Posets, sheaves, and their derived equivalences

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Posets, diagrams and sheaves

X – *poset* (finite partially ordered set)

$\mathcal{A}$ – abelian category

$\mathcal{A}^X$ – the category of *diagrams* over X with values in $\mathcal{A}$, or *functors* $F : X \to \mathcal{A}$ consisting of:

- An object F_x of $\mathcal{A}$ for each $x \in X$.
- A morphism $r_{xx'} \in \mathrm{Hom}_{\mathcal{A}}(F_x, F_{x'})$ for each $x \leq x'$.

such that $r_{xx''} = r_{x'x''} r_{xx'}$ for all $x \leq x' \leq x''$ (commutativity).

Natural *topology* on X: $U \subseteq X$ is open if $x \in U \, , \, x \leq x' \Rightarrow x' \in U$

Diagrams can be identified with *sheaves* over X with values in $\mathcal{A}$.

Universal derived equivalence

Two posets X and Y are *universally derived equivalent* ($X \overset{u}{\sim} Y$) if

$$\mathcal{D}^b(\mathcal{A}^X) \simeq \mathcal{D}^b(\mathcal{A}^Y)$$

for any abelian category $\mathcal{A}$.

Fix a field k, and specialize:
$\operatorname{mod} k$ – the category of finite dimensional vector spaces over k.

$(\operatorname{mod} k)^X$ can be identified with the category of finitely generated *right modules* over the *incidence algebra* of X over k.

X and Y are *derived equivalent* ($X \sim Y$) if

$$\mathcal{D}^b(\operatorname{mod} kX) \simeq \mathcal{D}^b(\operatorname{mod} kY)$$

Constructions of derived equivalent posets

Common theme: structured reversal of order relations.

- Generalized reflections (universal derived equivalences)
  - *Flip-Flops*, with application to posets of tilting modules
  - *Generalized BGP reflections*
  - Hybrid construction
- Mirroring with respect to a *bipartite* structure
  - *Mates* of triangular matrix algebras

Flip-Flops

Let $(X, \leq_X)$, $(Y, \leq_Y)$ be posets, $f : X \to Y$ order-preserving.

Define two partial orders $\leq^f_+$, $\leq^f_-$ on $X \sqcup Y$ as follows:

- Keep the original partial orders inside X and Y.
- Add the relations

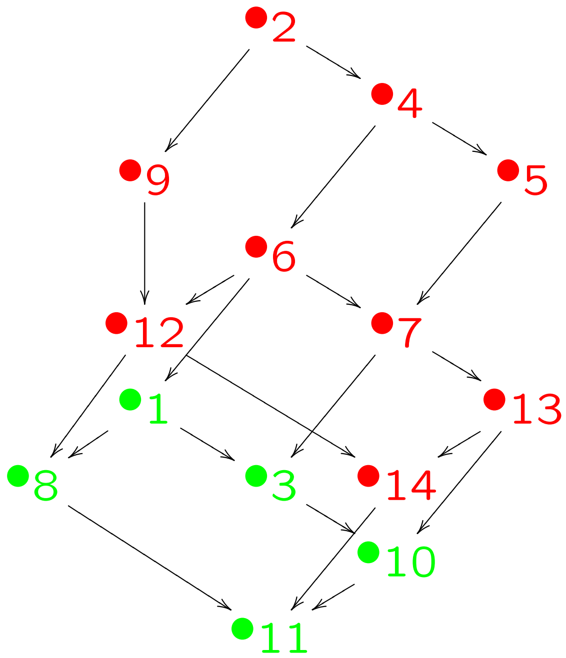
$$x \leq^f_+ y \iff f(x) \leq_Y y$$
$$y \leq^f_- x \iff y \leq_Y f(x)$$

for $x \in X$, $y \in Y$.

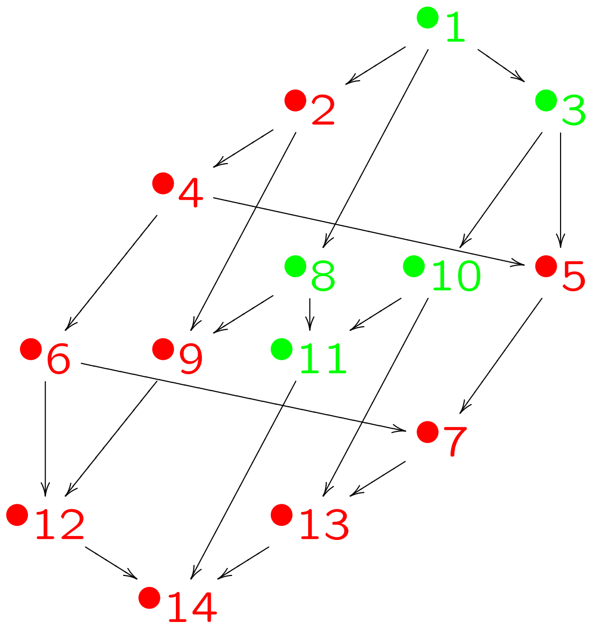
Theorem. $(X \sqcup Y, \leq^f_+) \overset{u}{\sim} (X \sqcup Y, \leq^f_-)$.

Flip-Flop – Example

$2 \mapsto 1 \quad 4 \mapsto 1 \quad 5 \mapsto 3 \quad 6 \mapsto 1 \quad 7 \mapsto 3 \quad 9 \mapsto 8 \quad 12 \mapsto 8 \quad 13 \mapsto 10 \quad 14 \mapsto 11$



$(X \sqcup Y, \leq^f_+)$



$(X \sqcup Y, \leq^f_-)$

Application – Posets of tilting modules

Q – quiver without oriented cycles, k – field
$\mathcal{T}_Q$ – poset of *tilting modules* of kQ [Riedtmann-Schofield, Happel-Unger]

x – a source in Q
Q' – the *BGP reflection* with respect to x.
$\mathcal{T}_Q^x$ – tilting modules containing the simple at x as summand

Theorem. $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ are related via a flip-flop.

$$\mathcal{T}_Q \simeq (\mathcal{T}_Q \setminus \mathcal{T}_Q^x \sqcup \mathcal{T}_Q^x, \leq_+^f) \qquad \mathcal{T}_{Q'} \simeq (\mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^x \sqcup \mathcal{T}_{Q'}^x, \leq_-^{f'})$$

Corollary. If $Q_1 \sim Q_2$ then $\mathcal{T}_{Q_1} \overset{u}{\sim} \mathcal{T}_{Q_2}$.

Generalized BGP reflections

Let $(Y, \leq)$ be poset, $Y_0 \subseteq Y$ a subset with the property

$$[y, \cdot] \cap [y', \cdot] = \phi = [\cdot, y] \cap [\cdot, y'] \quad \text{for all } y \neq y' \text{ in } Y_0$$

Define two partial orders $\leq_+^{Y_0}$, $\leq_-^{Y_0}$ on $\{*\} \cup Y$ as follows:

- Keep the original partial order inside Y.
- Add the relations

$$* <_+^{Y_0} y \iff \exists y_0 \in Y_0 \text{ with } y_0 \leq y$$
$$y <_-^{Y_0} * \iff \exists y_0 \in Y_0 \text{ with } y \leq y_0$$

  for $y \in Y$.

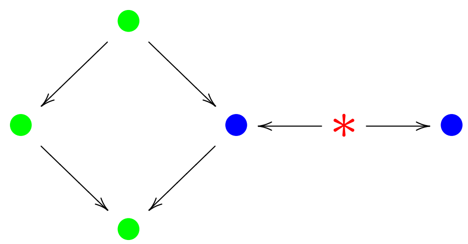
Generalized BGP reflections – continued

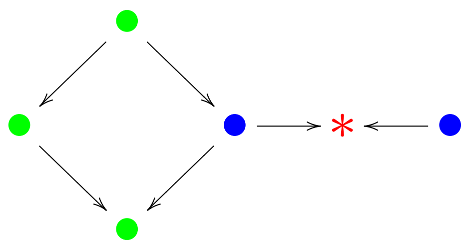
The vertex $*$ is a *source* in the Hasse diagram of $\leq_{+}^{Y_0}$, with arrows ending at the vertices of Y_0.

The Hasse diagram of $\leq_{-}^{Y_0}$ is obtained by reverting the orientations of the arrows from $*$, making it into a *sink*.

Theorem. $(\{*\} \cup Y, \leq_{+}^{Y_0}) \overset{u}{\sim} (\{*\} \cup Y, \leq_{-}^{Y_0})$.

Example.





Hybrid construction – setup

$(X, \leq_X)$, $(Y, \leq_Y)$ – posets, $\{Y_x\}_{x \in X}$ – collection of subsets $Y_x \subseteq Y$, with the properties:

- For all $x \in X$,

$$[y, \cdot] \cap [y', \cdot] = \phi = [\cdot, y] \cap [\cdot, y'] \quad \text{for all } y \neq y' \text{ in } Y_x$$

- For all $x \leq x'$, there exists an isomorphism $\varphi_{x,x'} : Y_x \xrightarrow{\sim} Y_{x'}$ with

$$y \leq_Y \varphi_{x,x'}(y) \qquad \text{for all } y \in Y_x$$

It follows that $\{Y_x\}_{x \in X}$ is a *local system* of subsets of Y:

$$\varphi_{x,x''} = \varphi_{x',x''} \varphi_{x,x'} \qquad \text{for all } x \leq x' \leq x''.$$

Hybrid construction – result

Define two partial orders on $\leq_+$, $\leq_-$ on $X \sqcup Y$ as follows:

- Keep the original partial orders inside X and Y.
- Add the relations

$$x \leq_+ y \iff \exists y_x \in Y_x \text{ with } y_x \leq_Y y$$
$$y \leq_- x \iff \exists y_x \in Y_x \text{ with } y \leq_Y y_x$$

for $x \in X$, $y \in Y$.

Theorem. $(X \sqcup Y, \leq_+) \overset{u}{\sim} (X \sqcup Y, \leq_-)$.

Remarks.

- When $X = \{*\}$, we recover the generalized BGP reflection.
- When $Y_x = \{*\}$ for all $x \in X$, we recover the flip-flop.

Mirroring with respect to a bipartite structure

Let S be *bipartite*. ($S = S_0 \sqcup S_1$ with $s < s' \Rightarrow s \in S_0$ and $s' \in S_1$)

Let $\mathfrak{X} = \{X_s\}_{s \in S}$ be a collection of posets indexed by S.

Define two partial orders $\leq_+$ and $\leq_-$ on $\bigsqcup_{s \in S} X_s$ as follows:

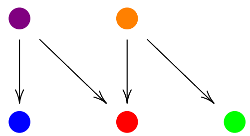
- Keep the original partial order inside each X_s.
- Add the relations

$$x_s <_+ x_t \iff s < t$$
$$x_t <_- x_s \iff t < s$$

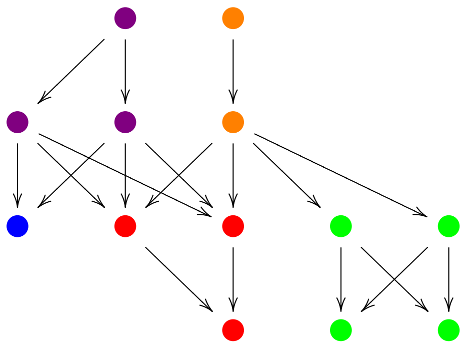
for $x_s \in X_s$, $x_t \in X_t$.

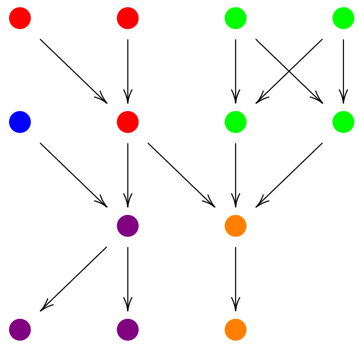
Theorem. $(\bigsqcup_{s \in S} X_s, \leq_+) \sim (\bigsqcup_{s \in S} X_s, \leq_-)$.

Bipartite structure – example

$S =$ 

$\mathfrak{X} =$ 

$(\bigsqcup_{s \in S} X_s, \leq_+)$ 

 $(\bigsqcup_{s \in S} X_s, \leq_-)$

Mates of triangular matrix algebras

Let k be a field, R and S k-algebras and ${}_RM_S$ bimodule. Consider the *triangular matrix algebras*

$$\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \text{and} \quad \widetilde{\Lambda} = \begin{pmatrix} S & DM \\ 0 & R \end{pmatrix}$$

where $DM = \mathrm{Hom}_k(M, k)$.

Theorem. $\mathcal{D}^b(\mathrm{mod}\,\Lambda) \simeq \mathcal{D}^b(\mathrm{mod}\,\widetilde{\Lambda})$, under the assumptions:

- $\dim_k R < \infty$, $\dim_k S < \infty$, $\dim_k M < \infty$
- $\mathrm{gl.dim}\, R < \infty$, $\mathrm{gl.dim}\, S < \infty$