INTRODUCTION TO REALIZABILITY OF MODULES OVER TATE COHOMOLOGY

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ABSTRACT. This paper is a companion to our paper, "Realizability of modules over Tate cohomology," in which we described an obstruction theory which applies in a number of contexts. This companion paper restricts attention to Tate cohomology, and gives constructions and proofs within that framework. A special case of our main theorem is as follows. Let k be a field and let G be a finite group. There is a canonical element in the Hochschild cohomology of the Tate cohomology $\gamma \in HH^{3,-1}\hat{H}^*(G,k)$ with the following property. Given a graded $\hat{H}^*(G,k)$-module X, the image of γ in $\mathrm{Ext}^{3,-1}_{\hat{H}^*(G,k)}(X,X)$ vanishes if and only if X is isomorphic to a direct summand of $\hat{H}^*(G,M)$ for some kG-module M. If X is realizable in this way, then the essentially different ways of realizing it form an affine space whose associated vector space is $\mathrm{Ext}^{2,-1}_{\hat{H}^*(G,k)}(X,X)$.

1. INTRODUCTION

This paper is a companion to our paper [2], in which we introduced an obstruction theory which applies in a number of different contexts.

Our investigation began with a study of the following question. Let k be a field and let G be a finite group. Given a graded module X over the Tate cohomology ring $\hat{H}^*(G,k)$, how do we decide whether there exists a kG-module M such that $\hat{H}^*(G,M) \cong X$? We described an element $\gamma \in HH^{3,-1}\hat{H}^*(G,k)$ with the property that its image $\bar{\gamma} \in \mathrm{Ext}^{3,-1}_{\hat{H}^*(G,k)}(X,X)$ is zero if and only if X is a direct summand of some module of the form $\hat{H}^*(G,M)$. The element γ in some sense encodes all possible information about Massey triple products of elements of Tate cohomology.

The theory was then generalized to various other contexts, including modules over the cohomology of a differential graded algebra, and a general obstruction theory in the context of triangulated categories. Unfortunately, in the process of generalization, it became hard for someone only interested in the case of Tate cohomology to extract from that paper a short, self contained version of the obstruction theory in that context. The purpose of this paper is to give the motivation, a description of the obstruction, and the proof of the theorem purely in the context of Tate cohomology, without any extra baggage. The main theorem of this paper is the following. The case

The first author is partly supported by a grant from the NSF.

where $N = k$ is the situation described above. We refer the reader to [2] for generalizations and further comments.

Theorem 1.1. *Let k be a field and let G be a finite group. Let N be a finite dimensional kG-module, and write $E = \widehat{\mathrm{Ext}}^*_{kG}(N,N)$ for the Tate Ext algebra of N, regarded as a graded k-algebra (so if $N = k$ then $E = \hat{H}^*(G,k)$). Then there exists a canonical element in Hochschild cohomology of E,*

$$\gamma \in HH^{3,-1}E,$$

with the following property. Given any graded E-module X, the following are equivalent:

(i) *The image $\bar{\gamma}$ of γ in $\mathrm{Ext}^{3,-1}_E(X,X)$ is zero.*

(ii) *There exists a kG-module M such that X is isomorphic to a direct summand of the graded E-module $\widehat{\mathrm{Ext}}^*_{kG}(N,M)$.*

The element γ has an interpretation as the degree three part of the A_∞-structure on $\hat{H}^*(G,k)$. The general context is that by a theorem of Kadeishvili [5], the cohomology of any differential graded algebra is an A_∞-algebra, well defined up to quasi-isomorphism. The Tate cohomology ring $\hat{H}^*(G,k)$ is the cohomology of a suitable A_∞-algebra. The structure map m_3 for the cohomology of a differential graded algebra is in general a Hochschild $(3,-1)$-cocycle, and changing the quasi-isomorphism class changes m_3 by a coboundary. The element γ is the Hochschild cohomology class represented by m_3. All this is explained further in our paper [2], but not here; we refer to Keller [6] for background material on A_∞-algebras and modules.

The element $\bar{\gamma}$ is the first of a sequence of obstructions in $\mathrm{Ext}^{n,2-n}_E(X,X)$ $(n \geq 3)$ which decide when X is realizable, not only as a summand. However, we have no nice interpretation of these higher obstructions and no predecessor in Hochschild cohomology. The obstruction $\bar{\gamma}$ works in the context of a triangulated category with arbitrary direct sums. In this context, N is a compact object in the category.

We also include a discussion of the parametrization of the realizations, in the case where $\bar{\gamma}$ is the zero element of $\mathrm{Ext}^{3,-1}_E(X,X)$.

Consider the collection $\mathsf{Split}(X)$ (too large to be a set) of ordered triples (M,i,π) consisting of a kG-module M and maps of E-modules

$$X \xrightarrow{i} \widehat{\mathrm{Ext}}^*_{kG}(N,M) \xrightarrow{\pi} X$$

whose composite is the identity map Id_X. We put an equivalence relation on this collection as follows. Two such triples (M,i,π) and (M',i',π') are said to be equivalent if there is a kG-module homomorphism $\rho\colon M \to M'$ making the following diagram commute.

$$\begin{array}{ccccc} & & \widehat{\mathrm{Ext}}^*_{kG}(N,M) & & \\ & \overset{i}{\nearrow} & & \overset{\pi}{\searrow} & \\ X & & \Big\downarrow \rho_* & & X \\ & \underset{i'}{\searrow} & & \underset{\pi'}{\nearrow} & \\ & & \widehat{\mathrm{Ext}}^*_{kG}(N,M') & & \end{array}$$

As it stands, this does not define an equivalence relation because it is not symmetric, but we complete it to one by taking the smallest equivalence relation on $\mathsf{Split}(X)$ containing all pairs of equivalent triples (M, i, π). It is not obvious that the equivalence classes in $\mathsf{Split}(X)$ form a set, but that is part of the content of the following theorem, whose proof can be found in Section 5.

Theorem 1.2. *If $\bar{\gamma} = 0$ then the equivalence classes in $\mathsf{Split}(X)$ form an affine space whose associated vector space is $\mathrm{Ext}^{2,-1}_E(X, X)$.*

To say that a set forms an affine space $\mathbb{A}$ with a given associated vector space V means that there is a free and transitive addition map $V \times \mathbb{A} \to \mathbb{A}$. So the difference between two elements of $\mathbb{A}$ is a well defined element of V, and a choice of which element to call zero in $\mathbb{A}$ defines a bijection with V.

As in the case of the obstruction $\bar{\gamma}$, this theorem works in any triangulated category with arbitrary direct sums. In algebraic topology, there are analogous appearances of Ext^2 groups in classification problems of realizable (co-)homology modules. Some comments on this can be found at the end of Section 5.

In a related paper, Beligiannis and Krause [1] investigate the question of when all maps from the Tate cohomology of a module are realizable. They develop an extended Milnor sequence which computes the obstructions in this context.

2. A motivating example

In this section, we motivate the discussion with an example. Let p be a prime, G be a cyclic group of order p, and k be a field of characteristic p.

In the case $p = 2$, the Tate cohomology ring is $\hat{H}^*(G, k) = k[x, x^{-1}]$ with $\deg(x) = 1$. In the graded sense, this is a field: every module over it is free. In particular, every module over $\hat{H}^*(G, k)$ can be realized as $\hat{H}^*(G, M)$ for a suitable kG-module M. Namely, M can be taken as a kG-module with trivial G-action, whose k-dimension is equal to the cardinality of a free generating set for the given $\hat{H}^*(G, k)$-module.

The case p odd is somewhat different. The Tate cohomology $\hat{H}^*(G, k)$ is the tensor product of $k[x]/(x^2)$ with a Laurent polynomial ring $k[y, y^{-1}]$. Here, the degrees are given by $\deg(x) = 1$ and $\deg(y) = 2$. The indecomposable kG-modules in this case correspond to Jordan blocks of length at most p. We write M_n $(1 \le n \le p)$ for the indecomposable module of length

n. It has a unique composition series, with just one submodule of each dimension from zero to n; we express this by saying that M_n is *uniserial.* The indecomposable projective kG-module is M_p, so the short exact sequence

$$0 \to M_{p-n} \to M_p \to M_n \to 0$$

shows that $\Omega(M_n) \cong M_{p-n}$. So $\Omega^i(M_n)$ is isomorphic to M_n if i is even and M_{p-n} if i is odd. So for all i we have

$$\hat{H}^i(G, M_n) \cong \underline{\mathrm{Hom}}_{kG}(k, \Omega^{-i}(M_n)) \cong k.$$

Multiplication by $y \in \hat{H}^2(G,k)$ is an isomorphism, but multiplication by x is harder to compute. For $2 \leq n \leq p-2$, x acts as zero. For $n = 1$ and $n = p-1$, M_n is k, respectively $\Omega(k)$, so $\hat{H}^*(G, M_n)$ is either $\hat{H}^*(G,k)$ or its shift in degree by one.

Let us consider the $\hat{H}^*(G,k)$-module $X = k[y, y^{-1}]$. As long as $p > 3$, we can choose a value of n satisfying $2 \leq n \leq p-2$, and then since x acts as zero, $\hat{H}^*(G, M_n)$ will decompose as a direct sum of X and its shift in degree by one. However, if $p = 3$ then the only nonprojective indecomposables are k and $\Omega(k)$, and there is not enough room to realize X as a direct summand of the Tate cohomology of a module. The problem is that the uniserial module of length three is already projective.

The relationship with Massey products is as follows. In order to build a module M_n whose cohomology has $k[y, y^{-1}]$ as a direct summand, we needed to be able to string together at least four copies of the trivial module to make a uniserial module of length four, so that we can form a module M_n with length at least two, and so that $\Omega(M_n)$ also has length at least two.

The obstruction theory for uniserial modules is well understood in terms of cup products and Massey products. If A, B, C and D are simple modules, then a uniserial extension with submodule B and quotient A (which we denote A/B) is represented by a nonzero element of $\mathrm{Ext}^1_{kG}(A,B)$. The cup product of an element $b \in \mathrm{Ext}^1_{kG}(B,C)$ with an element $a \in \mathrm{Ext}^1_{kG}(A,B)$ is zero in $\mathrm{Ext}^2_{kG}(A,C)$ precisely when the two extensions A/B and B/C can be fitted together to make a uniserial module of length three of the form $A/B/C$. If the cup product is zero, there may be a number of different ways of fitting together the twofold extensions to form a threefold extension. They are parametrized by elements of $\mathrm{Ext}^1_{kG}(A,C)$; but if A happens to be isomorphic to C then there can be unexpected isomorphisms between the resulting length three modules.

Given elements $a \in \mathrm{Ext}^1_{kG}(A,B)$, $b \in \mathrm{Ext}^1_{kG}(B,C)$, $c \in \mathrm{Ext}^1_{kG}(C,D)$, if the products ba and cb vanish so that there are uniserial modules $A/B/C$ and $B/C/D$, we may ask whether they can be fitted together to form a uniserial module of the form $A/B/C/D$. The obstruction to doing this is the Massey triple product $\langle c, b, a\rangle \in \mathrm{Ext}^2_{kG}(A,D)$. There is some indeterminacy involved; we can change the length three modules using elements of $\mathrm{Ext}^1_{kG}(A,C)$ and $\mathrm{Ext}^1_{kG}(B,D)$. So the Massey product is really only well defined up to adding an element of $c\,\mathrm{Ext}^1_{kG}(A,C) + \mathrm{Ext}^1_{kG}(B,D)\,a$. So in the case of the cyclic group of order three in characteristic three, there is a nonvanishing Massey triple product corresponding to the nonexistence of a uniserial module of length four. It is $\langle x, x, x\rangle = y$. For the cyclic group of order p in characteristic p with $p \neq 3$, there are no nonvanishing Massey triple products, but the p-fold Massey product $\langle x, \ldots, x\rangle$ is equal to y.

The preceding discussion should at least make some sense of the following theorem, which slightly generalizes the above setup and describes what happens in Theorem 1.1 if $N = k$ and G has cyclic Sylow p-subgroups. We refer to §7 of [2] for proofs.

Theorem 2.1. *Suppose that k has characteristic p and G has cyclic Sylow p-subgroups of order p^n. Then the following describes the obstruction $\gamma \in HH^{3,-1}\hat{H}^*(G,k)$.*

(i) *Unless $p^n = 3$, we have $\gamma = 0$.*

(ii) *If $p^n = 3$ and G is p-nilpotent then $\hat{H}^*(G,k) = k[y, y^{-1}, x]/(x^2)$ with* $\deg(x) = 1$, $\deg(y) = 2$. *In this case, γ is represented by the $(3,-1)$-cocycle m which satisfies*

$$m(y^i x \otimes y^j x \otimes y^\ell x) = y^{i+j+\ell+1} \qquad i,\ j,\ \ell \in \mathbb{Z}$$

and which vanishes on all other tensor products of monomials in x and y.

(iii) *If $p^n = 3$ and G is not p-nilpotent then $\hat{H}^*(G,k) = k[w, w^{-1}, v]/(v^2)$ with* $\deg(v) = 3$, $\deg(w) = 4$. *In this case, γ is represented by the $(3,-1)$-cocycle m which satisfies*

$$m(vw^i \otimes vw^j \otimes vw^\ell) = w^{i+j+\ell+2} \qquad i,\ j,\ \ell \in \mathbb{Z}$$

and which vanishes on all other tensor products of monomials in v and w. □

3. Notations and conventions.

Unless otherwise specified, when we talk about kG-modules, we mean *left* kG-modules. Let $\mathsf{Mod}(kG)$ be the category of all (not necessarily finitely generated) kG-modules. If M and N are left kG-modules, then $\widehat{\mathrm{Ext}}^*_{kG}(N,M)$ is a *right* $\widehat{\mathrm{Ext}}^*_{kG}(N,N)$-module by Yoneda composition. So unless otherwise specified, $\widehat{\mathrm{Ext}}^*_{kG}(N,N)$-modules are right modules.

We write $\mathsf{StMod}(kG)$ for the stable category of kG-modules. The objects in this category are the same as in $\mathsf{Mod}(kG)$, but the arrows are given by

$$\underline{\mathrm{Hom}}_{kG}(N,M) = \mathrm{Hom}_{kG}(N,M)/\mathrm{PHom}_{kG}(N,M)$$

where $\mathrm{PHom}_{kG}(N,M)$ is the linear subspace consisting of homomorphisms which factor through some projective module. The category $\mathsf{StMod}(kG)$ is a triangulated category, in which the triangles come from short exact sequences in $\mathsf{Mod}(kG)$. As usual, we denote by ΩM the kernel of a map from a projective module onto M, and by $\Omega^{-1}M$ the cokernel of an embedding of M into an injective module. These operations are well defined and mutually inverse on $\mathsf{StMod}(kG)$.

If V is a $\mathbb{Z}$-graded vector space, we write $V[n]$ for the graded vector space with $V[n]^i = V^{n+i}$. An element v of V^{n+i}, when regarded as an element of $V[n]^i$, is written $\Sigma^n v$. If V carries a differential $d\colon V^i \to V^{i+1}$ then the differential on $V[n]$ is defined by $d(\Sigma^n v) = (-1)^n \Sigma^n(dv)$. If a graded ring Λ acts on V on the left then it also acts on $V[n]$ via $\lambda(\Sigma^n v) = (-1)^{mn}\Sigma^n(\lambda v)$ where $m = \deg(\lambda)$. If Λ acts on V on the right then the action on $V[n]$ is given by $(\Sigma^n v)\lambda = \Sigma^n(v\lambda)$.

Cohomology of a graded module over a graded algebra is bigraded. The first index gives the cohomological degree and the second gives the internal degree. So for example if $E = \mathrm{Ext}^*_{kG}(N,N)$, we write $\mathrm{Ext}^{s,t}_E(X,Y)$ for $\mathrm{Ext}^s_E(X, Y[t])$. So an element of $\mathrm{Ext}^{3,-1}_E(X,Y)$ is represented by an exact sequence of graded modules

$$0 \to Y[-1] \to X_2 \to X_1 \to X_0 \to X \to 0.$$

Whenever convenient, we shall write a tensor product $\lambda_1 \otimes \cdots \otimes \lambda_n$ as an n-tuple $(\lambda_1, \dots, \lambda_n)$.

4. The obstruction $\bar\gamma$

We fix a kG-module N, and we write E for $\widehat{\mathrm{Ext}}^*_{kG}(N,N)$. In this section, given a graded E-module X, we define an obstruction

$$\bar\gamma \in \mathrm{Ext}^{3,-1}_E(X,X) = \mathrm{Ext}^3_E(X, X[-1]).$$

We prove that $\bar\gamma = 0$ if and only if X is isomorphic to a direct summand of an E-module of the form $\widehat{\mathrm{Ext}}^*_{kG}(N,M)$, for some kG-module M. In Section 7, we define the element

$$\gamma \in HH^{3,-1}E = HH^3(E, E[-1]),$$

and then in Section 8 we show that $\bar\gamma$ is the image of γ under the natural map from Hochschild cohomology to Ext.

Let X be a graded E-module, and let

$$0 \to K \to F_1 \xrightarrow{\rho} F_0 \xrightarrow{\varepsilon} X \to 0 \tag{4.1}$$

be the beginning of a free resolution of X over E. In other words, each of F_0 and F_1 is a direct sum of copies of E, with degree shifts as necessary to hit generators and relations for X, and K is defined as the kernel of ρ.

Since $\widehat{\mathrm{Ext}}^*_{kG}(N, \Omega^n N) \cong \widehat{\mathrm{Ext}}^*_{kG}(N,N)[-n]$, we can find kG-modules R_0 and R_1, each a direct sum of modules of the form $\Omega^n N$, $n \in \mathbb{Z}$, such that $\widehat{\mathrm{Ext}}^*_{kG}(N, R_0) = F_0$ and $\widehat{\mathrm{Ext}}^*_{kG}(N, R_1) = F_1$. Furthermore, there is a map $\alpha\colon R_1 \to R_0$ such that

$$\begin{array}{ccc}
\widehat{\mathrm{Ext}}^*_{kG}(N, R_1) & \xrightarrow{\alpha_*} & \widehat{\mathrm{Ext}}^*_{kG}(N, R_0) \\
\downarrow \cong & & \downarrow \cong \\
F_1 & \xrightarrow{\rho} & F_0
\end{array}$$

commutes. Complete $\alpha\colon R_1 \to R_0$ to a triangle in $\mathsf{StMod}(kG)$,

$$\Omega B \to R_1 \to R_0 \to B. \tag{4.2}$$

The long exact sequence in cohomology then gives

$$\cdots \to \widehat{\operatorname{Ext}}^*_{kG}(N,B)[-1] \to F_1 \xrightarrow{\rho} F_0 \to \widehat{\operatorname{Ext}}^*_{kG}(N,B) \to F_1[1] \to \cdots$$

so that we obtain a short exact sequence

$$0 \to X \xrightarrow{\sigma} \widehat{\operatorname{Ext}}^*_{kG}(N,B) \to K[1] \to 0.$$

Let $\bar{\gamma}$ be the element of

$$\operatorname{Ext}^1_E(K[1],X) \cong \operatorname{Ext}^1_E(K,X[-1]) \cong \operatorname{Ext}^3_E(X,X[-1])$$

defined by this short exact sequence. The second isomorphism here is the dimension shift given by Yoneda splice with the exact sequence (4.1). The element $\bar{\gamma}$ is well defined, by the following (more general) theorem.

Theorem 4.3. *Let*

$$M_2 \to M_1 \to M_0 \to \Omega^{-1}M_2$$

be any triangle in $\mathsf{StMod}(kG)$ *and let* X *be the cokernel of*

$$\widehat{\operatorname{Ext}}^*_{kG}(N,M_1) \to \widehat{\operatorname{Ext}}^*_{kG}(N,M_0).$$

Then the exact sequence of the triangle,

$$0 \to X[-1] \to \widehat{\operatorname{Ext}}^*_{kG}(N,M_2) \to \widehat{\operatorname{Ext}}^*_{kG}(N,M_1) \to \widehat{\operatorname{Ext}}^*_{kG}(N,M_0) \to X \to 0$$

represents the element $\bar{\gamma} \in \operatorname{Ext}^{3,-1}_E(X,X)$.

Proof. In the diagram of E-modules

$$\begin{array}{ccccccc}
F_1 & \longrightarrow & F_0 & \longrightarrow & X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \\
\widehat{\operatorname{Ext}}^*_{kG}(N,M_1) & \longrightarrow & \widehat{\operatorname{Ext}}^*_{kG}(N,M_0) & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

the vertical arrows, which have been constructed using the standard lifting argument, can be realized by a map of triangles

$$\begin{array}{ccccccc}
\Omega B & \longrightarrow & R_1 & \longrightarrow & R_0 & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_2 & \longrightarrow & M_1 & \longrightarrow & M_0 & \longrightarrow & \Omega^{-1}M_2.
\end{array}$$

This in turn gives rise to a map of extensions

$$\begin{array}{ccccccccccccc}
0 & \to & X[-1] & \to & \widehat{\operatorname{Ext}}^*_{kG}(N,\Omega B) & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & X & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \to & X[-1] & \to & \widehat{\operatorname{Ext}}^*_{kG}(N,M_2) & \to & \widehat{\operatorname{Ext}}^*_{kG}(N,M_1) & \to & \widehat{\operatorname{Ext}}^*_{kG}(N,M_0) & \to & X & \to & 0
\end{array}$$

which proves that these extensions represent the same element of the extension group $\operatorname{Ext}^{3,-1}_E(X,X)$, namely $\bar{\gamma}$; see Mac Lane [7], §III.5. □

Theorem 4.4. *The following are equivalent:*

(i) $\bar{\gamma} = 0$ *in* $\mathrm{Ext}_E^{3,-1}(X, X)$.

(ii) *The sequence*

$$0 \to X \xrightarrow{\sigma} \widehat{\mathrm{Ext}}^*_{kG}(N, B) \to K[1] \to 0 \tag{4.5}$$

splits.

(iii) *The* E*-module* X *is isomorphic to a direct summand of* $\widehat{\mathrm{Ext}}^*_{kG}(N, M)$ *for some* kG*-module* M.

Proof. For (i) $\Leftrightarrow$ (ii), we notice that by construction, under the isomorphism

$$\mathrm{Ext}_E^{3,-1}(X, X) \cong \mathrm{Ext}_E^1(K[1], X),$$

the class $\bar{\gamma}$ corresponds to the extension (4.5). So $\bar{\gamma} = 0$ if and only if this extension splits.

It is obvious that (ii) $\Rightarrow$ (iii), so we shall prove that (iii) $\Rightarrow$ (ii). Since the construction of $\bar{\gamma}$ is additive in X, we may assume that $X \cong \widehat{\mathrm{Ext}}^*_{kG}(N, M)$. Then the map $F_0 \to X$ is realized by a map $R_0 \to M$ whose composite with α is zero in $\mathsf{StMod}(kG)$, so this map lifts to a map $B \to M$. The induced map $\widehat{\mathrm{Ext}}^*_{kG}(N, B) \to \widehat{\mathrm{Ext}}^*_{kG}(N, M) = X$ splits the sequence (4.5). $\square$

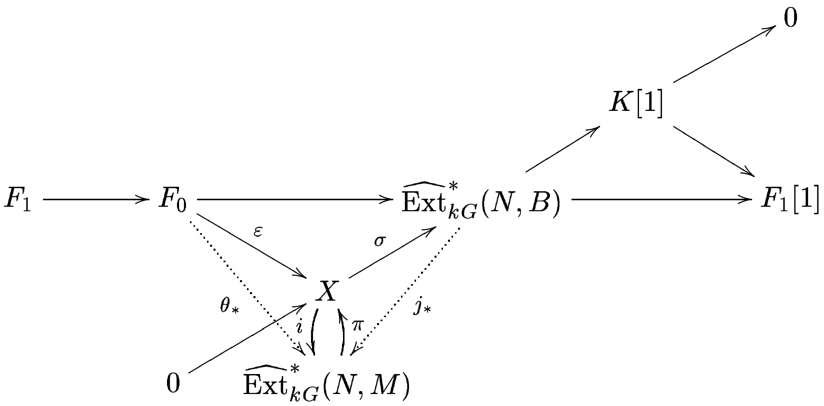
5. Parametrizing the realizations

In this section, we prove Theorem 1.2, parametrizing the realizations in the case where $\bar{\gamma}$ is the zero element of $\mathrm{Ext}_E^{3,-1}(X, X)$.

Recall from Section 1 that $\mathsf{Split}(X)$ consists of the ordered triples (M, i, π) consisting of a kG-module M and maps of E-modules

$$X \xrightarrow{i} \widehat{\mathrm{Ext}}^*_{kG}(N, M) \xrightarrow{\pi} X$$

whose composite is the identity map Id_X. There is an equivalence relation on $\mathsf{Split}(X)$ described Section 1, and Theorem 1.2 says that the equivalence classes form an affine space whose associated vector space is $\mathrm{Ext}_E^{2,-1}(X, X)$. In order to prove this theorem, we associate to two triples (M, i, π) and (M', i', π') a difference element in $d(M, i, \pi; M', i', \pi') \in \mathrm{Ext}_E^{2,-1}(X, X)$. To do this, we use the constructions described in Section 4. The composite $i \circ \varepsilon \colon F_0 \to \widehat{\mathrm{Ext}}^*_{kG}(N, M)$ can be realized by a map of kG-modules $\theta \colon R_0 \to M$ whose composite with $\alpha \colon R_1 \to R_0$ is zero in $\mathsf{StMod}(kG)$. It therefore lifts to a map $j \colon B \to M$ with the property that the composite $j_* \circ \sigma \colon X \to \widehat{\mathrm{Ext}}^*_{kG}(N, M)$ is equal to i. So composing with π gives the identity map on X.



It follows that $(\pi \circ j_* - \pi' \circ j'_*) \circ \sigma = 0$, so that $\pi \circ j_* - \pi' \circ j'_*$ defines a difference cocycle $K[1] \to X$. The difference element $d(M, i, \pi; M', i', \pi')$ is defined to be the image of this element of $\mathrm{Hom}_E(K[1], X)$ in $\mathrm{Ext}^{2,-1}_E(X, X)$. It is a routine exercise to check that this difference element does not depend on the choices involved in the construction, and that the following identities hold.

$$d(M', i', \pi'; M, i, \pi) = -d(M, i, \pi; M', i', \pi')$$
$$d(M, i, \pi; M', i', \pi') + d(M', i', \pi'; M'', i'', \pi'') = d(M, i, \pi; M'', i'', \pi'')$$

If two triples (M, i, π) and (M', i', π') are equivalent via a map $\rho \colon M \to M'$ then the map θ' may be chosen to be $\rho \circ \theta$ and j' can be chosen to be $\rho \circ j$. Then $\pi \circ j = \pi' \circ j'$ and so the difference element $d(M, i, \pi; M', i', \pi')$ is zero.

We can find a set of representatives of the equivalence classes as follows. A triple (M, i, π) is equivalent to the triple $(B, \sigma, \pi \circ j_*)$ via the map $j \colon B \to M$. So every triple is equivalent to a splitting of the sequence (4.5). It remains to examine when two such splittings are equivalent.

If $\lambda, \lambda' \colon \widehat{\mathrm{Ext}}^*_{kG}(N, B) \to X$ are splittings of (4.5), then $\lambda - \lambda'$ vanishes on X, and defines the map $K[1] \to X$ representing the difference element in $\mathrm{Ext}^{2,-1}_E(X, X)$. Conversely, any map $K[1] \to X$ can be used to adjust the splitting. If (B, σ, λ) and (B, σ, λ') are equivalent then the difference element is zero, so $\lambda - \lambda' \colon K[1] \to X$ factors as $K[1] \to F_1[1] \to X$. We can realize $F_1[1] \to X$ via a map $\Omega^{-1}R_1 \to B$. The composite $\phi \colon B \to \Omega^{-1}R_1 \to B$ then satisfies $\lambda \circ \phi_* = \lambda'$ and $\phi_* \circ \sigma = \sigma$. It follows that two splittings λ, λ' are equivalent if and only if there is a map $\phi \colon B \to B$ with $\lambda \circ \phi_* = \lambda'$ and $\phi_* \circ \sigma = \sigma$. If this holds, then $\lambda \circ (\phi - 1)_* = \lambda - \lambda'$ and $(\phi - 1)_* \circ \sigma = 0$. So $\phi - 1$ composes with $R_0 \to B$ to give zero, and it follows that $\phi - 1$ factors as $B \to \Omega^{-1}R_1 \to B$. Taking cohomology, we get a map $F_1[1] \to \widehat{\mathrm{Ext}}^*_{kG}(N, B)$ such that the composite $\mathrm{Ext}^*_{kG}(N, B) \to F_1[1] \to \widehat{\mathrm{Ext}}^*_{kG}(N, B)$ is $(\phi - 1)_*$. This vanishes on X, and so gives a factorization $K[1] \to F_1[1] \to X$ of the difference cocycle. This means that the difference element is zero in $\mathrm{Ext}^{2,-1}_E(X, X)$. This argument is reversible, so the splittings are equivalent if and only if the difference element is zero in $\mathrm{Ext}^{2,-1}_E(X, X)$. This completes the proof of Theorem 1.2.

Remarks 5.1. In algebraic topology, there are analogous appearances of Ext^2 groups in classification problems of realizations of (co-)homology modules. For example, given two p-local spectra X and Y and an isomorphism $f\colon H^*(X;\mathbb{F}_p) \to H^*(Y;\mathbb{F}_p)$ of modules over the Steenrod algebra $\mathcal{A}_p$, there is a corresponding element in $E_2^{0,0}$ of the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)) \Rightarrow [Y;X]_{t-s}.$$

Ignoring convergence issues, the first of an infinite sequence of obstructions to realizability as a map is the d_2 differential to $E_2^{2,1}$ in this spectral sequence.

Here we study a slightly different problem, namely realizations up to summands, and it turns out that then the only invariant is a class in Ext^2. This situation also comes up in stable homotopy theory in Bousfield's classification of K-local spectra at an odd prime [3]. Roughly speaking, Bousfield's classification is by the K-homology of a spectrum, including the Adams operations; these objects take values in an abelian category with injective dimension two. This implies that every algebraic homology object is realized as the K-homology of a spectrum, but there may be genuinely different realizations (i.e., by K-local spectra which are not stably equivalent).

Because the injective dimension is two, in the associated Adams spectral sequence, the only obstruction to realizing an algebraic morphism between the K-homologies by a geometric morphism is the d_2 differential, which lives in the respective Ext^2 group. Bousfield refers to this obstruction as the "K_*-k-invariant."

6. The map from Hochschild cohomology to Ext

If k is a field of coefficients and Λ is a k-algebra, we write Λ^e for $\Lambda \otimes_k \Lambda^{\mathrm{op}}$, so that Λ^e-modules are the same as Λ-Λ-bimodules with scalars from k acting the same way on both sides. If M is a Λ-Λ-bimodule, then Hochschild cohomology of Λ with coefficients in M is defined to be $H\!H^*(\Lambda, M) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda, M)$, In case $M = \Lambda$, we write $H\!H^*\Lambda$ for $H\!H^*(\Lambda,\Lambda)$.

If Λ is graded and M is a graded Λ-Λ-bimodule, then we use degree preserving maps in the above discussion and define

$$H\!H^{i,j}(\Lambda, M) = \mathrm{Ext}^i_{\Lambda^e}(\Lambda, M[j]),$$

and $H\!H^i(\Lambda, M) = H\!H^{i,0}(\Lambda, M)$. So the index i is the Hochschild cohomological index, and j comes from the internal grading. Our sign conventions for working with this definition are given in Section 3.

Hochschild cohomology can be calculated using the free resolution, often called the *bar resolution*,

$$\cdots \xrightarrow{d} \Lambda^{\otimes(n+2)} \xrightarrow{d} \Lambda^{\otimes(n+1)} \to \cdots \xrightarrow{d} \Lambda^{\otimes 2} \to \Lambda \to 0$$

where the nth term in the resolution is $\Lambda^{\otimes(n+2)}$. The Λ^e-module structure is given by

$$(\mu,\mu')(\lambda_0,\dots,\lambda_{n+1}) = (\mu\lambda_0,\lambda_1,\dots,\lambda_n,\lambda_{n+1}\mu').$$

The differential is defined by

$$d(\lambda_0,\dots,\lambda_{n+1}) = \sum_{i=0}^{n}(-1)^i(\lambda_0,\dots,\lambda_i\lambda_{i+1},\dots,\lambda_{n+1}).$$

For more details, see Cartan and Eilenberg [4], §IX.6.

If M is a Λ-Λ-bimodule, then we have

$$\mathrm{Hom}_{\Lambda^e}(\Lambda^{\otimes(n+2)}, M) \cong \mathrm{Hom}_k(\Lambda^{\otimes n}, M).$$

So a Hochschild n-cochain with coefficients in M is given by a function from n-tuples of elements of Λ to M. The differential is then given by

$$df(\lambda_1, \dots, \lambda_n) = (-1)^{|\lambda_1||f|}\lambda_1 f(\lambda_2, \dots, \lambda_n) + \sum_{i=1}^{n-1}(-1)^i f(\lambda_1, \dots, \lambda_i\lambda_{i+1}, \dots, \lambda_n) + (-1)^n f(\lambda_1, \dots, \lambda_{n-1})\lambda_n.$$

In the ungraded situation, all elements λ_i are interpreted as having degree zero, so the sign on the first term on the right of this equation disappears.

If X is a right Λ-module, then $\mathrm{Hom}_k(X, X)$ is a Λ-Λ-bimodule, there is a natural isomorphism

$$HH^*(\Lambda, \mathrm{Hom}_k(X, X)) \cong \mathrm{Ext}^*_\Lambda(X, X).$$

So the map of bimodules giving the Λ-action $\Lambda \to \mathrm{Hom}_k(X, X)$ induces a map

$$HH^*\Lambda \to \mathrm{Ext}^*_\Lambda(X, X).$$

An explicit way to describe the map is as follows. Given a projective resolution of Λ as a Λ^e-module and a Hochschild cochain α:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & P_n & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & \Lambda & \longrightarrow & 0 \\ & \big\downarrow{\scriptstyle \alpha} & & & & & & \\ & \Lambda & & & & & & \end{array}$$

we can apply $X \otimes_\Lambda -$ to obtain a cochain α_X:

$$\begin{array}{ccccccc} \cdots \longrightarrow & X \otimes_\Lambda P_n & \longrightarrow & X \otimes_\Lambda P_0 & \longrightarrow & X & \longrightarrow 0 \\ & \big\downarrow{\scriptstyle \alpha_X} & & & & & \\ & X & & & & & \end{array}$$

for this projective resolution of X. This construction commutes with the differential, so an element of Hochschild cohomology gives rise to a well defined element of $\mathrm{Ext}^*_\Lambda(X, X)$.

7. Description of the element γ

Let $\hat{P}_*$ be a complete resolution of N as a kG-module. In other words, $\hat{P}_*$ is obtained by splicing together a projective resolution and an injective resolution of N (recall that a kG-module is injective if and only if it is projective)

$$\begin{array}{ccccccccc} \cdots \to \hat{P}_2 \to \hat{P}_1 \to & \hat{P}_0 & & \longrightarrow & & \hat{P}_{-1} & \to \hat{P}_{-2} \to \cdots \\ & & \searrow & & \nearrow & & \\ & & & N & & & \\ & & \nearrow & & \searrow & & \\ & 0 & & & & 0 & \end{array}$$

We write $\Omega^n N$ for the kernel of $\hat{P}_{n-1} \to \hat{P}_{n-2}$ for $n \in \mathbb{Z}$.

Denote by A the cochain complex

$$A = \mathrm{Hom}^*_{kG}(\hat{P}_*, \hat{P}_*)$$

so that

$$A^n = \prod_j \mathrm{Hom}_{kG}(\hat{P}_{n+j}, \hat{P}_j)$$

with differential $d\colon A^n \to A^{n+1}$ defined by

$$(df)(x) = \partial(f(x)) - (-1)^n f(\partial(x))$$

where ∂ denotes the differential of $\hat{P}_*$. With this definition, the cocycles in A are the chain homomorphisms (negated if the degree is odd), and cocycles differ by a coboundary if and only if the corresponding chain maps are homologous. So A is a differential graded algebra whose cohomology $H^*(A)$ is Tate cohomology $\widehat{\mathrm{Ext}}^*_{kG}(N, N)$. So we can apply the construction described in §3.3 of Keller [6], which we repeat here for the convenience of the reader.

First, note that Keller writes m_2 for the multiplication maps,

$$m_2\colon A \otimes A \to A, \qquad m_2\colon H^*(A) \otimes H^*(A) \to H^*(A).$$

We regard $H^*(A)$ as a differential graded algebra with zero differential, and we choose a morphism of complexes $f_1\colon H^*(A) \to A$ which induces the identity in cohomology. This amounts to choosing a representative cocycle for each Tate cohomology class, in a linear fashion.

Now f_1 usually cannot be chosen to commute with multiplication, but at least it commutes up to coboundaries. So we choose a graded map of degree -1

$$f_2\colon H^*(A) \otimes H^*(A) \to A$$

satisfying

$$df_2(x, y) = f_1(xy) - f_1(x) f_1(y). \tag{7.1}$$

Given x, y, $z \in H^*(A)$, we have

$$d[(-1)^{|x|} f_1(x) f_2(y, z) - f_2(xy, z) + f_2(x, yz) - f_2(x, y) f_1(z)] = 0,$$

so the element inside the bracket is a cocycle. This means that we can find maps $m_3\colon H^*(A)^{\otimes 3} \to H^*(A)$ of degree -1 and $f_3\colon H^*(A)^{\otimes 3} \to A$ of degree -2, such that

$$f_1 m_3 - df_3 = m_2(f_1 \otimes f_2) - f_2(m_2 \otimes 1) + f_2(1 \otimes m_2) - m_2(f_2 \otimes f_1),$$

or

$$f_1(m_3(x,y,z)) - df_3(x,y,z) = \\ (-1)^{|x|} f_1(x) f_2(y,z) - f_2(xy,z) + f_2(x,yz) - f_2(x,y) f_1(z). \quad (7.2)$$

Lemma 7.3. *The map $m_3 \colon H^*(A)^{\otimes 3} \to H^*(A)$ of degree -1 described above is a Hochschild cocycle.*

Proof. This follows from the identity

$$m_2(1 \otimes m_3) - m_3(m_2 \otimes 1 \otimes 1) + m_3(1 \otimes m_2 \otimes 1) \\ - m_3(1 \otimes 1 \otimes m_2) + m_2(m_3 \otimes 1) = 0,$$

or

$$(-1)^{|w|} w m_3(x,y,z) - m_3(wx,y,z) + m_3(w,xy,z) \\ - m_3(w,x,yz) + m_3(w,x,y)z = 0. \quad (7.4)$$

To check this identity, apply f_1 to the left-hand side. Using equations 7.1 and 7.2, we see that the result is a coboundary. Since f_1 is the identity map on cohomology, the identity 7.4 is proved. □

Now the above construction of the Hochschild cocycle m_3 depends on some choices, but our next task is to prove that the Hochschild cohomology class it determines does not depend on these choices.

Proposition 7.5. *The Hochschild cohomology class*

$$\gamma \in HH^{3,-1} H^*(A)$$

determined by m_3 is independent of the choices made in defining m_3.

Proof. First let f_1' be another choice for the map f_1. Then there is a map $g_1 \colon H^*(A) \to A$ of degree -1 such that

$$dg_1(x) = f_1'(x) - f_1(x).$$

Setting

$$f_2'(x,y) = f_2(x,y) + g_1(xy) - (-1)^{|x|} f_1'(x) g_1(y) - g_1(x) f_1(y)$$
$$f_3'(x,y,z) = f_3(x,y,z) + g_1(m_3(x,y,z)) \\ - (-1)^{|x|} g_1(x) f_2(y,z) - (-1)^{|x|+|y|} f_2'(x,y) g_1(z)$$

it is easy to check that equations (7.1) and (7.2) hold for f_1', f_2', f_3' and m_3.

Next, we keep f_1 fixed and let f_2' be another choice for the map f_2. Then there are maps $n_2 \colon H^*(A) \otimes H^*(A) \to H^*(A)$ of degree -1 and $g_2 \colon H^*(A) \otimes H^*(A) \to A$ of degree -2 such that

$$f_1(n_2(x,y)) - dg_2(x,y) = f_2'(x,y) - f_2(x,y).$$

Setting

$$\begin{aligned}
f_3'(x,y,z) &= f_3(x,y,z) + (-1)^{|x|} f_2(x, n_2(y,z)) + f_1(x) g_2(y,z) \\
&\qquad - g_2(xy,z) + g_2(x,yz) - f_2(n_2(x,y),z) - g_2(x,y) f_1(z) \\
m_3'(x,y,z) &= m_3(x,y,z) + (-1)^{|x|} x n_2(y,z) \\
&\qquad - n_2(xy,z) + n_2(x,yz) - n_2(x,y)z,
\end{aligned} \tag{7.6}$$

we again check that equations (7.1) and (7.2) hold for f_1, f_2', f_3' and m_3'.

Finally, we keep f_1 and f_2 fixed, and let f_3' be another choice for the map f_3. Then f_3' differs from f_3 by a cocycle, so $df_3' = df_3$ and equations (7.1) and (7.2) hold for f_1, f_2, f_3' and m_3.

The upshot of this analysis is that the extent to which m_3 is not well defined is expressed by equation (7.6). This equation says that the difference between the cocycles m_3' and m_3 is the Hochschild coboundary of the Hochschild cochain $n_2 \colon H^*(A) \otimes H^*(A) \to H^*(A)$. $\square$

8. Comparing γ with $\bar{\gamma}$

In order to make the comparison between the definitions of γ and $\bar{\gamma}$, our goal is to construct a commutative diagram of E-modules

$$\begin{array}{ccccccccccccc}
 & & X \otimes_k E^{\otimes 4} & \xrightarrow{d} & X \otimes_k E^{\otimes 3} & \xrightarrow{d} & X \otimes_k E^{\otimes 2} & \xrightarrow{d} & X \otimes_k E & \to & X & \to & 0 \\
 & & \downarrow{\scriptstyle \gamma_X} & & \downarrow{\scriptstyle \lambda_*} & & \downarrow & & \downarrow & & \| & & \\
0 & \to & X[-1] & \to & \widehat{\operatorname{Ext}}^*_{kG}(N, \Omega B) & \to & F_1 & \xrightarrow{\rho} & F_0 & \to & X & \to & 0
\end{array} \tag{8.1}$$

where the top row is the Hochschild complex and the bottom row is the sequence defining $\bar{\gamma}$. An obvious simplification of this task is to choose F_1 and F_0 to be $X \otimes_k E^{\otimes 2}$ and $X \otimes_k E$ respectively, and to make the corresponding vertical maps the identity. The hard part of the proof is then to construct the map marked λ_* in this diagram in such a way that the two squares in which it is involved commute. Accomplishing this will complete the proof of Theorem 1.1.

We use the maps f_1, f_2, f_3 and m_3 defined in Section 7 to define explicit maps in $\mathsf{Mod}(kG)$ for use in the constructions which were used in Section 4 to define the bottom row of the above diagram. To this end, we define two functors R and Q from $\mathbb{Z}$-graded vector spaces to $\mathsf{Mod}(kG)$ via

$$R(V) = \bigoplus_{n \in \mathbb{Z}} V^n \otimes_k \Omega^n N, \qquad Q(V) = \bigoplus_{n \in \mathbb{Z}} V^n \otimes_k \hat{P}_n.$$

The G action is defined via the right tensor factor. There is a short exact sequence of kG-modules

$$0 \to R(V[-1]) \xrightarrow{i} Q(V) \xrightarrow{\partial} R(V) \to 0,$$

where

$$\partial(x, \zeta_1, \dots, \zeta_n, \alpha) = (-1)^{|x\zeta_1 \cdots \zeta_n|}(x, \zeta_1, \dots, \zeta_n, \partial(\alpha)).$$

We also write $\partial\colon Q(V) \to Q(V[1])$ for the map defined by the same formula. The module

$$R_n = R(X \otimes_k E^{\otimes n})$$

has the property that

$$\widehat{\operatorname{Ext}}^*_{kG}(N, R_n) \cong X \otimes_k E^{\otimes(n+1)}$$

which is the nth term of the complex described at the end of Section 6. We also write $R_n[m]$ for $R(X \otimes_k E^{\otimes n}[m])$, $Q_n[m]$ for $Q(X \otimes_k E^{\otimes n}[m])$ and Q_n for $Q_n[0]$. We define $\tilde{d}\colon R_n[m] \to R_{n-1}[m]$ by

$$\begin{aligned}\tilde{d}(x, \zeta_1, \dots, \zeta_n, \alpha) &= (x\zeta_1, \zeta_2, \dots, \zeta_n, \alpha) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i (x, \zeta_1, \dots, \zeta_i\zeta_{i+1}, \dots, \zeta_n, \alpha) + (-1)^n (x, \zeta_1, \dots, f_1(\zeta_n)(\alpha)).\end{aligned}$$

After applying $\widehat{\operatorname{Ext}}^*_{kG}(N, -)$, this induces the bar complex differential.

The same formula defines a map which we also denote $\tilde{d}$ from $Q_n[m]$ to $Q_{n-1}[m]$. The following diagram then commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_n & \xrightarrow{i} & Q_n[1] & \xrightarrow{\partial} & R_n[1] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{d}} & & \downarrow{\scriptstyle \tilde{d}} & & \downarrow{\scriptstyle \tilde{d}} & & \\ 0 & \longrightarrow & R_{n-1} & \xrightarrow{i} & Q_{n-1}[1] & \xrightarrow{\partial} & R_{n-1}[1] & \longrightarrow & 0 \end{array}$$

An easy calculation using equation (7.1) shows that we have

$$\begin{aligned}\tilde{d}\tilde{d}(x, \zeta_1, \dots, \zeta_n, \alpha) &= (x, \zeta_1, \dots, \zeta_{n-2}, (df_2(\zeta_{n-1}, \zeta_n))(\alpha)) \\ &= (x, \zeta_1, \dots, \zeta_{n-2}, \partial(f_2(\zeta_{n-1}, \zeta_n)(\alpha)) \\ &\qquad + (-1)^{|\zeta_{n-1}\zeta_n|} f_2(\zeta_{n-1}, \zeta_n)(\partial\alpha)).\end{aligned}$$

Define

$$\phi = 1 \otimes \dots \otimes 1 \otimes m_2(f_2 \otimes 1)\colon Q_n[1] \to Q_{n-2}$$

so that

$$\phi(x, \zeta_1, \dots, \zeta_n, \alpha) = (-1)^{|x\zeta_1 \dots \zeta_{n-2}|} (x, \zeta_1, \dots, \zeta_{n-2}, f_2(\zeta_{n-1}, \zeta_n)(\alpha)).$$

Then we have

$$\begin{aligned}\partial\phi(x, \zeta_1, \dots, \zeta_n, \alpha) &= (x, \zeta_1, \dots, \zeta_{n-2}, \partial(f_2(\zeta_{n-1}, \zeta_n)(\alpha))) \\ \phi\partial(x, \zeta_1, \dots, \zeta_n, \alpha) &= (-1)^{|x\zeta_1 \dots \zeta_n|} \phi(x, \zeta_1, \dots, \zeta_n, \partial\alpha) \\ &= (-1)^{|\zeta_{n-1}\zeta_n|} (x, \zeta_1, \dots, \zeta_{n-2}, f_2(\zeta_{n-1}, \zeta_n)(\partial\alpha)),\end{aligned}$$

and so

$$\tilde{d}\tilde{d} = \partial\phi + \phi\partial. \tag{8.2}$$

Next, define

$$\psi = 1 \otimes \dots \otimes 1 \otimes (f_1 m_3 - df_3)\colon Q_n[1] \to Q_{n-3}$$

so that

$$\begin{aligned}\psi(x,\zeta_1,\dots,\zeta_n,\alpha) = (-1)^{|x\zeta_1\cdots\zeta_{n-3}|}\big(&x,\zeta_1,\dots,\zeta_{n-3},\\ &[f_1(m_3(\zeta_{n-2},\zeta_{n-1},\zeta_n)) - df_3(\zeta_{n-2},\zeta_{n-1},\zeta_n)](\alpha)\big).\end{aligned}$$

Then a similar calculation using (7.2) shows that

$$\tilde{d}\phi - \phi\tilde{d} = (-1)^n\psi. \tag{8.3}$$

It is also easy to check that $\psi\partial = \partial\psi$, and so ψ induces a well defined map which we also denote $\psi\colon R_n[1] \to R_{n-3}$.

Define B to be the pushout of

$$\begin{array}{ccc} R_1 & \xrightarrow{\tilde{d}} & R_0 \\ \Big\downarrow{\scriptstyle i} & & \\ Q_1[1] & & \end{array}$$

in the category of kG-modules, so that there are exact sequences

$$0 \to R_1 \to R_0 \oplus Q_1[1] \to B \to 0 \tag{8.4}$$

and

$$0 \to R_0 \to B \to R_1[1] \to 0 \tag{8.5}$$

giving rise to a triangle (4.2).

In order to define a map

$$\lambda\colon Q_2[1] \to B,$$

we first define a map

$$\tilde{\lambda} = \begin{pmatrix}\phi\\ \tilde{d}\end{pmatrix} : Q_2[1] \to Q_0 \oplus Q_1[1]$$

or more explicitly,

$$\begin{aligned}\tilde{\lambda}(x,\zeta_0,\zeta_1,\alpha) = \big(&(-1)^{|x|}(x, f_2(\zeta_0,\zeta_1)(\alpha)),\\ &(x\zeta_0,\zeta_1,\alpha) - (x,\zeta_0\zeta_1,\alpha) + (x,\zeta_0,f_1(\zeta_1)(\alpha))\big).\end{aligned}$$

Equation (8.2) shows that the left hand square of the following diagram commutes, so that there is a map λ such that the right hand square also commutes.

$$\begin{array}{ccccccc} Q_2 & \xrightarrow{\partial} & Q_2[1] & \xrightarrow{\partial} & R_2[1] & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle \begin{pmatrix}-\phi\\ \tilde{d}\end{pmatrix}} & & \Big\downarrow{\scriptstyle \tilde{\lambda}=\begin{pmatrix}\phi\\ \tilde{d}\end{pmatrix}} & & \Big\downarrow{\scriptstyle \lambda} & & \\ Q_0[-1]\oplus Q_1 & \xrightarrow{\begin{pmatrix}\partial & \tilde{d}\\ 0 & \partial\end{pmatrix}} & Q_0\oplus Q_1[1] & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The map λ induces a map

$$\lambda_*\colon X \otimes_k E^{\otimes 3} \to \widehat{\mathrm{Ext}}^*_{kG}(N, B),$$

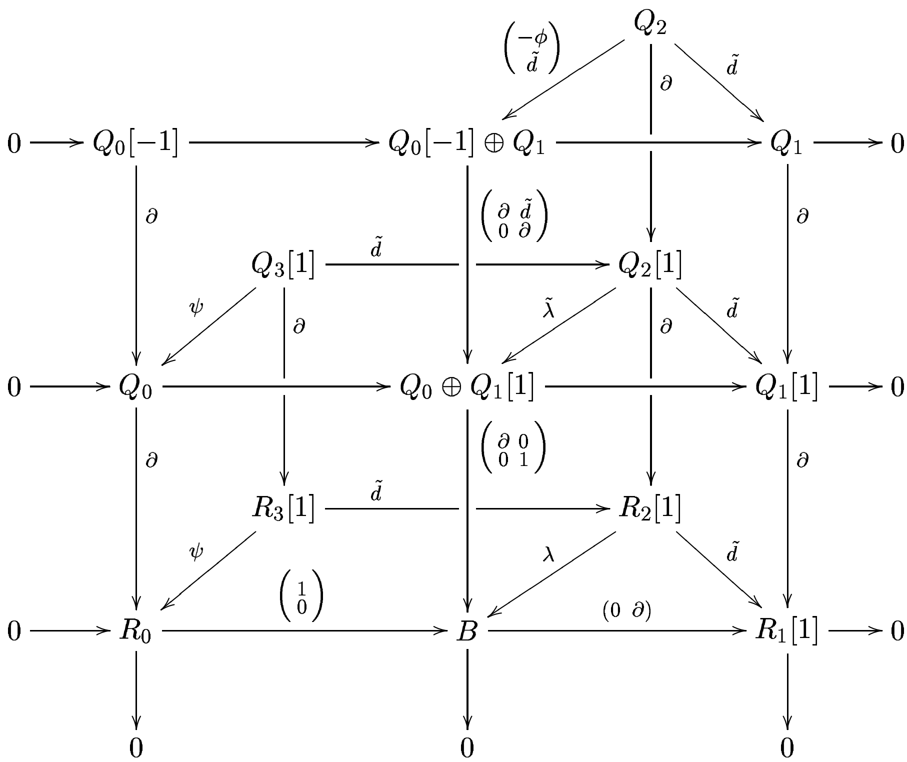


FIGURE 1

which is the desired map for diagram (8.1). It remains to check that the two squares in diagram (8.1) in which λ_* is involved commute. For this purpose, we examine the diagram in Figure 1. This diagram does not quite commute. The four front squares and the back square commute, the triangles commute, and the "side walls" commute. But the two "horizontal" squares only commute after applying $\widehat{\operatorname{Ext}}^*_{kG}(N,-)$. To see this, we calculate with maps from $R_3[1]$ to B. Using equations (8.2) and (8.3), we have

$$\begin{aligned}\lambda\tilde{d}\partial = \lambda\partial\tilde{d} &= \begin{pmatrix}\partial & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\phi\\ \tilde{d}\end{pmatrix}\tilde{d} = \begin{pmatrix}\partial & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\tilde{d}\phi+\psi\\ \partial\phi+\phi\partial\end{pmatrix}\\ &= \begin{pmatrix}\partial & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\partial & \tilde{d}\\ 0 & \partial\end{pmatrix}\begin{pmatrix}0\\ \phi\end{pmatrix} + \begin{pmatrix}\partial & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi\\ \phi\partial\end{pmatrix} = \begin{pmatrix}\partial & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi\\ \phi\partial\end{pmatrix} = \begin{pmatrix}\psi\partial\\ \phi\partial\end{pmatrix}.\end{aligned}$$

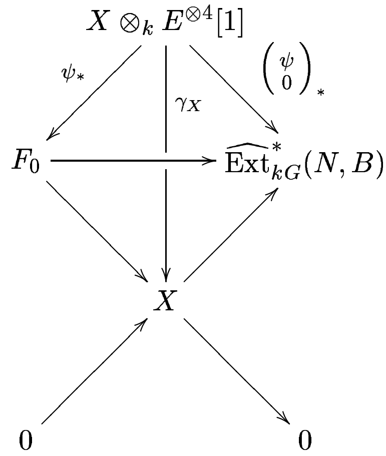
Since $\partial\colon Q_3[1] \to R_3[1]$ is surjective, it follows that

$$\lambda\tilde{d} = \begin{pmatrix}\psi\\ \phi\end{pmatrix}\colon R_3[1] \to B.$$

Now the map $\left(\begin{smallmatrix}0\\ \phi\end{smallmatrix}\right)$ factors through the projective module $Q_1[1]$. So the map in cohomology is zero, and it follows that the map

$$\lambda_* d\colon X \otimes_k E^{\otimes 4}[1] = \widehat{\operatorname{Ext}}^*_{kG}(N, R_3[1]) \to \widehat{\operatorname{Ext}}^*_{kG}(N, B)$$

is the same as the map induced by $\binom{\psi}{0}$. Finally, it is easy to see that the following diagram commutes.



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