UNIVERSAL DERIVED EQUIVALENCES OF POSETS OF CLUSTER TILTING OBJECTS

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Abstract. We show that for two quivers without oriented cycles related by a BGP reflection, the posets of their cluster tilting objects are related by a simple combinatorial construction, which we call a flip-flop.

We deduce that the posets of cluster tilting objects of derived equivalent path algebras of quivers without oriented cycles are universally derived equivalent. In particular, all Cambrian lattices corresponding to the various orientations of the same Dynkin diagram are universally derived equivalent.

1. Introduction

In this note we investigate the combinatorial relations between the posets of cluster tilting objects of derived equivalent path algebras, continuing our work [10] on the posets of tilting modules of such algebras.

Throughout this note, we fix an algebraically closed field k. Let Q be a finite quiver without oriented cycles and let $\operatorname{rep} Q$ denote the category of finite dimensional representations of Q over k. The associated cluster category $\mathcal{C}_Q$ was introduced in [2] (and in [3] for the A_n case) as a representation theoretic approach to the cluster algebras introduced and studied by Fomin and Zelevinsky [4]. It is defined as the orbit category [8] of the bounded derived category $\mathcal{D}^b(Q)$ of $\operatorname{rep} Q$ by the functor $S \cdot [-2]$ where $S : \mathcal{D}^b(Q) \to \mathcal{D}^b(Q)$ is the Serre functor and [1] is the suspension. The indecomposables of $\mathcal{C}_Q$ can be represented by the indecomposables of $\mathcal{D}^b(Q)$ in the fundamental domain of $S \cdot [-2]$, hence $\operatorname{ind} \mathcal{C}_Q = \operatorname{ind} \operatorname{rep} Q \cup \{P_y[1] : y \in Q\}$ where P_y are the indecomposable projectives in $\operatorname{rep} Q$.

Cluster tilting theory was investigated in [2]. A basic object $T \in \mathcal{C}_Q$ is a *cluster tilting object* if $\operatorname{Ext}^1_{\mathcal{C}_Q}(T, T) = 0$ and T is maximal with respect to this property, or equivalently, the number of indecomposable summands of T equals the number of vertices of Q. If $T = M \oplus U$ is cluster tilting and M is indecomposable, then there exist a unique indecomposable $M' \neq M$ such that $T' = M' \oplus U$ is cluster tilting. T' is called the *mutation* of T with respect to M.

Denote by $\mathcal{T}_{\mathcal{C}_Q}$ the set of all cluster tilting objects. In [7], a partial order on $\mathcal{T}_{\mathcal{C}_Q}$, extending the partial order on tilting modules introduced in [12], is defined by $T \leq T'$ if $\operatorname{fac} T \supseteq \operatorname{fac} T'$. Here, for $M \in \operatorname{rep} Q$, $\operatorname{fac} M$ denotes the full subcategory of $\operatorname{rep} Q$ consisting of all quotients of finite sums of copies of M, and for $T \in \mathcal{T}_{\mathcal{C}_Q}$, $\operatorname{fac} T = \operatorname{fac} \widehat{T}$ where $\widehat{T} \in \operatorname{rep} Q$ is the sum of all indecomposable summands of T which are not shifted projectives.

As shown in [7], the map $T \mapsto \operatorname{fac} T$ induces an order preserving bijection between $(\mathcal{T}_{\mathcal{C}_Q}, \leq)$ and the set of finitely generated torsion classes in $\operatorname{rep} Q$

ordered by reverse inclusion. Moreover, it is also shown that when Q is Dynkin, $(\mathcal{T}_{\mathcal{C}_Q}, \leq)$ is isomorphic to the corresponding Cambrian lattice defined in [11] as a certain lattice quotient of the weak order on the Coxeter group associated with Q.

For two partially ordered sets $(X, \leq_X)$, $(Y, \leq_Y)$ and an order preserving function $f : X \to Y$, define two partial orders $\leq_+^f$ and $\leq_-^f$ on the disjoint union $X \sqcup Y$ by keeping the original partial orders inside X and Y and setting

$$x \leq_+^f y \iff f(x) \leq_Y y \qquad\qquad y \leq_-^f x \iff y \leq_Y f(x)$$

with no other additional order relations. We say that two posets Z and Z' are related via a *flip-flop* if there exist X, Y and $f : X \to Y$ as above such that $Z \simeq (X \sqcup Y, \leq_+^f)$ and $Z' \simeq (X \sqcup Y, \leq_-^f)$.

Let x be a sink of Q and let Q' be the quiver obtained from Q by a BGP reflection [1] at x, that is, by reverting all the arrows ending at x. Our main result is the following.

Theorem 1.1. *The posets $\mathcal{T}_{\mathcal{C}_Q}$ and $\mathcal{T}_{\mathcal{C}_{Q'}}$ are related via a flip-flop.*

We give a brief outline of the proof. Let $\mathcal{T}_{\mathcal{C}_Q}^x$ denote the subset of cluster tilting objects containing the simple projective S_x at x as direct summand. Given $T \in \mathcal{T}_{\mathcal{C}_Q}^x$, let $f(T)$ be the mutation of T with respect to S_x. In Section 2 we prove that the function $f : \mathcal{T}_{\mathcal{C}_Q}^x \to \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}_{\mathcal{C}_Q}^x$ is order preserving and moreover

$$\mathcal{T}_{\mathcal{C}_Q} \simeq \left(\mathcal{T}_{\mathcal{C}_Q}^x \sqcup (\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}_{\mathcal{C}_Q}^x), \leq_+^f\right) \tag{1.1}$$

Similarly, let $\mathcal{T}_{\mathcal{C}_{Q'}}^{x[1]}$ be the subset of cluster tilting objects containing the shifted projective $P'_x[1]$ at x as direct summand. Given $T \in \mathcal{T}_{\mathcal{C}_{Q'}}^{x[1]}$, let $g(T)$ be the mutation of T with respect to $P'_x[1]$. In Section 3 we prove that the function $g : \mathcal{T}_{\mathcal{C}_{Q'}}^{x[1]} \to \mathcal{T}_{\mathcal{C}_{Q'}} \setminus \mathcal{T}_{\mathcal{C}_{Q'}}^{x[1]}$ is order preserving and moreover

$$\mathcal{T}_{\mathcal{C}_{Q'}} \simeq \left(\mathcal{T}_{\mathcal{C}_{Q'}}^{x[1]} \sqcup (\mathcal{T}_{\mathcal{C}_{Q'}} \setminus \mathcal{T}_{\mathcal{C}_{Q'}}^{x[1]}), \leq_-^g\right) \tag{1.2}$$

In Section 4 we relate the two isomorphisms given in (1.1) and (1.2) by considering, following [13], the action of the BGP reflection functor on the cluster tilting objects. We prove the existence of the following commutative diagram with horizontal isomorphisms of posets

$$\begin{tikzcd}
\mathcal{T}_{\mathcal{C}_Q}^x \arrow[r, "\simeq"] \arrow[d, "f"] & \mathcal{T}_{\mathcal{C}_{Q'}}^{x[1]} \arrow[d, "g"] \\
\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}_{\mathcal{C}_Q}^x \arrow[r, "\simeq"] & \mathcal{T}_{\mathcal{C}_{Q'}} \setminus \mathcal{T}_{\mathcal{C}_{Q'}}^{x[1]}
\end{tikzcd}$$

from which we deduce Theorem 1.1. An example demonstrating the theorem and its proof is given in Section 5.

In our previous work [10], we have shown a result analogous to Theorem 1.1 for the posets $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ of tilting modules, following a similar strategy of proof. However, there are some important differences.

First, the situation in the cluster tilting case is asymmetric, as the partition (1.1) for a sink involves the subset of cluster tilting objects containing the corresponding simple, while the corresponding partition of (1.2) at a source involves the subset of cluster tilting objects containing the shifted projective. In contrast, both partitions for the tilting case involve the subset of tilting modules containing the simple, either at a source or sink. This asymmetry is inherent in the proof of (1.2), which is not the dual of that of (1.1), and also in the analysis of the effect of the BGP reflection functor.

Second, in the cluster tilting case, the order preserving maps occurring in the flip-flop construction are from the set containing the simple (or shifted projective) to its complement, while in the tilting case, they are in the opposite direction, into the set containing the simple. As a consequence, a partition with respect to a sink in the cluster tilting case yields an order of the form $\leq_+$, while for the tilting case it gives $\leq_-$.

While two posets Z and Z' related via a flip-flop are in general not isomorphic, they are *universally derived equivalent* in the following sense; for any abelian category $\mathcal{A}$, the derived categories of the categories of functors $Z \to \mathcal{A}$ and $Z' \to \mathcal{A}$ are equivalent as triangulated categories, see [9].

It is known [5, (I.5.7)] that the path algebras of two quivers Q, Q' without oriented cycles are derived equivalent if and only if Q' can be obtained from Q by a sequence of BGP reflections (at sources or sinks). We therefore deduce the following theorem.

Theorem 1.2. *Let Q and Q' be two quivers without oriented cycles whose path algebras are derived equivalent. Then the posets $\mathcal{T}_{\mathcal{C}_Q}$ and $\mathcal{T}_{\mathcal{C}_{Q'}}$ are universally derived equivalent.*

Since for a Dynkin quiver Q, the poset $\mathcal{T}_{\mathcal{C}_Q}$ is isomorphic to the corresponding Cambrian lattice, the above theorem can be restated as follows.

Corollary 1.3. *All Cambrian lattices corresponding to the various orientations of the same Dynkin diagram are universally derived equivalent.*

In particular, the incidence algebras of the Cambrian lattices corresponding to the various orientations the same Dynkin diagram are derived equivalent, as the universal derived equivalence of two finite posets implies the derived equivalence of their incidence algebras.

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2. Cluster tilting objects containing P_x

Let $x \in Q$ be a vertex, and denote by $\mathcal{T}^x_{\mathcal{C}_Q}$ the subset of cluster tilting objects containing P_x as direct summand.

Lemma 2.1. *Let $M \in \operatorname{rep} Q$. Then $P_x \in \operatorname{fac} M$ if and only if M contains P_x as a direct summand.*

Proof. Assume that $P_x \in \operatorname{fac} M$, and let $q : M^n \twoheadrightarrow P_x$ be a surjection, for some $n \geq 1$. Since P_x is projective, there exists $j : P_x \to M^n$ such that

$qj = 1_{P_x}$. Let $N = \operatorname{Im} j = \operatorname{Im} jq$. As $(jq)^2 = jq$, we deduce that N is a direct summand of M^n and that $j : P_x \to N$ is an isomorphism. Since P_x is indecomposable, it is also a summand of M. □

Corollary 2.2. *Let $T \in \mathcal{T}_{\mathcal{C}_Q}$. Then $T \in \mathcal{T}^x_{\mathcal{C}_Q}$ if and only if $P_x \in \operatorname{fac} T$.*

Corollary 2.3. *If $T \in \mathcal{T}^x_{\mathcal{C}_Q}$ and $T' \leq T$, then $T' \in \mathcal{T}^x_{\mathcal{C}_Q}$,*

Proof. Let $T \in \mathcal{T}^x_{\mathcal{C}_Q}$ and $T' \in \mathcal{T}_{\mathcal{C}_Q}$. If $T' \leq T$, then $P_x \in \operatorname{fac} T \subseteq \operatorname{fac} T'$, hence $T' \in \mathcal{T}^x_{\mathcal{C}_Q}$. □

Define a map $f : \mathcal{T}^x_{\mathcal{C}_Q} \to \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q}$ as follows. Given $T \in \mathcal{T}^x_{\mathcal{C}_Q}$, write $T = P_x \oplus U$ and set $f(T) = M \oplus U$ where M is the unique other indecomposable complement of U such that $M \oplus U$ is a cluster tilting object.

Recall that for a tilting module $T \in \operatorname{rep} Q$, we have $\operatorname{fac} T = T^{\perp}$ where

$$T^{\perp} = \left\{ M \in \operatorname{rep} Q \,:\, \operatorname{Ext}^1_Q(T, M) = 0 \right\}$$

Lemma 2.4. *Let $T \in \mathcal{T}^x_{\mathcal{C}_Q}$. Then $f(T) > T$.*

Proof. One could deduce the claim from Lemma 2.32 of [7]. Instead, we shall give a direct proof. Write $T = P_x \oplus U$ and $f(T) = M \oplus U$. If M is a shifted projective, the claim is clear. Otherwise, by deleting the vertices of Q corresponding to the shifted projective summands of U, we may and will assume that $P_x \oplus U$ and $M \oplus U$ are tilting modules. Therefore

$$\operatorname{fac}(P_x \oplus U) = (P_x \oplus U)^{\perp} = U^{\perp}$$

where the last equality follows since P_x is projective. As $M \in U^{\perp}$, we get that $M \in \operatorname{fac}(P_x \oplus U)$, hence $\operatorname{fac}(M \oplus U) \subseteq \operatorname{fac}(P_x \oplus U)$. □

For the rest of this section, *we assume that the vertex x is a sink in Q*. In this case, $P_x = S_x$ and $\operatorname{ind} \operatorname{fac} S_x = \{S_x\}$. Moreover, $S_x \notin \operatorname{fac} M$ for any other indecomposable $M \neq S_x$, since $\operatorname{Hom}_Q(M, S_x) = 0$.

Lemma 2.5. *Let $T \in \mathcal{T}^x_{\mathcal{C}_Q}$. Then $\operatorname{ind} \operatorname{fac} f(T) = \operatorname{ind} \operatorname{fac} T \setminus \{S_x\}$.*

Proof. Write $T = S_x \oplus U$ and $f(T) = M \oplus U$. By the preceding remarks,

$$\operatorname{ind} \operatorname{fac} T = \operatorname{ind} \operatorname{fac}(S_x \oplus U) = \operatorname{ind} \operatorname{fac} S_x \cup \operatorname{ind} \operatorname{fac} U$$

is a disjoint union $\{S_x\} \amalg \operatorname{ind} \operatorname{fac} U$. By Lemma 2.4,

$$\operatorname{ind} \operatorname{fac} f(T) = \operatorname{ind} \operatorname{fac}(M \oplus U) \subseteq \operatorname{ind} \operatorname{fac}(S_x \oplus U) = \{S_x\} \amalg \operatorname{ind} \operatorname{fac} U,$$

therefore $\operatorname{ind} \operatorname{fac}(M \oplus U) = \operatorname{ind} \operatorname{fac} U$, as $S_x \notin \operatorname{fac} M$. □

Corollary 2.6. *Let $T \in \mathcal{T}^x_{\mathcal{C}_Q}$ and $T' \in \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q}$ be such that $T' > T$. Then $T' \geq f(T)$.*

Proof. By assumption, $\operatorname{fac} T' \subseteq \operatorname{fac} T$. Moreover, $S_x \notin \operatorname{fac} T'$, since $T' \notin \mathcal{T}^x_{\mathcal{C}_Q}$. Hence by Lemma 2.5, $\operatorname{ind} \operatorname{fac} T' \subseteq \operatorname{ind} \operatorname{fac} f(T)$, thus $T' \geq f(T)$. □

Corollary 2.7. *The map $f : \mathcal{T}^x_{\mathcal{C}_Q} \to \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q}$ is order preserving and*

$$\mathcal{T}_{\mathcal{C}_Q} \simeq \left(\mathcal{T}^x_{\mathcal{C}_Q} \sqcup (\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q}), \leq^f_+ \right)$$

Proof. If $T, T' \in \mathcal{T}^x_{\mathcal{C}_Q}$ are such that $T' \geq T$, then by Lemma 2.4, $f(T') > T' \geq T$, hence by Corollary 2.6, $f(T') \geq f(T)$, therefore f is order preserving. The other assertion follows from Corollaries 2.3, 2.6 and Lemma 2.4. □

3. Cluster tilting objects containing $P_x[1]$

For $M \in \operatorname{rep} Q$ and $y \in Q$, let $M(y)$ denote the vector space corresponding to y, and let $\operatorname{supp} M = \{y \in Q : M(y) \neq 0\}$ be the *support* of M.

Let $x \in Q$ be a vertex, and denote by $\mathcal{T}_{\mathcal{C}_Q}^{x[1]}$ the subset of cluster tilting objects containing the shifted indecomposable projective $P_x[1]$ as direct summand.

Lemma 3.1. *If $T \in \mathcal{T}_{\mathcal{C}_Q}^{x[1]}$ and $T' \geq T$, then $T' \in \mathcal{T}_{\mathcal{C}_Q}^{x[1]}$.*

Proof. Since T contains $P_x[1]$ as summand, we have $\operatorname{Ext}^1_{\mathcal{C}_Q}(P_x[1], T) = 0$, that is, $\operatorname{Hom}_Q(P_x, \widehat{T}) = 0$, or equivalently $x \notin \operatorname{supp} \widehat{T}$.

Now let $T' \geq T$. Then all the modules in $\operatorname{fac} T' \subseteq \operatorname{fac} T$ are not supported on x, and in particular $\operatorname{Hom}_Q(P_x, \widehat{T'}) = 0$, thus $\operatorname{Ext}^1_{\mathcal{C}_Q}(P_x[1], T') = 0$. The maximality of T' implies that it contains $P_x[1]$ as summand. □

Similarly to the previous section, define a map $g : \mathcal{T}_{\mathcal{C}_Q}^{x[1]} \to \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}_{\mathcal{C}_Q}^{x[1]}$ as follows. Given $T \in \mathcal{T}_{\mathcal{C}_Q}^{x[1]}$, write $T = P_x[1] \oplus U$ and set $g(T) = M \oplus U$ where M is the unique other indecomposable complement of U such that $M \oplus U$ is a cluster tilting object.

Lemma 3.2. *Let $T \in \mathcal{T}_{\mathcal{C}_Q}^{x[1]}$. Then $g(T) < T$.*

Proof. This is obvious. Indeed, write $T = P_x[1] \oplus U$ and $g(T) = M \oplus U$. Then $\operatorname{fac} g(T) = \operatorname{fac}(M \oplus U) \supsetneq \operatorname{fac} U = \operatorname{fac} T$. □

For the rest of this section, *we assume that the vertex x is a source.* In this case, for any module $M \in \operatorname{rep} Q$, we have that $S_x \in \operatorname{fac} M$ if and only if M is supported at x. Therefore we deduce the following lemma, which can be viewed as an analogue of Corollary 2.2.

Lemma 3.3. *Let $T \in \mathcal{T}_{\mathcal{C}_Q}$. Then $T \in \mathcal{T}_{\mathcal{C}_Q}^{x[1]}$ if and only if $S_x \notin \operatorname{fac} T$.*

Recall that a basic module $U \in \operatorname{rep} Q$ is an *almost complete tilting module* if $\operatorname{Ext}^1_Q(U, U) = 0$ and the number of indecomposable summands of U equals the number of vertices of Q less one. A *complement* to U is an indecomposable M such that $M \oplus U$ is a tilting module. It is known [6] that an almost complete tilting module U has at most two complements, and exactly two if and only if U is sincere, that is, $\operatorname{supp} U = Q$.

Proposition 3.4. *Let U be an almost complete tilting module of* $\operatorname{rep} Q$ *not supported on x, and let M be its unique indecomposable complement to a tilting module. Let $\mathcal{X}$ be a torsion class in* $\operatorname{rep} Q$ *satisfying* $\operatorname{fac} U \subseteq \mathcal{X}$ *and $S_x \in \mathcal{X}$. Then $M \in \mathcal{X}$.*

Proof. The natural inclusion $j : Q \setminus \{x\} \to Q$ induces a pair $(j_!, j^{-1})$ of exact functors

$$j^{-1} : \operatorname{rep} Q \to \operatorname{rep}(Q \setminus \{x\}) \qquad j_! : \operatorname{rep}(Q \setminus \{x\}) \to \operatorname{rep} Q$$

where j^{-1} is the natural restriction and $j_!$ is its left adjoint, defined as the extension of a representation of $Q \setminus \{x\}$ by zero at x.

Now $j_! j^{-1} U \simeq U$ since U is not supported on x. By adjunction and exactness,

$$\operatorname{Ext}^1_{Q\setminus\{x\}}(j^{-1}U, j^{-1}U) \simeq \operatorname{Ext}^1_Q(j_! j^{-1}U, U) = \operatorname{Ext}^1_Q(U, U)$$

thus $j^{-1}U$ is a (basic) tilting module of $\operatorname{rep}(Q \setminus \{x\})$. However, by [10, Proposition 2.6], $j^{-1}(M \oplus U)$ is also a tilting module of $\operatorname{rep}(Q \setminus \{x\})$, but not necessarily basic. It follows that $j^{-1}M \in \operatorname{add} j^{-1}U$, hence $j_! j^{-1}M \in \operatorname{add} j_! j^{-1}U = \operatorname{add} U$.

The adjunction morphism $j_! j^{-1}M \to M$ is injective, and we have an exact sequence

$$0 \to j_! j^{-1}M \to M \to S_x^n \to 0$$

for some $n \geq 0$. Now $S_x \in \mathcal{X}$ by assumption and $j_! j^{-1}M \in \operatorname{add} U \subseteq \mathcal{X}$, hence $M \in \mathcal{X}$ as $\mathcal{X}$ is closed under extensions. □

Corollary 3.5. *Let $T \in \mathcal{T}^{x[1]}_{\mathcal{C}_Q}$ and $T' \in \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_Q}$ be such that $T' < T$. Then $T' \leq g(T)$.*

Proof. Write $T = P_x[1] \oplus U$ and $g(T) = M \oplus U$. The assumptions on T' imply that $S_x \in \operatorname{fac} T'$ and $\operatorname{fac} U = \operatorname{fac} T \subseteq \operatorname{fac} T'$.

By deleting the vertices of Q corresponding to the shifted projective summands of U, we may and will assume that $M \oplus U$ is a tilting module, so that U is an almost complete tilting module. Applying Proposition 3.4 for $\mathcal{X} = \operatorname{fac} T'$, we deduce that $M \in \operatorname{fac} T'$, hence $\operatorname{fac} g(T) = \operatorname{fac}(M \oplus U) \subseteq \operatorname{fac} T'$. □

Corollary 3.6. *The map $g : \mathcal{T}^{x[1]}_{\mathcal{C}_Q} \to \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_Q}$ is order preserving and*

$$\mathcal{T}_{\mathcal{C}_Q} \simeq \left(\mathcal{T}^{x[1]}_{\mathcal{C}_Q} \sqcup (\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_Q}), \leq^g_-\right)$$

Proof. The claim follows from Lemmas 3.1, 3.2 and Corollary 3.5 as in the proof of Corollary 2.7. □

4. The effect of a BGP reflection

Let Q be a quiver without oriented cycles and let x be a sink. Let $y_1, \dots, y_m$ be the endpoints of the arrows ending at x, and denote by Q' the quiver obtained from Q by reflection at x. For a vertex $y \in Q$, denote by S_y, S'_y the simple modules corresponding to y in $\operatorname{rep} Q$, $\operatorname{rep} Q'$ and by P_y, P'_y their projective covers.

The categories $\operatorname{rep} Q$ and $\operatorname{rep} Q'$ are related by the BGP reflection functors, introduced in [1]. We recollect here the basic facts on these functors that will be needed in the sequel.

The BGP reflection functors are the functors

$$F^+ : \operatorname{rep} Q \to \operatorname{rep} Q' \qquad\qquad F^- : \operatorname{rep} Q' \to \operatorname{rep} Q$$

defined by

$$(F^+M)(x) = \ker\Big(\bigoplus_{i=1}^m M(y_i) \to M(x)\Big) \qquad (F^+M)(y) = M(y) \tag{4.1}$$

$$(F^-M')(x) = \operatorname{coker}\Big(M'(x) \to \bigoplus_{i=1}^m M'(y_i)\Big) \qquad (F^-M')(y) = M'(y)$$

for $M \in \operatorname{rep} Q$, $M' \in \operatorname{rep} Q'$ and $y \in Q \setminus \{x\}$, where the maps $(F^+M)(x) \to (F^+M)(y_i)$ and $(F^-M)(y_i) \to (F^-M)(x)$ are induced by the natural projection and inclusion.

It is clear that F^+ is left exact and F^- is right exact. The classical right derived functor of F^+ takes the form

$$(R^1F^+M)(x) = \operatorname{coker}\Big(\bigoplus_{i=1}^{m} M(y_i) \to M(x)\Big) \qquad (R^1F^+M)(y) = 0 \tag{4.2}$$

hence R^1F^+ vanishes for modules not containing S_x as direct summand.

The total derived functors

$$RF^+ : \mathcal{D}^b(Q) \to \mathcal{D}^b(Q') \qquad\qquad LF^- : \mathcal{D}^b(Q') \to \mathcal{D}^b(Q)$$

are triangulated equivalences, and their effect on the corresponding cluster categories has been analyzed in [13], where it is shown that RF^+ induces a triangulated equivalence $\mathcal{C}_Q \xrightarrow{\simeq} \mathcal{C}_{Q'}$ whose action on the indecomposables of $\mathcal{C}_Q$ is given by

$$S_x \mapsto P'_x[1] \qquad M \mapsto F^+M \qquad P_x[1] \mapsto S'_x \qquad P_y[1] \mapsto P'_y[1] \tag{4.3}$$

with an inverse given by

$$S'_x \mapsto P_x[1] \qquad M' \mapsto F^-M' \qquad P'_x[1] \mapsto S_x \qquad P'_y[1] \mapsto P_y[1] \tag{4.4}$$

for $M \neq S_x$, $M' \neq S'_x$ and $y \in Q \setminus \{x\}$. Moreover, this equivalence induces a bijection $\rho : \mathcal{T}_{\mathcal{C}_Q} \to \mathcal{T}_{\mathcal{C}_{Q'}}$ preserving the mutation graph [13, Proposition 3.2].

Lemma 4.1. *Let $T, T' \in \mathcal{T}_{\mathcal{C}_Q}$. If $\rho(T) \leq \rho(T')$, then $T \leq T'$.*

Proof. By (4.4), $\operatorname{fac} T = \operatorname{fac} F^-\widehat{\rho(T)}$ if $P'_x[1]$ is not a summand of $\rho(T)$, and $\operatorname{fac} T = \operatorname{fac}(S_x \oplus F^-\widehat{\rho(T)})$ if $P'_x[1]$ is a summand of $\rho(T)$. Note that by Lemma 3.1, the latter case implies that $P'_x[1]$ is also a summand of $\rho(T')$, hence in any case it is enough to verify that if $M, N \in \operatorname{rep} Q'$ satisfy $\operatorname{fac} N \subseteq \operatorname{fac} M$, then $\operatorname{fac} F^-N \subseteq \operatorname{fac} F^-M$. Indeed, since F^- is right exact, it takes an exact sequence $M^n \to N \to 0$ to an exact sequence $(F^-M)^n \to F^-N \to 0$. $\square$

Proposition 4.2. *ρ induces an isomorphism of posets $\mathcal{T}^x_{\mathcal{C}_Q} \xrightarrow{\simeq} \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}}$.*

Proof. Note that by (4.3), $\rho(\mathcal{T}^x_{\mathcal{C}_Q}) = \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}}$. In view of Lemma 4.1, it remains to show that if $T, T' \in \mathcal{T}^x_{\mathcal{C}_Q}$ satisfy $T \leq T'$, then $\rho(T) \leq \rho(T')$.

Write $\widehat{T} = S_x \oplus U$ and $\widehat{T'} = S_x \oplus U'$. Then $\operatorname{fac} \rho(T) = \operatorname{fac} F^+U$ and $\operatorname{fac} \rho(T') = \operatorname{fac} F^+U'$, and we need to show that $F^+U' \in \operatorname{fac} F^+U$.

Indeed, since $\operatorname{fac} T' \subseteq \operatorname{fac} T$, the proof of Lemma 2.5 shows that $U' \in \operatorname{fac} U$, hence there exists a short exact sequence

$$0 \to K \to U^n \xrightarrow{\varphi} U' \to 0$$

for some $n > 0$ and $K \in \operatorname{rep} Q$. Applying $\operatorname{Hom}_Q(-, S_x)$ to this sequence, noting that $\operatorname{Ext}^1_Q(U', S_x) = 0$ since $T' \in \mathcal{T}^x_{\mathcal{C}_Q}$, we get that $\operatorname{Hom}_Q(U^n, S_x) \to \operatorname{Hom}_Q(K, S_x)$ is surjective, hence K does not contain S_x as summand (otherwise U^n would contain S_x as summand). Therefore the exact sequence

$$F^+U^n \to F^+U' \to R^1F^+K = 0$$

shows that $F^+U' \in \operatorname{fac} F^+U$. $\square$

Proposition 4.3. *ρ induces an isomorphism of posets $\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q} \xrightarrow{\simeq} \mathcal{T}_{\mathcal{C}_{Q'}} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}}$.*

Proof. For a representation $M \in \operatorname{rep} Q$, let Q_M and Q'_M be the subquivers of Q and Q' obtained by deleting the vertices outside $\operatorname{supp} M \cup \{x\}$. The quivers Q'_M and Q_M are related via a BGP reflection at x, and we denote by $F^+_{Q_M} : \operatorname{rep} Q_M \to \operatorname{rep} Q'_M$ the corresponding reflection functor. The restriction functors $i^{-1} : \operatorname{rep} Q \to \operatorname{rep} Q_M$ and $j^{-1} : \operatorname{rep} Q' \to \operatorname{rep} Q'_M$ induced by the natural embeddings $i : Q_M \to Q$ and $j : Q'_M \to Q'$ satisfy

$$j^{-1}F^+M = F^+_{Q_M} i^{-1}M,$$

as can be easily verified using (4.1).

As in the proof of Proposition 4.2, it is enough to show that if $T, T' \in \mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q}$ satisfy $T \le T'$, then $\rho(T) \le \rho(T')$. In view of the preceding paragraph, we may assume that $Q = \operatorname{supp} \widehat{T} \cup \{x\}$.

We consider two cases. First, assume that $x \in \operatorname{supp} \widehat{T}$. Then $T = \widehat{T}$ is a tilting module, $\rho(T) = F^+T$ and $\operatorname{fac} \rho(T') = \operatorname{fac} F^+\widehat{T'}$ or $\operatorname{fac} \rho(T') = \operatorname{fac}(S'_x \oplus F^+\widehat{T'})$ according to whether $x \in \operatorname{supp} \widehat{T'}$ or not, hence it is enough to show that $S'_x \oplus F^+\widehat{T'} \in \operatorname{fac} F^+T$.

By assumption, $\widehat{T'} \in \operatorname{fac} T = T^{\perp}$. Since T does not contain S_x as summand, F^+T is a tilting module and $F^+\widehat{T'} \in (F^+T)^{\perp} = \operatorname{fac} F^+T$ [10, Corollary 4.3]. Moreover, $S'_x \in \operatorname{fac} F^+T$, as F^+T is sincere.

For the second case, assume that $x \notin \operatorname{supp} \widehat{T}$. Then $T = P_x[1] \oplus \widehat{T}$ and by Lemma 3.1, $T' = P_x[1] \oplus \widehat{T'} \oplus P[1]$ where P is a sum of projectives other than P_x. By (4.3), $\rho(T) = S'_x \oplus F^+\widehat{T}$ and $\rho(T') = S'_x \oplus F^+\widehat{T'} \oplus P'[1]$, hence it is enough to show that $F^+\widehat{T'} \in \operatorname{fac}(S'_x \oplus F^+\widehat{T})$.

Indeed, since $\widehat{T'} \in \operatorname{fac} \widehat{T}$, there exists a short exact sequence

$$0 \to K \to \widehat{T}^n \to \widehat{T'} \to 0$$

for some $n > 0$ and $K \in \operatorname{rep} Q$. Applying the functor F^+, noting that $\widehat{T}$ does not contain S_x as summand, we get

$$0 \to F^+K \to F^+\widehat{T}^n \to F^+\widehat{T'} \to R^1F^+K \to R^1F^+\widehat{T}^n = 0.$$

By (4.2), $R^1F^+K = {S'_x}^{n'}$ for some $n' \ge 0$, hence $F^+\widehat{T'}$ is an extension of ${S'_x}^{n'}$ with a quotient of $F^+\widehat{T}^n$. The result now follows, as $\operatorname{fac}(S'_x \oplus F^+\widehat{T})$ is closed under extensions. $\square$

Corollary 4.4. *We have a commutative diagram*

$$\begin{tikzcd}
\mathcal{T}^x_{\mathcal{C}_Q} \arrow[r, "\rho", "\simeq"'] \arrow[d, "f"] & \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}} \arrow[d, "g"] \\
\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^x_{\mathcal{C}_Q} \arrow[r, "\rho", "\simeq"'] & \mathcal{T}_{\mathcal{C}_{Q'}} \setminus \mathcal{T}^{x[1]}_{\mathcal{C}_{Q'}}
\end{tikzcd}$$

Proof. By Propositions 4.2 and 4.3, ρ induces the two horizontal isomorphisms. For $T \in \mathcal{T}^x_{\mathcal{C}_Q}$, $f(T)$ is defined as the mutation of T with respect to S_x and $g(\rho(T))$ is defined as the mutation of $\rho(T)$ with respect to $P'_x[1]$, which is, by (4.3), the image of S_x under the triangulated equivalence $\mathcal{C}_Q \to \mathcal{C}_{Q'}$. Therefore the commutativity of the diagram follows by the fact that ρ preserves the mutation graph [13, Proposition 3.2]. □

Theorem 4.5. *The posets $\mathcal{T}_{\mathcal{C}_Q}$ and $\mathcal{T}_{\mathcal{C}_{Q'}}$ are related via a flip-flop.*

Proof. Use Corollaries 2.7, 3.6 and 4.4. □

5. Example

Consider the following two quivers Q and Q' whose underlying graph is the Dynkin diagram A_3. The quiver Q' is obtained from Q by a BGP reflection at the sink 3.

$$Q: \bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \qquad\qquad Q': \bullet_1 \longrightarrow \bullet_2 \longleftarrow \bullet_3$$

We denote the indecomposables of the cluster categories $\mathcal{C}_Q$ and $\mathcal{C}_{Q'}$ by specifying their dimension vectors. These consist of the positive roots of A_3, which correspond to the indecomposable representations of the quivers, together with the negative simple roots $-e_1, -e_2, -e_3$ which correspond to the shifted projectives.

Figure 1 shows the Hasse diagrams of the posets $\mathcal{T}_{\mathcal{C}_Q}$ and $\mathcal{T}_{\mathcal{C}_{Q'}}$, where we used bold font to indicate the subsets $\mathcal{T}^3_{\mathcal{C}_Q}$ and $\mathcal{T}^{3[1]}_{\mathcal{C}_{Q'}}$ of cluster tilting objects containing the simple S_3 and the shifted projective $P'_3[1]$ as summand, respectively.

The posets $\mathcal{T}_{\mathcal{C}_Q}$ and $\mathcal{T}_{\mathcal{C}_{Q'}}$ are Cambrian lattices, and can be realized as sublattices of the weak order on the group of permutations on 4 letters, see [11, Section 6]. Moreover, the underlying graph of their Hasse diagrams is the 1-skeleton of the three-dimensional Stasheff associhedron. $\mathcal{T}_{\mathcal{C}_Q}$ is a Tamari lattice, corresponding to the linear orientation on A_3.

The BGP reflection at the vertex 3, whose action on the dimension vectors is given by

$$v \mapsto \begin{cases} v & \text{if } v \in \{-e_1, -e_2\} \\ s_3(v) & \text{otherwise} \end{cases}$$

where s_3 is the linear transformation specified by

$$s_3(v) = v \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix},$$

induces isomorphisms $\mathcal{T}^3_{\mathcal{C}_Q} \xrightarrow{\simeq} \mathcal{T}^{3[1]}_{\mathcal{C}_{Q'}}$ and $\mathcal{T}_{\mathcal{C}_Q} \setminus \mathcal{T}^3_{\mathcal{C}_Q} \xrightarrow{\simeq} \mathcal{T}_{\mathcal{C}_{Q'}} \setminus \mathcal{T}^{3[1]}_{\mathcal{C}_{Q'}}$ compatible with the mutations at S_3 and $P'_3[1]$.

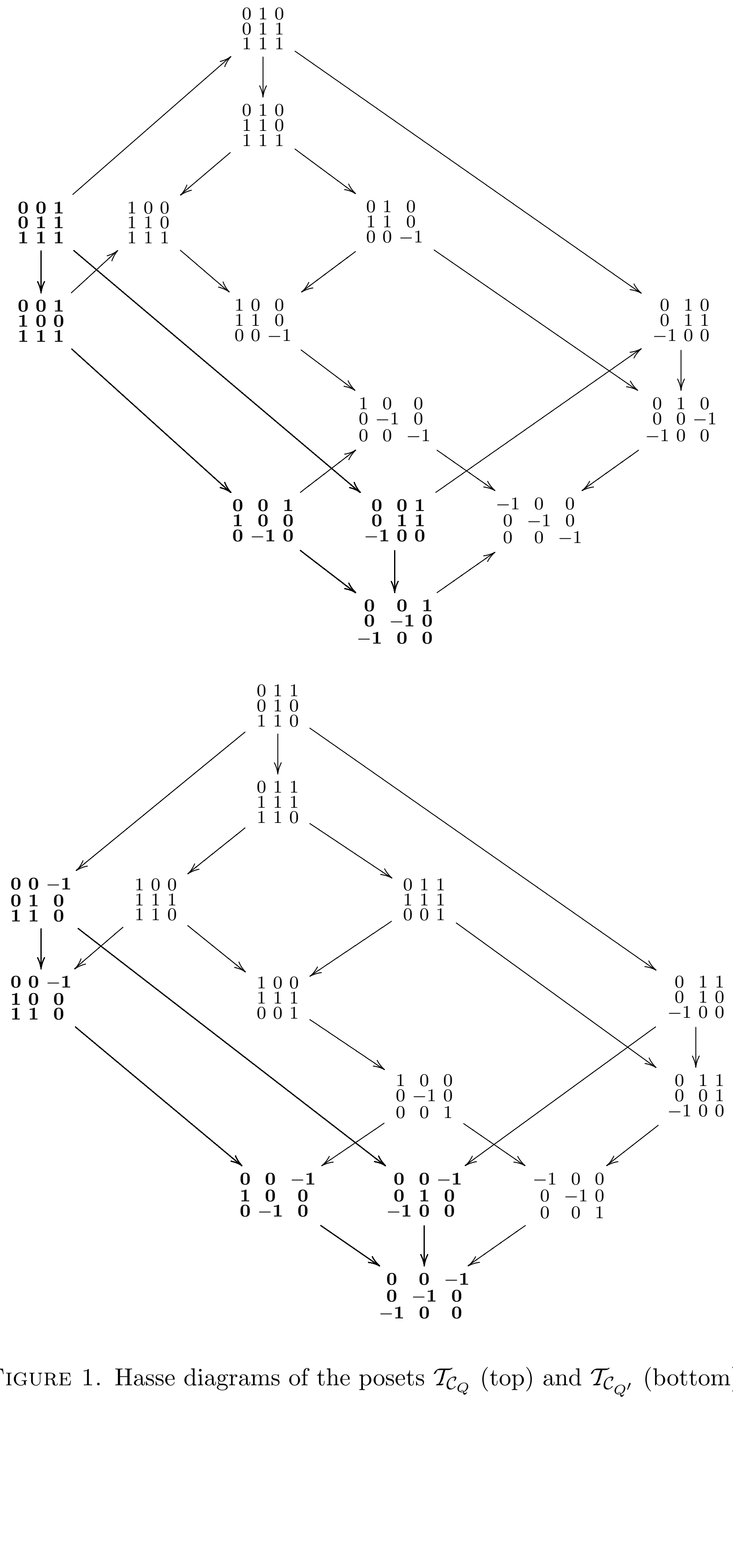

FIGURE 1. Hasse diagrams of the posets $\mathcal{T}_{\mathcal{C}_Q}$ (top) and $\mathcal{T}_{\mathcal{C}_{Q'}}$ (bottom).

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