 + 1)$ is a field with 361 elements,

c) show that the polynomial $p(t) = t^3 + t + 4$ over the field F_{11} is irreducible and show that the quotient ring $F_{11}[t]/(t^3 + t + 4)$ is a field with 11^3 elements.

Problem 2.5. Show that for any irreducible monic polynomial $p(t) \in K[t]$ of degree n there exists an extension $L \supset K$ such that $[L : K] \leq n!$ and $p(t)$ can be written in the ring $L[t]$ as a product of linear factors (i.e. polynomials of degree 1).