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An exact sequence interpretation of the Lie bracket in Hochschild cohomology

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Abstract. We give an interpretation of the Lie bracket and the divided squaring operations in the Hochschild cohomology of an associative algebra in terms of exact sequences of bimodules. We construct natural loops in the category of extensions whose homotopy classes represent the Lie bracket and squaring operations.

Introduction

Let k be a commutative ring, A a k-algebra and M an A-bimodule. The Hochschild cohomology groups $\mathrm{H}_k^*(A; M)$ of A with coefficients in M are defined as the cohomology of a certain cochain complex. If the bimodule M is the algebra A itself, one simply speaks of the Hochschild cohomology groups of A and writes $\mathrm{H}_k^*(A)$. In this case, there is a graded commutative cup product

$$\cup : \mathrm{H}_k^m(A) \otimes \mathrm{H}_k^n(A) \to \mathrm{H}_k^{m+n}(A),$$

as well as a Lie bracket

$$[-,-] : \mathrm{H}_k^m(A) \otimes \mathrm{H}_k^n(A) \to \mathrm{H}_k^{m+n-1}(A)$$

and divided square operations

$$Sq : \mathrm{H}_k^n(A) \to \mathrm{H}_k^{2n-1}(A),$$

Sq being defined when A is of characteristic 2 or n is even. The definitions of these operations are in terms of explicit formulae on the cochain level. The term Gerstenhaber-algebra is used for the kind of structure present in Hochschild cohomology, as a tribute to M. Gerstenhaber who introduced the Lie bracket and the squaring operations [G]. The Hochschild cohomology groups are naturally isomorphic to relative bimodule Ext groups and the standard isomorphism takes the cup product to the Yoneda product, defined by splicing of exact sequences. The purpose of this paper is to give an interpretation of the Lie bracket and the squaring operations in terms of exact sequences. For simplicity we will restrict to the case where A is projective as a k-module.

For any ring R and left R-modules M and N we denote by $\mathscr{E}xt^n_R(M,N)$ the category of n-fold extensions of M by N. The group $\operatorname{Ext}^n_R(M,N)$ can be defined as the path components of (the classifying space of) the category $\mathscr{E}xt^n_R(M,N)$ with group structure induced from Baer sum. V. Retakh [R] has determined the homotopy type of the classifying space of the category $\mathscr{E}xt^n_R(M,N)$: it is homotopy equivalent to a product of Eilenberg-MacLane spaces, and its i-th homotopy group is isomorphic to $\operatorname{Ext}^{n-i}_R(M,N)$ for $0 \leqq i \leqq n$ and trivial for $i > n$.

Our interpretation of the Lie bracket involves "loops of extensions". We construct natural elements in the fundamental group of the category $\mathscr{E}xt^{m+n}_{A^e}(A,A)$, where $A^e = A \otimes_k A^{op}$. By Retakh's theorem, this fundamental group is isomorphic to $\operatorname{Ext}^{m+n-1}_{A^e}(A,A)$. Given two bimodule extensions F and E of length m and n respectively, we denote by $F \# E$ the $(m+n)$-fold extension obtained by splicing. The tensor product of E and F, considered as augmented complexes of right (resp. left) A-modules, is another $(m+n)$-fold extension which comes with edge morphisms in $\mathscr{E}xt^{m+n}_{A^e}(A,A)$

$$F \# E \leftarrow E \otimes_A F \rightarrow (-1)^{mn} E \# F\,.$$

A mild flatness assumption is needed to ensure that the sequence $E \otimes_A F$ is exact. This pair of morphisms shows directly that the Yoneda ring structure on $\operatorname{Ext}^*_{A^e}(A,A)$ is graded commutative.

Instead of $E \otimes_A F$ one can also use the tensor product $(-1)^{mn} F \otimes_A E$ to relate the two spliced sequences. Since the tensor product of bimodules over A is not symmetric, we get two different paths between $F \# E$ and $(-1)^{mn} E \# F$. The upshot is a loop in the category $\mathscr{E}xt^{m+n}_{A^e}(A,A)$.

$$\begin{tikzcd}
& E \otimes_A F \arrow[dl] \arrow[dr] & \\
F \# E & & (-1)^{mn} E \# F \\
& (-1)^{mn} F \otimes_A E \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The construction of this loop is functorial in F and E, and we let

$$\Omega : \pi_0 \mathscr{E}xt^m_{A^e}(A,A) \times \pi_0 \mathscr{E}xt^n_{A^e}(A,A) \rightarrow \pi_1 \mathscr{E}xt^{m+n}_{A^e}(A,A)$$

be the induced map on components. Our main Theorem 3.1 says that the so defined *loop bracket* corresponds to the Lie bracket on Hochschild cohomology.

Theorem. *Assume that A is k-projective. Then the following diagram commutes up to the sign $(-1)^n$:*

$$\begin{tikzcd} \mathrm{H}^m_k(A) \times \mathrm{H}^n_k(A) \arrow[r, "{[-,-]}"] \arrow[d, "K \times K"'] & \mathrm{H}^{m+n-1}_k(A) \arrow[d, "\mu"] \\ \pi_0 \mathscr{E}xt^m_{A^e}(A,A) \times \pi_0 \mathscr{E}xt^n_{A^e}(A,A) \arrow[r, "\Omega"'] & \pi_1 \mathscr{E}xt^{m+n}_{A^e}(A,A) \end{tikzcd}$$

The vertical isomorphisms K and μ will be defined in Section 3. We give a similar interpretation of the divided squaring operations by loops of length 2.

The plan for this paper is as follows. In Section 1 we provide an explicit isomorphism of the fundamental group of $\mathscr{E}xt^n_R(M,N)$ (with arbitrary basepoint) with the group $\mathrm{Ext}^{n-1}_R(M,N)$. In Section 2 we define the loop operations for extensions. In Section 3 we review the Hochschild complex and the operations defined therein and compare them to the loop operations. Proofs are deferred to Section 4.

1. Loops of extensions

Every (small) category has a *classifying space*, defined as the geometric realization of its nerve (see [Q], §1). For any choice of object as base point, this classifying space has homotopy groups. In this paper we will be interested in the fundamental group of a category of extensions. We now recall how the fundamental group can be defined combinatorially, i.e., without reference to the classifying space. The fact that this group is (isomorphic to) the fundamental group of the classifying space is implicit in [Q], §1, Prop. 1.

Let X and Y be objects of a small category $\mathscr{C}$. A *path* Λ from X to Y is a zigzag of morphisms of $\mathscr{C}$

$$X = \Lambda_0 \xleftrightarrow{\alpha_1} \Lambda_1 \xleftrightarrow{\alpha_2} \cdots \xleftrightarrow{\alpha_{n-1}} \Lambda_{n-1} \xleftrightarrow{\alpha_n} \Lambda_n = Y.$$

Here the Λ_i are objects of $\mathscr{C}$ and the α_i are morphisms of $\mathscr{C}$. The symbol $\xleftrightarrow{\alpha_i}$ means that α_i either goes from Λ_{i-1} to Λ_i or it goes the other direction. The number n will be referred to as the length of the path. There is a unique path of length 0 from X to itself which we call the trivial path.

We call two paths Λ and Λ' of length n and $n+1$ *elementary homotopic* if there exist composable morphisms $f\colon C_0 \to C_1$ and $g : C_1 \to C_2$ in $\mathscr{C}$ such that Λ' is obtained from Λ by replacing a morphism occurring as one side of the triangle

$$\begin{tikzcd} & C_1 \arrow[dr, "g"] & \\ C_0 \arrow[ur, "f"] \arrow[rr, "gf"'] & & C_2 \end{tikzcd}$$

by the two other morphisms pointing in the appropriate directions. This means the following three types of modifications give rise to elementary homotopies:

- If $C_0 \xrightarrow{f} C_1$ occurs in Λ, it can be replaced by $C_0 \xrightarrow{gf} C_2 \xleftarrow{g} C_1$.
- If $C_1 \xrightarrow{g} C_2$ occurs in Λ, it can be replaced by $C_1 \xleftarrow{f} C_0 \xrightarrow{gf} C_1$.
- If $C_0 \xrightarrow{gf} C_2$ occurs in Λ, it can be replaced by $C_0 \xrightarrow{f} C_1 \xrightarrow{g} C_2$.

In addition, the trivial path at X is declared to be elementary homotopic to the path of length 1 consisting of the identity morphism of X. We say that two paths are *homotopic* relative to X and Y if they are equivalent under the equivalence relation generated by elementary homotopy.

A suitable combination of elementary homotopies shows that two paths are homotopic if one can be obtained from the other by inserting or deleting identity morphisms, replacing a morphism by any left or right inverse pointing in the opposite direction, or deleting or inserting pieces of the form

$$C \xrightarrow{f} C' \xleftarrow{f} C \quad \text{or} \quad C \xleftarrow{g} C'' \xrightarrow{g} C\,.$$

A *loop* at X is a path from X to itself. Concatenation of loops gives an associative monoid structure with the trivial loop as identity element. This concatenation respects the homotopy relation, so it passes to homotopy classes. On the level of homotopy classes, every loop has an inverse given by the same loop read from right to left. We denote the resulting group of relative homotopy classes of loops based at X by $\pi_1(\mathscr{C}, X)$ and refer to it as the fundamental group. We also use the notation $\pi_0\mathscr{C}$ for the set of components of the category $\mathscr{C}$. $\pi_0\mathscr{C}$ is the quotient of the set of objects of $\mathscr{C}$ by the equivalence relation generated by the existence of morphisms between objects.

In general, the fundamental group depends on the choice of basepoint. If two objects X and Y are in the same component of the category $\mathscr{C}$, the fundamental groups $\pi_1(\mathscr{C}, X)$ and $\pi_1(\mathscr{C}, Y)$ are non-canonically isomorphic. In fact, conjugation with a path between X and Y determines an isomorphism, and two such isomorphisms differ by an inner automorphism. Fundamental groups based at objects in different components can be non-isomorphic.

Now let R be a (unital and associative) ring and M and N left R-modules. We denote by $\mathscr{E}xt^n_R(M, N)$ the category of n-fold extensions of M by N ($n \geqq 1$). An object of $\mathscr{E}xt^n_R(M, N)$ is an exact sequence of left R-modules

$$E: 0 \to N \to E_{n-1} \to \cdots \to E_0 \to M \to 0\,.$$

We also use the notation $E_{-1} = M$ and $E_n = N$. A morphism $E \to F$ is a commutative diagram of R-module homomorphisms

$$\begin{array}{ccccccccc} N & \longrightarrow & E_{n-1} & \longrightarrow & \cdots & \longrightarrow & E_0 & \longrightarrow & M \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ N & \longrightarrow & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_0 & \longrightarrow & M \end{array}$$

For $n=1$, every morphism is an isomorphism, i.e., the category $\mathscr{E}xt^1_R(M,N)$ is a groupoid.

There is a standard way of identifying Ext groups, defined via projective resolutions, with the components of the extension categories (see [MacL], III, Thm. 6.4). We recall this identification in some detail in order to set up notation and extend it to an identification of the fundamental group of the extension category. We let $P \to M$ be a projective resolution, so that the cohomology groups of the cochain complex $\mathrm{Hom}_R(P,N)$ serve as the groups $\mathrm{Ext}^*_R(M,N)$. For any n-cocycle $\varphi : P_n \to N$ (i.e., $\varphi \cdot d = 0$ where d denotes the differential of P), an object $K(\varphi)$ of the category $\mathscr{E}xt^n_R(M,N)$ is defined as follows. First we let $K(\varphi)_{n-1}$ be the pushout of the diagram

$$P_{n-1} \xleftarrow{\ d\ } P_n \xrightarrow{\ \varphi\ } N\,.$$

So $K(\varphi)_{n-1}$ is the quotient module of $P_{n-1} \oplus N$ by the submodule consisting of the elements of the form $\big(-d(x), \varphi(x)\big)$ for all $x \in P_n$. We set $K(\varphi)_i = P_i$ for $0 \leqq i < n-1$ and $K(\varphi)_{-1} = M$. The maps $N \to K(\varphi)_{n-1} \to K(\varphi)_{n-2} = P_{n-2}$ are induced by $n \mapsto (0,n)$ and $(p,n) \mapsto d(p)$ respectively. All other differentials are the ones from P. Since $\varphi \cdot d = 0$, the map $N \to K(\varphi)_{n-1}$ is injective. Furthermore, the whole sequence $K(\varphi)$ is exact, defining an object of $\mathscr{E}xt^n_R(M,N)$.

An $(n-1)$-cochain of the complex $\mathrm{Hom}_R(P,N)$ is simply a homomorphism $\kappa : P_{n-1} \to N$. The cochain determines a morphism

$$\mu(\kappa) : K(\varphi) \to K(\varphi + \kappa \cdot d)$$

in the category $\mathscr{E}xt^n_R(M,N)$ for any n-cocycle φ. The morphism $\mu(\kappa)$ is the identity except in dimension $n-1$. There,

$$\mu(\kappa)_{n-1} : K(\varphi)_{n-1} \to K(\varphi + \kappa \cdot d)_{n-1}$$

is induced on quotients by $(p,n) \mapsto \big(p, n - \kappa(p)\big)$. Then $\mu(\kappa)$ is a map of extensions and μ satisfies

$$\mu(\kappa + \kappa') = \mu(\kappa') \cdot \mu(\kappa) \quad \text{and} \quad \mu(0) = \mathrm{id}\,.$$

The existence of the morphisms $\mu(\kappa)$ shows that the assignment $\varphi \mapsto K(\varphi)$ induces a well defined map

$$K : \mathrm{H}^n\big(\mathrm{Hom}_R(P,N)\big) \to \pi_0 \mathscr{E}xt^n_R(M,N)\,.$$

This map is a bijection and a homomorphism with respect to Baer sum of extensions. In fact, K is inverse to the isomorphism ζ of [MacL], III, Thm. 6.4.

Now suppose the $(n-1)$-cochain $\kappa : P_{n-1} \to N$ is actually a cocycle, so $\kappa \cdot d = 0$. Then $\mu(\kappa)$ is an endomorphism of the extension $K(\varphi)$, i.e., a loop based at $K(\varphi)$.

Theorem 1.1. *The homotopy class of the loop $\mu(\kappa)$ depends only on the cohomology class of κ. The induced map*

$$\mu : \mathrm{H}^{n-1}\big(\mathrm{Hom}_R(P, N)\big) \to \pi_1\big(\mathcal{E}xt^n_R(M, N), K(\varphi)\big)$$

is an isomorphism of groups for all n-cocycles φ.

The left hand side of Theorem 1.1 is the group $\mathrm{Ext}^{n-1}_R(M, N)$, so we recover a special case of Retakh's Theorem 1 of [R]. For the convenience of the reader, and because certain tools will be needed for the poof of Theorem 3.1, we reprove this result in Section 4.

The cautious reader will have noticed that the category $\mathcal{E}xt^n_R(M, N)$ is not small, nor even equivalent to a small category. There are various ways to avoid this problem, depending on the underlying framework for set theory. We will ignore this point and treat $\mathcal{E}xt^n_R(M, N)$ as if it were a small category.

2. The loop bracket

In this section we define the loop bracket of bimodule extensions of an algebra A by itself, as well as divided square loops. We will show in Theorem 3.1 that the loop bracket corresponds to the Lie bracket on Hochschild cohomology groups, and that the divided square loop corresponds to the divided square operation. We fix a commutative ring k and a k-algebra A. We assume for simplicity that A is projective as a k-module. In terms of the previous section, we now take the ring R to be $A^{\mathrm{e}} = A \otimes_k A^{\mathrm{op}}$, so that the R-modules are precisely the k-symmetric A-bimodules.

We consider objects F of $\mathcal{E}xt^m_{A^{\mathrm{e}}}(A, A)$ and E of $\mathcal{E}xt^n_{A^{\mathrm{e}}}(A, A)$ with $m, n \geqq 1$. We denote by $i_F : A \to F_{m-1}$ the injection and by $p_F : F_0 \to A$ the surjection at the two ends of F, and similarly for E. $E \# F$ then denotes spliced sequence

$$A \to F_{m-1} \to \cdots \to F_0 \xrightarrow{i_E \cdot p_F} E_{n-1} \to \cdots \to E_0 \to A\,.$$

We denote by $(-1)E$ the sequence obtained from E by replacing p_E by $-p_E$. The sequence $(-1)E$ represents the inverse with respect to Baer sum of the sequence E in $\pi_0\mathcal{E}xt^n_{A^{\mathrm{e}}}(A, A)$. With this convention, the sequences $\big((-1)E\big) \# F$ and $(-1)(E \# F)$ are *equal*, so we can leave out the parenthesis. Both sequences are *isomorphic* to $E \# \big((-1)F\big)$. Splicing induces a product on the components of the extension category, and we want to see that this so called *Yoneda product* is graded commutative. In other words, we are looking for a path from $F \# E$ to $(-1)^{mn}E \# F$ in $\mathcal{E}xt^{m+n}_{A^{\mathrm{e}}}(A, A)$.

For this we view the exact sequence E as a truncated complex

$$\cdots \to 0 \to A \to E_{n-1} \to \cdots \to E_0 \to 0 \to \cdots$$

together with the map p_E as an augmentation to A, and similarly for F. The augmentation is a quasi-isomorphism when considered as a map of complexes. Then the tensor product of E and F as augmented complexes of right resp. left A-modules is another $(m+n)$-fold extension $E \otimes_A F$ which comes with morphisms

$$F \# E \xleftarrow{\lambda_{E,F}} E \otimes_A F \xrightarrow{\varrho_{E,F}} (-1)^{mn} E \# F .$$

In detail,

$$(E \otimes_A F)_i = \bigoplus_{k+l=i;\, k,l \geqq 0} E_k \otimes_A F_l$$

for $i = 0, \ldots, m+n$ and $(E \otimes_A F)_{-1} = A$. The differential in this complex is given as usual by

$$d(e \otimes f) = d(e) \otimes f + (-1)^{\dim(e)} e \otimes d(f) ,$$

and $p_{E \otimes_A F}$ is equal to $p_E \otimes_A p_F : E_0 \otimes_A F_0 \to A \otimes_A A \cong A$. Note that the three sequences

$$\big((-1)\, E\big) \otimes_A F, \quad (-1)(E \otimes_A F) \quad \text{and} \quad E \otimes_A \big((-1)\, F\big)$$

are *equal*, so again the parenthesis can be omitted.

We want the tensor product to induce an operation on the components of the extension categories. For this the sequence $E \otimes_A F$ has to be exact, so that it is an object of the category $\mathscr{E}xt^{m+n}_{A^e}(A, A)$. Although $E \otimes_A F$ will not always be exact, it is so on certain big enough subcategories of the extension categories. Here "big" means that the subcategory has the same components as the entire extension category. We could restrict to extensions E for which all the E_i are flat as right A-modules, but we can even arrange that all modules occurring are *left-right projective*, i.e., that they are projective when considered as left A-modules or right A-modules separately. The bimodule A is an example which is left-right projective, but usually not projective as a bimodule. We let $\underline{\mathscr{E}xt}^n_{A^e}(A, A)$ denote the full subcategory of the extension category consisting of those sequences in which all modules are left-right projective. For $n = 1$, every extension splits as a short sequence of either left or right A-modules, hence $\underline{\mathscr{E}xt}^n_{A^e}(A, A)$ is the entire extension category. For $n > 1$, however, $\underline{\mathscr{E}xt}^n_{A^e}(A, A)$ is in general a proper subcategory. Since we assumed that A is k-projective, the inclusion of the subcategory $\underline{\mathscr{E}xt}^n_{A^e}(A, A)$ into the extension category induces a homotopy equivalence on classifying spaces (compare [Q], §1, Prop. 2) by the following lemma.

Lemma 2.1. *There exists a functor Q from $\mathscr{E}xt^n_{A^e}(A, A)$ to itself with values in the full subcategory $\underline{\mathscr{E}xt}^n_{A^e}(A, A)$ and a natural transformation from Q to the identity functor. In particular, the inclusion functor induces a bijection of components*

$$\pi_0 \underline{\mathscr{E}xt}^n_{A^e}(A, A) \cong \pi_0 \mathscr{E}xt^n_{A^e}(A, A) .$$

Proof. We consider an n-fold extension E as above and we define $Q(E)_0$ as the free A^e-module on the underlying set of E_0. The map $Q(E)_0 \to E_0$ is the A^e-linear extension of the identity. For $0 < i < n-1$ we define $Q(E)_i$ inductively as the free A^e-module on the underlying set of the pullback $K_{i-1} \times_{E_{i-1}} E_i$ where $K_{i-1} = \operatorname{Ker}\big(Q(E)_{i-1} \to Q(E)_{i-2}\big)$. The map $Q(E)_i \to E_i$ and the differential $Q(E)_i \to Q(E)_{i-1}$ are the composites of the A^e-linear extension of the identity followed by the projection

$$K_{i-1} \times_{E_{i-1}} E_i \to E_i$$

and the map

$$K_{i-1} \times_{E_{i-1}} E_i \xrightarrow{\text{projection}} K_{i-1} \xrightarrow{\text{inclusion}} Q(E)_{i-1}$$

respectively. The module $Q(E)_{n-1}$ is defined as the pullback $K_{n-2} \times_{E_{n-2}} E_{n-1}$ with similar maps to E_{n-1} and $Q(E)_{n-2}$. The map $A \to Q(E)_{n-1} = K_{n-2} \times_{E_{n-2}} E_{n-1}$ takes a to $(0, i_E(a))$. We omit the verification that the sequence $Q(E)$ is exact and that $Q(E) \to E$ is a morphism in $\mathscr{E}xt^n_{A^e}(A, A)$. By construction, the modules $Q(E)_0, \ldots, Q(E)_{n-2}$ are free over A^e. Since A was assumed to be projective over k, every projective A^e-module is also left-right projective. It remains to show that $Q(E)_{n-1}$ is left-right projective. By induction the short exact sequences that make up $Q(E)$ are split as sequences of right A-modules and the kernels K_i are projective as right modules for $i = 0, \ldots, n-2$. Hence

$$Q(E)_{n-1} \cong A \oplus K_{n-2}$$

as right A-modules, thus $Q(E)_{n-1}$ is right-projective. The proof that $Q(E)_{n-1}$ is left-projective is analogous. □

Next we define the edge morphisms $\lambda_{E,F}$ and $\varrho_{E,F}$. In dimensions i with $m \leqq i \leqq m+n$,

$$\lambda_{E,F} : E \otimes_A F \to F \# E$$

is given by the projection $(E \otimes_A F)_i \to E_{i-m} \otimes_A F_m$ followed by the identification

$$E_{i-m} \otimes_A F_m \cong E_{i-m} .$$

In dimensions $0 \leqq i < m$ it is given by the projection $(E \otimes_A F)_i \to E_0 \otimes_A F_i$ followed by the map $p_E \otimes \mathrm{id}_{F_i}$. The morphism

$$\varrho_{E,F} : E \otimes_A F \to (-1)^{mn} E \# F$$

is defined in a similar way. In dimensions i with $n \leqq i \leqq m+n$ it is given by $(-1)^{n(m+n-i)}$ times the projection $(E \otimes_A F)_i \to E_n \otimes_A F_{i-n}$ followed by the identification

$$E_n \otimes_A F_{i-n} \cong F_{i-n} .$$

In dimensions $0 \leqq i < n$ it is given by $(-1)^{mn}$ times the projection $(E \otimes_A F)_i \to E_i \otimes_A F_0$ followed by the map $\mathrm{id}_{E_i} \otimes p_F$ to $E_i \otimes_A A \cong E_i$. We omit the verification that $\lambda_{E,F}$ and $\varrho_{E,F}$ are in fact morphisms in the category $\mathscr{E}xt^{n+m}_{A^e}(A, A)$.

Instead of $E \otimes_A F$ we could have used the complex $(-1)^{mn} F \otimes_A E$ to relate the two spliced sequences $F \# E$ and $(-1)^{mn} E \# F$. Since the tensor product of A-bimodules is *not* symmetric, these two paths between the spliced sequences may be quite different, and this is what leads to the loop bracket of F and E. In fact, if F and E consist of left-right projective modules, we obtain a loop in $\mathscr{E}xt^{m+n}_{A^e}(A, A)$

$$\begin{tikzcd}[column sep=small]
 & E \otimes_A F \arrow[dl, "\lambda_{E,F}"'] \arrow[dr, "\varrho_{E,F}"] & \\
F \# E & & (-1)^{mn} E \# F \\
 & (-1)^{mn} F \otimes_A E \arrow[ul, "\varrho_{F,E}"] \arrow[ur, "\lambda_{F,E}"'] &
\end{tikzcd}$$

It is important to note that although two of the objects in the loop have a sign, none of the morphisms has. This is because any morphism between two objects in $\mathcal{E}xt_R^{m+n}(M,N)$ is also a morphism between the negated objects without introduction of extra signs.

We denote the above loop of extensions, oriented counter-clockwise, by $\Omega(F,E)$. Since the construction of the loop is functorial in F and E, the homotopy class of $\Omega(F,E)$ only depends on the components of F and E in the respective extension categories. Hence the loop construction induces a well-defined map

$$\Omega : \pi_0 \underline{\mathcal{E}xt}^m_{A^e}(A,A) \times \pi_0 \underline{\mathcal{E}xt}^n_{A^e}(A,A) \to \pi_1 \mathcal{E}xt^{m+n}_{A^e}(A,A) .$$

In view of Lemma 2.1 we can (and will) identify the components of the subcategories $\underline{\mathcal{E}xt}^m_{A^e}(A,A)$ and $\underline{\mathcal{E}xt}^n_{A^e}(A,A)$ with the components of the whole extension categories. We refer to the resulting operation as the *loop bracket*.

We have completely suppressed basepoints in the construction of the loop bracket. The justification for this is that although the fundamental group can in general depend on the choice of basepoint, it is independent of such choices in the case of the extension category. In fact, since the fundamental group of the extension category is abelian (by Retakh's theorem [R] or Theorem 1.1), it is independent up to canonical isomorphism of the choice of basepoint within a component. Furthermore, all components have canonically isomorphic fundamental groups, so we can simply speak of the fundamental group of the extension category. For the same reason, a closed oriented loop without choice of basepoint whatsoever determines an element in the fundamental group. To calculate it, we can choose any object as basepoint and the result is independent of this choice.

If we assume that A is of characteristic 2 (i.e., $2 = 0$ in A) or that n is even, then $\lambda_{E,E}$ and $\varrho_{E,E}$ give two different maps from $E \otimes_A E$ to $E \# E$. We thus get a loop of length 2

$$E \otimes_A E \xrightarrow{\lambda_{E,E}} E \# E \xleftarrow{\varrho_{E,E}} E \otimes_A E$$

which we denote by $Sq(E)$ and refer to as the *divided square loop*. After all, the loop bracket $\Omega(E,E)$ is the loop product of two copies of $Sq(E)$. Again, since the divided square loop is functorial in E, it induces a well defined map

$$Sq : \pi_0 \mathcal{E}xt^n_{A^e}(A,A) \cong \pi_0 \underline{\mathcal{E}xt}^n_{A^e}(A,A) \to \pi_1 \mathcal{E}xt^{2n}_{A^e}(A,A) .$$

3. Review of the Hochschild complex

In this section we recall the Hochschild complex of an algebra. More details, background and applications can be found in [G], [GS], [MacL], X. 3 or [W], Chpt. 9. We use the notation and sign conventions of [GS]. We also recall the definition of the Lie bracket and the divided square operations in Hochschild cohomology and compare them to the loop bracket and divided square loop constructions introduced in the previous section.

Again k is a commutative ring, A an associative and unital k-algebra and M an A-bimodule (always k-symmetric). Unspecified tensor products are taken over k. For $m \geqq 0$, the Hochschild m-cochains $C^m(A;M)$ are defined as the k-module of k-linear maps from $A^{\otimes m}$ to M. The Hochschild coboundary map

$$\delta^m : C^m(A;M) \to C^{m+1}(A;M)$$

is defined by

$$\begin{aligned}(\delta^m f)&(a_1 \otimes \cdots \otimes a_{m+1}) \\ &= a_1 f(a_2 \otimes \cdots \otimes a_{m+1}) + \sum_{i=1}^{m} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{m+1}) \\ &\quad + (-1)^{m+1} f(a_1 \otimes \cdots \otimes a_m)\, a_{m+1}\,.\end{aligned}$$

The coboundary satisfies $\delta^{m+1}\delta^m = 0$ and the Hochschild cohomology groups $\mathrm{H}^*_k(A;M)$ are defined as the cohomology groups of the cochain complex $\big(C^*(A;M),\delta\big)$.

We want to view the cup product as an external pairing

$$\cup : C^m(A;M) \otimes C^n(A;N) \to C^{m+n}(A; M \otimes_A N)$$

defined by

$$(f \cup g)(a_1 \otimes \cdots \otimes a_{m+n}) = f(a_1 \otimes \cdots \otimes a_m) \otimes g(a_{m+1} \otimes \cdots \otimes a_{m+n})\,.$$

The cup product is associative and unital. This means that it satisfies $(f \cup g) \cup h = f \cup (g \cup h)$ for $f \in C^m(A;M)$, $g \in C^n(A;N)$, $h \in C^p(A;P)$ with respect to the natural isomorphism $(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P)$. Also $f \cup 1 = f = 1 \cup f$ where $1 : k \to A$ is the unit map of A, viewed as an element of $C^0(A;A)$ and where we identify $A \otimes_A M \cong M \cong M \otimes_A A$ in the standard fashion. The cup product furthermore satisfies

$$\delta(f \cup g) = (\delta f) \cup g + (-1)^m f \cup (\delta g)\,.$$

When the cochain g takes values in A, the graded commutator with respect to cup product is defined as

$$[f,g]^{\cup} = f \cup g - (-1)^{mn} g \cup f$$

where again we identified $M \otimes_A A$ and $A \otimes_A M$ with M. $[f,g]^{\cup}$ is defined similarly when f takes values in A and g takes values in an arbitrary A-bimodule. Then the equality $[g,f]^{\cup} = -(-1)^{mn}[f,g]^{\cup}$ holds.

The next piece of structure is the circle product, which was introduced by M. Gerstenhaber in [G]. For $f \in C^m(A;M)$ and $g \in C^n(A;A)$ ($m \geqq 1, n \geqq 0$), and every $i = 1, \dots, m$, there is a partial product $f \circ_i g \in C^{m+n-1}(A;M)$ defined by substitution in the i-th place

$$(f \circ_i g)(a_1 \otimes \cdots \otimes a_{m+n-1})$$
$$= f\big(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes a_{i+n} \otimes \cdots \otimes a_{m+n-1}\big).$$

The circle product is then defined as the alternating sum

$$f \circ g = \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} f \circ_i g .$$

This definition is extended to 0-cochains by setting $f \circ g = 0$ if $m = 0$. Note that f can have values in any A-bimodule, but g has to take values in A for the circle product to make sense. The coboundary of the circle product satisfies the fundamental formula

$$(*) \qquad \delta(f \circ g) = (-1)^{(n-1)}(\delta f) \circ g + f \circ (\delta g) + (-1)^n [g, f]^{\cup} .$$

We also need the functoriality of the various cochain operations with respect to change of coefficient modules. Since the symbol $\circ$ is reserved for the circle product, we use a dot to denote composition of homomorphisms. If $h : M \to M'$ is a homomorphism of A-bimodules, we then have the equalities

$$\delta(h \cdot f) = h \cdot \delta(f), \quad (h \cdot f) \cup g = (h \otimes \mathrm{id}) \cdot (f \cup g) \quad \text{and} \quad h \cdot (f \circ g) = (h \cdot f) \circ g .$$

In the case where the bimodule M is the algebra A itself, $C^*(A; A)$ is a differential graded algebra with respect to the cup product, and $\cup$ induces a product on the Hochschild cohomology groups $\mathrm{H}^*_k(A; A)$, which are abbreviated to $\mathrm{H}^*_k(A)$. The circle product does not directly pass to a product in cohomology, but the formula $(*)$ for the coboundary of a circle product gives rise to further relations and operations, due to M. Gerstenhaber [G]. For example, if $f \in C^m(A; A)$ and $g \in C^n(A; A)$ are cocycles, then $(*)$ specializes to $\delta(f \circ g) = (-1)^n [g, f]^{\cup}$, which means that the cup product is graded commutative on the level of cohomology. Also, if f and g are cocycles as above, then their graded commutator with respect to the circle product,

$$[f, g]^{\circ} = f \circ g - (-1)^{(m-1)(n-1)} g \circ f ,$$

is a cocycle whose cohomology class only depends on the classes of f and g. The Hochschild cohomology groups thus have induced operations

$$[-, -] : \mathrm{H}^m_k(A) \otimes \mathrm{H}^n_k(A) \to \mathrm{H}^{m+n-1}_k(A)$$

which make them into a Lie algebra, graded by *degree*, which is one less than the dimension of a cohomology class.

Finally, we consider a cocycle $g \in C^n(A; A)$ and assume that n is even or A is of characteristic 2 (i.e., $2 = 0$ in A). Then the formula $(*)$ shows that $g \circ g$ is also a cocycle whose cohomology class only depends on that of g. This defines the divided squaring operations

$$Sq : \mathrm{H}^n_k(A) \to \mathrm{H}^{2n-1}_k(A) .$$

The standard identification of the Hochschild cohomology groups with bimodule Ext groups uses the (unnormalized) bar resolution $\mathscr{B}(A)$ of the algebra A (see [W], Cor. 9.1.5 or the note on p. 287 of [MacL], X. 3). The latter is a complex of A-bimodules defined by $\mathscr{B}(A)_n = A^{\otimes(n+2)}$ (where A acts through the two outermost tensor factors) and with differential given by

$$d(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}\,.$$

The multiplication map $\mathscr{B}(A)_0 = A \otimes A \to A$ is an augmentation of $\mathscr{B}(A)$ to A. Extension of scalars from k to $A^{\mathrm{e}} = A \otimes_k A^{\mathrm{op}}$ gives an isomorphism of cochain complexes

$$C^*(A; M) \cong \mathrm{Hom}_{A^{\mathrm{e}}}\big(\mathscr{B}(A), M\big)$$

for any A-bimodule M. Since we assume that A is projective as a k-module, $\mathscr{B}(A)$ is a projective resolution of A by A-bimodules. The constructions K and μ of Section 1 thus give natural isomorphisms

$$K\colon \mathrm{H}^n_k(A; M) \to \pi_0 \mathscr{E}xt^n_{A^{\mathrm{e}}}(A, M)$$

and

$$\mu\colon \mathrm{H}^{n-1}_k(A; M) \to \pi_1 \mathscr{E}xt^n_{A^{\mathrm{e}}}(A, M)\,.$$

Specializing to the case $M = A$, there are cohomology operations on the one side and loop operations on the other. The following main theorem of this paper says that these operations correspond to each other under the natural isomorphisms.

Theorem 3.1. *Assume that A is k-projective. Then the following diagram commutes up to the sign $(-1)^n$:*

$$\begin{tikzcd}
\mathrm{H}^m_k(A) \times \mathrm{H}^n_k(A) \arrow[r, "{[-,-]}"] \arrow[d, "K \times K"'] & \mathrm{H}^{m+n-1}_k(A) \arrow[d, "\mu"] \\
\pi_0 \mathscr{E}xt^m_{A^{\mathrm{e}}}(A, A) \times \pi_0 \mathscr{E}xt^n_{A^{\mathrm{e}}}(A, A) \arrow[r, "\Omega"'] & \pi_1 \mathscr{E}xt^{m+n}_{A^{\mathrm{e}}}(A, A)
\end{tikzcd}$$

If n is even or A of characteristic 2, then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}^n_k(A) \arrow[r, "Sq"] \arrow[d, "K"'] & \mathrm{H}^{2n-1}_k(A) \arrow[d, "\mu"] \\
\pi_0 \mathscr{E}xt^n_{A^{\mathrm{e}}}(A, A) \arrow[r, "Sq"'] & \pi_1 \mathscr{E}xt^{2n}_{A^{\mathrm{e}}}(A, A)
\end{tikzcd}$$

4. Proofs

This section is the technical part of the paper. Here we calculate the fundamental group of the extension category (Theorem 1.1) and carry out the comparison of the Hochschild cohomology operations with the loop operations (Theorem 3.1). We start in the situation of Theorem 1.1. So R is a ring and M and N are left R-modules, and we want to identify the fundamental group of the category $\mathcal{E}xt^n_R(M,N)$. We first need a notion of chain homotopy which has meaning inside the category of extensions and which implies that chain homotopic maps give homotopic paths.

Definition 4.1. Let $f, g: F \to E$ be morphisms in $\mathcal{E}xt^n_R(M,N)$. We say that f and g are chain homotopic *relative to M and N* if there exist homomorphisms $s_i: F_i \to E_{i+1}$ for $i = 0, \dots, n-2$ satisfying

$$\begin{aligned} d \cdot s_0 \quad &= \quad g_0 - f_0, \\ d \cdot s_i + s_{i-1} \cdot d = &\quad g_i - f_i \quad \text{for } 0 < i < n-1, \\ s_{n-2} \cdot d &= g_{n-1} - f_{n-1}. \end{aligned}$$

Lemma 4.2. *Let $f, g: F \to E$ be morphisms in $\mathcal{E}xt^n_R(M,N)$ which are chain homotopic relative to M and N. Then the paths represented by f and g are homotopic relative to the endpoints.*

Proof. We adapt the notion of a cylinder object. We will define an object $F \times I$ of $\mathcal{E}xt^n_R(M,N)$ together with morphisms $i_0, i_1: F \to F \times I$ and $p: F \times I \to F$ satisfying $p \cdot i_0 = \mathrm{id}_F = p \cdot i_1$. Furthermore, a relative chain homotopy between f and g gives rise to a morphism $S: F \times I \to E$ in $\mathcal{E}xt^n_R(M,N)$ such that $S \cdot i_0 = g$ and $S \cdot i_1 = f$. We first show how this implies the lemma: since $f = S \cdot i_1$, the path represented by f is homotopic, relative endpoints, to the path

$$F \xrightarrow{i_1} F \times I \xrightarrow{S} E.$$

Since the morphisms i_0 and i_1 have p as common left inverse, i_1 can then be replaced by i_0 without changing the relative homotopy class. Because of $S \cdot i_0 = g$, the resulting path is then relatively homotopic to the one represented by g.

The cylinder $F \times I$ is a quotient of the usual cylinder object of the complex F, but modified so that it becomes an object of the category $\mathcal{E}xt^n_R(M,N)$. We define

$$(F \times I)_i = \begin{cases} F_0 \oplus F_0 & \text{if } i = 0, \\ F_i \oplus F_{i-1} \oplus F_i & \text{if } 0 < i < n-1, \\ (F_{n-1} \oplus F_{n-2} \oplus F_{n-1})/W & \text{if } i = n-1. \end{cases}$$

Here W is the submodule of $F_{n-1} \oplus F_{n-2} \oplus F_{n-1}$ consisting of the elements of the form $(x, dx, -x)$ for $x \in F_{n-1}$. For $i > 1$, the differential $d: F_i \oplus F_{i-1} \oplus F_i \to F_{i-1} \oplus F_{i-2} \oplus F_{i-1}$ is given by the formula

$$d(x,y,z) = \big(dx + (-1)^{n-i} y, dy, (-1)^{n-i-1} y + dz\big).$$

For $i = n-1$, this map passes to the quotient by W. In dimension 1,

$$d(x, y, z) = \big(dx - (-1)^n y, (-1)^n y + dz\big).$$

The inclusion $N \to (F_{n-1} \oplus F_{n-2} \oplus F_{n-1})/W$ sends n to the class of $\big(i_F(n), 0, 0\big)$, and the projection $F_0 \oplus F_0 \to M$ is the original projection $F_0 \to M$ on each summand. The two inclusion morphisms $i_0, i_1 : F \to F \times I$ are given by the first and last direct summand inclusions, the projection morphism $p : F \times I \to F$ is defined by $p_i(x, y, z) = x + z$ (mutatis mutandis when $i = 0$ or $n-1$). A relative chain homotopy $s_i : F_i \to E_{i+1}$, $i = 0, \ldots, n-2$ gives rise to a morphism $S : F \times I \to E$ via

$$\begin{aligned} S_0(x, z) &= g_0(x) + && f_0(z), \\ S_i(x, y, z) &= g_i(x) + (-1)^{n-i} s_{i-1}(y) + && f_i(z) \qquad \text{if } 0 < i \leqq n-1, \end{aligned}$$

the latter formula being well defined on the quotient when $i = n-1$. We omit the verification that the complex $F \times I$ is acyclic, that S is a chain map and that i_0, i_1 and p have the properties claimed. □

Now we can prove the first part of Theorem 1.1, namely that μ induces a well defined group homomorphism

$$\mu : \mathrm{H}^{n-1}\big(\mathrm{Hom}_R(P, N)\big) \to \pi_1\big(\mathscr{E}xt^n_R(M, N), K(\varphi)\big).$$

We consider a projective resolution $P \to M$ and cocycles $\varphi : P_n \to N$ and $\kappa : P_{n-1} \to N$ of the complex $\mathrm{Hom}_R(P, N)$. If $\sigma : P_{n-2} \to N$ is any homomorphism, a relative chain homotopy between $\mu(\kappa)$ and $\mu(\kappa + \sigma \cdot d)$ is obtained by setting $s_i = 0$ for $i \neq n-2$ and by setting $s_{n-2}(y)$ equal to the class of $\big(0, -\sigma(y)\big)$ in $K(\varphi)_{n-1} = (P_{n-1} \oplus N)/\big(-d(x), \varphi(x)\big)$ for $y \in P_{n-2}$. Lemma 4.2 thus shows that the loops based at $K(\varphi)$ given by $\mu(\kappa)$ and $\mu(\kappa + \sigma \cdot d)$ are homotopic. Since $\mu(\kappa + \kappa')$ is equal to the composite of $\mu(\kappa)$ and $\mu(\kappa')$ as an endomorphism of $K(\varphi)$, μ induces a group homomorphism after passage to homotopy classes of loops.

The next ingredient is a recipe for calculating the homotopy classes of loops of length 2. This lemma will be the main tool in the comparison of loop operations and Hochschild cohomology operations. Let $f, g : F \to E$ be two morphisms in $\mathscr{E}xt^n_R(M, N)$ and $P \to M$ a projective resolution. We choose a chain map $\Phi : P \to F$ covering the identity of M. Then $\Phi_n : P_n \to N$ is a cocycle and the chain map induces a morphism $\tilde{\Phi} : K(\Phi_n) \to F$. Since both $f \cdot \Phi$ and $g \cdot \Phi : P \to E$ cover the identity of M and since P is a complex of projectives, we can choose a chain homotopy $s_i : P_i \to E_{i+1}$ from $f \cdot \Phi$ to $g \cdot \Phi$ over M. In general, this chain homotopy is *not* in any sense relative to N. The failure to be a relative chain homotopy is measured by the homomorphism $s_{n-1} : P_{n-1} \to N$, which is a cocycle in the complex $\mathrm{Hom}_R(P, N)$. However, the homomorphisms s_i for $i = 0, \ldots, n-2$ are a chain homotopy, relative to M and N, between the composites

$$K(\Phi_n) \xrightarrow{\tilde{\Phi}} F \xrightarrow{f} E$$

and

$$K(\Phi_n) \xrightarrow{\mu(s_{n-1})} K(\Phi_n) \xrightarrow{\tilde{\Phi}} F \xrightarrow{g} E.$$

So an application of Lemma 4.2 gives

Lemma 4.3. *The loop*

$$F \xrightarrow{f} E \xleftarrow{g} F$$

is homotopic, relative to F, to the loop

$$F \xleftarrow{\tilde{\Phi}} K(\Phi_n) \xrightarrow{\mu(s_{n-1})} K(\Phi_n) \xrightarrow{\tilde{\Phi}} F. \quad \square$$

Now we proceed to show that the map μ is surjective. To prepare for this, we need a lemma about factorization of morphisms in the category $\mathscr{E}xt^n_R(M,N)$.

Lemma 4.4 ([R], Lemmas 1, 2). (1) *Every morphism in $\mathscr{E}xt^n_R(M,N)$ can be factored as the composite of two morphisms such that the first one is injective and the second one admits a section.*

(2) *Pushouts along injective morphisms exist in $\mathscr{E}xt^n_R(M,N)$.*

Proof. (1) Let $f: F \to E$ be a morphism in $\mathscr{E}xt^n_R(M,N)$. We choose an acyclic complex Z concentrated in dimensions 0 through $n-1$ with a chain map $b: F \to Z$ which is injective in dimensions 0 through $n-2$. For example, Z can be taken to be

$$Z_i = \begin{cases} F_{n-2} & \text{if } i = n-1, \\ F_i \oplus F_{i-1} & \text{if } 0 < i < n-1, \\ F_0 & \text{if } i = 0 \end{cases}$$

with differential $d(x_i, x_{i-1}) = (x_{i-1}, 0)$ and $b(x_i) = (x_i, d(x_i))$. The desired factorization is then given by

$$F \xrightarrow{(f,b)} E \oplus Z \xrightarrow{\binom{\mathrm{id}}{0}} E,$$

where the first map is also injective in dimension $n-1$.

(2) Let $E \xleftarrow{f} F \xrightarrow{g} G$ be a diagram in $\mathscr{E}xt^n_R(M,N)$ with f injective in every dimension. Let P be the pushout of this diagram in the category of chain complexes of R-modules. Then P is concentrated between dimensions -1 and n, and the groups in degree -1 and n are canonically identified with M and N respectively. So in order for P to be an object of $\mathscr{E}xt^n_R(M,N)$, it suffices to show that P is acyclic as a chain complex. But this holds because P contains the acyclic subcomplex G and P/G is also acyclic. $\square$

Lemma 4.5. *μ is surjective.*

Proof. We start with an arbitrary loop Λ based at $K(\varphi)$ representing a given class in the fundamental group. If the length of Λ is greater than 2, we show how to find a loop of shorter length in the same homotopy class. We can assume that no two adjacent arrows point in the same direction. This means that some piece of the loop looks like

$$\cdots \Lambda_{i-1} \xrightarrow{\alpha_i} \Lambda_i \xleftarrow{\alpha_{i+1}} \Lambda_{i+1} \xrightarrow{\alpha_{i+2}} \Lambda_{i+2} \cdots$$

(or with all arrows pointing in the opposite direction). We can factor $\alpha_{i+1} = \pi \cdot \iota$ according to Lemma 4.4 (1), replace π by its section pointing in the opposite direction and compose that section with α_i. This way we can assume that α_{i+1} is injective. By Lemma 4.4 (2), the pushout of α_{i+1} and α_{i+2} then exists, so these two morphisms can simultaneously be replaced by morphisms pointing in the opposite directions. But then the arrow replacing α_{i+1} can be composed with α_i, which reduces the length.

We can thus assume that Λ is of the form

$$K(\varphi) \xrightarrow{f} E \xleftarrow{g} K(\varphi) .$$

To apply Lemma 4.3, we can take $\Phi : P \to K(\varphi)$ to be the unique chain map which satisfies $\Phi_n = \varphi$ and $\tilde{\Phi} = \mathrm{id}_{K(\varphi)}$. We choose a chain homotopy $s_i : P_i \to E_{i+1}$ between $f \cdot \Phi$ and $g \cdot \Phi$. The loop Λ is then homotopic, relative $K(\varphi)$, to the loop of length 1 given by $\mu(s_{n-1})$. □

To establish the injectivity of μ we recall a general procedure for constructing maps out of the fundamental group of a category. Suppose $\tau : \mathscr{C} \to \mathscr{D}$ is a functor which takes all morphisms to isomorphisms. A loop

$$\Lambda : X = \Lambda_0 \xleftrightarrow{\alpha_1} \Lambda_1 \xleftrightarrow{\alpha_2} \cdots \xleftrightarrow{\alpha_{n-1}} \Lambda_{n-1} \xleftrightarrow{\alpha_n} \Lambda_n = X$$

based at X gives an element $\tau_X(\Lambda)$ in the automorphism group of the object $\tau(X)$ by applying τ to all the morphisms α_i, then inverting those morphisms which point backwards, and then composing. τ_X respects elementary homotopies and composition, so it passes to a group homomorphism

$$\tau_X : \pi_1(\mathscr{C}, X) \to \mathrm{Aut}_{\mathscr{D}}\big(\tau(X)\big) .$$

We apply this construction with $\mathscr{C} = \mathscr{E}xt_R^n(M,N)$ and with $\mathscr{D} = \mathscr{D}(R)$, the derived category of chain complexes of R-modules (see e.g. [W], Chpt. 10). A functor

$$\tau : \mathscr{E}xt_R^n(M,N) \to \mathscr{D}(R)$$

is defined by taking an extension E to the doubly truncated complex

$$\tau(E) : \cdots 0 \to 0 \to E_{n-1} \to \cdots \to E_0 \to 0 \to 0 \to \cdots$$

considered as an object in the derived category. This functor takes all morphisms to isomorphisms hence for every object E of $\mathscr{E}xt_R^n(M,N)$, it induces a group homomorphism

$$\tau_E : \pi_1\big(\mathscr{E}xt_R^n(M,N), E\big) \to \mathrm{Aut}_{\mathscr{D}(R)}\big(\tau(E)\big) .$$

Lemma 4.6. *The composite*

$$\tau_{K(\varphi)} \cdot \mu : \mathrm{H}^{n-1}(\mathrm{Hom}_R(P,N)) \to \mathrm{Aut}_{\mathscr{D}(R)}(\tau(K(\varphi)))$$

is injective. Hence μ is injective.

Proof. For any chain complex C and integer j, we denote by $C[j]$ the j-fold shift suspension of C. We denote by $\tilde{M}$ the cokernel of $i_{K(\varphi)} : N[n-1] \to \tau(K(\varphi))$, considered as a map of chain complexes. $\tilde{M}$ is quasi-isomorphic to the complex consisting only of the module M in dimension 0. The short exact sequence of chain complexes

$$0 \to N[n-1] \xrightarrow{i_{K(\varphi)}} \tau(K(\varphi)) \xrightarrow{\pi} \tilde{M} \to 0$$

induces long exact sequences of homomorphism groups in the derived category. Since $\mathrm{Hom}_{\mathscr{D}(R)}(\tilde{M}, \tilde{M}[-1]) = 0$, the map

$$\mathrm{Hom}_{\mathscr{D}(R)}(\tilde{M}, N[n-1]) \to \mathrm{Hom}_{\mathscr{D}(R)}(\tilde{M}, \tau(K(\varphi)))$$

induced by $i_{K(\varphi)}$ is injective. Similarly, since $\mathrm{Hom}_{\mathscr{D}(R)}(N[n], \tau(K(\varphi))) = 0$, the map

$$\mathrm{Hom}_{\mathscr{D}(R)}(\tilde{M}, \tau(K(\varphi))) \to \mathrm{Hom}_{D(R)}(\tau(K(\varphi)), \tau(K(\varphi)))$$

induced by π is injective.

Let $\kappa : P_{n-1} \to N$ be a cocycle representing a cohomology class in the kernel of the map $\mathrm{H}^{n-1}(\mathrm{Hom}_R(P,N)) \to \mathrm{Aut}_{\mathscr{D}(R)}(\tau(K(\varphi)))$. Since $\kappa \cdot d = 0$, κ factors over a chain map $\tilde{\kappa} : \tilde{M} \to N[n-1]$ that sends the residue class of (p,n) in $K(\varphi)_{n-1}$ to $\kappa(p)$. Since the cohomology class of κ is in the kernel, $\mathrm{id} - \mu(\kappa)$ is trivial as an endomorphism of $\tau(K(\varphi))$ in the derived category. But this difference factors as $\mathrm{id} - \mu(\kappa) = i_{K(\varphi)} \cdot \tilde{\kappa} \cdot \pi$ on the chain complex level. By the injectivity properties we derived in the previous paragraph, $\tilde{\kappa}$ is thus trivial in the derived category, so the cohomology class of κ is trivial. □

Now we can proceed to prove Theorem 3.1. To identify the homotopy classes of loop bracket and divided square loop we use Lemma 4.3 with the bar resolution $\mathscr{B}(A)$ as the projective resolution of A. We fix components of the extension categories $\mathscr{E}xt^m_{A^e}(A,A)$ and $\mathscr{E}xt^n_{A^e}(A,A)$. By Lemma 2.1 we can choose representing bimodule extensions F and E which consist entirely of left-right projective modules, so that their tensor products are objects of $\mathscr{E}xt^{m+n}_{A^e}(A,A)$. We choose chain maps

$$\psi : \mathscr{B}(A) \to F \quad \text{and} \quad \varphi : \mathscr{B}(A) \to E$$

covering the identity of A. It will be convenient to use the same name for the A-bimodule homomorphism $\psi_i : \mathscr{B}(A)_i = A^{\otimes(i+2)} \to F_i$ and the associated Hochschild cochain in $C^i(A; F_i)$. The fact that ψ and φ are chain maps can then be rephrased as

$$d \cdot \psi_i = \delta(\psi_{i-1}) \quad \text{and} \quad d \cdot \varphi_i = \delta(\varphi_{i-1}) .$$

The maps $\varphi_k \cup \psi_l \in C^{k+l}(A; E_k \otimes_A F_l)$ (for $0 \leqq i \leqq n+m$, $k, l \geqq 0$) provide a chain map

$$\varphi \cup \psi : \mathscr{B}(A) \to E \otimes_A F$$

covering the identity of A. The homomorphisms φ_n and ψ_m are Hochschild cocycles whose cohomology classes are mapped by K to the components of E and F respectively. Similarly, the cohomology class of $\varphi_n \cup \psi_m$ is mapped by K to the component of the sequence $E \otimes_A F$.

We start with the slightly easier second part of Theorem 3.1. So we consider a single n-fold extension E where A is of characteristic 2 or n is even. To apply Lemma 4.3 to the morphisms

$$\lambda_{E,E}, \varrho_{E,E} : E \otimes_A E \to E \# E,$$

we choose the bar resolution $\mathscr{B}(A)$ as the projective resolution of A, so that

$$\varphi \cup \varphi : \mathscr{B}(A) \to E \otimes_A E$$

covers the identity of A. The following lemma, combined with Lemma 4.3, identifies the homotopy class of the divided square loop $Sq(E)$ with the divided square cohomology class.

Lemma 4.7. *There exists a chain homotopy $s_i : \mathscr{B}(A)_i \to (E \# E)_{i+1}$ between*

$$\lambda_{E,E} \cdot (\varphi \cup \varphi) \quad \textit{and} \quad \varrho_{E,E} \cdot (\varphi \cup \varphi)$$

satisfying $s_{2n-1} = \varphi_n \circ \varphi_n$.

Proof. We calculate the composite map

$$\mathscr{B}(A) \xrightarrow{\varphi \cup \varphi} E \otimes_A E \xrightarrow{\varrho_{E,E} - \lambda_{E,E}} E \# E$$

from the definitions. This map is trivial in dimension $i < n$ and equal to

$$[\varphi_n, \varphi_{i-n}]^{\cup} : \mathscr{B}(A)_i \to E_{i-n}$$

in dimensions i for $n \leqq i \leqq 2n$. We obtain the desired chain zero homotopy of this composite by setting $s_i = \varphi_{i+1-n} \circ \varphi_n$ for $n \leqq i < 2n$ and $s_i = 0$ else. In fact, for $n \leqq i < 2n$ we have

$$\begin{aligned} d \cdot s_i + s_{i-1} \cdot d &= (d \cdot \varphi_{i+1-n}) \circ \varphi_n + \delta(\varphi_{i-n} \circ \varphi_n) \\ &= \delta(\varphi_{i-n}) \circ \varphi_n + (-1)^{n-1} \delta(\varphi_{i-n}) \circ \varphi_n + (-1)^n [\varphi_n, \varphi_{i-n}]^{\cup} \\ &= [\varphi_n, \varphi_{i-n}]^{\cup}. \end{aligned}$$

The second equality is the formula (*) for the coboundary of a circle product, the third equality uses the assumption that A has characteristic 2 or n is even. □

The loop bracket has length 4, so we cannot apply Lemma 4.3 directly to calculate it. Instead we consider the diagram in $\mathscr{E}xt^{m+n}_{A^e}(A, A)$

$$\begin{tikzcd}
 & E \otimes_A F \arrow[dl, "\lambda_{E,F}"'] \arrow[dr, "\varrho_{E,F}"] & & \\
F \# E & & (-1)^{mn} E \# F & K(\varphi_n \cup \psi_m) \arrow[ull, "\varphi \cup \psi"'] \arrow[dll, "(-1)^{mn}\psi \cup \varphi + \varepsilon"] \\
 & (-1)^{mn} F \otimes_A E \arrow[ul, "\varrho_{F,E}"] \arrow[ur, "\lambda_{F,E}"'] & &
\end{tikzcd}$$

whose left part is the loop bracket $\Omega(F, E)$, and whose right part commutes. The important new ingredient comes from a chain map $\varepsilon : \mathscr{B}(A) \to F \otimes_A E$ that is added to $(-1)^{mn}\psi \cup \varphi$ to make up the lower right map. ε is trivial in dimensions $< n$, and in higher dimensions it has only two non-trivial components (with respect to the direct sum decomposition of $F \otimes_A E$). These two non-trivial components are

$$(-1)^{mn+1}[\psi_{i-n}, \varphi_n]^{\cup} : \mathscr{B}(A)_i \to F_{i-n} \cong F_{i-n} \otimes_A E_n \quad \text{for } n \leqq i \leqq m+n\,,$$

and

$$(-1)^{mn+ni+i+1}(\psi_{i-n+1} \circ \varphi_n) \cup i_E^k : \mathscr{B}(A)_i \to F_{i-n+1} \otimes_A E_{n-1} \quad \text{for } n \leqq i \leqq m+n-1\,.$$

The map $i_E^k : k \to E_{n-1}$ is the composite of the unit map $k \to A$ and the injection $i_E : A \to E_{n-1}$, viewed as a cocycle in $C^0(A; E_{n-1})$. The fact that ε is a chain map uses the coboundary formula $(*)$. The equality of $\varrho_{E,F} \cdot (\varphi \cup \psi)$ and $\lambda_{F,E} \cdot \big((-1)^{mn}\psi \cup \varphi + \varepsilon\big)$ can be verified directly from the definitions.

The chain maps $\varphi \cup \psi$ and $(-1)^{mn}\psi \cup \varphi + \varepsilon$ with source $\mathscr{B}(A)$ factor over morphisms with source $K(\varphi_n \cup \psi_m)$, for which we used the same names in the above diagram. Since the right part of the diagram commutes, the loop bracket represents the same homotopy class as the loop of length 2 made up from the non-commutative outer square of the diagram. Now we are in the situation of Lemma 4.3. We again use the bar resolution as the projective resolution of A, so we only need a suitable chain homotopy between the two ways around the outer square in the above diagram. The following lemma provides such a homotopy and concludes the proof of Theorem 3.1.

Lemma 4.8. *There exists a chain homotopy* $s_i : \mathscr{B}(A)_i \to (F \# E)_{i+1}$ *between* $\lambda_{E,F} \cdot (\varphi \cup \psi)$ *and* $\varrho_{F,E} \cdot \big((-1)^{mn}\psi \cup \varphi + \varepsilon\big)$ *satisfying* $s_{m+n-1} = (-1)^n[\psi_m, \varphi_n]^{\circ}$.

Proof. Another look at the definitions shows that the difference

$$\varrho_{F,E} \cdot \big((-1)^{mn}\psi \cup \varphi\big) - \lambda_{E,F} \cdot (\varphi \cup \psi) : \mathscr{B}(A) \to F \# E$$

is trivial in dimensions $<m$ and equal to $-[\varphi_{i-m}, \psi_m]^{\cup}$ in dimensions $m \leqq i \leqq m+n$. The map $\varrho_{F,E} \cdot \varepsilon$ is trivial except in dimension $m+n$ and $m+n-1$, where it is equal to $[\varphi_n, \psi_m]^{\cup}$ and $(-1)^n i_E \cdot (\psi_m \circ \varphi_n)$ respectively. We define the desired chain homotopy as follows:

$$s_i = \begin{cases} (-1)^n [\psi_m, \varphi_n]^{\circ} & \text{if } i = m+n-1, \\ (-1)^{m(i+1)} \varphi_{i-m+1} \circ \psi_m & \text{if } m \leqq i \leqq m+n-2, \\ 0 & \text{else}. \end{cases}$$

In fact, we have for $m \leqq i \leqq m+n-2$

$$\begin{aligned} d \cdot s_i + s_{i-1} \cdot d &= (-1)^{m(i+1)} \delta(\varphi_{i-m}) \circ \psi_m + (-1)^{mi} \delta(\varphi_{i-m} \circ \psi_m) \\ &= (-1)^{m(i+1)} \delta(\varphi_{i-m}) \circ \psi_m + (-1)^{mi} \{(-1)^{m-1} \delta(\varphi_{i-m}) \circ \psi_m \\ &\quad + (-1)^m [\psi_m, \varphi_{i-m}]^{\cup}\} \\ &= (-1)^{m(i-m)} [\psi_m, \varphi_{i-m}] = -[\varphi_{i-m}, \psi_m]^{\cup} \end{aligned}$$

where the second equality uses the formula (*) for the coboundary of a circle product, and the third equality uses that m^2 is congruent to m modulo 2. Similarly,

$$\begin{aligned} d \cdot s_{m+n-1} + s_{m+n-2} \cdot d &= (-1)^n i_E \cdot [\psi_m, \varphi_n]^{\circ} + (-1)^{m(m+n-1)} \delta(\varphi_{n-1} \circ \psi_m) \\ &= (-1)^n i_E \cdot (\psi_m \circ \varphi_n) + (-1)^{m(n-1)} i_E \cdot (\varphi_n \circ \psi_m) \\ &\quad + (-1)^{m(m+n-1)} \{(-1)^{m-1} \delta(\varphi_{n-1}) \circ \psi_m + (-1)^m [\psi_m, \varphi_{n-1}]^{\cup}\} \\ &= (-1)^n i_E \cdot (\psi_m \circ \varphi_n) + (-1)^{m(n-1)} [\psi_m, \varphi_{n-1}]^{\cup} \\ &= (-1)^n i_E \cdot (\psi_m \circ \varphi_n) - [\varphi_{n-1}, \psi_m]^{\cup}. \quad \square \end{aligned}$$

References

[G] *M. Gerstenhaber*, The cohomology structure of an associative ring, Ann. Math. **78** (1963), 267–288.

[GS] *M. Gerstenhaber*, *S.D. Schack*, Algebraic cohomology and deformation theory, in: Deformation theory of algebras and applications, Kluwer ASI ser. **247** (1988), 11–264.

[MacL] *S. Mac Lane*, Homology, Grundl. math. Wiss. **114**, Springer, 1963.

[Q] *D. Quillen*, Higher Algebraic *K*-theory, I, in: Algebraic *K*-Theory I: Higher *K*-Theories, Lect. Notes Math. **341**, Springer (1973), 85–147.

[R] *V.S. Retakh*, Homotopical properties of categories of extensions, Russ. Math. Surv. **41** (1986), 217–218.

[W] *C.A. Weibel*, An introduction to homological algebra, Cambridge Stud. Adv. Math. **38**, Cambridge University Press, 1994.

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