PERVERSE EQUIVALENCES, BB-TILTING, MUTATIONS AND APPLICATIONS

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Abstract. We relate the notions of BB-tilting and perverse derived equivalence at a vertex. Based on these notions, we define mutations of algebras, leading to derived equivalent ones.

We present applications to endomorphism algebras of cluster-tilting objects in 2-Calabi-Yau categories and to algebras of global dimension at most 2.

Introduction

In [12, §2], Brenner and Butler introduced a construction of tilting modules, known as BB-tilting modules, generalizing APR-tilts [6], which are themselves generalizations of the BGP reflection functors introduced in [11]. Roughly speaking, for a finite dimensional algebra A over a field, the associated BB-tilting modules are parameterized by the simple A-modules satisfying certain homological conditions (to be recalled in Section 1.1 below).

On the other hand, Chuang and Rouquier introduced the notion of perverse Morita equivalences, which are certain derived equivalences that are controlled by filtrations $\mathcal{S} = (\phi = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_r)$ of the set of isomorphism classes of simple A-modules together with perversity functions $p : \{1, \dots, r\} \to \mathbb{Z}$, see [42, §2.6].

In this paper we focus on the special case where the filtration $\mathcal{S}$ has only two levels and $|p(2) - p(1)| = 1$. This case leads to complexes of A-modules consisting of only two projective terms concentrated in consecutive degrees. Such complexes have already been discussed in the literature, see for example [24] and the discussion of the associated torsion theory in [25].

Many of the constructions of tilting complexes introduced in the literature to show derived equivalences for various kinds of algebras are in fact such perverse equivalences. Examples are [37, Theorem 11.5.1] for symmetric algebras; [7, Theorems 29, 32] for gentle algebras; [8], [10, §3.1] and [38, Theorems 3.12, 3.15] for cluster-tilted algebras; and [44] for certain Jacobian algebras.

We start in Sections 1 and 2 by reviewing the relations between these two notions of tilting for a finite dimensional algebra A given by a quiver with relations. To any vertex k without loops in the quiver, there are two complexes T_k^- and T_k^+ attached. We list equivalent formulations, in homological as well as in combinatorial language, that these complexes are tilting

Date: January 26, 2010.
This work was supported by a European Postdoctoral Institute (EPDI) fellowship.

complexes, hence leading to perverse equivalences with respect to the two-level filtration $\mathcal{S}$ with $\mathcal{S}_1 = \{k\}$. Furthermore, we show that BB-tilting can always be regarded as a perverse equivalence induced by a complex T_k^-, and conversely, under certain additional assumptions, the perverse equivalence given by T_k^- is given by BB-tilting.

This leads us to define, in Section 3, mutations of finite dimensional algebras over an algebraically closed field, leading to derived equivalent ones. To any vertex k without loops in the quiver of such an algebra A, the negative mutation $\mu_k^-(A)$ is defined as the endomorphism algebra of T_k^- when it is a tilting complex. Similarly, the positive mutation $\mu_k^+(A)$ is defined as the endomorphism algebra of T_k^+ when it is a tilting complex. Thus, a vertex k leads to at most two mutations of A, and they are derived equivalent to A. In addition, we define the BB-mutation $\mu_k^{\mathrm{BB}}(A)$ as the endomorphism algebra of the BB-tilting module associated with k, when it is defined. Hence, when $\mu_k^{\mathrm{BB}}(A)$ is defined, so is $\mu_k^-(A)$ and they coincide. Analog notions for Calabi-Yau algebras have been introduced in [30].

From a K-theoretical viewpoint, we also show that the change-of-basis transformation of the Grothendieck groups induced by a mutation of an algebra coincides with that occurring in the matrix mutation, in the sense of Fomin and Zelevinsky [21], of the skew-symmetric matrix corresponding to the quiver of the algebra, provided that it has no loops and 2-cycles.

Mutations of algebras arise naturally when considering endomorphism algebras of objects related by approximation sequences in additive categories. This has been essentially observed by Hu and Xi in [26]. In Section 4 we show that in an additive Hom-finite category over an algebraically closed field with split idempotents, if Λ and Λ' are the endomorphism algebras of two basic objects related by replacing the indecomposable k-th summand by another through an approximation sequence, then the mere existence of the two BB-mutations $\mu_k^{\mathrm{BB}}(\Lambda)$ and $\mu_k^{\mathrm{BB}}(\Lambda'^{op})$ automatically implies that they take the "correct" values, namely Λ' and Λ^{op}, respectively.

Such approximation sequences, known as exchange sequences, appear in relation with mutations of cluster-tilting objects in 2-Calabi-Yau (2-CY) categories [13, 15, 22, 31], studied in connection with representation theoretical interpretation of cluster algebras [21]. In Section 5 we apply the results of the previous section to the study of endomorphism algebras of cluster-tilting objects in Hom-finite idempotent split Frobenius stably 2-CY categories, as well as in such triangulated 2-CY categories, where these algebras are known as 2-CY-tilted algebras. The latter algebras include the cluster-tilted algebras [17] and more generally [1, 14] finite dimensional Jacobian algebras of quivers with potentials [20].

We generalize a result in [22] and show that in the Frobenius case, the endomorphism algebra of a cluster-tilting object U admits all the BB-, negative and positive mutations at any vertex corresponding to a non projective-injective summand of U, and moreover all these mutations coincide with the endomorphism algebra of the mutation of U at that summand.

In the triangulated case, the picture is more complicated, and mutation of cluster-tilting objects does not always lead to a mutation of their endomorphism algebras. Indeed, neighboring two 2-CY-tilted algebras, that is, endomorphism algebras of two cluster-tilting objects related by a mutation, are always near-Morita equivalent [17, 34, 40], but not necessarily derived equivalent. However, several derived equivalence classifications of cluster-tilted algebras [8, 10, 18] have revealed that there are far less derived equivalence classes than isomorphism classes of such algebras (at least when their number is finite). Moreover, these classifications rely on showing that sufficiently many pairs of neighboring algebras are in fact derived equivalent.

In an attempt to provide a conceptual explanation of these facts, we study the conditions that two neighboring 2-CY-tilted algebras Λ and Λ' are related by BB-mutation. Following [10], we call in this case the corresponding mutation of the cluster-tilting objects a good mutation, since it leads to a mutation of their endomorphism algebras.

Obviously, a necessary condition for $\Lambda' = \mu_k^{\mathrm{BB}}(\Lambda)$ is that $\mu_k^{\mathrm{BB}}(\Lambda)$, hence by symmetry also $\mu_k^{\mathrm{BB}}(\Lambda'^{op})$, are defined. We show that this condition is also sufficient, that is, the existence of these two BB-mutations automatically implies that they take the "correct" values, namely Λ' and Λ^{op}, respectively, yielding a good mutation between Λ and Λ'.

We present several applications of this result. Firstly, by combining it with [16], we deduce an efficient algorithm that determines whether two neighboring cluster-tilted algebras of any Dynkin type (given by their quivers) are related by a BB-tilt. Secondly and more generally, building on their Gorenstein property [34], we give a numerical criterion for the derived equivalence via BB-mutation of neighboring 2-CY-tilted algebras, stated only in terms of their Cartan matrices, provided that they are invertible over $\mathbb{Q}$.

Another interesting class of algebras for which mutation can be related to the quiver mutation of [21] consists of the algebras of global dimension at most 2. Following [3, 32], we associate to such an algebra A a quiver $\widetilde{Q}_A$ called the extended quiver, and show that when it has no loops and 2-cycles, then the extended quiver of a mutation $\mu_k^-(A)$ or $\mu_k^+(A)$ that has global dimension at most 2 is obtained from $\widetilde{Q}_A$ by mutation at k.

We also provide an interpretation in terms of the generalized cluster category $\mathcal{C}_A$ of A introduced in [1]. Namely, when $\mathcal{C}_A$ is Hom-finite, then the image in $\mathcal{C}_A$ of any of the complexes T_k^- and T_k^+ equals the mutation at k of the canonical cluster-tilting object in $\mathcal{C}_A$, if the corresponding complex is tilting with endomorphism algebra of global dimension at most 2.

1. Preliminaries

Let K be a field and let $A = KQ/I$ be a finite dimensional algebra over K given as a quotient of the path algebra of a finite quiver Q by an admissible ideal I. For an arrow α in Q, let $s(\alpha)$ and $t(\alpha)$ denote its start and end vertices. Our convention is that two arrows α, β can be composed (as $\alpha\beta$) if $t(\alpha) = s(\beta)$.

Let $\operatorname{mod} A$ be the category of finite dimensional right A-modules and denote by $\mathcal{D}^b(A)$ its bounded derived category. Let $D = \operatorname{Hom}_k(-, k)$ be the duality on k-vector spaces. For a vertex i of Q, denote by S_i the simple module corresponding to i and by P_i its projective cover, which is spanned by all (non-zero) paths starting at i. Thus an arrow α gives rise to a map $P_{t(\alpha)} \xrightarrow{\alpha} P_{s(\alpha)}$.

Throughout this section, we fix a vertex k of Q which has no loops, that is, $\operatorname{Ext}^1_A(S_k, S_k) = 0$.

1.1. **BB-tilting modules.** Let τ denote the Auslander-Reiten translation in $\operatorname{mod} A$.

Definition 1.1. We say that the BB-tilting module is *defined* at the vertex k if the A-module

$$T_k^{\mathrm{BB}} = \tau^{-1} S_k \oplus \big(\bigoplus_{i \neq k} P_i\big)$$

is a tilting module of projective dimension at most 1. In this case, T_k^{BB} is called the *BB-tilting module associated with* k.

It is shown in [43, Theorem 2.5] that the condition in Definition 1.1 is equivalent to the conditions that $\operatorname{Hom}_A(DA, S_k) = 0$ and $\operatorname{Ext}^1_A(S_k, S_k) = 0$. In particular, S_k is not injective, meaning that k is not a source. See also [12] for the original construction and the survey article [4, §2.8].

1.2. **Perverse equivalence at a vertex.** We describe the construction of [42, §2.6.2] (and its dual) for the filtration $\phi \subset \{k\} \subset \{1, \dots, n\}$ with perversities differing by 1.

Let k be a vertex of Q without loops, and consider the maps

$$P_k \xrightarrow{f} \bigoplus_{\alpha \,:\, t(\alpha)=k} P_{s(\alpha)}, \qquad\qquad \bigoplus_{\beta \,:\, s(\beta)=k} P_{t(\beta)} \xrightarrow{g} P_k$$

where $f = \bigoplus \alpha$ is induced by the arrows α ending at k and $g = \bigoplus \beta$ is induced by the arrows β starting at k. Note that f is a minimal left $\operatorname{add}(A/P_k)$-approximation of P_k and g is a minimal right $\operatorname{add}(A/P_k)$-approximation of P_k (these notions are defined later in Section 4).

Let L_k be the cone of f and R_k be the cone of g shifted one place to the right. In a shortened notation, these are the complexes

$$L_k = P_k \xrightarrow{f} \bigoplus_{j \to k} P_j, \qquad\qquad R_k = \bigoplus_{k \to j} P_j \xrightarrow{g} P_k$$

where P_k is in degree -1 in L_k and in degree 1 in R_k.

Define complexes T_k^- and T_k^+ of projective A-modules by

$$T_k^- = L_k \oplus \big(\bigoplus_{i \neq k} P_i\big), \qquad\qquad T_k^+ = R_k \oplus \big(\bigoplus_{i \neq k} P_i\big). \tag{1.1}$$

In other words, T_k^- and T_k^+ are obtained by replacing the summand P_k in the (trivial) tilting complex A by the complex L_k or R_k, respectively. When we want to stress the dependency of these complexes on the algebra A, we shall use the notation $T_k^-(A)$ and $T_k^+(A)$.

2. BB-tilting vs. perverse equivalence at a vertex

We keep the assumptions of the previous section. In particular, k is a vertex without loops.

2.1. **Conditions for tilting.** In general, T_k^- and T_k^+ need not be tilting complexes. Therefore we start by giving the necessary and sufficient conditions on T_k^- and T_k^+ to be tilting, see also [25], [44] and [30, Theorem 4.1].

Observe that the summands of each of the complexes T_k^- and T_k^+ always generate $\operatorname{per} A$, the triangulated subcategory of $\mathcal{D}^b(A)$ consisting of the perfect complexes (that is, bounded complexes of finitely generated projectives). This follows from the two short exact sequences of complexes

$$0 \to \bigoplus_{j \to k} P_j \to L_k \to P_k[1] \to 0, \qquad 0 \to P_k[-1] \to R_k \to \bigoplus_{k \to j} P_j \to 0,$$

recalling that none of the summands P_j equals P_k.

Proposition 2.1. *Let T_k^-, T_k^+ be the complexes defined in* (1.1)*. Then:*

(a) *T_k^- is a tilting complex if and only if*

$$\operatorname{Hom}_{\mathcal{D}^b(A)}(P_i, L_k[-1]) = 0$$

for all vertices $i \neq k$.

(b) *T_k^- is isomorphic in $\mathcal{D}^b(A)$ to a tilting module if and only if*

$$\operatorname{Hom}_{\mathcal{D}^b(A)}(P_i, L_k[-1]) = 0$$

for all vertices i.

(c) *T_k^+ is a tilting complex if and only if*

$$\operatorname{Hom}_{\mathcal{D}^b(A)}(R_k[1], P_i) = 0$$

for all vertices $i \neq k$.

(d) *T_k^+ is never isomorphic in $\mathcal{D}^b(A)$ to a tilting module.*

Proof. The conditions in (a) and (c) are obviously necessary. We show now claim (a). The proof of (c) is similar.

Since T_k^- is concentrated in only two consecutive degrees it is enough to verify that

$$\operatorname{Hom}_{\mathcal{D}^b(A)}(T_k^-, T_k^-[-1]) = 0 = \operatorname{Hom}_{\mathcal{D}^b(A)}(T_k^-, T_k^-[1]).$$

As we are dealing with complexes whose terms are projective, morphisms in the derived category can be computed as morphisms in the homotopy category of complexes. Obviously, the morphism spaces

$$\operatorname{Hom}(P_i, P_{i'}[1]), \quad \operatorname{Hom}(P_i, P_{i'}[-1]), \quad \operatorname{Hom}(P_i, L_k[1]), \quad \operatorname{Hom}(L_k, P_i[-1])$$

vanish for all $i, i' \neq k$.

By assumption, $\operatorname{Hom}_{\mathcal{D}^b(A)}(P_i, L_k[-1]) = 0$ for every $i \neq k$. It follows that $\operatorname{Hom}_{\mathcal{D}^b(A)}(L_k, L_k[-1]) = 0$ as well, by considering the right square of a commutative diagram

$$\begin{tikzcd} P_k \arrow[r, "f"] \arrow[d] & \bigoplus P_j \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & P_k \arrow[r, "f"] & \bigoplus P_j \end{tikzcd}$$

and using the assumption for each of the middle vertical maps $P_j \to P_k$, recalling that none of the P_j equals P_k.

In addition, $\mathrm{Hom}(L_k, P_i[1]) = 0$ for any $i \neq k$, as every path from i to k (which corresponds to a map $P_k \to P_i$) factorizes through one of the arrows ending at k, so that one can always define the diagonal dotted homotopy in the diagram

$$\begin{tikzcd} P_k \arrow[r, "f"] \arrow[d] & \bigoplus P_j \arrow[d] \arrow[dl, dotted] \\ P_i \arrow[r] & 0 \end{tikzcd}$$

This also shows that $\mathrm{Hom}(L_k, L_k[1]) = 0$, by considering the right square in a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & P_k \arrow[r, "f"] \arrow[d] & \bigoplus P_j \arrow[d] \arrow[dl, dotted] \\ P_k \arrow[r, "f"] & \bigoplus P_j \arrow[r] & 0 \end{tikzcd}$$

and applying the above argument for each of the middle vertical maps $P_k \to P_j$, recalling that none of the P_j equals P_k. The proof of (a) is thus complete.

For the proof of (b), note that $\mathrm{H}^{-1}(T_k^-) \simeq \mathrm{Hom}_{\mathcal{D}^b(A)}(A, L_k[-1])$. Finally, for (d) observe that the map $\bigoplus_{k \to j} P_j \to P_k$ can never be surjective, hence $\mathrm{H}^1(T_k^+) \neq 0$. $\square$

One can restate the conditions for T_k^- to be a tilting complex in terms of the kernel of the map f whose cone is L_k.

Corollary 2.2. *Let T_k^- be the complex of* (1.1). *Then:*

(a) *T_k^- is a tilting complex if and only if* $\ker f \simeq S_k^m$ *for some* $m \geq 0$.
(b) *T_k^- is isomorphic in $\mathcal{D}^b(A)$ to a tilting module if and only if f is a monomorphism.*

Proof. The claims follow from the isomorphisms

$$\mathrm{Hom}_{\mathcal{D}^b(A)}(P_i, L_k[-1]) \simeq \mathrm{Hom}_A(P_i, \ker f)$$

for any vertex i and the corresponding claims in Proposition 2.1. Note that our assumption that are no loops at k implies that any module whose composition factors consist only of S_k is of the form S_k^m for some $m \geq 0$. $\square$

The conditions in Proposition 2.1 can also be conveniently rephrased in terms of (non-vanishing of) paths, as follows. This is useful in practical calculations.

Proposition 2.3. *Let k be a vertex without loops and T_k^-, T_k^+ as in* (1.1).

(a) *T_k^- is a tilting complex if and only if for any non-zero linear combination $\sum a_r p_r$ of paths starting at k and ending at some vertex $i \neq k$, there exists at least one arrow α ending at k such that the composition $\sum a_r \alpha p_r$ is not zero.*
(b) *T_k^- is isomorphic in $\mathcal{D}^b(A)$ to a tilting module if and only if the condition of (a) holds for any vertex i (including k).*

(c) *T_k^+ is a tilting complex if and only if for any non-zero linear combination of paths $\sum a_r p_r$ starting at some vertex $i \neq k$ and ending at k, there exists at least one arrow β starting at k such that the composition $\sum a_r p_r \beta$ is not zero.*

Proof. We show only (a), as the proof of the other assertions is similar. By considering the diagram below,

$$\begin{tikzcd} P_i \arrow[r] \arrow[d] & 0 \arrow[d] \\ P_k \arrow[r, "f"] & \bigoplus_{j\to k} P_j \end{tikzcd}$$

we see that the condition $\mathrm{Hom}_{\mathcal{D}^b(A)}(P_i, L_k[-1]) = 0$ is equivalent to the condition that for any non-zero morphism $\rho : P_i \to P_k$, there exists an arrow $\alpha : P_k \to P_j$ such that the composition of ρ with α gives a non-zero map $P_i \to P_j$. □

Remark 2.4. From the proposition we immediately see that if k is a sink, then T_k^- is always a tilting complex while T_k^+ is never one, whereas if k is a source, then T_k^- is never a tilting complex while T_k^+ is always one.

Let Q^{op} be the opposite quiver of Q. Namely, it has the same set of vertices as Q, with an (opposite) arrow $\alpha^* : j \to i$ for any arrow $\alpha : i \to j$ of Q. If $A = KQ/I$, then the opposite algebra A^{op} can be written as $A^{op} = Q^{op}/I^{op}$ where I^{op} is generated by the paths opposite to those generating I. The simple and indecomposable projective A^{op}-modules corresponding to a vertex i of Q^{op} are then $D(S_i)$ and $P_i^* = \mathrm{Hom}_A(P_i, A)$, respectively.

From the criteria in Proposition 2.3, we immediately deduce the following.

Corollary 2.5. *$T_k^-(A)$ is a tilting complex for A if and only if $T_k^+(A^{op})$ is a tilting complex for A^{op}.*

We conclude this section by giving another simple criterion for the BB-tilting module to be defined at the vertex k. It will be used in Section 5.3.

Lemma 2.6. *Let k be a vertex without loops and let $\cdots \to R^{-1} \to R^0 \to 0 \to \dots$ be a minimal projective resolution of DA. Then T_k^{BB} is defined if and only if P_k does not occur as summand in R^0.*

Proof. Recall that T_k^{BB} is defined if and only if $\mathrm{Hom}_A(DA, S_k)$ vanishes. The claim now follows from the fact that for any module M, $\mathrm{Hom}_A(M, S_k) \simeq \mathrm{Hom}_A(P(M), S_k)$ where $P(M)$ is the projective cover of M. □

2.2. **BB-tilt as perverse equivalence at a vertex and vice versa.**

Lemma 2.7. *The sequence*

$$P_k \xrightarrow{f} \bigoplus_{j\to k} P_j \to \tau^{-1} S_k \to 0 \tag{2.1}$$

is a projective presentation of $\tau^{-1} S_k$.

Proof. Since $\tau^{-1} = \operatorname{Tr} D$, we start by writing a projective presentation of the simple A^{op}-module $D(S_k)$ as

$$\bigoplus_{k \to j} P_j^* \xrightarrow{f^*} P_k^* \to D(S_k) \to 0$$

where the sum goes over the arrows $\alpha^* : k \to j$ in Q^{op}, and apply Tr to get (2.1). □

The relations between BB-tilting and perverse equivalences at a vertex are summarized in the following proposition.

Proposition 2.8. *Let k be a vertex (without loops) of Q.*

(a) *If the BB-tilting module is defined at k, then T_k^- is a tilting complex and $T_k^{\mathrm{BB}} \simeq T_k^-$ in $\mathcal{D}^b(A)$.*
(b) *Conversely, if T_k^- is isomorphic in $\mathcal{D}^b(A)$ to a tilting module, then the BB-tilting module is defined at k and $T_k^- \simeq T_k^{\mathrm{BB}}$ in $\mathcal{D}^b(A)$.*

Proof. (a) The assumption implies that $\operatorname{pd}_A \tau^{-1} S_k \leq 1$. It follows that the projective presentation of (2.1) is actually a resolution. In other words, $\tau^{-1} S_k \simeq L_k$ in $\mathcal{D}^b(A)$. Hence $T_k^{\mathrm{BB}} \simeq T_k^-$.
(b) By Corollary 2.2, $\ker f = 0$, hence (2.1) is a projective resolution of $\tau^{-1} S_k$. Therefore $\operatorname{pd}_A \tau^{-1} S_k \leq 1$ and $\tau^{-1} S_k \simeq L_k$ in $\mathcal{D}^b(A)$. Hence $T_k^{\mathrm{BB}} \simeq T_k^-$ is a tilting module of projective dimension at most 1.

□

Under certain additional assumptions, any perverse equivalence given by a tilting complex T_k^- can be regarded as a BB-tilting module.

Lemma 2.9. *Assume that S_k is not a submodule of the radical of P_k. If T_k^- is a tilting complex, then the BB-tilting module is defined at k and $T_k^- \simeq T_k^{\mathrm{BB}}$ in $\mathcal{D}^b(A)$.*

Proof. By Corollary 2.2(a), $\ker f \simeq S_k^m$ for some $m \geq 0$. On the other hand, $\ker f \subseteq \operatorname{rad} P_k$. Therefore $\ker f = 0$ and the result follows from Corollary 2.2(b) and Proposition 2.8(b). □

Remark 2.10. When $\operatorname{End}_A(P_k) \simeq K$, the assumption of the lemma holds. This happens, in particular, when the quiver Q is acyclic (i.e. A is *triangular*), or when the algebra A is *schurian*, that is, $\dim_K \operatorname{Hom}_A(P_i, P_{i'}) \leq 1$ for any two vertices i, i' of Q.

Remark 2.11. When A is a symmetric algebra which is not semi-simple, T_k^- is always a tilting complex [42, §2.6], but the assumption of Lemma 2.9 does not hold. Observe that T_k^{BB} is never defined since A has no non-trivial tilting modules.

Lemma 2.12. *Assume that $\operatorname{pd}_A \tau^{-1} S_k \leq 2$. If T_k^- is a tilting complex, then the BB-tilting module is defined at k and $T_k^- \simeq T_k^{\mathrm{BB}}$ in $\mathcal{D}^b(A)$.*

Proof. By Corollary 2.2(a), the sequence (2.1) yields an exact sequence

$$0 \to S_k^m \to P_k \xrightarrow{f} \bigoplus_{j \to k} P_j \to \tau^{-1} S_k \to 0$$

for some $m \geq 0$. If $m > 0$, then by our assumption, S_k must be projective, hence $P_k = S_k$, so that f is a monomorphism, a contradiction. Thus $m = 0$ and the result follows. $\square$

3. Mutations of algebras

3.1. **Operations.** We assume now that the field K is algebraically closed. Let $A = KQ/I$ be a finite dimensional K-algebra given as a quiver Q with relations.

Definition 3.1. Let k be a vertex of Q without loops.

(a) We say that the negative mutation is *defined* at the vertex k if $T_k^-(A)$ is a tilting complex. In this case, we call

$$\mu_k^-(A) = \operatorname{End}_{\mathcal{D}^b(A)} T_k^-(A)$$

the *negative mutation of A at the vertex k.*

(b) We say that the positive mutation is *defined* at the vertex k if $T_k^+(A)$ is a tilting complex. In this case,

$$\mu_k^+(A) = \operatorname{End}_{\mathcal{D}^b(A)} T_k^+(A)$$

is called the *positive mutation of A at the vertex k.*

(c) We say that the BB-mutation is *defined* at the vertex k if $T_k^{\mathrm{BB}}(A)$ is defined. In this case, we call

$$\mu_k^{\mathrm{BB}}(A) = \operatorname{End}_A T_k^{\mathrm{BB}}(A)$$

the *BB-mutation of A at the vertex k.*

Obviously, when $\mu_k^{\mathrm{BB}}(A)$ is defined, so is $\mu_k^-(A)$, and moreover $\mu_k^{\mathrm{BB}}(A) \simeq \mu_k^-(A)$. The following proposition justifies the name "mutations" for these operations.

Proposition 3.2. *Let k be a vertex of Q without loops.*

(a) *If $\mu_k^-(A)$ is defined, then $\mu_k^+(\mu_k^-(A))$ is defined and isomorphic to A.*
(b) *If $\mu_k^+(A)$ is defined, then $\mu_k^-(\mu_k^+(A))$ is defined and isomorphic to A.*
(c) *If $\mu_k^{\mathrm{BB}}(A)$ is defined, then $\mu_k^{\mathrm{BB}}\big((\mu_k^{\mathrm{BB}}(A))^{op}\big)$ is defined and isomorphic to A^{op}.*

Given a vertex k, precisely one of the following situations can occur:

- Both $\mu_k^-(A)$ and $\mu_k^+(A)$ are defined; or
- One of them is defined; or
- None of them is defined.

When k is a sink or a source in Q, then exactly one of the mutations is defined, namely, $\mu_k^-(A)$ for a sink k and $\mu_k^+(A)$ for a source k.

Example 3.3. Let A be the path algebra of the quiver

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \ .$$

Using Proposition 2.3, we see that at the vertex 1, only $\mu_1^+(A)$ is defined, whereas at the vertex 3, only $\mu_3^-(A)$ is defined. At the vertex 2, both $\mu_2^-(A)$

and $\mu_2^+(A)$ are defined, and are given by the following quivers with zero relations:

$$\begin{tikzcd} & \bullet_2 \arrow[dl] \arrow[dr, dash, dotted] & \\ \bullet_1 \arrow[rr] & & \bullet_3 \end{tikzcd} \qquad \begin{tikzcd} & \bullet_2 \arrow[dl, dash, dotted] & \\ \bullet_1 \arrow[rr] & & \bullet_3 \arrow[ul] \end{tikzcd}$$

Let $A' = \mu_2^-(A)$ be as in the left picture. Then at the vertex 1, none of $\mu_1^-(A')$ and $\mu_1^+(A')$ is defined.

3.2. **K-theoretical interpretation.** Let Q be a quiver with n vertices. For a vertex $1 \leq k \leq n$, define the $n \times n$ matrices $r_k^- = r_k^-(Q)$ and $r_k^+ = r_k^+(Q)$ by

$$\left(r_k^-\right)_{ij} = \begin{cases} -\delta_{ij} + |\{\text{arrows } j \to k \text{ in } Q\}| & \text{if } i = k, \\ \delta_{ij} & \text{otherwise,} \end{cases}$$

and

$$\left(r_k^+\right)_{ij} = \begin{cases} -\delta_{ij} + |\{\text{arrows } k \to j \text{ in } Q\}| & \text{if } i = k, \\ \delta_{ij} & \text{otherwise.} \end{cases}$$

As already observed in [22, Lemma 7.1], these matrices are closely related to the Fomin-Zelevinsky matrix mutation [21] of the skew-symmetric matrix corresponding to Q. Namely, recall that for a quiver Q there is an associated skew-symmetric matrix b_Q defined by

$$(b_Q)_{ij} = \big|\{\text{arrows } j \to i\}\big| - \big|\{\text{arrows } i \to j\}\big|, \tag{3.1}$$

and one can recover Q from b_Q as long as it has no loops or 2-cycles.

Fomin and Zelevinsky have defined the mutation $\mu_k(b_Q)$, which is a again a skew-symmetric matrix, for any vertex k. Then $\mu_k(b_Q)$ is obtained from b_Q by viewing the latter as a matrix of a bilinear form and applying a change-of-basis transformation given by the matrix r_k^- or r_k^+. More precisely, the following lemma is a reformulation of [22, Lemma 7.1].

Lemma 3.4. *Assume that Q has no loops and no 2-cycles. Then for any vertex k, we have*

$$\mu_k(b_Q) = \left(r_k^-\right)^T b_Q r_k^- = \left(r_k^+\right)^T b_Q r_k^+.$$

Now we relate the matrices r_k^- and r_k^+ to mutations of algebras via the notion of the Euler form. Let $A = KQ/I$ be a finite dimensional K-algebra given as a quiver with relations and let n be the number of vertices of Q. The *Cartan matrix* of A is the $n \times n$ integral matrix C_A whose entries are

$$(C_A)_{ij} = \dim_K \operatorname{Hom}_A(P_i, P_j)$$

for $1 \leq i, j \leq n$.

Recall that the Grothendieck group $K_0(\operatorname{per} A)$ is free abelian on the generators $[P_1], \dots, [P_n]$, and the expression

$$\langle X, Y \rangle_A = \sum_{r \in \mathbb{Z}} (-1)^r \dim_K \operatorname{Hom}_{\mathcal{D}^b(A)}(X, Y[r]) \tag{3.2}$$

is well defined for any $X, Y \in \operatorname{per} A$ and induces a bilinear form on $K_0(\operatorname{per} A)$, known as the *Euler form*, whose matrix with respect to the basis of projectives is C_A.

The following lemma is briefly mentioned in [10].

Lemma 3.5. *Let T be a (basic) tilting complex in $\mathcal{D}^b(A)$ with endomorphism algebra $A' = \operatorname{End}_{\mathcal{D}^b(A)}(T)$ and let $T_1, \dots, T_n$ be the indecomposable summands of T. Then the Cartan matrix of A' is given by $C_{A'} = rC_Ar^T$, where $r = (r_{ij})_{i,j=1}^n$ is the matrix defined by*

$$[T_i] = \bigoplus_{j=1}^{n} r_{ij}[P_j]$$

(that is, its i-th row is the class of the summand T_i in $K_0(\operatorname{per} A)$ written in the basis $[P_1], \dots, [P_n]$).

Proof. As the indecomposable projectives of A' correspond to the indecomposable summands of T, we have

$$(C_{A'})_{ij} = \dim_K \operatorname{Hom}_{\mathcal{D}^b(A)}(T_i, T_j) = \langle [T_i], [T_j] \rangle_A$$

since T is a tilting complex. The result now follows. □

Proposition 3.6. *Let $A = KQ/I$ be a finite dimensional K-algebra.*

(a) *If $\mu_k^-(A)$ is defined, then $C_{\mu_k^-(A)} = r_k^- \cdot C_A \cdot \left(r_k^-\right)^T$.*
(b) *If $\mu_k^+(A)$ is defined, then $C_{\mu_k^+(A)} = r_k^+ \cdot C_A \cdot \left(r_k^+\right)^T$.*

Proof. Use Lemma 3.5 and the definition of $T_k^-(A)$ and $T_k^+(A)$ in (1.1). □

When the algebra A has finite global dimension, $\operatorname{per} A = \mathcal{D}^b(A)$ and the classes $[S_1], \dots, [S_n]$ form a basis of $K_0(\operatorname{per} A)$. The matrix of the Euler form (3.2) with respect to that basis is then given by $c_A = C_A^{-T}$, that is, the inverse of the transpose of C_A.

Corollary 3.7. *Assume that A has finite global dimension.*

(a) *If $\mu_k^-(A)$ is defined, then $c_{\mu_k^-(A)} = \left(r_k^-\right)^T \cdot c_A \cdot r_k^-$.*
(b) *If $\mu_k^+(A)$ is defined, then $c_{\mu_k^+(A)} = \left(r_k^+\right)^T \cdot c_A \cdot r_k^+$.*

Proof. By Proposition 3.6 we have

$$\begin{aligned} c_{\mu_k^-(A)} &= \left(C_{\mu_k^-(A)}\right)^{-T} = \left(r_k^- C_A \left(r_k^-\right)^T\right)^{-T} = \left(r_k^-\right)^{-T} c_A \left(r_k^-\right)^{-1} \\ &= \left(r_k^-\right)^T c_A r_k^- \end{aligned}$$

where for the last equality we used the fact that $\left(r_k^-\right)^2$ is the identity matrix. □

4. Mutations of endomorphism algebras

Mutations of algebras arise naturally when considering endomorphism algebras of objects related by approximation sequences in additive categories.

This has been essentially observed by Hu and Xi in [26], where they introduced the notion of almost $\mathcal{D}$-split sequences. We recall their result, formulating it in a form which will be convenient for our applications.

Let $\mathcal{C}$ be a category, $\mathcal{D}$ a full subcategory and X an object of $\mathcal{C}$. A morphism $f : X \to D$ is called a *left $\mathcal{D}$-approximation* if $D \in \mathcal{D}$ and any morphism $f' : X \to D'$ with $D' \in \mathcal{D}$ can be completed to a commutative diagram as in the left picture.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f'"'] & D \arrow[d, dotted] \\ & D' \end{tikzcd} \qquad\qquad \begin{tikzcd} D \arrow[r, "g"] & X \\ D' \arrow[u, dotted] \arrow[ur, "g'"'] & \end{tikzcd}$$

f is a *minimal left $\mathcal{D}$-approximation* if furthermore, when considering the left diagram with $f' = f$, any dotted arrow making it commutative is an automorphism of D. The notions of a *right $\mathcal{D}$-approximation* and a *minimal right $\mathcal{D}$-approximation* are defined similarly using the right diagram, see [5].

Let $\mathcal{C}$ be an additive category with split idempotents. For an object $M \in \mathcal{C}$, denote by $\operatorname{add} M$ the full subcategory consisting of finite direct sums of direct summands of M. It is equivalent to the additive category of finitely generated projective $\operatorname{End}_{\mathcal{C}}(M)$-modules, via the functor $\operatorname{Hom}_{\mathcal{C}}(M, -)$.

Proposition 4.1 (Lemma 3.4 of [26])**.** *Let $M \in \mathcal{C}$ and let*

$$X \xrightarrow{f} B \xrightarrow{g} X'$$

be a sequence of morphisms satisfying the following conditions:

(i) *$f : X \to B$ is a left $\operatorname{add} M$-approximation of X and $g : B \to X'$ is a right $\operatorname{add} M$-approximation of X',*

(ii) *The induced sequences*

$$0 \to \operatorname{Hom}_{\mathcal{C}}(U, X) \xrightarrow{f_*} \operatorname{Hom}_{\mathcal{C}}(U, B) \xrightarrow{g_*} \operatorname{Hom}_{\mathcal{C}}(U, X')$$

$$0 \to \operatorname{Hom}_{\mathcal{C}}(X', U') \xrightarrow{g^*} \operatorname{Hom}_{\mathcal{C}}(B, U') \xrightarrow{f^*} \operatorname{Hom}_{\mathcal{C}}(X, U')$$

are exact, where $U = X \oplus M$ and $U' = X' \oplus M$.

Then the rings $\Lambda = \operatorname{End}_{\mathcal{C}}(U)$ and $\Lambda' = \operatorname{End}_{\mathcal{C}}(U')$ are derived equivalent.

Moreover, examining the proof in [26], we see that the following complex of projective Λ-modules

$$\Big(\operatorname{Hom}_{\mathcal{C}}(U, X) \xrightarrow{f_*} \operatorname{Hom}_{\mathcal{C}}(U, B)\Big) \oplus \operatorname{Hom}_{\mathcal{C}}(U, M), \tag{4.1}$$

where $\operatorname{Hom}_{\mathcal{C}}(U, X)$ is in degree -1 and the other terms are in degree 0, is a tilting complex whose endomorphism ring is isomorphic to Λ'. The exactness of the first sequence implies that this complex is quasi-isomorphic to a tilting module.

The similarity of (4.1) and (1.1) suggests that one can replace some of the conditions on f_* and g^* by intrinsic conditions expressed only in terms of the algebras Λ and Λ'.

Theorem 4.2. *Let K be an algebraically closed field and $\mathcal{C}$ be a K-linear,* Hom*-finite, additive category with split idempotents. Let $U_1, U_2, \dots, U_{n-1}$ be non-isomorphic indecomposable objects of $\mathcal{C}$ and set $M = U_1 \oplus \cdots \oplus U_{n-1}$.*

Let U_n and U'_n be indecomposable objects of $\mathcal{C}$ not in $\operatorname{add} M$. *Set $U = M \oplus U_n$ and $U' = M \oplus U'_n$ and consider the endomorphism algebras*

$$\Lambda = \operatorname{End}_{\mathcal{C}}(U) \quad \text{and} \quad \Lambda' = \operatorname{End}_{\mathcal{C}}(U').$$

Number the vertices of their quivers in accordance with the numbering of the summands U_i. Assume that:

(i) *There exist morphisms*

$$f : U_n \to B \quad \text{and} \quad g : B \to U'_n$$

such that f is a minimal left $\operatorname{add} M$*-approximation and g is a minimal right* $\operatorname{add} M$*-approximation;*

(ii) *The induced sequences*

$$\operatorname{Hom}_{\mathcal{C}}(U, U_n) \xrightarrow{f_*} \operatorname{Hom}_{\mathcal{C}}(U, B) \xrightarrow{g_*} \operatorname{Hom}_{\mathcal{C}}(U, U'_n)$$

$$\operatorname{Hom}_{\mathcal{C}}(U'_n, U') \xrightarrow{g^*} \operatorname{Hom}_{\mathcal{C}}(B, U') \xrightarrow{f^*} \operatorname{Hom}_{\mathcal{C}}(U_n, U')$$

are exact.

Then the following are equivalent:

(a) *The BB-tilting modules $T_n^{\mathrm{BB}}(\Lambda)$ and $T_n^{\mathrm{BB}}(\Lambda'^{op})$ are defined (in particular, the quivers of Λ and Λ' have no loops at the vertex n).*

(b) *The induced maps f_* and g^* are monomorphisms.*

(c) *$\Lambda' \simeq \mu_n^{\mathrm{BB}}(\Lambda)$ (in particular, Λ and Λ' are derived equivalent).*

Proof. The indecomposable projectives in $\operatorname{mod} \Lambda$ are precisely the modules $\operatorname{Hom}_{\mathcal{C}}(U, U_i)$. The assumption that $f : U_n \to B$ is a minimal left $\operatorname{add} M$-approximation implies that $f_* : \operatorname{Hom}_{\mathcal{C}}(U, U_n) \to \operatorname{Hom}_{\mathcal{C}}(U, B)$ is a minimal left $\operatorname{add} \operatorname{Hom}_{\mathcal{C}}(U, M)$-approximation in $\operatorname{mod} \Lambda$, and therefore the complex $T_n^-(\Lambda)$ can be written as

$$T_n^-(\Lambda) \simeq \Big(\operatorname{Hom}_{\mathcal{C}}(U, U_n) \xrightarrow{f_*} \operatorname{Hom}_{\mathcal{C}}(U, B)\Big) \oplus \operatorname{Hom}_{\mathcal{C}}(U, M).$$

From Proposition 2.8 and Corollary 2.2(b) we deduce that f_* is a monomorphism if and only if $T_n^{\mathrm{BB}}(\Lambda)$ is defined.

Similarly, the indecomposable projectives in $\operatorname{mod} \Lambda'^{op}$ are the modules $\operatorname{Hom}_{\mathcal{C}}(U_i, U')$ for $i < n$ together with $\operatorname{Hom}_{\mathcal{C}}(U'_n, U')$. The assumption that $g : B \to U'_n$ is a minimal right $\operatorname{add} M$-approximation implies that $g^* : \operatorname{Hom}_{\mathcal{C}}(U'_n, U') \to \operatorname{Hom}_{\mathcal{C}}(B, U')$ is a minimal left $\operatorname{add} \operatorname{Hom}_{\mathcal{C}}(M, U')$-approximation in $\operatorname{mod} \Lambda'^{op}$, and therefore the complex $T_n^-(\Lambda'^{op})$ can be written as

$$T_n^-(\Lambda'^{op}) \simeq \Big(\operatorname{Hom}_{\mathcal{C}}(U'_n, U') \xrightarrow{g^*} \operatorname{Hom}_{\mathcal{C}}(B, U')\Big) \oplus \operatorname{Hom}_{\mathcal{C}}(M, U').$$

A similar reasoning to the above shows that g^* is a monomorphism if and only if $T_n^{\mathrm{BB}}(\Lambda'^{op})$ is defined. This shows the equivalence of (a) and (b).

Now (c) follows from (b) by Proposition 4.1 and the discussion afterwards. Finally, if $\Lambda' \simeq \mu_n^{\mathrm{BB}}(\Lambda)$, then $\Lambda^{op} \simeq \mu_n^{\mathrm{BB}}(\Lambda'^{op})$ and hence (c) implies (a). $\square$

We see that if Λ and Λ' are the endomorphism algebras of two objects related by replacing an indecomposable summand by another through an approximation sequence, then the existence of the two BB-mutations $\mu_n^{\mathrm{BB}}(\Lambda)$

and $\mu_n^{\mathrm{BB}}(\Lambda'^{op})$ automatically implies that they take the "correct" values, namely Λ' and Λ^{op}, respectively.

In the applications, the condition (ii) in the theorem is usually automatically satisfied. Indeed, consider an approximation sequence

$$U_n \xrightarrow{f} B \xrightarrow{g} U'_n \tag{4.2}$$

as in (i) of the theorem.

Corollary 4.3. *If $\mathcal{C}$ is exact and* (4.2) *is an exact sequence, then (ii) is satisfied. Moreover, in this case f_* and g^* are monomorphisms so that always $\Lambda' \simeq \mu_n^{\mathrm{BB}}(\Lambda)$.*

Remark 4.4. If $\mathcal{C}$ is triangulated and (4.2) is a triangle, then (ii) is satisfied. In this case, the BB-tilting modules $T_n^{\mathrm{BB}}(\Lambda)$, $T_n^{\mathrm{BB}}(\Lambda'^{op})$ are not always defined and the algebras Λ, Λ' are not necessarily derived equivalent.

5. Mutations of endomorphism algebras in 2-CY categories

Cluster categories were introduced in [15] as a categorical framework for the cluster algebras of Fomin and Zelevinsky [21]. Particular role is played by the so-called cluster-tilting objects and their endomorphism algebras, known as cluster-tilted algebras [17]. Cluster categories are 2-Calabi-Yau triangulated categories, and the theory has been extended to such categories [1, 13, 14, 31, 34] as well as to Frobenius categories which are stably 2-Calabi-Yau [13, 22].

The results of the preceding section can be applied in the study of endomorphism algebras of cluster-tilting objects in these categories. We begin by recalling the relevant notions.

Let K be an algebraically closed field and $\mathcal{C}$ a K-linear Hom-finite additive category with split idempotents which is either:

- triangulated 2-Calabi-Yau (2-CY) category, that is, there exist functorial isomorphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y[2]) \simeq D\,\mathrm{Hom}_{\mathcal{C}}(Y, X)$ for $X, Y \in \mathcal{C}$; or
- a Frobenius category whose stable category, which is triangulated by [23], is 2-CY.

When $\mathcal{C}$ is triangulated, we denote $\mathrm{Hom}_{\mathcal{C}}(X, Y[1])$ by $\mathrm{Ext}^1_{\mathcal{C}}(X, Y)$.

An object $U \in \mathcal{C}$ is *rigid* if $\mathrm{Ext}^1_{\mathcal{C}}(U, U) = 0$. It is a *cluster-tilting object* if it is rigid and for any $X \in \mathcal{C}$, $\mathrm{Ext}^1_{\mathcal{C}}(U, X) = 0$ implies that $X \in \operatorname{add} U$. When $\mathcal{C}$ is Frobenius stably 2-CY, any indecomposable projective-injective is a summand of any cluster-tilting object.

The operation of *mutation* of [21] is categorified by mutation of cluster-tilting objects, see [15, 31]. Let $U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ be a basic cluster-tilting object written as a sum of n non-isomorphic indecomposables. When $\mathcal{C}$ is triangulated, set $m = n$. When $\mathcal{C}$ is Frobenius, assume that $U_{m+1}, \dots, U_n$ are all the projective-injective summands of U. Then for any $1 \le k \le m$ there exists a unique U'_k non-isomorphic to U_k such that $\mu_k(U) = (U/U_k) \oplus U'_k$ is a cluster-tilting object (here, U/U_k denotes the sum of all U_i for $i \ne k$). Moreover, there are so-called *exchange triangles*

$$U_k \xrightarrow{f} B \xrightarrow{g} U'_k \qquad\qquad U'_k \xrightarrow{f'} B' \xrightarrow{g'} U_k \tag{5.1}$$

when $\mathcal{C}$ is triangulated, and (exact) *exchange sequences*

$$0 \to U_k \xrightarrow{f} B \xrightarrow{g} U'_k \to 0 \qquad 0 \to U'_k \xrightarrow{f'} B' \xrightarrow{g'} U_k \to 0 \tag{5.2}$$

when $\mathcal{C}$ is Frobenius, such that the maps f, f' are minimal left $\mathrm{add}(U/U_k)$-approximations and g, g' are minimal right $\mathrm{add}(U/U_k)$-approximations.

5.1. **Endomorphism algebras in 2-CY Frobenius categories.** Let K be an algebraically closed field and $\mathcal{E}$ a K-linear, Hom-finite, Frobenius category which is stably 2-CY.

It is known by [39] that the derived equivalence class of the endomorphism algebra of a cluster-tilting object U in $\mathcal{E}$ does not depend on U, see also [27, §5]. It is still of interest to relate the endomorphism algebras of two cluster-tilting objects related by mutation. This has been done in [22] for Frobenius categories arising from preprojective algebras. The following is a generalization, essentially saying that the mutation operations of algebras "commute" with mutation of cluster-tilting objects.

Theorem 5.1. *Let U be a cluster-tilting object in $\mathcal{E}$. Then for any non projective-injective summand U_k of U, all the corresponding BB-, negative and positive mutations of* $\mathrm{End}_{\mathcal{C}}(U)$ *are defined and moreover,*

$$\mathrm{End}_{\mathcal{E}}\big(\mu_k(U)\big) \simeq \mu_k^{\mathrm{BB}}\big(\mathrm{End}_{\mathcal{E}}(U)\big) = \mu_k^{-}\big(\mathrm{End}_{\mathcal{E}}(U)\big) \simeq \mu_k^{+}\big(\mathrm{End}_{\mathcal{E}}(U)\big).$$

Proof. Using Corollary 4.3 for the left exchange sequence in (5.2), we get that $\mu_k^{\mathrm{BB}}(\mathrm{End}_{\mathcal{C}}(U))$ is defined and

$$\mathrm{End}_{\mathcal{E}}\big(\mu_k(U)\big) \simeq \mu_k^{\mathrm{BB}}\big(\mathrm{End}_{\mathcal{E}}(U)\big) = \mu_k^{-}\big(\mathrm{End}_{\mathcal{E}}(U)\big).$$

Using this for $\mu_k(U)$ instead of U, we get that

$$\mathrm{End}_{\mathcal{E}}(U) \simeq \mathrm{End}_{\mathcal{E}}\big(\mu_k(\mu_k(U))\big) \simeq \mu_k^{\mathrm{BB}}\big(\mathrm{End}_{\mathcal{E}}(\mu_k(U))\big) = \mu_k^{-}\big(\mathrm{End}_{\mathcal{E}}(\mu_k(U))\big),$$

hence $\mu_k^{+}(\mathrm{End}_{\mathcal{E}}(U))$ is also defined and isomorphic to $\mathrm{End}_{\mathcal{C}}(\mu_k(U))$. $\square$

5.2. **Endomorphism algebras in 2-CY triangulated categories.** Let K be an algebraically closed field and $\mathcal{C}$ a K-linear, Hom-finite, triangulated 2-CY category with split idempotents.

Let $U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ a cluster-tilting object. According to [17, 34], the neighboring 2-CY-tilted algebras $\Lambda = \mathrm{End}_{\mathcal{C}}(U)$ and $\Lambda' = \mathrm{End}_{\mathcal{C}}(\mu_k(U))$ are related by what is called in [40] *near-Morita equivalence*, namely if S_k and S'_k are the simple modules in Λ and Λ' corresponding to the indecomposables U_k and U'_k, then

$$\mathrm{mod}\,\Lambda/\langle \mathrm{add}\, S_k\rangle \simeq \mathrm{mod}\,\Lambda'/\langle \mathrm{add}\, S'_k\rangle.$$

This generalizes APR-tilting [6], which corresponds to the case where the vertex k is a sink.

Another feature of APR-tilting is that the algebras related by an APR-tilt are derived equivalent. However, it is easily seen (and well known) that two nearly-Morita equivalent 2-CY-tilted algebras are in general not derived equivalent.

Example 5.2. The quivers of two cluster-tilted algebras of type A_3 are shown below. One is obtained from the other by mutation at the vertex 2.

$$\begin{tikzcd} & \bullet_2 \arrow[dr] & \\ \bullet_1 \arrow[ur] & & \bullet_3 \end{tikzcd} \qquad \begin{tikzcd} & \bullet_2 \arrow[dl] & \\ \bullet_1 \arrow[rr] & & \bullet_3 \arrow[ul] \end{tikzcd} \tag{5.3}$$

The corresponding algebras, where in the right one the composition of any pair of consecutive arrows is zero, are nearly Morita equivalent but not derived equivalent.

Nevertheless, several derived equivalence classifications of cluster-tilted algebras [8, 10, 18] have revealed that, at least in finite mutation type, the number of derived equivalence classes is much smaller than the number of isomorphism classes of algebras (which equals the number of quivers in the mutation class). For example, in type E_8 there are 1574 algebras but only 15 derived equivalence classes [10]. Moreover, many of the classifications rely on showing that sufficiently many pairs of near Morita equivalent cluster-tilted algebras are also derived equivalent.

In an attempt to provide a conceptual explanation of these phenomena, it is therefore interesting to formulate conditions that will guarantee the derived equivalence of neighboring 2-CY-tilted algebras. A similar problem for Jacobian algebras was studied in [44].

Theorem 5.3. *Let U be a cluster-tilting object in $\mathcal{C}$ and let $\Lambda = \operatorname{End}_{\mathcal{C}}(U)$ and $\Lambda' = \operatorname{End}_{\mathcal{C}}(\mu_k(U))$ be two neighboring 2-CY-tilted algebras.*

Then $\Lambda' \simeq \mu_k^{\mathrm{BB}}(\Lambda)$ if and only if the BB-tilting modules $T_k^{\mathrm{BB}}(\Lambda)$ and $T_k^{\mathrm{BB}}(\Lambda'^{op})$ are defined. In particular, in this case Λ and Λ' are derived equivalent.

Proof. Use the left exchange triangle in (5.1) and Theorem 4.2. □

Example 5.4. Looking again at Example 5.2, denote by Λ and Λ' the left and right cluster-tilted algebras in (5.3). Then $T_2^{\mathrm{BB}}(\Lambda)$ is defined and $\mu_2^{\mathrm{BB}}(\Lambda)$ is the algebra given by the left quiver of Example 3.3 (with zero relation), hence $\Lambda' \not\simeq \mu_2^{\mathrm{BB}}(\Lambda)$. Observe that for any $1 \leq i \leq 3$, none of the mutations $\mu_i^+(\Lambda')$, $\mu_i^-(\Lambda')$ is defined.

Theorem 5.3 motivates the following definition, see also [10].

Definition 5.5. Let U be a cluster-tilting object of $\mathcal{C}$. The mutation $\mu_k(U)$ is *good* if $\operatorname{End}_{\mathcal{C}}(\mu_k(U)) \simeq \mu_k^{\mathrm{BB}}(\operatorname{End}_{\mathcal{C}}(U))$. In this case, we say that the algebra $\operatorname{End}_{\mathcal{C}}(\mu_k(U))$ is obtained from $\operatorname{End}_{\mathcal{C}}(U)$ by a *good mutation*.

Remark 5.6. For certain classes of 2-CY-tilted algebras, such as cluster-tilted algebras of Dynkin type, there exist algorithms to determine the relations from their quivers [16]. Moreover, the nature of these relations allows to compute bases of paths for these algebras and hence, thanks to Proposition 2.3, to efficiently decide for any vertex whether the BB-tilting module is defined.

The author has implemented these algorithms in a computer program that determines whether two (neighboring) cluster-tilted algebras of Dynkin type (given by their quivers) are related by a good mutation. In particular, this allowed to verify the results of [10] on a computer.

Inspired by the connectivity of the mutation graph of cluster-tilting objects in cluster categories [15], the following question naturally arises.

Question 5.7. Let Λ and Λ' be two derived equivalent 2-CY-tilted algebras (for a fixed 2-CY category $\mathcal{C}$). Are they connected by a sequence of good mutations (in $\mathcal{C}$)?

In other words, does there exist a sequence of 2-CY-tilted algebras $\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_t = \Lambda'$ such that for any $0 \leq i < t$, either Λ_{i+1} is obtained from Λ_i by a good mutation or Λ_i is obtained from Λ_{i+1} by a good mutation?

The results in [8, 10, 18] show that the answer to this question is positive for the cluster-tilted algebras of types A, $\tilde{A}$ and E. However, there are cluster-tilted algebras of type D which are derived equivalent but not connected by a sequence of good mutations, as shown by the following example, see also the forthcoming paper [9].

Example 5.8. The quivers of two derived equivalent cluster-tilted algebras of type D_6 are shown below.

Since these algebras are self-injective [41], they have no non-trivial tilting modules and in particular no good mutations.

5.3. **A numerical criterion for derived equivalence.** By a result of Keller and Reiten [34], 2-CY-tilted algebras are Gorenstein of dimension at most one. We now use this fact to obtain a criterion for the derived equivalence via BB-mutation of neighboring 2-CY-tilted algebras in terms of their Cartan matrices, under the assumption that these matrices are invertible over $\mathbb{Q}$.

For a finite dimensional algebra A with n non-isomorphic simples, we denote by I_i the (indecomposable) injective envelope of S_i.

Definition 5.9. Let A be a finite dimensional algebra with $\operatorname{pd}_A DA < \infty$. We define the (integral) matrix S_A by the following equalities in $K_0(\operatorname{per} A)$:

$$[I_i] = \sum_{j=1}^{n} (S_A)_{ij}\, [P_j], \qquad\qquad 1 \leq i \leq n.$$

When A has finite global dimension, the matrix $-S_A$ is the matrix of the *Coxeter transformation* of A with respect to the basis of $K_0(\operatorname{per} A)$ given by the indecomposable projectives, see [36].

Lemma 5.10. *Let A be a finite dimensional algebra with* $\operatorname{pd}_A DA < \infty$.

(a) *If* $\operatorname{id}_A A < \infty$*, then S_A is invertible over $\mathbb{Z}$.*
(b) *Let C_A be the Cartan matrix of A. Then $S_A C_A^T = C_A$.*
(c) *In particular, if C_A is invertible over $\mathbb{Q}$, then S_A equals the* asymmetry $C_A C_A^{-T}$.

Proof. (a) Let w be the integral matrix defined by the equalities $[P_i] = \sum_{j=1}^{n} w_{ij}[I_j]$ in $K_0(\operatorname{per} A)$ for $1 \leq i \leq n$. Then $w \cdot S_A$ is the identity matrix.

(b) By duality, we have $\operatorname{Hom}(P_l, I_i) \simeq D\operatorname{Hom}(P_i, P_l)$ for $1 \leq i, l \leq n$. Thus

$$\sum_{j=1}^{n} (S_A)_{ij}(C_A)_{lj} = \langle [P_l], [I_i] \rangle_A = \dim_K \operatorname{Hom}(P_l, I_i) = \dim_K \operatorname{Hom}(P_i, P_l)$$
$$= (C_A)_{il}$$

and so $S_A \cdot C_A^T = C_A$.

□

As before, let $U = U_1 \oplus \cdots \oplus U_n$ be a basic cluster-tilting object in the 2-CY category $\mathcal{C}$ and let $\Lambda = \operatorname{End}_{\mathcal{C}}(U)$ be the corresponding 2-CY-tilted algebra. Since Λ is Gorenstein, the matrix S_Λ is defined. We list some of its properties, based on the notion of *index* from [19].

As the Gorenstein dimension of Λ is at most one [34], there exists a minimal projective resolution of $D\Lambda$ of the form

$$\bigoplus_{j=1}^{n} P_j^{e_j} \to \bigoplus_{j=1}^{n} P_j^{d_j} \tag{5.4}$$

with integers $d_j, e_j \geq 0$ for $1 \leq j \leq n$.

Lemma 5.11. *Let $1 \leq j \leq n$. Then one of d_j, e_j vanishes.*

Proof. By [34], the resolution (5.4) arises from a triangle

$$U^1 \to U^0 \to \nu U \to U^1[1]$$

with $U^0, U^1 \in \operatorname{add} U$, where $\nu \simeq [2]$ is the Serre functor in $\mathcal{C}$. Since νU is a rigid object of $\mathcal{C}$, the objects U^0 and U^1 have no common summand by [19, Prop. 2.1]. This means that d_j and e_j cannot be both positive. □

It follows that one can read the terms of the minimal projective resolution of each I_i from the entries of the matrix S_Λ.

Proposition 5.12. *For $1 \leq i \leq n$, let*

$$\bigoplus_{j=1}^{n} P_j^{e_{ij}} \to \bigoplus_{j=1}^{n} P_j^{d_{ij}}$$

be a minimal projective resolution of I_i. Then:

(a) *For any $1 \leq j \leq n$,*

$$d_{ij} = \begin{cases} (S_\Lambda)_{ij} & \text{if } (S_\Lambda)_{ij} > 0, \\ 0 & \text{otherwise;} \end{cases} \qquad e_{ij} = \begin{cases} -(S_\Lambda)_{ij} & \text{if } (S_\Lambda)_{ij} < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *A column of S_Λ cannot contain both positive and negative entries.*

Proof. Let $1 \leq j \leq n$. We have $d_j = \sum_{i=1}^{n} d_{ij}$ and $e_j = \sum_{i=1}^{n} e_{ij}$ as sums of non-negative integers. Since one of d_j, e_j vanishes by Lemma 5.11, we have that all the entries d_{ij} or all the entries e_{ij} vanish.

In particular, since one of d_{ij}, e_{ij} vanishes, the first assertion follows from $(S_\Lambda)_{ij} = d_{ij} - e_{ij}$.

If the j-th column of S_Λ contained entries of different signs, this would mean that d_{ij} and $e_{i'j}$ are non-zero for some i, i', which is impossible. $\square$

By using the triangle $U \to U^0 \to U^1 \to U[1]$ with $U^0, U^1 \in \operatorname{add} \nu U$ we obtain the dual statement.

Proposition 5.13. *For $1 \leq i \leq n$, let*

$$\bigoplus_{j=1}^{n} I_j^{d_{ij}} \to \bigoplus_{j=1}^{n} I_j^{e_{ij}}$$

be a minimal injective resolution of P_i. Then:

(a) *For any $1 \leq j \leq n$,*

$$d_{ij} = \begin{cases} (S_\Lambda^{-1})_{ij} & \text{if } (S_\Lambda^{-1})_{ij} > 0, \\ 0 & \text{otherwise;} \end{cases} \qquad e_{ij} = \begin{cases} -(S_\Lambda^{-1})_{ij} & \text{if } (S_\Lambda^{-1})_{ij} < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *A column of S_Λ^{-1} cannot contain both positive and negative entries.*

Corollary 5.14. *Let k be a vertex without loops in the quiver of Λ. Then:*

(a) *$T_k^{\mathrm{BB}}(\Lambda)$ is defined if and only if $(S_\Lambda)_{ik} \leq 0$ for all $1 \leq i \leq n$.*
(b) *$T_k^{\mathrm{BB}}(\Lambda^{op})$ is defined if and only if $(S_\Lambda^{-1})_{ik} \leq 0$ for all $1 \leq i \leq n$.*

Proof. By Lemma 2.6, $T_k^{\mathrm{BB}}(\Lambda)$ is defined if and only if $d_k = 0$. Now use Proposition 5.12. The proof of the second statement is dual. $\square$

Theorem 5.15. *Let U be a cluster-tilting object in $\mathcal{C}$ and let $\Lambda = \operatorname{End}_{\mathcal{C}}(U)$ and $\Lambda' = \operatorname{End}_{\mathcal{C}}(\mu_k(U))$ be two neighboring 2-CY-tilted algebras. Assume that their quivers have no loops at k.*

Then $\Lambda' = \mu_k^{\mathrm{BB}}(\Lambda)$ if and only if $(S_\Lambda)_{ik} \leq 0$ and $\left(S_{\Lambda'}^{-1}\right)_{ik} \leq 0$ for any $1 \leq i \leq n$.

Proof. Use Theorem 5.3 and Corollary 5.14. $\square$

Remark 5.16. When the Cartan matrices C_Λ and $C_{\Lambda'}$ are invertible over $\mathbb{Q}$, the theorem gives an effective criterion to decide if $\Lambda' = \mu_k^{\mathrm{BB}}(\Lambda)$, by examining the signs of the entries in the k-th columns of the asymmetries $S_\Lambda = C_\Lambda C_\Lambda^{-T}$ and $S_{\Lambda'}^{-1} = C_{\Lambda'}^T C_{\Lambda'}^{-1}$.

Example 5.17. Consider the cluster-tilted algebras Λ and Λ' of type D_5 whose quivers, related by a mutation at the vertex 3, are shown below.

$$\begin{tikzcd}[column sep=small]
\bullet_1 \arrow[rr] & & \bullet_2 \arrow[rr] \arrow[dl] & & \bullet_3 \arrow[dl] \\
& \bullet_4 \arrow[ul] \arrow[rr] & & \bullet_5 \arrow[ul] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\bullet_1 \arrow[rr] & & \bullet_2 \arrow[dl] & & \bullet_3 \arrow[ll] \\
& \bullet_4 \arrow[ul] \arrow[rr] & & \bullet_5 \arrow[ur] &
\end{tikzcd}$$

Using the description of the relations given in [16], one computes their Cartan matrices

$$C_\Lambda = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad C_{\Lambda'} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

and the corresponding asymmetries

$$S_\Lambda = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad S_{\Lambda'}^{-1} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since the entries in the third column of both matrices are non-positive, the corresponding BB-tilting modules $T_3^{\mathrm{BB}}(\Lambda)$ and $T_3^{\mathrm{BB}}(\Lambda'^{op})$ are defined, and we deduce that $\Lambda' \simeq \mu_3^{\mathrm{BB}}(\Lambda)$ is obtained from Λ by a good mutation at the vertex 3.

6. Mutations of algebras of global dimension at most 2

We start with the following motivating example. Consider the two skew-symmetric matrices

$$b = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad b' = \mu_2(b) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

whose associated quivers Q and Q' via (3.1) are given by those in (5.3). So by considering the matrices c_Q and $c_{Q'}$ defined by

$$(c_Q)_{ij} = -\bigl|\{\text{arrows } i \to j \text{ in } Q\}\bigr|,$$

one can think of b and b' as the skew-symmetrizations $b = c_Q - c_Q^T$ and $b' = c_{Q'} - c_{Q'}^T$.

As Q is acyclic while Q' is not, the path algebra KQ is finite dimensional while KQ' is infinite dimensional, so they cannot be derived equivalent. Thus, at a first stage one must introduce some relations on Q' so that by dividing its path algebra by the ideals they generate, one obtains a finite dimensional algebra. The notion of *quivers with potentials* and their mutations, introduced in [20], provides a systematic way of doing this, and the resulting algebras are known as *Jacobian algebras*.

The Jacobian algebras corresponding to Q and Q' are precisely the cluster-tilted algebras of Example 5.2, which, despite being closely related via near Morita equivalence, are not derived equivalent. One approach to get a derived equivalence is to replace the Jacobian algebras by suitable dg-algebras (the *Ginzburg algebras*). Then one is always able to interpret mutation as derived equivalence, see [35].

Another approach is inspired from the fact that cluster-tilted algebras are relation-extension of tilted algebras [3]. Therefore, by cleverly deleting arrows and interpreting them as relations in the opposite direction, one should obtain algebras of global dimension at most 2, see also [2], and (sometimes) interpret quiver mutation as mutation of these algebras in the sense of Section 3.

Indeed, if $A = KQ$, then by deleting the arrows in Q' from 3 to 2 or from 2 to 1, one gets the algebras $\mu_2^-(A)$ and $\mu_2^+(A)$ respectively, see Example 3.3, thus interpreting the quiver mutation at the vertex 2 as a mutation of algebras. Moreover, one can view b and b' as the skew-symmterizations of the corresponding Euler forms,

$$b = c_A - c_A^T, \qquad b' = c_{\mu_2^-(A)} - c^T_{\mu_2^-(A)} = c_{\mu_2^+(A)} - c^T_{\mu_2^+(A)}.$$

6.1. **Mutations of algebras as mutations of quivers.** Let K be an algebraically closed field and let A be a finite dimensional algebra of global dimension at most 2. Denote by Q_A the quiver of A and recall that c_A denotes the matrix of the Euler form on $\mathcal{D}^b(A)$ with respect to the basis of simples. Motivated by [3, 32], we make the following definition.

Definition 6.1. The *extended quiver* of A, denoted $\widetilde{Q}_A$, has the same vertices as Q_A, with the number of arrows from i to j equal to

$$\dim_K \operatorname{Ext}^1_A(S_i, S_j) + \dim_K \operatorname{Ext}^2_A(S_j, S_i).$$

When A is hereditary, $\widetilde{Q}_A$ coincides with Q_A.

Lemma 6.2. $b_{\widetilde{Q}_A} = c_A - c_A^T$.

Proof. We have

$$\begin{aligned}\left(b_{\widetilde{Q}_A}\right)_{ij} &= -\dim_K \operatorname{Ext}^1_A(S_i, S_j) - \dim_K \operatorname{Ext}^2_A(S_j, S_i) \\ &\quad + \dim_K \operatorname{Ext}^1_A(S_j, S_i) + \dim_K \operatorname{Ext}^2_A(S_i, S_j) \\ &= (c_A)_{ij} - (c_A)_{ji}\end{aligned}$$

since $\operatorname{gl.dim} A \leq 2$. ☐

Lemma 6.3. *Let k be a vertex of $\widetilde{Q}_A$ without loops.*

(a) *If $\mu_k^-(A)$ is defined and* $\operatorname{gl.dim} \mu_k^-(A) \leq 2$, *then* $r_k^-(\widetilde{Q}_A) = r_k^-(Q_A)$.
(b) *If $\mu_k^+(A)$ is defined and* $\operatorname{gl.dim} \mu_k^+(A) \leq 2$, *then* $r_k^+(\widetilde{Q}_A) = r_k^+(Q_A)$.

Proof. We show only the first claim, as the proof of the second is similar. By the definition of the matrices r_k^-, we only need to show that $\operatorname{Ext}^2_A(S_k, S_i) = 0$ for any $i \neq k$.

Let $A' = \mu_k^-(A)$ and denote by $F = \mathbf{R}\mathrm{Hom}(T_k^-, -)$ the equivalence from $\mathcal{D}^b(A)$ to $\mathcal{D}^b(A')$. From (1.1) and the definition of L_k we see that there exist A'-modules M'_i such that $F(S_i) \simeq M'_i$ for $i \neq k$ and $F(S_k) \simeq M'_k[-1]$ (in fact, M'_k is the simple A'-module corresponding to k, but we do not need this here). It follows that

$$\begin{aligned}\operatorname{Ext}^2_A(S_k, S_i) &= \operatorname{Hom}_{\mathcal{D}^b(A)}(S_k, S_i[2]) \simeq \operatorname{Hom}_{\mathcal{D}^b(A')}(M'_k[-1], M'_i[2]) \\ &= \operatorname{Ext}^3_{A'}(M'_k, M'_i) = 0\end{aligned}$$

by our assumption that $\operatorname{gl.dim} A' \leq 2$. ☐

The following proposition shows that mutations between algebras of global dimension at most 2 can be interpreted as mutations of the corresponding extended quivers, as long as these quivers contain neither loops nor 2-cycles.

Proposition 6.4. *Assume that $\widetilde{Q}_A$ does not contain loops nor 2-cycles, and let k be a vertex of $\widetilde{Q}_A$.*

(a) *If* $\mu_k^-(A)$ *is defined and* $\operatorname{gl.dim} \mu_k^-(A) \leq 2$, *then* $b_{\widetilde{Q}_{\mu_k^-(A)}} = \mu_k(b_{\widetilde{Q}_A})$. *Moreover, if* $\widetilde{Q}_{\mu_k^-(A)}$ *does not have loops or 2-cycles, then*

$$\widetilde{Q}_{\mu_k^-(A)} = \mu_k(\widetilde{Q}_A).$$

(b) *If* $\mu_k^+(A)$ *is defined and* $\operatorname{gl.dim} \mu_k^+(A) \leq 2$, *then* $b_{\widetilde{Q}_{\mu_k^+(A)}} = \mu_k(b_{\widetilde{Q}_A})$. *Moreover, if* $\widetilde{Q}_{\mu_k^+(A)}$ *does not have loops or 2-cycles, then*

$$\widetilde{Q}_{\mu_k^+(A)} = \mu_k(\widetilde{Q}_A).$$

Proof. By Lemma 3.4, Lemma 6.3 and Corollary 3.7,

$$\begin{aligned}
\mu_k(b_{\widetilde{Q}_A}) &= \left(r_k^-(\widetilde{Q}_A)\right)^T b_{\widetilde{Q}_A} \left(r_k^-(\widetilde{Q}_A)\right) \\
&= \left(r_k^-(Q_A)\right)^T b_{\widetilde{Q}_A} \left(r_k^-(Q_A)\right) = \left(r_k^-(Q_A)\right)^T \left(c_A - c_A^T\right) \left(r_k^-(Q_A)\right) \\
&= c_{\mu_k^-(A)} - c_{\mu_k^-(A)}^T = b_{\widetilde{Q}_{\mu_k^-(A)}}.
\end{aligned}$$

This proves (a). The proof of (b) is similar. □

6.2. **Interpretation in cluster categories.** For certain algebras A with $\operatorname{gl.dim} A \leq 2$, the above results can be refined to have an interpretation in terms of the generalized cluster category $\mathcal{C}_A$ associated with A that was introduced in [1]. Recall that $\mathcal{C}_A$ is the triangulated hull of the orbit category $\mathcal{D}^b(A)/\nu_A[-2]$, where ν_A denotes the Serre functor on $\mathcal{D}^b(A)$.

Consider the A-A-bimodule $\operatorname{Ext}_A^2(DA, A)$ and the tensor algebra $\widetilde{A} = T_A(\operatorname{Ext}_A^2(DA, A))$ known as the *3-preprojective algebra* of A, see [29, 32]. In this section we assume that $\widetilde{A}$ is finite dimensional over K. It is shown in [1] that under this condition the triangulated category $\mathcal{C}_A$ is Hom-finite and 2-Calabi-Yau, the image $\pi_A(A)$ of A under the canonical projection

$$\pi_A : \mathcal{D}^b(A) \to \mathcal{D}^b(A)/\nu_A[-2] \to \mathcal{C}_A$$

is a cluster-tilting object in $\mathcal{C}_A$ with endomorphism algebra

$$\operatorname{End}_{\mathcal{C}_A}(\pi_A(A)) \simeq \widetilde{A},$$

and the quiver of $\widetilde{A}$ is $\widetilde{Q}_A$.

Lemma 6.5 (Amiot). *Let T be a tilting complex in $\mathcal{D}^b(A)$ and let $B = \operatorname{End}_{\mathcal{D}^b(A)}(T)$. If $\operatorname{gl.dim} B \leq 2$, then $\pi_A(T)$ is a cluster-tilting object in $\mathcal{C}_A$ whose endomorphism ring is isomorphic to $\widetilde{B}$.*

Proof. The assumptions imply that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{D}^b(A) \arrow[r, "\Phi=\mathbf{R}\mathrm{Hom}(T,-)", "\simeq"'] \arrow[d, "\pi_A"'] & \mathcal{D}^b(B) \arrow[d, "\pi_B"] \\
\mathcal{C}_A \arrow[r, "\widetilde{\Phi}", "\simeq"'] & \mathcal{C}_B.
\end{tikzcd}$$

Since Φ maps T to B, the isomorphism $\widetilde{\Phi}$ maps $\pi_A(T)$ to $\pi_B(B)$, and the claim follows. □

Lemma 6.6. *Let X be a complex in $\mathcal{D}^b(A)$ of the form*

$$\cdots \to 0 \to P^{-1} \to P^0 \to P^1 \to 0 \to \dots$$

where P^i are projectives. If $\pi_A(X) \simeq \pi_A(P)$ in $\mathcal{C}_A$ for a projective A-module P, then $X \simeq P$ already in $\mathcal{D}^b(A)$.

Proof. For any $n \in \mathbb{Z}$, denote by $\mathcal{D}^{\geq n}$ the full subcategory of $\mathcal{D}^b(A)$ consisting of complexes whose cohomology vanishes in degrees smaller than n. Let $F = \nu_A[-2]$. Since $\operatorname{gl.dim} A \leq 2$, we have $F(\mathcal{D}^{\geq n}) \subseteq \mathcal{D}^{\geq n}$ for any $n \in \mathbb{Z}$.

Since $\nu_A P$ is an injective A-module, we get that $FP \in \mathcal{D}^{\geq 2}$. Hence $F^m P \in \mathcal{D}^{\geq 2}$, thus $\operatorname{Hom}_{\mathcal{D}^b(A)}(X, F^m P) = 0$ for any $m > 0$. Similarly, FX is the complex of injectives $\nu_A P^{-1} \to \nu_A P^0 \to \nu_A P^1$ concentrated in degrees 1, 2 and 3, so it lies in $\mathcal{D}^{\geq 1}$. It follows that $\operatorname{Hom}_{\mathcal{D}^b(A)}(P, F^m X) = 0$ for any $m > 0$.

If $\pi_A(X) \simeq \pi_A(P)$, then by the definition of the orbit category [33], there exist morphisms $f_m : P \to F^m X$ and $g_m : X \to F^m P$ in $\mathcal{D}^b(A)$ for $m \in \mathbb{Z}$, all but finitely many are zero, such that

$$\sum_{m\in\mathbb{Z}} F^m(g_{-m}) \circ f_m = 1_P, \qquad \sum_{m\in\mathbb{Z}} F^m(f_{-m}) \circ g_m = 1_X. \tag{6.1}$$

But we have just shown that $f_m = 0$ and $g_m = 0$ for any $m > 0$, so the equalities in (6.1) simplify to $g_0 f_0 = 1_P$ and $f_0 g_0 = 1_X$, giving that $X \simeq P$ in $\mathcal{D}^b(A)$. ☐

Proposition 6.7. *Let k be a vertex of Q_A.*

(a) *Assume that $\mu_k^-(A)$ is defined and* $\operatorname{gl.dim} \mu_k^-(A) \leq 2$. *Then $\pi_A(T_k^-)$ is the mutation of the canonical cluster-tilting object $\pi_A(A)$ in $\mathcal{C}_A$ at the vertex k.*

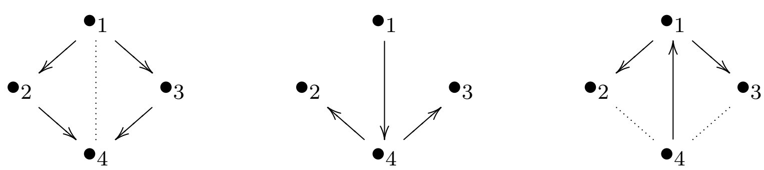
(b) *Assume that $\mu_k^+(A)$ is defined and* $\operatorname{gl.dim} \mu_k^+(A) \leq 2$. *Then $\pi_A(T_k^+)$ is the mutation of the canonical cluster-tilting object $\pi_A(A)$ in $\mathcal{C}_A$ at the vertex k.*

Proof. $\pi_A(T_k^-)$ is a cluster-tilting object in $\mathcal{C}_A$ by Lemma 6.5. By (1.1), it is obtained from $\pi_A(A)$ by replacing the summand $\pi_A(P_k)$ by $\pi_A(L_k)$. As $\pi_A(L_k) \not\simeq \pi_A(P_k)$ by Lemma 6.6, we get the first claim by the uniqueness of mutation [31]. The proof of the second claim is similar. ☐

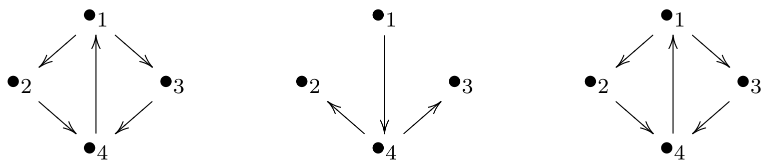
Thus, when $\mathcal{C}_A$ is Hom-finite, by combining Proposition 6.7 with [13, Theorem II.1.6] we get an alternative proof of Proposition 6.4.

6.3. **Examples.** We give examples showing that starting with the extended quiver $\widetilde{Q}_A$ of an algebra A with $\operatorname{gl.dim} A \leq 2$, there are cases where a mutation of $\widetilde{Q}_A$ at a vertex has two interpretations as mutations of algebras, as well as other cases where is has no such interpretation.

Example 6.8. Let A^+, A and A^- be the algebras given by the quivers with relations



(A^+ has a commutativity relation while A^- has zero relations). These algebras are of global dimension at most 2 and their extended quivers $\widetilde{Q}_{A^+}$, $\widetilde{Q}_A$ and $\widetilde{Q}_{A^-}$ are given by



We see that $A^+ = \mu_4^+(A)$ and $A^- = \mu_4^-(A)$ and correspondingly $\widetilde{Q}_{A^+} = \mu_4(\widetilde{Q}_A) = \widetilde{Q}_{A^-}$, hence the mutation from $\widetilde{Q}_A$ to $\mu_4(\widetilde{Q}_4)$ carries two different interpretations as mutations of algebras, leading to different, yet derived equivalent algebras.

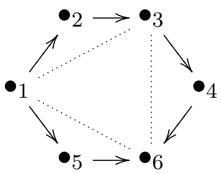
The passage from A^+ to A^- can be viewed as a composition of the following two perverse equivalences at the vertex 4,

$$\mathcal{D}^b(A^+) \xrightarrow[\simeq]{\mathbf{R}\mathrm{Hom}(T_4^{\mathrm{BB}}(A^+),-)} \mathcal{D}^b(A) \xrightarrow[\simeq]{\mathbf{R}\mathrm{Hom}(T_4^{\mathrm{BB}}(A),-)} \mathcal{D}^b(A^-)$$

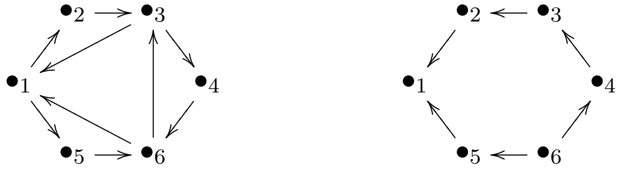
arising from the BB-tilting modules $T_4^{\mathrm{BB}}(A^+)$ and $T_4^{\mathrm{BB}}(A)$. This composition coincides with the 2-APR-tilt [28] of A^+ at the sink 4. In fact, one can show that any 2-APR-tilt is a composition of the corresponding two BB-tilts.

The next example shows that there are cases where the mutation equivalence of the quivers $\widetilde{Q}_A$ and $\widetilde{Q}_B$ can not be interpreted as a derived equivalence of the algebras A and B, even if the cluster categories $\mathcal{C}_A$ and $\mathcal{C}_B$ are Hom-finite (Claire Amiot, private communication).

Example 6.9. Let A be the algebra given by the quiver with zero relations as in the following picture.



The quiver $\widetilde{Q}_A$ is given by the left picture below. If we mutate it at the vertices 2, 4 and 5, we arrive at quiver Q given in the right picture.



Nevertheless, A is not derived equivalent to the algebra $B = KQ$, despite the fact that the quivers $\widetilde{Q}_A$ and $\widetilde{Q}_B = Q$ are mutation equivalent. Indeed, even the corresponding Euler bilinear forms c_A and c_B are not equivalent.

Note that the above sequence of mutations cannot be interpreted as a sequence of perverse derived equivalences at the corresponding vertices. Indeed, none of the complexes T_2^-, T_2^+, T_4^-, T_4^+, T_5^-, T_5^+ is a tilting complex over A.

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