## Correction to:

### The $p$-order of topological triangulated categories

Journal of Topology 6 (2013), 868–914

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Zhi-Wei Li has pointed out a gap in the proof of Proposition A.4 and a missing argument in Proposition A.14. Vincent Gajda has pointed out an embarrassing omission in my formulation of axioms of a triangulated category. The purpose of this note is to fix these issues. All statements in the paper are correct as they stand, so none of the results are affected. The omissions already occur in the earlier preprint version of the paper that appeared on the arXiv as under the title *'Topological triangulated categories'*.

We start with the correction to axiom (T1). In the formulation of the axioms of a triangulated category (page 908), I forgot to include the requirement that every morphism participates in a distinguished triangle. So axiom (T1) should be amended to:

**(T1)** For every object $X$ the triangle $0 \to X \xrightarrow{\mathrm{Id}} X \to 0$ is distinguished. For every morphism $f$ there exists a distinguished triangle $(f, g, h)$ starting with $f$.

While this omission in the formulation of (T1) is embarrassing, the verification of the extra requirement is straightforward for the homotopy category of a stable cofibration category. The following arguments should simply be added to the proof of Theorem A.12 in the paragraph on (T1):

We let $f : A \longrightarrow B$ be a morphism in $\mathbf{Ho}(\mathcal{C})$. Theorem A.1 (i) lets us choose $\mathcal{C}$-morphisms $\varphi : A \longrightarrow B'$ and $s : B \longrightarrow B'$ such that $s$ is an acyclic cofibration and $f = \gamma(s)^{-1} \circ \gamma(\varphi)$, where $\gamma : \mathcal{C} \longrightarrow \mathbf{Ho}(\mathcal{C})$ is the localization functor. We factor $\varphi = t \circ j$ as a cofibration $j : A \longrightarrow \bar{B}$ followed by a weak equivalence $t : \bar{B} \longrightarrow B'$. Then the following diagram commutes in $\mathbf{Ho}(\mathcal{C})$:

$$\begin{array}{ccccccc}
A & \xrightarrow{\quad f \quad} & B & \xrightarrow{\gamma(q)\circ\gamma(t)^{-1}\circ\gamma(s)} & \bar{B}/A & \xrightarrow{\quad \delta(j) \quad} & \Sigma A \\
\Vert & & \gamma(t)^{-1}\circ\gamma(s) \downarrow \cong & & \Vert & & \Vert \\
A & \xrightarrow[\gamma(j)]{} & \bar{B} & \xrightarrow[\gamma(q)]{} & \bar{B}/A & \xrightarrow[\delta(j)]{} & \Sigma A
\end{array}$$

The triangle $(f, \gamma(q) \circ \gamma(t)^{-1} \circ \gamma(s), \delta(j))$ is thus isomorphic to the elementary distinguished triangle of the cofibration $j$, and hence itself distinguished.

Now we turn to corrections of the proofs of Propositions A.4 and Proposition A.14. Proposition 1 below patches the argument in the proof of Proposition A.4. Proposition 2 provides the missing argument in Proposition A.14, namely that the preferred isomorphism $\tau_{F,A} : F(\Sigma A) \longrightarrow \Sigma(FA)$ that comes with any exact functor between pointed cofibration categories is natural.

We quickly recall the setup and the definition of the suspension construction. We consider a pointed cofibration category $\mathcal{C}$ and denote by $\gamma : \mathcal{C} \longrightarrow \mathbf{Ho}(\mathcal{C})$ the localization functor. An object of $\mathcal{C}$ is *weakly contractible* if the unique morphism to a zero object is a weak equivalence. We choose a cone for every object $A$ of $\mathcal{C}$, i.e., a cofibration $i_A : A \longrightarrow CA$ with weakly contractible target. The *suspension* $\Sigma A$ of $A$ is then a cokernel of the chosen cone inclusion, i.e., a pushout:

$$\begin{array}{ccc} A & \xrightarrow{i_A} & CA \\ \downarrow & & \downarrow{\scriptstyle p} \\ * & \longrightarrow & \Sigma A \end{array}$$

Lemma A.3 guarantees the existence of *cone extensions*: Let $i : A \longrightarrow C$ be any cofibration with weakly contractible target, and $\alpha : A \longrightarrow B$ a morphism in $\mathcal{C}$. Then there exists a *cone extension* of $\alpha$, i.e., a pair $(\bar{\alpha}, s)$ consisting of a morphism $\bar{\alpha} : C \longrightarrow \bar{C}$ and an acyclic cofibration $s : CB \longrightarrow \bar{C}$ such that $\bar{\alpha} i = s i_B \alpha$ and such that the induced morphism $\bar{\alpha} \cup s : C \cup_A CB \longrightarrow \bar{C}$ is a cofibration, where the source is a pushout of $i$ and $i_B \alpha$. Moreover, the composite morphism in $\mathbf{Ho}(\mathcal{C})$

$$C/A \xrightarrow{\gamma(\bar{\alpha}/\alpha)} \bar{C}/B \xrightarrow{\gamma(s/B)^{-1}} CB/B = \Sigma B$$

is independent of the cone extension $(\bar{\alpha}, s)$. Given a $\mathcal{C}$-morphism $\alpha : A \longrightarrow B$, we choose a cone extension $(\bar{\alpha}, s)$ with respect to the chosen cone $i_A : A \longrightarrow CA$. We define $\Sigma\alpha$ as the composite in $\mathbf{Ho}(\mathcal{C})$

$$\Sigma A = CA/A \xrightarrow{\gamma(\bar{\alpha}/\alpha)} \bar{C}/B \xrightarrow{\gamma(s/B)^{-1}} CB/B = \Sigma B \ .$$

Lemma A.3 guarantees that this definition is independent of the cone extension.

The following proposition says that for calculating the suspension of a $\mathcal{C}$-morphism, we can use something slightly weaker than a cone extension: the requirement that the induced morphism $\bar{\alpha} \cup s : C \cup_A CB \longrightarrow \bar{C}$ is a cofibration is not necessary for calculating $\Sigma\alpha$.

**Proposition 1.** *Let $\alpha : A \longrightarrow B$, $\hat{\alpha} : CA \longrightarrow \hat{C}$ and $t : CB \longrightarrow \hat{C}$ be morphisms in $\mathcal{C}$ such that $t$ is an acyclic cofibration and*

$$\hat{\alpha} \circ i_A \ = \ t \circ i_B \circ \alpha \ : \ A \ \longrightarrow \ \hat{C} \ .$$

*Then*

$$\gamma(t/B)^{-1} \circ \gamma(\hat{\alpha}/\alpha) \ = \ \Sigma\alpha \ : \ \Sigma A \ \longrightarrow \ \Sigma B$$

*as morphisms in the homotopy category $\mathbf{Ho}(\mathcal{C})$.*

*Proof.* We factor the morphism

$$\hat{\alpha} \cup t \ : \ CA \cup_A CB \ \longrightarrow \ \hat{C}$$

as a cofibration followed by a weak equivalence

$$CA \cup_A CB \ \xrightarrow{\bar{\alpha}\cup s} \ \bar{C} \ \xrightarrow{q} \ \hat{C} \ .$$

Since $i_A : A \longrightarrow CA$ is a cofibration, so is its cobase change, the canonical morphism from $CB$ to $CA \cup_A CB$. So the morphism $s : CB \longrightarrow \bar{C}$ is a cofibration; since source and target of $s$ are weakly contractible, $s$ is even an acyclic cofibration. Altogether, we have obtained a cone extension $(\bar{\alpha}, s)$ of $\alpha$.

We have

$$\gamma(\hat{\alpha}/\alpha) \ = \ \gamma((q\bar{\alpha})/\alpha) \ = \ \gamma(q/B) \circ \gamma(\bar{\alpha}/\alpha) \ : \ \Sigma A = CA/A \ \longrightarrow \ \hat{C}/B \ ,$$

and similarly

$$\gamma(t/B) \ = \ \gamma((qs)/B) \ = \ \gamma(q/B) \circ \gamma(s/B) \ : \ \Sigma B = CB/B \ \longrightarrow \ \hat{C}/B \ .$$

Combining these two formulas gives the desired relation

$$\gamma(t/B)^{-1} \circ \gamma(\hat{\alpha}/\alpha) \ = \ \gamma(t/B)^{-1} \circ \gamma(q/B) \circ \gamma(\bar{\alpha}/\alpha) \ = \ \gamma(s/B)^{-1} \circ \gamma(\bar{\alpha}/\alpha) \ = \ \Sigma\alpha \ . \qquad \square$$

Proposition 1 now allows for quick correction of the proof of Proposition A.4:

**Proposition A.4.** *The suspension construction is a functor* $\Sigma : \mathcal{C} \longrightarrow \mathbf{Ho}(\mathcal{C})$. *The suspension functor takes weak equivalences to isomorphisms and preserves coproducts.*

*Proof.* The problem in the published proof is in the verification that $\Sigma$ takes the identity of $A$ in $\mathcal{C}$ to the identity of $\Sigma A$ in $\mathbf{Ho}(\mathcal{C})$; contrary to what I claim, the pair $(\mathrm{Id}_{CA}, \mathrm{Id}_{CA})$ is typically not a cone extension of the identity of $A$, because the fold map $\mathrm{Id} \cup \mathrm{Id} : CA \cup_A CA \longrightarrow CA$ will usually not be a cofibration. Proposition 1 exactly fixes this problem: we can take $\alpha = \mathrm{Id}_A$ and $\hat{\alpha} = t = \mathrm{Id}_{CA}$, and conclude that

$$\mathrm{Id}_{\Sigma A} \;=\; \gamma(\mathrm{Id}_{CA} / A)^{-1} \circ \gamma(\mathrm{Id}_{CA} / \mathrm{Id}_A) \;=\; \Sigma\, \mathrm{Id}_A \ .$$

For compatibility with composition we consider two composable morphisms

$$\alpha \;:\; A \;\longrightarrow\; B \qquad \text{and} \qquad \beta \;:\; B \;\longrightarrow\; D$$

in $\mathcal{C}$. We choose a cone extension $(\bar{\alpha} : CA \longrightarrow \bar{C}, s : CB \longrightarrow \bar{C})$ of $\alpha$ and a cone extension $(\bar{\beta} : CB \longrightarrow \bar{C}', t : CD \longrightarrow \bar{C}')$ of $\beta$. Then we choose a pushout:

$$\begin{array}{ccc} CB & \xrightarrow{\;s\;} & \bar{C} \\ {\scriptstyle \bar{\beta}}\downarrow & & \downarrow{\scriptstyle \bar{\beta}'} \\ \bar{C}' & \xrightarrow[\;\bar{s}\;]{} & \hat{C} \end{array}$$

The morphism $\bar{s}$ is an acyclic cofibration since $s$ is. We apply Proposition 1 to the triple of morphisms $\beta\alpha : A \longrightarrow D$,

$$\bar{\beta}'\bar{\alpha} \;:\; CA \;\longrightarrow\; \hat{C} \qquad \text{and} \qquad \bar{s}t \;:\; CD \;\longrightarrow\; \hat{C}$$

and conclude that

$$\gamma(\bar{s}t/D)^{-1} \circ \gamma(\bar{\beta}'\bar{\alpha}/\beta\alpha) \;=\; \Sigma(\beta\alpha) \ .$$

The relation

$$\gamma(\bar{\beta}'/\beta) \circ \gamma(s/B) \;=\; \gamma(\bar{\beta}'s/\beta) \;=\; \gamma(\bar{s}\bar{\beta}/\beta) \;=\; \gamma(\bar{s}/D) \circ \gamma(\bar{\beta}/\beta)$$

is equivalent to

$$\gamma(\bar{s}/D)^{-1} \circ \gamma(\bar{\beta}'/\beta) \;=\; \gamma(\bar{\beta}/\beta) \circ \gamma(s/B)^{-1}$$

because both $\gamma(\bar{s}/D) : \bar{C}'/D \longrightarrow \hat{C}/D$ and $\gamma(s/B) : CB/B \longrightarrow \bar{C}/B$ are invertible in $\mathbf{Ho}(\mathcal{C})$. So we get

$$\begin{aligned} \Sigma(\beta\alpha) \;&=\; \gamma(\bar{s}t/D)^{-1} \circ \gamma(\bar{\beta}'\bar{\alpha}/\beta\alpha) \;=\; \gamma(t/D)^{-1} \circ \gamma(\bar{s}/D)^{-1} \circ \gamma(\bar{\beta}'/\beta) \circ \gamma(\bar{\alpha}/\alpha) \\ &=\; \gamma(t/D)^{-1} \circ \gamma(\bar{\beta}/\beta) \circ \gamma(s/B)^{-1} \circ \gamma(\bar{\alpha}/\alpha) \;=\; (\Sigma\beta) \circ (\Sigma\alpha) \ . \end{aligned}$$

So the suspension construction is functorial. The remaining parts of the proposition work as in the published proof. □

Since the suspension functor takes weak equivalences to isomorphisms, it descends to a unique functor

$$\Sigma \;:\; \mathbf{Ho}(\mathcal{C}) \;\longrightarrow\; \mathbf{Ho}(\mathcal{C})$$

such that $\Sigma \circ \gamma = \Sigma$. Since coproducts in $\mathcal{C}$ are coproducts in $\mathbf{Ho}(\mathcal{C})$, this induced suspension functor again preserves coproducts.

Now we recall how exact functors between cofibration categories give rise to exact functors between the triangulated homotopy categories. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between cofibration categories is *exact* if it preserves initial objects, cofibrations, weak equivalences and the particular pushouts along cofibrations that are guaranteed by axiom (C3). Since $F$ preserves weak equivalences, the composite functor $\gamma^{\mathcal{D}} \circ F : \mathcal{C} \longrightarrow \mathbf{Ho}(\mathcal{D})$ takes weak equivalences to isomorphisms and the universal property of the homotopy category provides a unique *derived functor* $\mathbf{Ho}(F) : \mathbf{Ho}(\mathcal{C}) \longrightarrow \mathbf{Ho}(\mathcal{D})$ such that $\mathbf{Ho}(F) \circ \gamma^{\mathcal{C}} = \gamma^{\mathcal{D}} \circ F$.

We will now explain that for pointed cofibration categories $\mathcal{C}$ and $\mathcal{D}$ the derived functor $\mathbf{Ho}(F)$ commutes with suspension up to a preferred natural isomorphism

$$\tau_F \ : \ \mathbf{Ho}(F) \circ \Sigma \ \xrightarrow{\cong} \ \Sigma \circ \mathbf{Ho}(F)$$

of functors from $\mathbf{Ho}(\mathcal{C})$ to $\mathbf{Ho}(\mathcal{D})$. If $A$ is any object of $\mathcal{C}$, then the cofibration $F(i_A) : FA \longrightarrow F(CA)$ is a cone since $F$ is exact. Lemma A.3 provides a cone extension of the identity of $FA$, i.e., a morphism $\bar{\alpha} : F(CA) \longrightarrow \bar{C}$, necessarily a weak equivalence, and an acyclic cofibration $s : C(FA) \longrightarrow \bar{C}$ such that $si_{FA} = \bar{\alpha}F(i_A)$. The composite in $\mathbf{Ho}(\mathcal{D})$

$$\tau_{F,A} \ : \ F(\Sigma A) = F(CA)/(FA) \ \xrightarrow{\gamma(\bar{\alpha}/(FA))} \ \bar{C}/(FA) \ \xrightarrow{\gamma(s/(FA))^{-1}} \ \Sigma(FA)$$

is then an isomorphism, and independent (by Lemma A.3) of the cone extension $(\bar{\alpha}, s)$.

The next proposition supplies the missing justification in Proposition A.14 for why the isomorphism $\tau_{F,A}$ is natural.

**Proposition 2.** *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an exact functor between pointed cofibration categories. Then the isomorphism $\tau_{F,A} : F(\Sigma A) \longrightarrow \Sigma(FA)$ is natural in $A$.*

*Proof.* Every morphism in $\mathbf{Ho}(\mathcal{C})$ is a fraction, i.e., a composite $\gamma(s)^{-1} \circ \gamma(\alpha)$ for two morphisms $\alpha, s$ in $\mathcal{C}$ with common target, and such that $s$ is a weak equivalence. Naturality of the isomorphism $\tau_F$ for $\gamma(s)$ implies naturality for the inverse $\gamma(s)^{-1}$, so it suffices to show naturality for morphisms in the image of the localization functor. In other words, we need to show that the following square commutes in $\mathbf{Ho}(\mathcal{C})$ for every $\mathcal{C}$-morphism $\alpha : A \longrightarrow B$:

$$\begin{array}{ccc} F(\Sigma A) & \xrightarrow{\mathbf{Ho}(F)(\Sigma\alpha)} & F(\Sigma B) \\ \downarrow{\scriptstyle \tau_{F,A}} & & \downarrow{\scriptstyle \tau_{F,B}} \\ \Sigma(FA) & \xrightarrow[\Sigma\mathbf{Ho}(F)(\gamma(\alpha))]{} & \Sigma(FB) \end{array}$$

To attack this we go back to the definitions. We choose a cone extension

$$(\bar{\alpha} : CA \longrightarrow \bar{C}, \ s : CB \longrightarrow \bar{C})$$

for $\alpha$. Then

$$\begin{aligned} \mathbf{Ho}(F)(\Sigma\alpha) &= \mathbf{Ho}(F)(\gamma(s/B)^{-1} \circ \gamma(\bar{\alpha}/\alpha)) \\ &= \mathbf{Ho}(F)(\gamma(s/B))^{-1} \circ \mathbf{Ho}(F)(\gamma(\bar{\alpha}/\alpha)) \\ &= \gamma((Fs)/(FB))^{-1} \circ \gamma((F\bar{\alpha})/(F\alpha)) \ : \ F(\Sigma A) \ \longrightarrow \ F(\Sigma B) \ . \end{aligned} \tag{3}$$

Now we choose cone extensions for the identity of $FA$ and for the identity of $FB$. In particular, this provides commutative diagrams

$$\begin{array}{ccc} FA & \xrightarrow{i_{FA}} & C(FA) \\ \downarrow{\scriptstyle F(i_A)} & & \downarrow{\scriptstyle t} \\ F(CA) & \xrightarrow[I_A]{} & \tilde{C} \end{array} \qquad\qquad \begin{array}{ccc} FB & \xrightarrow{i_{FB}} & C(FB) \\ \downarrow{\scriptstyle F(i_B)} & & \downarrow{\scriptstyle u} \\ F(CB) & \xrightarrow[I_B]{} & C' \end{array}$$

in $\mathcal{D}$ such that $t$ and $u$ are acyclic cofibrations and the morphism

$$I_A \cup t \ : \ F(CA) \cup_{FA} C(FA) \ \longrightarrow \ \tilde{C}$$

is a cofibration. Since $i_{FA}$ is a cofibration, so is its cobase change, the canonical morphism from $F(CA)$ to $F(CA) \cup_{FA} C(FA)$. Hence the morphism $I_A : F(CA) \longrightarrow \tilde{C}$ is a cofibration.

Since $F$ is exact the morphism $Fs : F(CB) \longrightarrow F(\bar{C})$ is an acyclic cofibration in $\mathcal{D}$. So we can choose a pushout square in $\mathcal{D}$:

$$\text{(4)} \qquad \begin{array}{ccc} F(CB) & \xrightarrow{I_B} & C' \\ {\scriptstyle Fs}\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle g} \\ F(\bar{C}) & \xrightarrow[J]{} & D \end{array}$$

and the cobase change $g$ is again an acyclic cofibration. Since $I_A$ is a cofibration, we can choose another pushout square in $\mathcal{D}$

$$\text{(5)} \qquad \begin{array}{ccc} F(CA) & \xrightarrow{J\circ(F\bar{\alpha})} & D \\ {\scriptstyle I_A}\downarrow & & \downarrow{\scriptstyle h} \\ \tilde{C} & \xrightarrow[K]{} & E \end{array}$$

and $h$ is a cofibration as a cobase change of a cofibration. We choose a cone of $E$, i.e., an acyclic cofibration $\mu : E \longrightarrow \hat{C}$ with weakly contractible target.

Applying Proposition 1 to the triple of morphisms $F\alpha : FA \longrightarrow FB$,

$$\mu K t \ : \ C(FA) \ \longrightarrow \ \hat{C} \qquad \text{and} \qquad \mu h g u \ : \ C(FB) \ \longrightarrow \ \hat{C}$$

yields the relation

$$\text{(6)} \qquad \begin{aligned} \Sigma(F\alpha) &= \gamma((\mu hgu)/(FB))^{-1} \circ \gamma((\mu Kt)/(F\alpha)) \\ &= \gamma((gu)/(FB))^{-1} \circ \gamma((\mu h)/(FB))^{-1} \circ \gamma((\mu K)/(F\alpha)) \circ \gamma(t/(FA)) \ . \end{aligned}$$

So we conclude that

$$\begin{aligned} \tau_{F,B} \circ (\mathbf{Ho}(F)(\Sigma\alpha)) \ \overset{(3)}{=}& \ \gamma(u/(FB))^{-1} \circ \gamma(I_B/(FB)) \circ \gamma((Fs)/(FB))^{-1} \circ \gamma((F\bar{\alpha})/(F\alpha)) \\ \overset{(4)}{=}& \ \gamma(u/(FB))^{-1} \circ \gamma(g/(FB))^{-1} \circ \gamma(J/(FB)) \circ \gamma((F\bar{\alpha})/(F\alpha)) \\ =& \ \gamma((gu)/(FB))^{-1} \circ \gamma((\mu h)/(FB))^{-1} \circ \gamma((\mu h)/(FB)) \circ \gamma((J(F\bar{\alpha}))/(F\alpha)) \\ \overset{(5)}{=}& \ \gamma((gu)/(FB))^{-1} \circ \gamma((\mu h)/(FB))^{-1} \circ \gamma((\mu K)/(F\alpha)) \circ \gamma(I_A/(FA)) \\ \overset{(6)}{=}& \ \Sigma(F\alpha) \circ \gamma(t/(FA))^{-1} \circ \gamma(I_A/(FA)) \\ =& \ \Sigma(\mathbf{Ho}(F)(\gamma(\alpha))) \circ \tau_{F,A} \end{aligned}$$

as morphisms

$$F(\Sigma A) = F(CA/A) \ \longrightarrow \ C(FB)/(FB) = \Sigma(FB)$$

in the homotopy category of $\mathcal{D}$. □