The moduli space of curves

Johannes Schmitt

September 30, 2020

The following are the lecture notes for a course about the moduli space of stable curves given in the Summer of 2020 at the University of Bonn. In the course, we define the moduli space and discuss its basic properties. We present many of the basic constructions related to it, e.g. of gluing and forgetful maps, and explain a number of examples of these moduli spaces in more details. Towards the end of the course we also cover the basics about the intersection theory on the space and the definition of tautological classes inside its cohomology groups.

Special thanks go to Bence Forrás, Sjoerd de Vries, Isabell Hellmann, Thibaud van den Hove and Pim Spelier for pointing out typos and giving feedback for this script, greatly improving the quality!

Contents

1 Introduction and Motivation

This course is an introduction to the moduli spaces of algebraic curves. The idea behind these spaces is that they allow us to classify algebraic curves up to isomorphism.

To get some intuition what this could mean, let's start with a more basic and familiar example: finite-dimensional vector spaces V over a field K. These mathematical objects are uniquely classified by their dimension, since $V \cong V'$ if and only if $\dim V = \dim V'$. Thus it makes sense to write the following:

$$\begin{aligned} \{\text{fin. dim. vector spaces over } K\}/\text{iso.} &\xrightarrow{\sim} \{0, 1, 2, \ldots\}, \\ [V] &\mapsto \dim V. \end{aligned} \tag{1}$$

This is a great classification: the left side looks complicated and scary but we understand the right-hand side (the natural numbers) very well!

The above was an example from linear algebra, but for an algebraic geometer, a natural question is to try classifying algebraic varieties up to isomorphism. In this course we will look at algebraic curves, i.e. algebraic varieties of dimension one. Why dimension one? It turns out that varieties of dimension zero are too easy: over an algebraically closed field K they are disjoint unions of points (i.e. $\mathrm{Spec}(K)$). On the other hand, varieties of dimension greater than one turn out to be more complicated to classify, so dimension one is the natural place to start.

To fix some conventions, we will say that a *curve* is a variety of pure dimension 1 and a *variety* is a reduced, separated scheme of finite type over our base field (not necessarily irreducible). For now, let us fix the base field to be the complex numbers $\mathbb{C}$ and let us try to classify smooth, irreducible, projective curves over $\mathbb{C}$, i.e. look at the set

$$\mathcal{M} = \{C \text{ smooth, irreducible, complex projective curve}\}/\text{iso.} \tag{2}$$

Why did we require C to have this list of properties? Some motivation:

- We ask C to be *smooth* since singularities add complications and can destroy some of the nice structures that smooth curves have, which will help us classify them. We will relax this condition below and allow so-called nodal singularities.

- Once we assume C to be smooth, being *irreducible* is equivalent to being connected[1]. A possibly disconnected curve is the union of connected ones, so it makes sense to classify these first.

- The assumption that C is *complex* is mostly for our convenience, much can be extended to working over algebraically closed fields (sometimes of characteristic 0). However, having $\mathbb{C}$ as the base field will allow us to draw some nice pictures, see below.

- As for the assumption that the curve is *projective*, note that for every smooth, irreducible, not-necessarily projective curve C' there exists an embedding $C' \hookrightarrow C$ into a smooth, irreducible and projective curve C (see [Vak17, Theorem 17.4.2]). By dimension reasons, the complement $C \setminus C'$ is a finite union of points and it will be easier to simply classify the data of C together with these points (from which we can reconstruct C'). Again, this is something we will see later, for today we just consider curves without the additional data of marked points.

[1] For a connected, reducible scheme there must be a point contained in two irreducible components (otherwise all these components would be disjoint, contradicting connectedness). Then one checks that such a point cannot be smooth. This statement is true in more generality, see [Sta13, Tag 033M].

What can we say about $\mathcal{M}$? Is it finite or infinite? Can we hope to give a list of elements, as in (1)?

Well, let's start getting our hands on some elements of $\mathcal{M}$. Below we will use that apart from the Zariski topology on a complex curve C, the set $C(\mathbb{C})$ of complex (or closed) points of the scheme C also has a *complex*[2] topology. We'll treat this in more detail later, so if you haven't seen it, dont worry; for now it just allows us to draw some nice pictures of curves.

Example 1.1 (The projective line)**.** The most basic example is the curve $\mathbb{P}^1$, the projective line. It is covered by $U_0 = \mathbb{P}^1 \setminus \{[0:1]\} \cong \mathbb{A}^1$ and $U_1 = \mathbb{P}^1 \setminus \{[1:0]\} \cong \mathbb{A}^1$, overlapping in $U_0 \cap U_1 = \mathbb{A}^1 \setminus \{0\}$. This allows us to see that its complex points $\mathbb{P}^1(\mathbb{C})$ are exactly given by the *two-sphere* S^2 (see Figure 1).

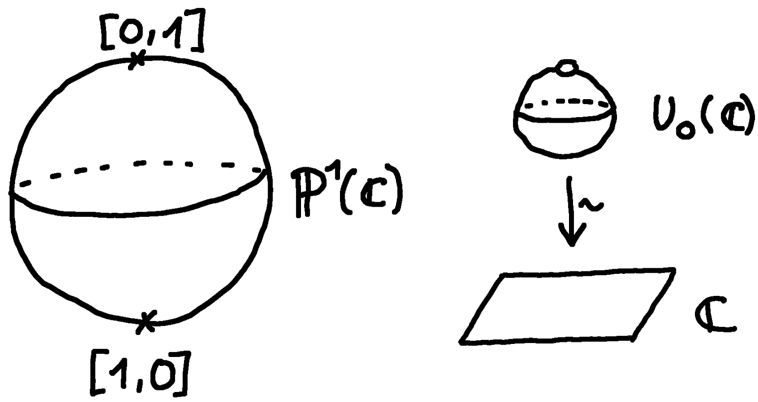


Figure 1: The complex points of the projective line and its chart U_0

A second source of examples are curves which are subvarieties of the projective plane $\mathbb{P}^2$ cut out by a homogeneous equation. It turns out that smooth curves cut out by equations of degree one and two (lines and quadrics) are actually isomorphic to $\mathbb{P}^1$, so they don't give new examples. Thus, let's go to degree three.

Example 1.2 (Plane cubics)**.** Consider the family E_t of cubic curves in $\mathbb{P}^2$ defined by

$$E_t = \left\{[X:Y:Z] \in \mathbb{P}^2 : Y^2Z + X(X-Z)(X-tZ) = 0\right\}, \quad t \in \mathbb{C}. \tag{3}$$

One checks that for $t \neq 0, 1$, these curves are indeed smooth and with some work one can show that their complex points $E_t(\mathbb{C})$ are isomorphic to a *torus* $T = S^1 \times S^1$, for all values $t \in \mathbb{C} \setminus \{0,1\} = U$ (see Figure 2).

In the above examples, we saw that the complex points of the curves we considered had a nice structure, being a sphere and a torus. This generalizes to arbitrary curves C in $\mathcal{M}$: the complex points $C(\mathbb{C})$ are a smooth, oriented compact real surface (also known as the surface of a donut with g holes). This number g of holes[3] is called the *genus* of the surface (and also of the corresponding curve C). See Figure 3 for an illustration.

[2]The idea behind the complex topology is that for $U \subset \mathbb{A}^n$ an affine variety, the complex topology on $U(\mathbb{C}) \subset \mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$ is the relative topology coming from $\mathbb{C}^n$. For a general complex variety X with an affine cover of U_i, the set $X(\mathbb{C})$ is covered by the $U_i(\mathbb{C})$ and their complex topologies glue together.

[3]To make this mathematically precise, one can define g as half of the dimension of the first singular cohomology group of the surface.

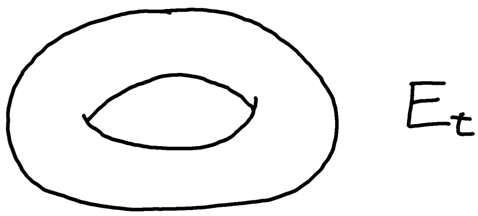


Figure 2: The complex points of E_t

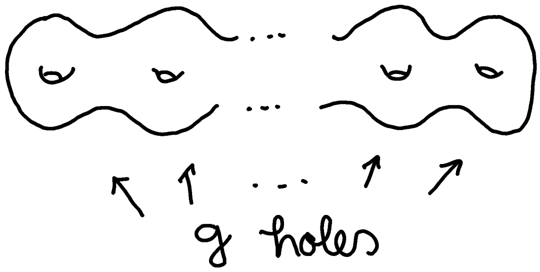


Figure 3: The complex points a curve C of genus g

From an algebraic geometry point of view, the genus of a curve $C \in \mathcal{M}$ can be defined as the dimension[4]

$$g(C) = \dim H^0(C, \Omega_C^1) \tag{4}$$

of the space of sections of the cotangent line bundle Ω_C^1 of C.

Exercise 1.3. Use this definition to verify that the genus of $\mathbb{P}^1$ is 0 and the genus of E_t is 1.

For the problem of classifying curves up to isomorphism, the genus is important since two curves of different genera clearly cannot be isomorphic (in particular $\mathbb{P}^1 \not\cong E_t$). Indeed we can write down a well-defined map

$$\mathcal{M} \to \{0, 1, 2, \ldots\}, \ [C] \mapsto g(C) \tag{5}$$

and given $g \in \{0, 1, 2, \ldots\}$ define $\mathcal{M}_g$ as the preimage of g under this map. In other words, we have that $\mathcal{M}_g$ is the set of isomorphism classes of genus g curves.

Returning to our examples above, it turns out that in genus zero our classification is already complete:

Fact 1.4. Every smooth, irreducible complex projective curve C of genus 0 is isomorphic to $\mathbb{P}^1$.

In other words $\mathcal{M}_0$ is the set with unique element $[\mathbb{P}^1]$. However, the situation is already more complicated in genus one:

[4] Silly remark: note how we use the classification (1) via dimension for the vector space $H^0(C, \Omega_C^1)$!

Fact 1.5. Every smooth, irreducible complex projective curve C of genus 1 is isomorphic to one of the curves E_t ($t \in \mathbb{C} \setminus \{0,1\} = U$). Moreover, for $t_1, t_2 \in U$ we have $E_{t_1} \cong E_{t_2}$ if and only if t_2 satisfies

$$t_2 \in \left\{ t_1,\ \frac{1}{t_1},\ 1-t_1,\ \frac{1}{1-t_1},\ \frac{t_1-1}{t_1},\ \frac{t_1}{t_1-1} \right\}. \tag{6}$$

So while the topological space $E_t(\mathbb{C})$ is independent of t, it turns out that the algebraic curves E_t are *not* all isomorphic. In fact, we see that there are uncountably infinitely many isomorphism classes of curves in $\mathcal{M}_1$! It turns out that the story continues similarly for higher genus: for each $g \geq 2$ the set $\mathcal{M}_g$ is uncountable (of the same cardinality as $\mathbb{C}$).

So are we done yet? Is the course complete at this point? After all, we have classified $\mathcal{M}_g$ as abstract sets. It turns out that we can ask for more!

For this, we look back at the family E_t parametrized by $t \in U$. This is a very nice family of curves (e.g. the equations defining E_t depend continuously, in fact algebraically, on the parameter t). Fact 1.5 tells us that every genus 1 curve appears as a member of this family and gives a criterion which members of the family are pairwise isomorphic. Starting with the family over U, we can try to get rid of this redundancy. One can define an action of the symmetric group S_3 on U such that the generators $(12), (23) \in S_3$ act by

$$(12) \cdot t = \frac{1}{t}, (23) \cdot t = 1 - t. \tag{7}$$

Then the orbit of $t_1 \in U$ under S_3 is exactly the set given in (6). Thus we have a natural identification $\mathcal{M}_1 = U/S_3$, where U/S_3 is the set of orbits in U under the S_3-action. But this shows that we can hope to have more structure on $\mathcal{M}_1$! After all, the set $U = \mathbb{C} \setminus \{0,1\}$ is naturally a topological space, and in fact the set of complex points of the scheme $\mathbb{A}^1 \setminus \{0,1\}$. Also, the action of S_3 is algebraic (e.g. the map

$$\mathbb{A}^1 \setminus \{0,1\} \to \mathbb{A}^1 \setminus \{0,1\}, t \mapsto 1/t$$

is an algebraic morphism) and in fact it turns out that the quotient $(\mathbb{A}^1 \setminus \{0,1\})/S_3$ also makes sense as an scheme. This scheme is given by the affine line $\mathbb{A}^1$ and the quotient morphism

$$j : \mathbb{A}^1 \setminus \{0,1\} \to \mathbb{A}^1, t \mapsto 2^8 \frac{(t^2-t+1)^3}{t^2(t-1)^2} \tag{8}$$

is known as the *j-invariant*[5]. This is an algebraic morphism and the fibres (of closed points in $\mathbb{A}^1$) are exactly the orbits of the action of S_3 on U.

Looking back, we see that the scheme $\mathbb{A}^1$ together with the data of the j-invariant and the family E_t is a much more satisfying answer to the problem of "classifying genus 1 curves" than just saying that $\mathcal{M}_1$ is an infinite set. In particular it allows us to make the following statements about curves of genus $g = 1$:

- There exists an algebraic variety $U = U_g$ and a family C_t of genus g curves parametrized by $t \in U_g$ such that every smooth, irreducible projective genus g curve appears as a fibre C_t.
- The variety U_g is smooth and connected, so any two genus g curves can be deformed into each other using this family.
- There exists a variety M_g and a surjective morphism $U_g \to M_g$ which identifies two closed points t_1, t_2 iff $C_{t_1} \cong C_{t_2}$, so the closed points of M_g are in bijection with the smooth, irreducible projective genus g curves up to isomorphism.

[5]There are good reasons for putting the factor 2^8, see the reference section below.

It turns out that *all of these statements are still true for $g \geq 2$*. The spaces M_g appearing in the statement then deserve to be called the moduli spaces of curves and those are the objects of study in this course. However, there is also a surprising set of things which go wrong or unexpected:

- For $g \geq 2$, the variety M_g is *not* smooth. For $g \geq 4$ its singular locus corresponds to the isomorphism classes of curves C having a nontrivial automorphism.

- For $g \geq 1$, the family C_t, $t \in U_g$, above does *not* descend to a family over M_g. In other words, we cannot write down a (reasonable) family of curves parametrized by M_g such that the member of the family associated to $[C] \in M_g$ is isomorphic to C. However, there *does* exist such a family over the complement of the locus of curves having a nontrivial automorphism.

Thus in both cases automorphisms of curves are the source of trouble. In the course, we will see how the theory of *algebraic stacks*, a generalization of schemes, resolves this problem by viewing M_g as a stack instead of a scheme.

The first part of our course will be to make precise what we mean by words like "moduli space" and "reasonable family of curves" above and then to study the spaces M_g that can be defined this way. We will see how modern tools of algebraic geometry (e.g. deformation theory) can be used in this study (e.g. to determine the dimension of M_g).

The second part of the course will focus on studying the cohomology groups of the moduli spaces of curves. Here, we will encounter a second issue, related to a quote by Angelo Vistoli:

"Working with noncompact spaces is like trying to
keep change with holes in your pockets."

Indeed, for $g \geq 1$ the spaces M_g are not proper, so their sets of $\mathbb{C}$-points are not compact. This means that their cohomology groups lack some nice properties, such as Poincaré duality, which hold for compact spaces. We already saw an example above: identifying $\mathcal{M}_1$ with $\mathbb{A}^1$, its complex points are $\mathbb{C}$, which is contractible and thus has the cohomology of a point.

This issue can be resolved by finding a *compactification* of $\mathcal{M}_g$, i.e. a space $\overline{\mathcal{M}}_g$ containing $\mathcal{M}_g$ as an open subset and working on $\overline{\mathcal{M}}_g$ instead. For this compactification to be helpful, it should itself be a moduli space of geometric objects generalizing the smooth genus g curves which $\mathcal{M}_g$ classifies.

Going back to the family E_t of genus 1 curves parametrized by $t \in \mathbb{C} \setminus \{0, 1\}$ can give us a hint what these more general geometric objects could be. Indeed, why did we restrict to $t \neq 0, 1$ in this family? The reason was that putting $t = 0, 1$ in the defining equation of E_t would give us a *singular* curve in $\mathbb{P}^2$. For instance, for $t = 0$ we obtain

$$E_0 = \left\{ [X : Y : Z] \in \mathbb{P}^2 : Y^2 Z + X^2 (X - Z) = 0 \right\}.$$

The singular point is $[X : Y : Z] = [0 : 0 : 1]$. Let's see how the curve looks like in a neighborhood of this point: going to the chart $\{Z \neq 0\}$ of $\mathbb{P}^2$ with coordinates $[X : Y : Z] = [x : y : 1]$, the singular point is $(x, y) = (0, 0)$ and the equation becomes

$$y^2 + x^2 (x - 1) = 0.$$

Around $(0, 0)$ we have $x - 1 \approx -1$ and so in a small neighbourhood of this point[6] the equation looks like

$$y^2 - x^2 = 0 \Longleftrightarrow (y - x)(y + x) = 0,$$

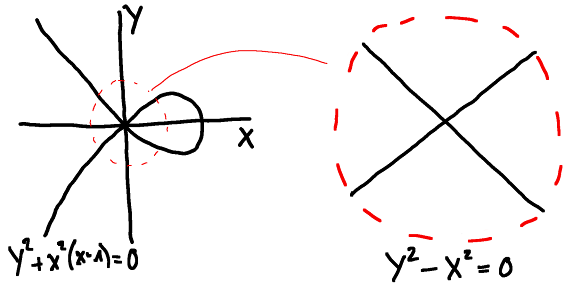


Figure 4: A neighbourhood of the singular point in E_0 looks like the union of the lines $y = x$ and $y = -x$

which is simply two lines meeting transversally.

Such a singular point is called a *nodal singularity*. Using the pictures of real surfaces from above, we can illustrate how the set of complex points $E_t(\mathbb{C})$ changes as t approaches 0.

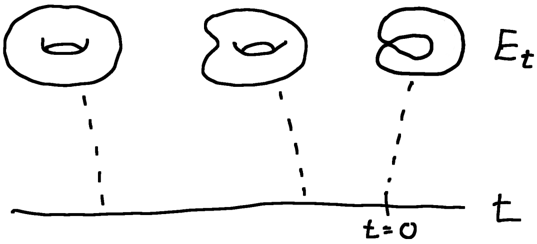


Figure 5: The degeneration of the surface $E_t(\mathbb{C})$ as $t \to 0$

Generalizing the moduli space $\mathcal{M}_g$ of smooth curves C we will study a moduli space $\overline{\mathcal{M}}_g$ which allows the curve C to have such nodal singularities. This space $\overline{\mathcal{M}}_g$ then turns out to be proper (or compact), fixing the holes that we had in our pockets.

The goal of the second half of this lecture series is to show the beautiful structures that appear in the cohomology groups of the spaces $\overline{\mathcal{M}}_g$. This is an area of active research, with connections to many parts of mathematics such as graph theory, enumerative geometry and the theory of integrable systems (to name but a few).

References and further reading (and watching)

A nice and mostly elementary discussion of the Facts about curves of genus zero and one stated above can be found in Sections 19.3 and 19.9 of [Vak17]. The result that the set of complex points of a plane cubic is isomorphic to a torus (which can and should be seen as a quotient $\mathbb{C}/\Lambda$ of the complex numbers by a lattice $\Lambda \subset \mathbb{C}$) has a beautiful connection to the so-called Weierstrass $\wp$-function, e.g. explained in [Hai08, Section 5].

There is a video recording of a great one-hour introductory talk to the moduli spaces of curves given by my PhD advisor Rahul Pandharipande at the ICM in Rio in 2018. It does not only cover essentially everything that we are going to treat in this course, but it

[6] This can be made precise using formal algebraic geometry and completions of local rings, let's not worry about this now.

also contains the Portuguese translation of a quote by Riemann (at 10:53) and a picture of a passion flower taken by a PhD brother of mine (at 13:39), among other things.

2 Fine and coarse moduli spaces

2.1 Motivation

Before we discuss the moduli space of curves, we should first make sure we understand what we even mean by "moduli space". To start off, let us consider a familiar example: the projective space $\mathbb{P}^n$. It classifies lines through the origin (i.e. sub-vector spaces of dimension 1) in $\mathbb{C}^{n+1}$. However, just writing

$$\mathbb{P}^n = \{\ell : \ell \text{ line through the origin in } \mathbb{C}^{n+1}\} \tag{9}$$

is not enough: the right-hand side of (9) is a set, equal to the set $\mathbb{P}^n(\mathbb{C})$ of $\mathbb{C}$-points of $\mathbb{P}^n$, but it does not have the structure of a scheme.

So what makes the usual scheme structure on $\mathbb{P}^n$ the right one to make it a moduli space of lines in $\mathbb{C}^{n+1}$? Let's start with the topology: intuitively, for the lines

$$\ell_n = \left\langle \begin{pmatrix} 1 \\ 1/n \end{pmatrix} \right\rangle, \ \ell_0 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

the sequence ℓ_n of lines in $\mathbb{C}^2$ "converges" to the line ℓ_0.

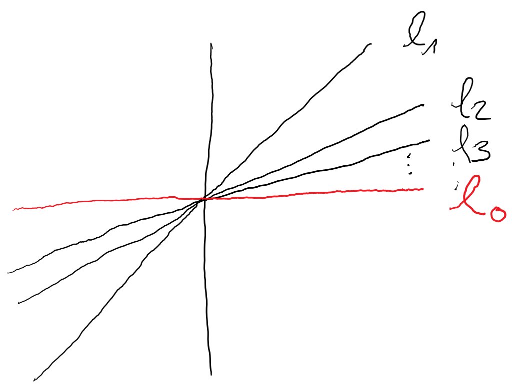


Figure 6: The sequence of lines ℓ_n converging to the line ℓ_0

And indeed, the (complex) topology on $\mathbb{P}^n(\mathbb{C})$ satisfies $\ell_n \xrightarrow{n\to\infty} \ell_0$. Similarly, for the scheme structure on $\mathbb{P}^n$ note that the family of lines

$$\left(L_t = \left\langle \begin{pmatrix} 1 \\ t \end{pmatrix} \right\rangle : t \in \mathbb{A}^1 \right)$$

parametrized by the variety $\mathbb{A}^1$ has defining equations which are algebraic in the coordinate t on $\mathbb{A}^1$. Because of this, we expect that the scheme structure on $\mathbb{P}^2$ should satisfy that the map

$$\mathbb{A}^1 \to \mathbb{P}^2, t \mapsto L_t \tag{10}$$

is an algebraic morphism, which again is true for the standard structure on $\mathbb{P}^2$. Generalizing this example from $X = \mathbb{A}^1$ to arbitrary schemes X, there should be a bijective correspondence

$$\begin{aligned} \text{Families of lines parametrized by } X &\longleftrightarrow \text{Morphisms } X \to \mathbb{P}^n, \\ (L_t : t \in X) &\longleftrightarrow (X \to \mathbb{P}^n, t \mapsto L_t). \end{aligned} \tag{11}$$

The crucial insight is that knowing the sets of morphisms $X \to \mathbb{P}^n$ for *every* scheme X uniquely determines the scheme structure on $\mathbb{P}^n$ (see Lemma 2.1 below). Thus, by the equivalence (11) above, we can uniquely specify the scheme $\mathbb{P}^n$ by making precise what we mean with a "family of lines parametrized by X".

2.2 Moduli functors and fine moduli spaces

The correct way to do this is using category theory. Denote by $\mathbf{Sch}_{\mathbb{C}}$ the category of schemes over $\mathbb{C}$, where for schemes $X, Y \in \mathrm{Ob}(\mathbf{Sch}_{\mathbb{C}})$ the set $\mathrm{Mor}(X, Y)$ of morphisms from X to Y is the set of algebraic morphisms $X \to Y$ over $\mathbb{C}$. Then, for any scheme $M \in \mathrm{Ob}(\mathbf{Sch}_{\mathbb{C}})$, we have a functor

$$h^M : \mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Sets},\ X \mapsto \mathrm{Mor}(X, M). \tag{12}$$

Why the "op" above? This is because $\mathrm{Mor}(-, M)$ is a contravariant functor, i.e. given $g : X' \to X$ we get a natural map $h^M(X) \to h^M(X')$ in the opposite direction, given by

$$\mathrm{Mor}(X, M) \to \mathrm{Mor}(X', M), (X \xrightarrow{f} M) \mapsto (X' \xrightarrow{g} X \xrightarrow{f} M).$$

For $M = \mathbb{P}^n$, the set $h^M(X)$ is exactly the right-hand side of (11). The statement of the Yoneda Lemma is that the functor h^M uniquely determines the scheme M.

Lemma 2.1 (Yoneda's Lemma)**.** The functor

$$\begin{aligned} h^- : \mathbf{Sch}_{\mathbb{C}} &\to \mathrm{Functors}(\mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Sets}), \\ M &\mapsto h^M \end{aligned} \tag{13}$$

is a fully faithful embedding. In other words

- given schemes M, N, the morphisms $M \to N$ are in bijection with natural transformations $h^M \to h^N$,
- in particular $M \cong N$ iff $h^M \cong h^N$.

By this Lemma, we can see the category of schemes as sitting inside the category of functors $\mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Sets}$. Thus we will give these functors their own name.

Definition 2.2. A *moduli functor* h is a functor $h : \mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Sets}$. In other words, the data we need to specify is the following:

- for every scheme X over $\mathbb{C}$ a set $h(X)$ (the *families parametrized by/over* X),
- for every morphism $f : X' \to X$ a map $h(f) : h(X) \to h(X')$ (the *pullback of families under* f),
- satisfying that $h(\mathrm{id}_X) = \mathrm{id}_{h(X)}$ and that for $X'' \xrightarrow{g} X' \xrightarrow{f} X$ we have that the composition $h(X) \xrightarrow{h(f)} h(X') \xrightarrow{h(g)} h(X'')$ equals $h(X) \xrightarrow{h(f \circ g)} h(X'')$ (the *compatibility of pullback for identity and compositions*).

Definition 2.3. A moduli functor h is called *representable* if it is of the form $h \cong h^M$ for a scheme M. This scheme M (which is unique up to isomorphism) is then called a *fine moduli space* for h.

Example 2.4. We return to the example of projective space $\mathbb{P}^n$, showing that it is a fine moduli space for "lines through the origin in $\mathbb{C}^{n+1}$". Define a moduli functor $h : \mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Sets}$ by the following data:

- for every scheme X over $\mathbb{C}$ the set $h(X)$ is given by

$$h(X) = \left\{ \begin{tikzcd} L \arrow[r, "i"] \arrow[d] & X \times \mathbb{C}^{n+1} \arrow[dl] \\ X & \end{tikzcd} \right\} / \sim \tag{14}$$

  where $L \to X$ is (the total space of) a line bundle on X which is a subbundle of the trivial bundle $X \times \mathbb{C}^{n+1} \to X$ (i.e. there is an injective map $i : L \to X \times \mathbb{C}^{n+1}$ of vector bundles such that the quotient is also a vector bundle). We take this set up to isomorphisms, i.e. $(L \xrightarrow{i} X \times \mathbb{C}^{n+1}) \sim (L' \xrightarrow{i'} X \times \mathbb{C}^{n+1})$ iff there is an isomorphism $L \xrightarrow{\sim} L'$ of line bundles on X making the obvious diagram commute,

- for every morphism $f : X' \to X$ we define the pullback by

$$\begin{aligned} h(f) &: h(X) \to h(X'), \\ (L \xrightarrow{i} X \times \mathbb{C}^{n+1}) &\mapsto (f^*L \xrightarrow{f^*i} f^*(X \times \mathbb{C}^{n+1}) = X' \times \mathbb{C}^{n+1}). \end{aligned}$$

The compatibility conditions of the pullback are satisfied since clearly $h(\mathrm{id}_X) = \mathrm{id}_{h(X)}$ and since for $X'' \xrightarrow{g} X' \xrightarrow{f} X$ we have a canonical isomorphism $g^*f^*L \cong (g \circ f)^*L$.

Finally we want to show that h is representable by $\mathbb{P}^n$, i.e. that we have a natural equivalence $h \cong h^{\mathbb{P}^n}$ of functors. This exactly makes precise the equivalence (11) we claimed before. Recall that to specify a natural equivalence $h \cong h^{\mathbb{P}^n}$ we must give, for every scheme X, a bijection $h(X) \to h^{\mathbb{P}^n}(X) = \mathrm{Mor}(X, \mathbb{P}^n)$ such that for $f : X' \to X$ the diagram

$$\begin{tikzcd} h(X) \arrow[r] \arrow[d, "h(f)"] & \mathrm{Mor}(X, \mathbb{P}^n) \arrow[d, "h^{\mathbb{P}^n}(f)"] \\ h(X') \arrow[r] & \mathrm{Mor}(X', \mathbb{P}^n) \end{tikzcd} \tag{15}$$

commutes. What should the map $h(X) \to \mathrm{Mor}(X, \mathbb{P}^n)$ do again? Given

$$(i : L \to X \times \mathbb{C}^{n+1}) \in h(X), \tag{16}$$

for any $x \in X$ we have that $i(L_x) \subset \{x\} \times \mathbb{C}^{n+1}$ is a line through the origin of $\mathbb{C}^{n+1}$ and so (16) should be sent to the morphism $X \to \mathbb{P}^n$ mapping x to $[i(L_x)] \in \mathbb{P}^n$. But a priori it is only clear what to do at closed points $x \in X$ and we should find a more algebraic way to phrase this, allowing us to deal with more complicated X.

We take a small but interesting detour to do this. Fix a scheme X and an element $(L \xrightarrow{i} X \times \mathbb{C}^{n+1})$ in $h(X)$. The injective map i of total spaces of vector bundles (whose cokernel is a vector bundle) corresponds to a short exact sequence

$$0 \to \mathcal{L} \xrightarrow{\iota} \mathcal{O}_X^{n+1} \to \mathcal{Q} \to 0$$

of locally free sheaves on X[7]. Taking the dual, this is equivalent to an exact sequence

$$0 \leftarrow \mathcal{L}^\vee \xleftarrow{\iota^\vee} \mathcal{O}_X^{n+1} \leftarrow \mathcal{Q}^\vee \leftarrow 0$$

[7]Alternatively, we can say it corresponds to a *locally split injective* morphism $\iota : \mathcal{L} \to \mathcal{O}_X^{n+1}$.

of locally free sheaves. Since the kernel of a map of locally free sheaves is automatically locally free, this is equivalent to just specifying the surjection $\iota^\vee : \mathcal{O}_X^{n+1} \to \mathcal{L}^\vee$. Such a surjection is determined by specifying sections $s_0, \ldots, s_n \in H^0(X, \mathcal{L}^\vee)$ without common zero, i.e. not vanishing simultaneously anywhere on X. Then writing $\mathcal{M} = \mathcal{L}^\vee$ and defining

$$h'(X) = \left\{ (\mathcal{M}, s_0, \ldots, s_n) : \begin{array}{c} \mathcal{M} \text{ a locally free sheaf on } X \\ s_0, \ldots, s_n \in H^0(X, \mathcal{M}) \text{ sections} \\ \text{without common zero} \end{array} \right\} / \sim \tag{17}$$

we have just described a map $h(X) \to h'(X)$. One checks that this map is a bijection, that h' is in fact a moduli functor and that the maps $h(X) \to h'(X)$ define a natural equivalence $h \cong h'$. Thus we have reduced the problem to showing that $h' \cong h^{\mathbb{P}^n}$.

The fact that a map $\varphi : X \to \mathbb{P}^n$ is equivalent to the data of a line bundle $\mathcal{M}$ on X together with $n+1$ sections without common zero (up to isomorphism) is proven in many textbooks on Algebraic Geometry (see e.g. [Sch17, Corollary 12.10], [Har77, II, Theorem 7.1], [Vak17, Important Theorem 16.4.1.]). Let us recall the argument: starting with an element $(\mathcal{M}, s_0, \ldots, s_n) \in h'(X)$ we can write down a map $X \to \mathbb{P}^n$ by

$$\varphi : X \to \mathbb{P}^n, \ x \mapsto [s_0(x) : s_1(x) : \ldots : s_n(x)]. \tag{18}$$

A priori, the expression $[s_0(x) : s_1(x) : \ldots : s_n(x)]$ does not make sense, since the $s_j(x)$ are not functions but elements in the fibre $\mathcal{M}_x$ of the line bundle $\mathcal{M}$ on X. But we can choose any cover $X = \bigcup_i U_i$ of X trivializing the line bundle and any isomorphism $\mathcal{M}|_{U_i} \cong \mathcal{O}_{U_i}$. Then on U_i we can identify the sections s_j with regular functions and the element $[s_0(x) : s_1(x) : \ldots : s_n(x)]$ does not depend on the choice of trivialization. Indeed, a different choice of trivialization differs by multiplying all components of $[s_0(x) : s_1(x) : \ldots : s_n(x)]$ with the *same* nonvanishing function, so the corresponding element of $\mathbb{P}^n$ does not change.

Conversely, given a map $\varphi : X \to \mathbb{P}^n$ we define $\mathcal{M} = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$. For the sections, observe that $\mathcal{O}_{\mathbb{P}^n}(1)$ has a space of global sections of rank $n+1$ given by

$$H^0(\mathbb{P}^n, \mathcal{O}(1)) = \langle x_0, \ldots, x_n \rangle$$

and we define $s_j = \varphi^*(x_j) \in H^0(X, \mathcal{M})$. One checks that the two correspondences between $(\mathcal{M}, s_0, \ldots, s_n)$ (up to isomorphism) and $\varphi : X \to \mathbb{P}^n$ are inverse to each other and "functorial" (i.e. making the diagrams (15) commute). Thus they define a natural equivalence $h' \cong h^{\mathbb{P}^n}$, finishing the proof.

Remark 2.5. Instead of constructing the equivalence $h \cong h^{\mathbb{P}^n}$ via the functor h', a direct construction is also possible: given an element $(L \xrightarrow{i} X \times \mathbb{C}^{n+1})$ in $h(X)$ let $L' \subset L$ be the open set that is the complement of the zero section. Due to the assumption that i is injective, it maps L' to $X \times (\mathbb{C}^{n+1} \setminus 0)$. Consider the diagram

$$\begin{tikzcd} L' \arrow[r, "i"] \arrow[d] & X \times (\mathbb{C}^{n+1} \setminus 0) \arrow[r, "\pi"] & \mathbb{P}^n \\ X \arrow[urr, dashed, "\varphi"'] & & \end{tikzcd}$$

where π is the composition of the projection $X \times (\mathbb{C}^{n+1} \setminus 0) \to \mathbb{C}^{n+1} \setminus 0$ with the quotient map $\mathbb{C}^{n+1} \setminus 0 \to \mathbb{P}^n$. One checks that the composition $\pi \circ i : L' \to \mathbb{P}^n$ is constant on the fibres of $L' \to X$. Intuitively, this should allow us to define a map $\varphi : X \to \mathbb{P}^n$ completing the diagram above by $\varphi(x) = (\pi \circ i)(\ell)$ for any $\ell \in L'$ mapping to x. This intuition indeed works, using the so-called fpqc descent (see [Sta13, Tag 023Q]). So starting from $(L \xrightarrow{i} X \times \mathbb{C}^{n+1}) \in h(X)$ we constructed $\varphi \in \mathrm{Mor}(X, \mathbb{P}^n)$ and one checks that this induces the same natural transformation $h \cong h^{\mathbb{P}^n}$ as in Example 2.4.

Definition 2.6. Given a moduli functor h and a natural isomorphism $h \cong h^M$ for some scheme M, define the *universal family* $U \in h(M)$ to be the element in $h(M)$ corresponding to the canonical element $\mathrm{id}_M \in \mathrm{Mor}(M, M) = h^M(M)$.

Exercise 2.7. Let h be a moduli functor with fine moduli space M and universal family $U \in h(M)$.

a) Show that for any scheme X and any family $F \in h(X)$ parametrized by X there exists a unique morphism $f : X \to M$ such that the pullback of U under f is F, i.e. $h(f)(U) = F$.

b) Show that for the moduli functor h from Example 2.4 with moduli space $\mathbb{P}^n$, the universal family is given by the "tautological" line bundle

$$\begin{tikzcd} \{(\ell, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : v \in \ell\} = L \arrow[r, "i"] \arrow[d] & \mathbb{P}^n \times \mathbb{C}^{n+1} \arrow[dl] \\ \mathbb{P}^n & \end{tikzcd}$$

The line bundle L above is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-1)$.

c) Show that for the moduli functor h' from Example 2.4 with moduli space $\mathbb{P}^n$, the universal family is given by

$$(\mathcal{O}_{\mathbb{P}^n}(1), x_0, \ldots, x_n).$$

Easy exercise 2.8 (Fibre products as moduli spaces)**.** Let $\pi_X : X \to Z$, $\pi_Y : Y \to Z$ be morphisms of schemes. For any scheme S define

$$h(S) = (h^X \times_{h^Z} h^Y)(S) = \left\{ (\sigma_X, \sigma_Y) : \begin{array}{c} \sigma_X : S \to X, \sigma_Y : S \to Y \\ \text{such that } \pi_X \circ \sigma_X = \pi_Y \circ \sigma_Y \end{array} \right\}$$

a) Show that h defines a moduli functor.

b) Prove that the fibre product $X \times_Z Y$ is a fine moduli space for h. What is its universal family?

2.3 Application : The Picard scheme

Until now we only used the theory of moduli spaces to interpret known schemes as a fine moduli space for some functor. Now let's use it to actually define a new space. For this, recall the definition of the Picard group of a scheme[8] X:

$$\mathrm{Pic}(Y) = \{\mathcal{L} : \mathcal{L} \text{ an invertible sheaf on } Y\}/\mathrm{iso}. \tag{19}$$

It classifies line bundles up to isomorphisms. A priori, the definition (19) only makes sense as a set (or a group with respect to tensor product $\otimes$). But it turns out that in many situations we can find a nice scheme structure underlying $\mathrm{Pic}(Y)$ by seeing it as a fine moduli space.

What should be the moduli functor? Given a scheme X we need to define what we mean by a "family of line bundles on Y over the base X". The natural thing is to take the product $X \times Y$ (the trivial family with fibre Y over X) and look at the data of line bundles up to isomorphism on $X \times Y$.

[8]In this section, all schemes will be schemes over $\mathbb{C}$, all morphisms will be morphisms over $\mathbb{C}$, all fibre products will be fibre products over $\mathbb{C}$ etc.

Definition 2.9 (Picard functor - first attempt)**.** We define the (absolute) Picard functor $\mathrm{Pic}_Y : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Sets}$ of the scheme Y as

$$\mathrm{Pic}_Y(X) = \mathrm{Pic}(X \times Y).$$

Given $f : X' \to X$ the corresponding morphism $\mathrm{Pic}_Y(X) \to \mathrm{Pic}_Y(X')$ is given by the pullback $\mathcal{L} \mapsto f^*\mathcal{L}$ of line bundles under $f \times \mathrm{id}_Y$.

This is a perfectly nice and reasonable moduli functor, except for the fact that it is *never* representable (for Y nonempty).

Proposition 2.10. For $Y \neq \emptyset$ the functor Pic_Y is not representable.

Proof. Let X be any scheme with a nontrivial line bundle $\mathcal{M}$ (e.g. $X = \mathbb{P}^1$ with $\mathcal{M} = \mathcal{O}_{\mathbb{P}^1}(1)$). Let $\pi_X : X \times Y \to X$ be the projection, then we have the object

$$\mathcal{M}_X = \pi_X^*\mathcal{M} \in \mathrm{Pic}(X \times Y) = \mathrm{Pic}_Y(X).$$

Note that this object is nontrivial (i.e. not isomorphic to $\mathcal{O}_{X\times Y}$). Indeed, since Y is nonempty (and since $\mathbb{C}$ is algebraically closed) it has a $\mathbb{C}$-point and so π_X has a section. Then the pullback of $\mathcal{M}_X$ under this section is $\mathcal{M}$, which is not trivial.

Assume that Pic_Y were representable by a scheme P with a universal object $\mathcal{U} \in \mathrm{Pic}_Y(P)$. Then by definition there exists a unique morphism $g : X \to P$ with $(g \times \mathrm{id}_Y)^*\mathcal{U} = \mathcal{M}_X$. Similarly, there is a unique morphism $p : \mathrm{pt} = \mathrm{Spec}(\mathbb{C}) \to P$ associated to the trivial line bundle $\mathcal{O}_{\mathrm{pt}\times Y}$, i.e. satisfying $(p \times \mathrm{id}_Y)^*\mathcal{U} = \mathcal{O}_{\mathrm{pt}\times Y}$.

Now let $X = \bigcup_i U_i$ be an open cover on which $\mathcal{M}$ is trivial, then the pullbacks $\mathcal{M}_{U_i}$ of $\mathcal{M}_X$ to $U_i \times Y$ *are* trivial. In other words, they are pullbacks of $\mathcal{O}_{\mathrm{pt}\times Y}$ under the maps $U_i \to \mathrm{pt}$ to a point. This implies that the unique map $U_i \to P$ associated to $\mathcal{M}_{U_i}$ must factor through $p : \mathrm{pt} \to P$. We then have a diagram of maps as follows

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, dashed, "g'"] & P \\ \bigsqcup_i U_i \arrow[u] \arrow[r] & \mathrm{pt} \arrow[u, "p"'] \end{tikzcd} \tag{20}$$

Since the restrictions of the map $g : X \to P$ to the covering opens U_i all factor through p, the map g itself must factor through this morphism. But this is impossible since then we have

$$\mathcal{M}_X = (g \times \mathrm{id}_Y)^*\mathcal{U} = (g' \times \mathrm{id}_Y)^*(p \times \mathrm{id}_Y)^*\mathcal{U} = (g' \times \mathrm{id}_Y)^*\mathcal{O}_{\mathrm{pt}\times Y} = \mathcal{O}_{X\times Y},$$

a contradiction to the statement that $\mathcal{M}_X$ is not trivial. $\square$

Remark 2.11. The fact that Pic_Y does not have a moduli space is a first example of a general slogan[9] in the theory of moduli spaces:

For moduli functors h, the presence of nontrivial automorphisms
often prevents the existence of a fine moduli space. (21)

As above, the rough idea is that automorphisms allow us to find a scheme X with an open cover $X = \bigcup_i U_i$ such that we can construct a nontrivial family $F_X \in h(X)$ by gluing trivial families $F_{U_i} \in h(U_i)$ along nontrivial automorphisms $F_{U_i}|_{U_i\cap U_j} \cong F_{U_j}|_{U_i\cap U_j}$. We then reach a contradiction by using a variant of the diagram (20). As we will see later, the theory of algebraic stacks was specifically invented to be able to deal with these kinds of moduli problems.

[9]As with many one-sentence slogans, this is a bit too general to be true in all contexts and hard to make precise anyway. It is more a guideline to build some first intutions.

In the proof of Proposition 2.10 we were able to reach a contradiction for the representability of Pic_Y by using the nontrivial family $\mathcal{M}_X \in \mathrm{Pic}(X \times Y)$ obtained by pulling back a line-bundle $\mathcal{M} \in \mathrm{Pic}(X)$ from the base. Surprisingly, in this case we can solve all our problems by just dividing out such pullback bundles in our moduli functor.

Definition 2.12 (Picard functor - second attempt)**.** We define the (relative) Picard functor $\mathrm{Pic}_{Y/\mathbb{C}} : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Sets}$ of the scheme Y as

$$\mathrm{Pic}_{Y/\mathbb{C}}(X) = \mathrm{Pic}(X \times Y)/\pi_X^* \mathrm{Pic}(X),$$

where $\pi_X : X \times Y \to X$ is the projection on the first factor and $\pi_X^* \mathrm{Pic}(X) \subset \mathrm{Pic}(X \times Y)$ is the set of line bundles that are pullbacks from X. Given $f : X' \to X$ the corresponding morphism $\mathrm{Pic}_{Y/\mathbb{C}}(X) \to \mathrm{Pic}_{Y/\mathbb{C}}(X')$ is still given by the pullback $\mathcal{L} \mapsto f^*\mathcal{L}$ of line bundles under $f \times \mathrm{id}_Y$ (and this respects the equivalence relation we divided out).

Theorem 2.13. *If Y is an integral, projective variety over $\mathbb{C}$, then the functor $\mathrm{Pic}_{Y/\mathbb{C}}$ is representable by a separated, locally finite type scheme $\mathbf{Pic}_{Y/\mathbb{C}}$.*

Proof. See e.g. [Kle05, Theorem 4.8]. □

Remark 2.14. Assume we are given Y such that $\mathrm{Pic}_{Y/\mathbb{C}}$ is representable by a scheme $\mathbf{Pic}_{Y/\mathbb{C}}$.

a) A line bundle $\mathcal{L}$ on $\mathbf{Pic}_{Y/\mathbb{C}} \times Y$ representing the universal family on the moduli space is called a *Poincaré line bundle*. Given a $\mathbb{C}$-point $[\mathcal{M}] \in \mathbf{Pic}_{Y/\mathbb{C}}$, the restriction of $\mathcal{L}$ to $[\mathcal{M}] \times Y \subset \mathbf{Pic}_{Y/\mathbb{C}} \times Y$ is isomorphic to $\mathcal{M}$. The line bundle $\mathcal{L}$ is unique up to tensoring with line bundles pulled back from $\mathbf{Pic}_{Y/\mathbb{C}}$.

b) The set $\mathrm{Pic}(Y)$ of line bundles on Y has a natural structure of an abelian group, where multiplication is given by tensor product of line bundles. From this one can show that $\mathbf{Pic}_{Y/\mathbb{C}}$ has the structure of an *algebraic group*, i.e. the natural maps

$$\begin{aligned} \mathbf{Pic}_{Y/\mathbb{C}} \times \mathbf{Pic}_{Y/\mathbb{C}} &\to \mathbf{Pic}_{Y/\mathbb{C}}, \\ ([\mathcal{L}_1], [\mathcal{L}_2]) &\mapsto [\mathcal{L}_1 \otimes \mathcal{L}_2] \end{aligned} \qquad \begin{aligned} \mathbf{Pic}_{Y/\mathbb{C}} &\to \mathbf{Pic}_{Y/\mathbb{C}} \\ \mathcal{L} &\mapsto \mathcal{L}^\vee \end{aligned}$$

are algebraic morphisms which together with the inclusion $[\mathcal{O}_Y] \in \mathbf{Pic}_{Y/\mathbb{C}}$ define multiplication, inverse and neutral element of a group-structure on $\mathbf{Pic}_{Y/\mathbb{C}}$.

c) We denote by $\mathbf{Pic}^0_{Y/\mathbb{C}} \subset \mathbf{Pic}_{Y/\mathbb{C}}$ the connected component of $\mathbf{Pic}_{Y/\mathbb{C}}$ containing the trivial line bundle $[\mathcal{O}_Y]$. It gives a subgroup of $\mathbf{Pic}_{Y/\mathbb{C}}$. For $Y = C$ a smooth, projective algebraic curve, the scheme $\mathbf{Pic}^0_{C/\mathbb{C}} = \mathrm{Jac}(C)$ is called the *Jacobian* of C.

Example 2.15. a) For a point $Y = \mathrm{pt} = \mathrm{Spec}(\mathbb{C})$ we have that for any scheme S we obtain

$$\mathrm{Pic}_{\mathrm{pt}/\mathbb{C}}(S) = \mathrm{Pic}(\mathrm{pt} \times S)/\mathrm{Pic}(S) = \{[\mathcal{O}_S]\}.$$

This implies that

$$\mathbf{Pic}_{\mathrm{pt}/\mathbb{C}} = \mathrm{Spec}(\mathbb{C}).$$

However, surprisingly for $Y = \mathbb{A}^n$ the same is *not* true (even though $\mathrm{Pic}(\mathbb{A}^n) = \{\mathcal{O}_{\mathbb{A}^n}\}$). This follows from the existence [Liu] of a scheme X such that the pullback $\mathrm{Pic}(X) \to \mathrm{Pic}(X \times \mathbb{A}^1)$ is not an isomorphism.

b) For any $n \geq 1$ we have that

$$\mathbf{Pic}_{\mathbb{P}^n/\mathbb{C}} = \coprod_{m \in \mathbb{Z}} \{\mathcal{O}_{\mathbb{P}^n}(m)\}$$

is a countable union of isolated points.

c) Given an elliptic curve (E, p_0), we have

$$\mathbf{Pic}^0_{E/\mathbb{C}} = E,$$

and for $\Delta \subset E \times E$ the diagonal, the line bundle

$$\mathcal{L} = \mathcal{O}_{E\times E}(\Delta - E \times p_0) \in \mathrm{Pic}(\mathbf{Pic}^0_{E/\mathbb{C}} \times E)$$

is a Poincaré line bundle over $\mathbf{Pic}^0_{E/\mathbb{C}}$.

2.4 Coarse moduli spaces

As we have seen, even reasonable functors like Pic_Y can fail to have fine moduli spaces. While in this particular example we were able to fix this by considering the relative Picard functor, in general (and in particular for the moduli functors of curves that we will consider soon) a fine moduli space is too much to ask. Thus we need a weaker notion for a scheme to "approximately represent" a moduli functor.

Definition 2.16. Given a moduli functor h, a *coarse moduli space* is a pair (M, Φ) of a scheme M together with a natural transformation $\Phi : h \to h^M$ such that

a) (M, Φ) is initial among all such pairs, i.e. for any other scheme M' and natural transformation $\Phi' : h \to h^{M'}$ there exists a unique natural transformation $\Psi : h^M \to h^{M'}$ such that the following diagram commutes

$$\begin{tikzcd} h \arrow[rr, "\Phi"] \arrow[dr, "\Phi'"'] & & h^M \arrow[dl, "\exists!\Psi"] \\ & h^{M'} & \end{tikzcd} \tag{22}$$

b) the map Φ induces a bijection

$$\Phi(\mathrm{Spec}(\mathbb{C})) : h(\mathrm{Spec}(\mathbb{C})) \xrightarrow{\sim} h^M(\mathrm{Spec}(\mathbb{C})) = M(\mathbb{C}) \tag{23}$$

on $\mathbb{C}$-points.

Easy exercise 2.17. a) Show that every fine moduli space is also a coarse moduli space (in particular, make precise what this statement means).

b) Show that given a moduli functor h, if a pair (M, Φ) satisfying condition a) in Definition 2.16 exists, this pair is unique up to isomorphism. In particular, coarse moduli spaces are unique up to isomorphism.

In the next section we are going to introduce the moduli functors for families of curves and we will see that they have a coarse moduli space, but not a fine one. A second example of this are the absolute Picard functors Pic_Y introduced in Section 2.3.

Proposition 2.18. For any scheme Y such that the relative Picard functor $\mathrm{Pic}_{Y/\mathbb{C}}$ has a fine moduli space $\mathbf{Pic}_{Y/\mathbb{C}}$, the natural map $\Phi : \mathrm{Pic}_Y \to \mathrm{Pic}_{Y/\mathbb{C}} \cong h^{\mathbf{Pic}_{Y/\mathbb{C}}}$ makes $(\mathbf{Pic}_{Y/\mathbb{C}}, \Phi)$ into a coarse moduli space for Pic_Y.

Proof. To show property a) from Definition 2.16, let $\Phi' : \mathrm{Pic}_Y \to h^{M'}$ be a second natural transformation. We need to show that given X, the map $\mathrm{Pic}_Y(X) \to \mathrm{Mor}(X, M')$ induced by Φ' factors through $\mathrm{Pic}_Y(X)/\pi_X^* \mathrm{Pic}(X)$. So let $\mathcal{L}_1, \mathcal{L}_2$ be line bundles on $X \times Y$ with $\mathcal{L}_2 = \mathcal{L}_1 \otimes \pi_X^* \mathcal{M}$ for a line bundle $\mathcal{M}$ on X. Let $g_1, g_2 : X \to M'$ be the maps induced from $\mathcal{L}_1, \mathcal{L}_2$ via Φ'. We are finished if we can show $g_1 = g_2$.

For this note that given for any open cover $X = \bigcup_i U_i$ trivializing $\mathcal{M}$ on X, the pullbacks of $\mathcal{L}_1, \mathcal{L}_2$ to $U_i \times Y$ coincide for all i. By functoriality of Φ', this implies that $g_1|_{U_i} = g_2|_{U_i}$. But since morphisms are determined by their restriction to an open cover, we conclude $g_1 = g_2$.

Property b) of Definition 2.16 follows since

$$\begin{aligned} \mathrm{Pic}_Y(\mathrm{Spec}(\mathbb{C})) &= \mathrm{Pic}(\mathrm{Spec}(\mathbb{C}) \times Y) \\ &= \mathrm{Pic}(\mathrm{Spec}(\mathbb{C}) \times Y)/\mathrm{Pic}(\mathrm{Spec}(\mathbb{C})) = \mathrm{Pic}_{Y/\mathbb{C}}(\mathrm{Spec}(\mathbb{C})) = \mathbf{Pic}(\mathbb{C}). \end{aligned}$$ □

Challenge 2.19. Prove or disprove that $\mathrm{pt} = \mathrm{Spec}(\mathbb{C})$ is a *coarse* moduli space for $\mathrm{Pic}_{\mathbb{A}^1/\mathbb{C}}$. See also my question on math.stackexchange.

Update: The question has been answered affirmatively within a day by two students from the class! You can check out their answers under the link above.

Exercise 2.20. We want to study the moduli problem of classifying "two points on $\mathbb{P}^1$, not necessarily distinct, up to projective linear transformations". For this, consider the moduli functor

$$h : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Sets}, X \mapsto \{s : X \to \mathbb{P}^1 \times \mathbb{P}^1\}/\sim\ , \tag{24}$$

where $s \sim s'$ if there exists $G : X \to \mathrm{PGL}_2$ such that

$$G(x) \cdot s(x) = s'(x),$$

where PGL_2 acts diagonally on $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $\mathrm{pt} = \mathrm{Spec}(\mathbb{C})$, when we want to show that (pt, Φ) for the unique natural transformation $\Phi : h \to h^{\mathrm{pt}}$ satisfies condition a) from Definition 2.16, but not condition b).

a) Assume we have a scheme U and $s \in h(U)$ such that $s : U \to \mathbb{P}^1 \times \mathbb{P}^1$ factors through the complement $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$ of the diagonal Δ. Show that then there exists an open cover $U = \bigcup_i U_i$ of U such that the restrictions $s|_{U_i}$ are equivalent under $\sim$ to the constant section
$$s_0 = ([1:0],[0:1]) : U_i \to \mathbb{P}^1 \times \mathbb{P}^1.$$

b) Let (M', Φ') be a pair of a scheme and a natural transformation $\Phi' : h \to h^{M'}$. Let $\psi : \mathrm{pt} \to M'$ be the morphism associated via Φ' to
$$(i : \mathrm{pt} \xrightarrow{([1:0],[0:1])} \mathbb{P}^1 \times \mathbb{P}^2) \in h(\mathrm{pt}).$$
Consider the family
$$(s_{\mathbb{P}^1 \times \mathbb{P}^1} = \mathrm{id}_{\mathbb{P}^1 \times \mathbb{P}^1} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1) \in h(\mathbb{P}^1 \times \mathbb{P}^1)$$
and use the previous exercise part to show that the associated morphism $\Phi'(s_{\mathbb{P}^1 \times \mathbb{P}^1}) : \mathbb{P}^1 \times \mathbb{P}^1 \to M'$ factors through $\psi : \mathrm{pt} \to M'$. (*Hint* : Use the continuity of $\Phi'(s_{\mathbb{P}^1 \times \mathbb{P}^1})$.)

c) Show that given any scheme X and $s \in h(X)$ there exists a morphism $f : X \to \mathbb{P}^1 \times \mathbb{P}^1$ such that $f^*(s_{\mathbb{P}^1 \times \mathbb{P}^1}) = s$. Use this to finish the proof that (pt, Φ) satisfies condition a) from Definition 2.16.

d) Why does it not satisfy condition b)?

To summarize, we learned that the functor h^- embeds the category of schemes into the category of moduli functors. Functors in the image are called representable and have a fine moduli space. However, there are bigger classes of moduli functors, having only a coarse moduli space or even just a pair (M, Φ) satisfying part a) of Definition 2.16. We illustrate this in Figure 7.

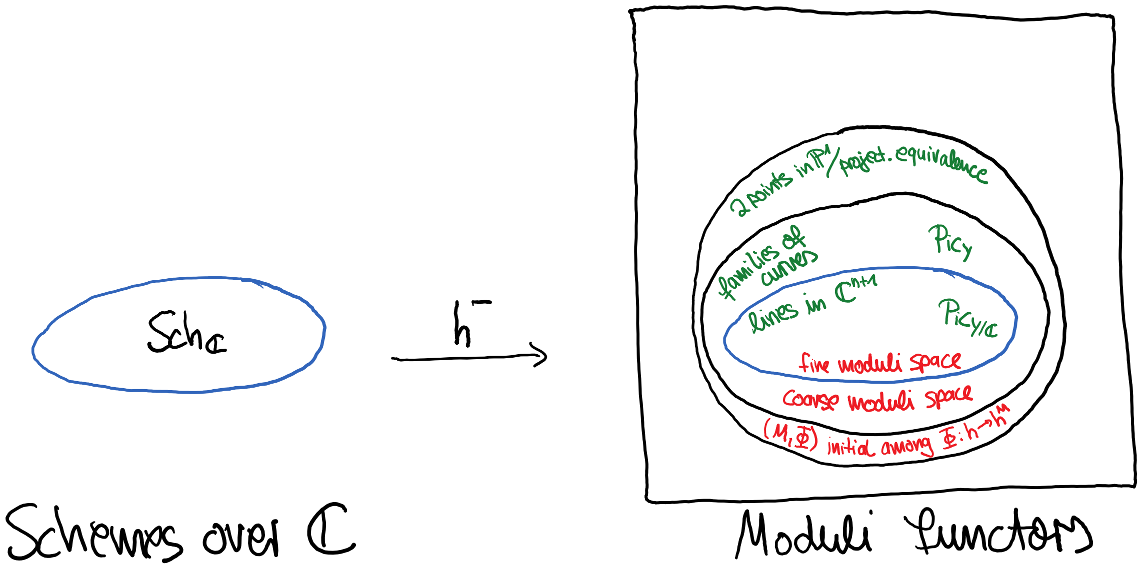


Figure 7: The Yoneda embedding of the category of schemes to the category of moduli functors

Exercises

Exercise 2.21. As in Example 2.4, define a moduli functor h for "families of k-dimensional subspaces of $\mathbb{C}^n$" and show that h is representable by the Grassmannian $\mathrm{Gr}(k, n)$.

References and further reading

A nice introduction to moduli spaces in general, with applications to moduli spaces of genus 0 curves, can be found in [KV07, Section 0.2]. A short introduction can also be found in [HM98, Chapter 1A].

See [Kle05] for further discussions of Picard functors and schemes.

3 Families of curves and their moduli

3.1 Smooth and nodal curves

Now we want to define moduli functors and spaces for algebraic curves. We start by recalling some basics for curves over $\mathrm{Spec}(\mathbb{C})$, which will correspond to the $\mathbb{C}$-points of our moduli space.

Definition 3.1. A *(complex) curve* is a one-dimensional variety[10] $C \to \mathrm{Spec}(\mathbb{C})$. In other words it is a reduced, separated scheme of finite type over $\mathrm{Spec}(\mathbb{C})$ such that all irreducible components have dimension 1.

In the following, we will mostly be concerned with projective curves. For these, the most important invariant is their genus, coming in two flavours.

Definition 3.2. Let C be a complex projective curve.

a) If C is smooth, its *geometric genus* is defined to be

$$p_g(C) = h^{1,0}(C) = \dim H^0(C, \Omega_C^1),$$

where Ω_C^1 is the cotangent line bundle of C. For a singular curve C, its geometric genus is defined to be the geometric genus of its normalization.

b) The *arithmetic genus* of C is defined as

$$p_a(C) = 1 - \chi(C, \mathcal{O}_C) = 1 - \dim H^0(C, \mathcal{O}_C) + \dim H^1(C, \mathcal{O}_C).$$

Proposition 3.3. For a smooth, irreducible projective curve we have $p_g(C) = p_a(C)$.

Proof. We have $H^0(C, \mathcal{O}_C) = \mathbb{C}$ since C is irreducible projective and $H^1(C, \mathcal{O}_C) \cong H^0(C, \Omega_C^1)^\vee$ by Serre duality. This implies

$$p_a(C) = 1 - 1 + \dim H^0(C, \Omega_C^1)^\vee = p_g(C). \qquad \square$$

Exercise 3.4. Let $C \subset \mathbb{P}^2$ be a nodal cubic curve, e.g.

$$C = \{[X : Y : Z] \in \mathbb{P}^2 : ZY^2 + X^3 - ZX^2 = 0\}. \tag{25}$$

Show that $p_g(C) = 0$ and $p_a(C) = 1$.

Digression 3.5 (Riemann surfaces)**.** There is a second approach for studying algebraic curves, going via *Riemann surfaces*. While this is not necessary to build the theory, it is often useful to have in mind.

To start, let C be a smooth, projective, irreducible curve. Consider its set $S = C(\mathbb{C})$ of complex points with the complex topology[11]. Then it turns out that S is a compact, connected complex manifold of complex dimension 1. Thus, seen as a real manifold, it is a compact, connected oriented real surface without boundary.

To give an idea why this is true:

[10] We take the convention that varieties are not necessarily irreducible.

[11] For any scheme X of finite type over $\mathbb{C}$, there is a natural *complex topology* (or even the structure of a complex analytic space) on its set $X(\mathbb{C})$ of $\mathbb{C}$-points. For an affine scheme $X = \mathrm{Spec}\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ it is given by the topology on the common zero set $V(f_1, \ldots, f_m) \subset \mathbb{C}^n$ induced from the complex topology on $\mathbb{C}^n$. The construction for general X goes via gluing these topological spaces for an affine cover of X. See [Har77, Appendix B.1] or [Vak17, Exercise 5.3.G] for more details.

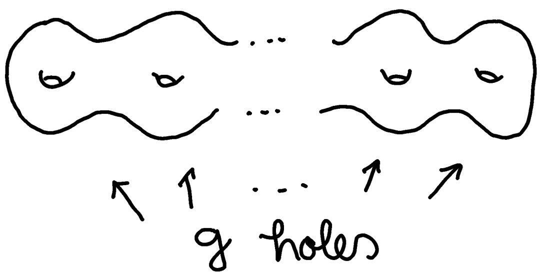


Figure 8: The complex points a curve C of genus g give a compact, connected oriented real surface S without boundary with "g holes"

- S is compact since C was assumed to be projective.
- The curve C is 1-dimensional over $\mathbb{C}$ and hence $S = C(\mathbb{C})$ is 2-dimensional over $\mathbb{R}$ ($\dim_{\mathbb{R}} \mathbb{C} = 2$).
- S is a smooth manifold since C is smooth as an algebraic variety.
- S is oriented since in fact it is a complex manifold.

There is a complete classification of compact, connected oriented real surfaces[12] without boundary. They are all of the form "a donut with g holes" as in Figure 8 and the number $g \geq 0$ is called the *topological genus* of the surface. For $S = C(\mathbb{C})$ it turns out that the topological genus of S is equal to the geometric (or arithmetic) genus of the algebraic curve C.

As mentioned in the introduction, it will turn out that moduli spaces of smooth curves are not compact. To find a larger moduli space which will be compact, we allow curves to have *nodal* singularities.

Definition 3.6. Let C be a complex curve.

a) A closed point $q \in C$ is a *node*[13] if it satisfies one of the following two equivalent conditions:

   - There exists a neighbourhood of $q \in C(\mathbb{C})$ which is complex-analytically isomorphic to a neighbourhood of the origin in the locus $\{(x, y) : x \cdot y = 0\} \subset \mathbb{C}^2$.
   - The completion $\widehat{O}_{C,q}$ of the local ring of C at q is isomorphic to $\mathbb{C}[[x, y]]/(x \cdot y)$.

b) The curve C is called *nodal* if every closed point $q \in C$ is either a smooth point or a node.

Exercise 3.7. Show that the "nodal cubic curve" (25) from Exercise 3.4 is indeed a nodal curve.

Let C be a complex, projective and nodal curve. Then its normalization $\nu : \widetilde{C} \to C$ is a complex, projective curve which is smooth (but possibly disconnected). The morphism

[12] In this context, surface means a 2-dimensional topological manifold.

[13] Alternative names are *nodal singularity* or *ordinary double point.*

ν is an isomorphism over the smooth locus of C and every node $q \in C$ has exactly two preimages $q', q'' \in \widetilde{C}$. We have the *normalization exact sequence*[14]

$$0 \to \mathcal{O}_C \to \nu_* \mathcal{O}_{\widetilde{C}} \to \bigoplus_{q \text{ node of } C} \mathbb{C}_q \to 0, \tag{26}$$

where the first map sends a (local) function f on C to $f \circ \nu$ on $\widetilde{C}$ and the second map sends a (local) function g on $\widetilde{C}$ to $(g(q') - g(q''))_q$ where q runs through nodes of C and q', q'' are the preimages[15] of q under ν.

Easy exercise 3.8. Use the sequence (26) to show that for a complex, projective, nodal curve C with normalization $\widetilde{C} \to C$ we have

$$p_a(C) = p_a(\widetilde{C}) + \#\{\text{nodes of } C\}, \tag{27}$$

or equivalently

$$p_a(C) = p_g(\widetilde{C}) + 1 - \#\{\text{components of } \tilde{C}\} + \#\{\text{nodes of } C\}. \tag{28}$$

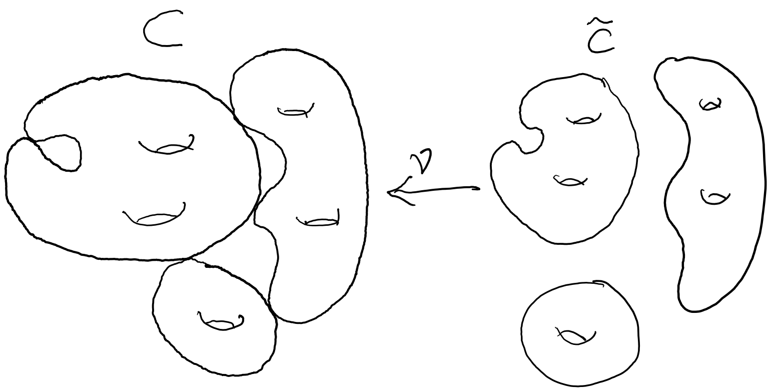


Figure 9: A nodal curve C and its normalization $\widetilde{C}$, with $p_g(C) = p_g(\widetilde{C}) = 5$ and $p_a(C) = 8$

Fact 3.9. The data of a complex, projective, nodal curve C together with the tuple $(q_i)_{i=1}^{\ell}$ of its nodes, is equivalent to the data of its normalization $\widetilde{C}$ together with the sets $(\{q_i', q_i''\})_{i=1}^{\ell}$. In other words there is a unique way to "glue" the components of $\widetilde{C}$ together by identifying the pairs q_i', q_i'' to form nodes.

$$C, (q_i)_{i=1}^{\ell} \underset{\text{gluing}}{\overset{\text{normalization}}{\rightleftarrows}} \widetilde{C}, (\{q_i', q_i''\})_{i=1}^{\ell} \tag{29}$$

In particular, the data of a morphism $\varphi : C \to X$ to some scheme X is equivalent to the data of $\widetilde{\varphi} : \widetilde{C} \to X$ such that $\widetilde{\varphi}(q_i') = \widetilde{\varphi}(q_i'')$ for all i.

$$\varphi : C \to X \underset{\text{gluing}}{\overset{\text{normalization}}{\rightleftarrows}} \widetilde{\varphi} : \widetilde{C} \to X \text{ s.t. } \widetilde{\varphi}(q_i') = \widetilde{\varphi}(q_i'') \ \forall i \tag{30}$$

[14]To show exactness of this sequence, one uses that a) it suffices to check it stalkwise, i.e. after tensoring with $\mathcal{O}_{C,p}$ for $p \in C$ and b) that the map $\mathcal{O}_{C,p} \to \widehat{\mathcal{O}}_{C,p}$ is faithfully flat. This last point is how the definition of a node comes into play.

[15]For this we need to choose an order on the preimages, alternatively we need to replace $\mathbb{C}_q$ by $\nu_*(\mathbb{C}_{q'} \oplus \mathbb{C}_{q''})/\mathbb{C}_q$.

In the next subsection, we will define families and then moduli spaces of nodal curves. It turns out (see Remark 3.16) that in a family of nodal curves, the arithmetic genus of the curves stays constant, while the geometric genus can change. For instance, in the family E_t of smooth plane cubic curves from Example 1.2, the general E_t was smooth of (arithmetic and geometric) genus 1, and the nodal cubic E_0 still has arithmetic genus 1 but geometric genus 0 (by Exercise 3.4).

So we see that smooth curves of genus g degenerate to nodal curves of arithmetic genus g. However, it turns out that there are *too many* nodal curves of arithmetic genus g to obtain a compact moduli space! More precisely, consider the following sequence of nodal curves of genus 2.

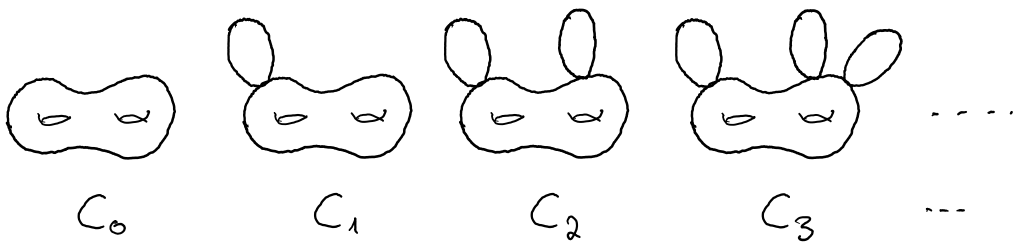


Figure 10: An infinite sequence of nodal curves of arithmetic genus 2

If we had a compact moduli space of such curves, this would be a sequence of closed points in this space. After taking a subsequence, this would need to converge (in the complex topology) to some point of the moduli space, but intuitively the above sequence of curves cannot possibly converge to a (finite type) curve! So allowing *all* nodal curves is too much, but it turns out that things work out very well if we restrict ourself to so-called *stable* curves. Here is the definition, including a variant for curves together with (marked) points.

Definition 3.10. A connected, nodal, complex projective curve C is called *stable* if the group

$$\mathrm{Aut}(C) = \{\varphi : C \to C : \varphi \text{ isomorphism}\} \tag{31}$$

of its automorphisms is finite. Moreover, given $p_1, \ldots, p_n \in C$ distinct smooth points of C, we say that $(C, p_1, \ldots, p_n)$ is stable if the group

$$\mathrm{Aut}(C, p_1, \ldots, p_n) = \{\varphi : C \to C : \varphi \text{ isomorphism with } \varphi(p_i) = p_i\} \tag{32}$$

of automorphisms of C fixing all p_i is finite.

This definition makes some sense: we have seen before that automorphisms create trouble when trying to find moduli spaces, so infinite automorphisms can create infinite trouble! More seriously, let's see that this definition prevents the counterexample from Figure 10. For this we need some facts about automorphisms of curves, starting with the case of smooth curves.

Fact 3.11. Let C be a smooth, complex, irreducible projective curve of genus g.

a) For $g = 0$ we have $C \cong \mathbb{P}^1$ and $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$, where

$$\mathrm{PGL}_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}(\mathrm{Mat}_{2\times 2,\mathbb{C}}) = \mathbb{P}^3 : ad - bc \neq 0 \right\}$$

is the projective linear group. The action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{P}^1$ is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} .[X : Y] = [aX + bY : cX + dY].$$

Moreover, this action is 3-*transitive*, i.e. for $p_1, p_2, p_3 \in \mathbb{P}^1$ pairwise distinct closed points there exists a unique element of $\mathrm{PGL}_2(\mathbb{C})$ sending them to $0, 1, \infty \in \mathbb{P}^1$, respectively. Or, formulated differently, the morphism

$$\begin{aligned} \mathrm{PGL}_2 &\to (\mathbb{P}^1)^3 \setminus \Delta, \\ A &\mapsto (A.[0:1],\ A.[1:1],\ A.[1:0]) \end{aligned}$$

to the complement of the big diagonal $\Delta \subset (\mathbb{P}^1)^3$ is an isomorphism.

b) For $g = 1$ the curve $C = E$ has automorphism group is isomorphic to $\mathrm{Aut}(E) \cong E(\mathbb{C}) \rtimes G$ for G one of the finite groups $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$. The normal subgroup $E(\mathbb{C}) \subset \mathrm{Aut}(E)$ acts *simply transitively*, i.e. for $p, p' \in E$ closed points there exists a unique element of $E(\mathbb{C})$ sending p to p'.

c) For $g \geq 2$ the automorphism group $\mathrm{Aut}(C)$ is finite, of order at most $84(g-1)$.

Easy exercise 3.12. Let C be a smooth, complex, irreducible projective curve of genus g and $p_1, \ldots, p_n \in C$ be distinct points. Show that $\mathrm{Aut}(C, p_1, \ldots, p_n)$ is finite if and only if $2g - 2 + n > 0$.

From this we can get a very explicit criterion for a nodal and pointed curve to be stable.

Proposition 3.13. Let C be a connected, nodal, complex projective curve C and let $p_1, \ldots, p_n \in C$ be distinct smooth points. Then $(C, p_1, \ldots, p_n)$ is stable if and only if every irreducible component $\widetilde{C}_v \subset \widetilde{C}$ of the normalization of C satisfies

- $\widetilde{C}_v$ has genus 0 and contains at least 3 special points, i.e. preimages of nodes of C or markings p_i, or
- $\widetilde{C}_v$ has genus 1 and contains at least 1 special point, or
- $\widetilde{C}_v$ has genus at least 2.

Proof. By Fact 3.9 together with the universal property of the normalization, an automorphism $\varphi : C \to C$ of C is equivalent to an automorphism $\widetilde{\varphi} : \widetilde{C} \to \widetilde{C}$ mapping each pair $q_i', q_i'' \in \widetilde{C}$ of preimages of nodes to some other such pair. In particular, we get a group morphism

$$\mathrm{Aut}(C, p_1, \ldots, p_n) \to \mathrm{Sym}(\{\text{components } \widetilde{C}_v \text{ of } \widetilde{C}\}) \times \mathrm{Sym}(\{q_j', q_j'' : j = 1, \ldots, \ell\}) \quad (33)$$

sending an automorphism to the permutation on the set of components of $\widetilde{C}$ and the set of preimages of nodes. Since this permutation group is finite, the group $\mathrm{Aut}(C, p_1, \ldots, p_n)$ is finite if and only if the kernel K of the above map is finite.

But an element of the kernel is precisely a collection of automorphisms $\widetilde{\varphi}_v : \widetilde{C}_v \to \widetilde{C}_v$ of the components of $\widetilde{C}$ which fix all special points (i.e. the points q_j', q_j'' and the points p_i). Comparing with Fact 3.11 (or with Easy Exercise 3.12), we see that the group of automorphisms of $\widetilde{C}_v$ fixing a number m of distinct points of $\widetilde{C}_v$ is finite if and only if $\widetilde{C}_v$ is of genus 0 with $m \geq 3$, of genus 1 with $m \geq 1$ or of genus at least 2. $\square$

Example 3.14. For the nodal curve in Figure 11, we see that its normalization has three components of genus $0, 1, 2$ respectively. They have $2, 1$ and 3 preimages of nodes, respectively, and the genus 0 component has also a preimage of the marked point $p_1 \in C$. Checking with the criterion above, we see that C itself is *not* stable (the genus 0 component only has 2 special points), but (C, p_1) *is* stable (together with the preimage of p_1 it now has 3 special points).

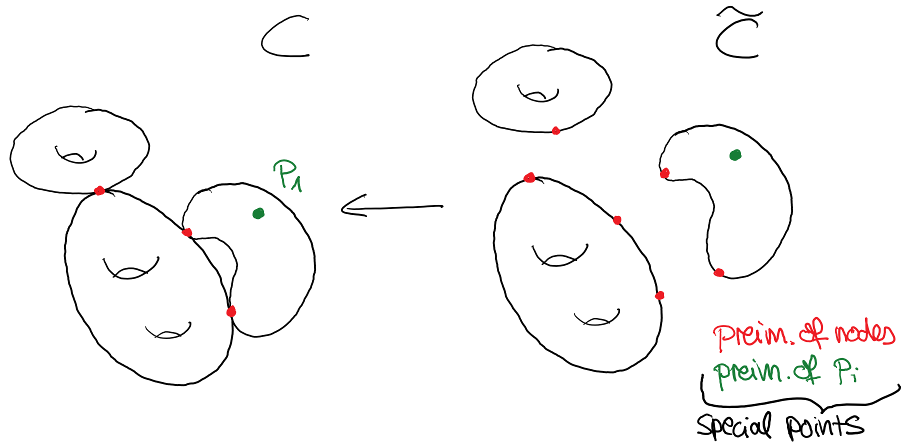


Figure 11: The nodal curve C and its normalization $\widetilde{C}$ together with the special points on $\widetilde{C}$

Looking back at Figure 10 we see that only the first curve in the sequence is stable: all others have components of genus 0 with only one special point on them. Thus this sequence will *not* give a counterexample to the compactness of the moduli space of stable curves!

3.2 Families and moduli spaces of smooth and stable curves

Now we define the notion of a family of curves. As we will see later, it is very natural to not just consider the the curve C itself, but the additional data of $p_1, \ldots, p_n \in C$ of n distinct points. Moreover, we will treat families of smooth and stable curves simultaneously.

Definition 3.15. Given $g, n \geq 0$, an *n-pointed family of (smooth/stable) genus g curves* over a scheme S is a tuple

$$(\pi : C \to S; p_1, \ldots, p_n : S \to C) \tag{34}$$

where

- π is a (smooth/flat), proper, surjective, finitely presented morphism of schemes such that the fibre C_s over any geometric point $s \in S$ is a (smooth/stable), projective, connected curve of arithmetic genus g,

- the morphisms $p_1, \ldots, p_n$ are pairwise disjoint sections of π, with image in the smooth locus of π.

We say that a second family $(C'/S; p'_1, \ldots, p'_n)$ is *isomorphic* to $(C/S; p_1, \ldots, p_n)$ if there exists an isomorphism $\varphi : C \to C'$ over S such that $\varphi \circ p_i = p'_i$, i.e. such that the

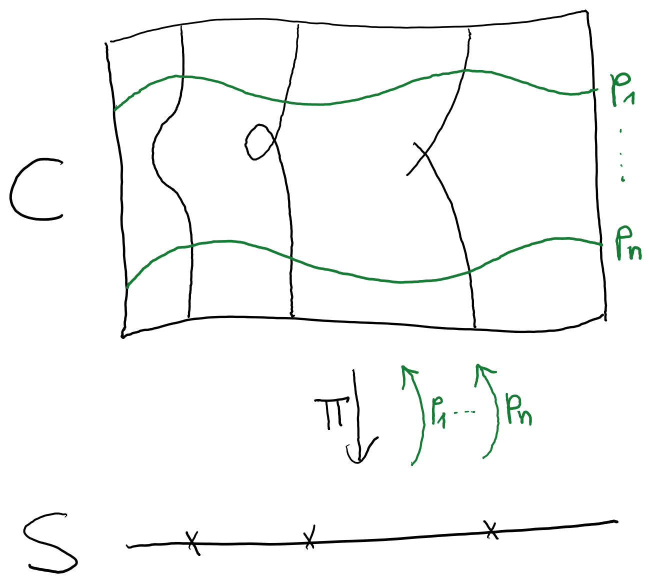


Figure 12: A family of stable curves

diagram

$$\begin{tikzcd}
C \arrow[rr, "\varphi"] \arrow[dr, "\pi"] & & C' \arrow[dl, "\pi'"'] \\
& S \arrow[ul, bend left, "p_i"] \arrow[ur, bend right, "p_i'"'] &
\end{tikzcd}$$

commutes. Given a map $f : T \to S$, we define the *pullback of* $(C/S; p_1, \dots, p_n)$ *under* f to be the family $(C_T/T; p_{1,T}, \dots, p_{n,T})$ for the fibre product $C_T = C \times_S T$ with induced sections $p_{i,T} = (p_i \circ f) \times \mathrm{id}_T$.

$$\begin{tikzcd}
C_T \arrow[r] \arrow[d] & C \arrow[d] \\
T \arrow[u, bend left, "p_{i,T}"] \arrow[r, "f"] & S \arrow[u, bend right, "p_i"']
\end{tikzcd}$$

Remark 3.16. Most of the above definition should be very reasonable, but I want to comment on two small points.

a) Over reasonable bases (i.e. locally Noetherian), the assumption of π being finitely presented already follows from the map being proper. But we need it for technical reasons (see point c) below).

b) Given that we ask all fibres C_s of the morphism π to be projective, it is tempting to just require π being projective instead. However, this would lead to some technical difficulties (e.g. being projective cannot be checked on a Zariski open cover of S). However, it turns out to be true that for a stable family of curves, the morphism π is *locally projective*, i.e. there exists a Zariski open cover of S such that π is projective restricted to the open sets in this cover[16].

[16] Idea of proof: we will later see that there is a so-called *canonical line bundle* ω_π of π and that the family being stable guarantees that $\omega_\pi(\sum_i p_i)$ is ample on the fibres C_s. By EGA IV_3, 9.6.4 this implies that this line bundle is π-relatively ample. Then this line bundle makes π projective locally on S.

c) For the family of stable curves, we require the map π to be flat. In the case of smooth curves we did not have to add this extra assumption since every smooth morphism is automatically flat. One nice consequence is that for a flat, proper, locally finitely presented morphism the Euler characteristic of the fibres is locally constant ([Vak17, ,Exercise 28.2.M]). This means that even if we did not ask all fibres to have arithmetic genus g, we would at least know that their arithmetic genus is constant on connected components of the base.

Definition 3.17. Let $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ be the moduli functors sending a scheme $S \in \mathbf{Sch}_{\mathbb{C}}$ to the sets

$$\mathcal{M}_{g,n}(S) = \{(\pi : C \to S; p_1, \ldots, p_n : S \to C) : \text{ smooth curve over } S\}/\text{iso},$$
$$\overline{\mathcal{M}}_{g,n}(S) = \{(\pi : C \to S; p_1, \ldots, p_n : S \to C) : \text{ stable curve over } S\}/\text{iso}.$$

of n-pointed families of smooth (or stable) genus g curves over S, up to isomorphism. Given a morphism $f : T \to S$, the induced maps

$$\mathcal{M}_{g,n}(S) \to \mathcal{M}_{g,n}(T),\ \overline{\mathcal{M}}_{g,n}(S) \to \overline{\mathcal{M}}_{g,n}(T)$$

are defined by the pullback of the families of curves over S to T.

Easy exercise 3.18. Convince yourself that this defines a moduli functor. In particular, check that the pullback of a family of (smooth/stable) curves is again (smooth/stable).

Theorem 3.19 (see [DM69, Knu83b])**.** Let $g, n \geq 0$ with $2g - 2 + n > 0$.

a) There exist coarse moduli spaces $M_{g,n}$ and $\overline{M}_{g,n}$ of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$.

b) They are normal algebraic varieties of dimension $3g - 3 + n$ and there is a natural inclusion $M_{g,n} \subset \overline{M}_{g,n}$ as a nonempty, open and dense subvariety.

c) The variety $\overline{M}_{g,n}$ is irreducible, projective and has quotient singularities[17].

d) The complement $\partial\overline{M}_{g,n} = \overline{M}_{g,n} \setminus M_{g,n}$ of the locus of smooth curves, called the *boundary* of $\overline{M}_{g,n}$, is a (Weil) divisor[18].

e) The locus $\overline{M}^0_{g,n} \subset \overline{M}_{g,n}$ of $[(C, p_1, \ldots, p_n)]$ with trivial automorphism group

$$\mathrm{Aut}(C, p_1, \ldots, p_n) = \{\mathrm{id}_C\}$$

is an open and smooth subvariety. It is a fine moduli space for the moduli functor $\overline{\mathcal{M}}^0_{g,n}$ of stable curves with trivial automorphism group and thus has a universal family

$$\begin{array}{ccc} \overline{C}^0_{g,n} & & \\ {\scriptstyle\pi}\big\downarrow \big\uparrow{\scriptstyle p_i} & & \\ \overline{M}^0_{g,n} & \subset & \overline{M}_{g,n} \end{array} \tag{35}$$

[17]That means it has a Zariski open cover by varieties V/G for V smooth affine and G a finite group acting on V. Here, if V is the spectrum of a ring R, the action of G on V induces an action of G on R and the variety V/G is the spectrum of the ring R^G of G-invariant elements in R.

[18]In general, the divisor $\overline{M}_{g,n} \setminus M_{g,n}$ is not Cartier (see here). However, since $\overline{M}_{g,n}$ has quotient singularities it is $\mathbb{Q}$-factorial so a multiple of the divisor is Cartier.

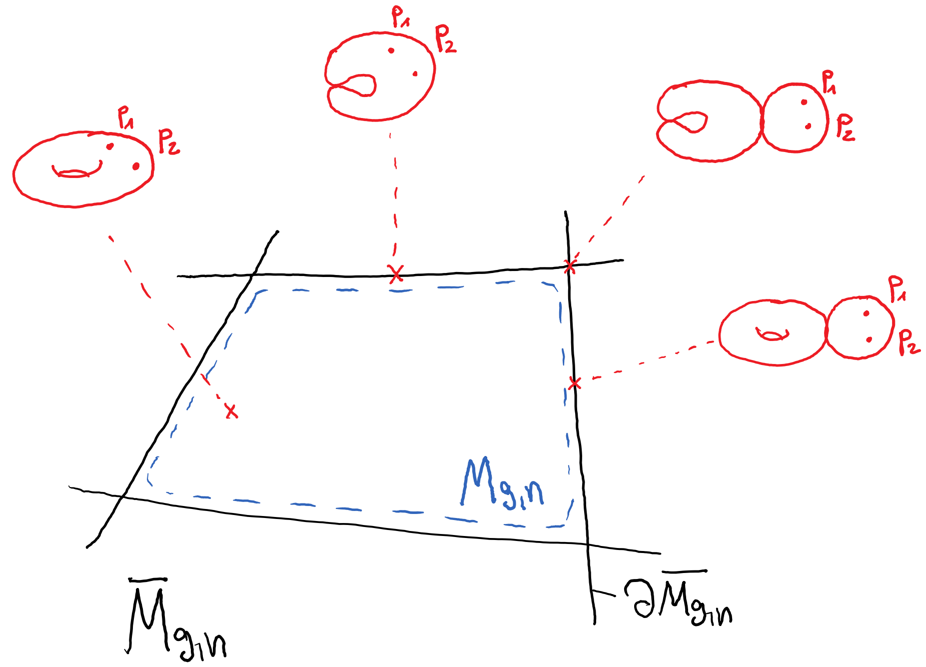


Figure 13: The moduli space of stable curves $\overline{M}_{g,n}$ with the open set $M_{g,n}$ of smooth curves and the boundary $\partial\overline{M}_{g,n}$ illustrated

Remark 3.20. Using Easy exercise 3.12 we see that the reason for the requirement $2g-2+n>0$ of the theorem is precisely that it ensures that any smooth curve $(C,p_1,\ldots,p_n)$ has finite automorphism group (so that indeed $\emptyset \neq M_{g,n} \subset \overline{M}_{g,n}$). In the end, it is just a concise way to exclude the finite list of cases

$$(g,n) = (0,0),(0,1),(0,2),(1,0).$$

Concerning part e) of the Theorem, one can check that for $2g-2+n>0$ the set $\overline{M}_{g,n}^0$ of curves with trivial automorphism group is nonempty if and only if $(g,n) \neq (1,1),(2,0)$.

In order to help us digest this big theorem, we will start by looking at some concrete examples of moduli spaces $\overline{M}_{g,n}$ in low genus in the next section. Along the way we will also introduce some general concepts, like the dual graph of a stable curve. Later we will discuss the main ideas of how to prove the various properties of $\overline{M}_{g,n}$ stated above.

References and further reading

A great introduction to (smooth) algebraic curves with an overview of the behaviour for small genus is given in [Vak17, Chapter 19]. See also [Vak17, Exercise 16.4.C] for the automorphism group of $\mathbb{P}^1$ and [Vak17, Section 21.7.8] for the finiteness of automorphisms for $g \geq 2$ from Fact 3.11. See this great poster about automorphism groups of smooth, complete curves over more general algebraically closed fields.

For an introduction to nodal curves see [ACG11, Chapter X, §2]. For some more technical results about (nodal) curves, see [Sta13, Tag 0BRV], in particular subsections 14, 15 and 19.

4 Examples of moduli of curves and basic constructions

4.1 Smooth curves of genus 0

Let's start with the moduli spaces $M_{0,n}$ of smooth curves, which is defined for

$$2g - 2 + n = -2 + n > 0 \iff n \geq 3.$$

From Fact 3.11 a) we know that every smooth genus 0 curve C is isomorphic to $\mathbb{P}^1$. Thus, beginning with the simplest case $n = 3$, every curve $(C, p_1, p_2, p_3) \in \mathcal{M}_{0,3}(\mathrm{Spec}(\mathbb{C}))$ is isomorphic to $(\mathbb{P}^1, p_1, p_2, p_3) \in \mathcal{M}_{0,3}(\mathrm{Spec}(\mathbb{C}))$. Again by Fact 3.11 a) there exists an automorphism of $\mathbb{P}^1$, an element of $\mathrm{PGL}_2(\mathbb{C})$, sending p_1, p_2, p_3 to $0, 1, \infty$. Thus we have

$$(C, p_1, p_2, p_3) \cong (\mathbb{P}^1, p_1, p_2, p_3) \cong (\mathbb{P}^1, 0, 1, \infty).$$

Thus, up to isomorphism, there exists a *unique* smooth genus 0 curve with three distinct marked points! Therefore, we expect that the moduli space $M_{0,3}$ is a point. This turns out to be true, but the proof (without using Theorem 3.19) is actually quite involved!

Proposition 4.1. The variety $M_{0,3} = \mathrm{pt} = \mathrm{Spec}(\mathbb{C})$ is a fine moduli space for the functor $\mathcal{M}_{0,3}$ and the universal family is given by

$$(\pi : \mathbb{P}^1 \to \mathrm{Spec}(\mathbb{C}); p_1 = 0, p_2 = 1, p_3 = \infty : \mathrm{Spec}(\mathbb{C}) \to \mathbb{P}^1) \tag{36}$$

For the proof we are going to use the following result. Proving it is quite technical, so we only give the proof in the appendix in the optional Section 4.6.

Proposition 4.2. Let $\pi : C \to B$ be a smooth, proper, surjective, locally finitely presented morphism of relative dimension ≤ 1 with geometric fibres isomorphic to $\mathbb{P}^1$.

a) If π admits a section $p_1 : B \to C$, then there exists a rank 2 vector bundle $\mathcal{E}$ on B such that C is isomorphic to the projective bundle $C \cong \mathbb{P}(\mathcal{E})$ over B.

b) If π admits two disjoint sections $p_1, p_2 : B \to C$, then the bundle $\mathcal{E}$ splits as a direct sum $\mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$ of line bundles.

c) If π admits three disjoint sections $p_1, p_2, p_3 : B \to C$, then we can take $\mathcal{E} = \mathcal{O}_B \oplus \mathcal{O}_B$ above, so that C is isomorphic to the trivial projective bundle $C \cong B \times \mathbb{P}^1$.

Proof of Proposition 4.1. By definition, we need to show that the functor $\mathcal{M}_{0,3}$ is isomorphic to h^{pt}. Since for any scheme S the set $h^{\mathrm{pt}}(S) = \{S \to \mathrm{pt}\}$ has a unique element, this means we need to show that any family of smooth genus 0 curves

$$(\pi : C \to S; p_1, p_2, p_3 : S \to C) \in \mathcal{M}_{0,3}(S) \tag{37}$$

is isomorphic to the trivial family

$$(S \times \mathbb{P}^1 \to S; 0, 1, \infty : S \to S \times \mathbb{P}^1), \tag{38}$$

so that really there exists a *unique* element of $\mathcal{M}_{0,3}(S)$. By Proposition 4.2 we see that indeed $C \cong S \times \mathbb{P}^1$ over S. Then since the three sections $p_1, p_2, p_3 : S \to S \times \mathbb{P}^1$ are assumed to be disjoint, they induce a map

$$A = (p_1, p_2, p_3) : S \to (\mathbb{P}^1)^3 \setminus \Delta \cong \mathrm{PGL}_2 \tag{39}$$

to the complement of the big diagonal $\Delta \subset (\mathbb{P}^1)^3$, which by Fact 3.11 a) is isomorphic to PGL_2. Then an isomorphism $S \times \mathbb{P}^1 \to S \times \mathbb{P}^1$ sending the sections p_1, p_2, p_3 to $0, 1, \infty$ is given by

$$S \times \mathbb{P}^1 \to S \times \mathbb{P}^1, (s, p) \mapsto (s, A(s)^{-1}.p).$$

This gives the desired isomorphism of the families (37) and (38). □

It turns out that once we understand the case $n = 3$, the case of general $n \geq 4$ is actually not much more difficult. Indeed, given any curve $(C, p_1, \ldots, p_n) \in \mathcal{M}_{0,n}(\mathbb{C})$ we still have $C \cong \mathbb{P}^1$ and there exists a unique element $B \in \mathrm{PGL}_2(\mathbb{C})$ of the automorphism group of $\mathbb{P}^1$ sending p_1, p_2, p_3 to $0, 1, \infty$. Let $p_j' = B.p_j$, then we have

$$(C, p_1, \ldots, p_n) \cong (\mathbb{P}^1, 0, 1, \infty, p_4', \ldots, p_n').$$

The elements $p_4', \ldots, p_n' \in \mathbb{P}^1$ are pairwise distinct and also distinct from $0, 1, \infty$ and uniquely determine the isomorphism class of $(C, p_1, \ldots, p_n)$[19]. This gives the following result.

Proposition 4.3. For $n \geq 3$ the moduli functor $\mathcal{M}_{0,n}$ is representable by the variety

$$M_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta, \tag{40}$$

where $\Delta = \{(q_i)_i : \exists i \neq j \text{ with } q_i = q_j\}$ is the big diagonal.

Exercise 4.4. Give a proof of Proposition 4.3, following the proof of Proposition 4.1. What is the universal family over $M_{0,n}$?

From the concrete description above we can actually now verify several of the statements from Theorem 3.19:

- The space $M_{0,n}$ exists as a coarse (even fine!) moduli space.
- It is normal (even smooth!) and irreducible of dimension $3g - 3 + n = n - 3$.
- In fact one easily checks that every smooth genus 0 curve $(C, p_1, \ldots, p_n)$ has trivial automorphism group, so that $M_{0,n} \subset \overline{M}_{0,n}^0$. This explains the fact that it is a fine moduli space and smooth instead of just normal!

Next we want to look at moduli spaces of *stable* curves in genus 0, to see if the nice properties above (being smooth and a fine moduli space) extend. For this, we will now define a very useful tool (working for any genus g), the *dual/stable graph* of a stable curve.

4.2 Stable graphs and gluing morphisms

Given a stable curve $(C, p_1, \ldots, p_n)$, we want to define a combinatorial object Γ_C that allows us to describe its shape, i.e. how many components it has, of which genus they are and how they intersect among themselves. Consider Figure 14 for illustration.

We see there that the combinatorial object is a graph together with some decorations. In the graph

- the vertices v correspond to irreducible components C_v of C and are decorated with the *geometric* genus of the component,

[19] For $n = 4$, the value p_4' is called the *cross ratio* of the points $p_1, \ldots, p_4$.

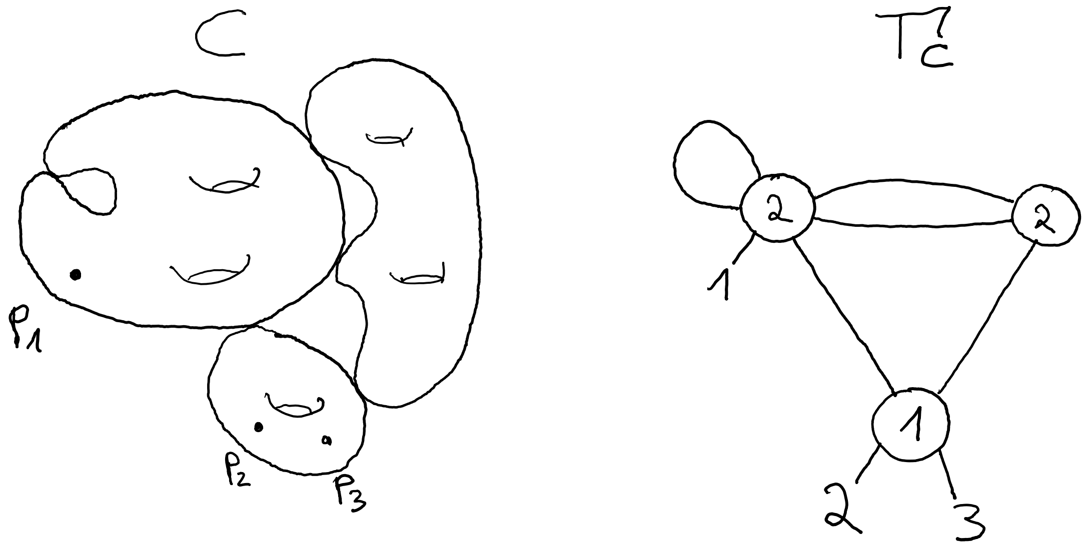


Figure 14: A stable curve and the associated dual graph

- the edges correspond to nodes of the curve, where a node connecting two components C_v, C_w (or C_v with itself) gives an edge between v and w (or from v to itself),
- there are legs attached to the vertices v, numbered $1, \ldots, n$, describing in which components C_v the markings $p_1, \ldots, p_n \in C$ are contained.

Now we set up the notation how to formally encode all this data. This involves a bunch of sets and maps satisfying properties and as you go through the definition you should check back to Figure 14 to match the parts of the definition to the picture.

Definition 4.5. A *stable graph* Γ is a tuple

$$\Gamma = (V, H, L, g : V \to \mathbb{Z}_{\geq 0}, v : H \to V, \iota : H \to H, \ell : L \to \{1, \ldots, n\}) \tag{41}$$

where

a) $V = V(\Gamma)$ is a finite set (the *vertices* of Γ) and $g : V \to \mathbb{Z}_{\geq 0}$ is a map associating a *genus* $g(v)$ to each vertex v,

b) $H = H(\Gamma)$ is a finite set (the *half-edges* of Γ). The map $v : H \to V$ associates to each half-edge h a vertex $v(h)$ (the vertex *incident* to h). We denote by

$$H(v) = \{h \in H : v(h) = v\}$$

the half-edges incident at v and by $n(v) = \#H(v)$ the number of these half-edges. The map $\iota : H \to H$ is an involution (i.e. $\iota \circ \iota = \mathrm{id}_H$). Thus H decomposes into pairs of half-edges switched by ι and fixed points of ι.

c) The pairs $e = \{h, h'\}$ of distinct half-edges exchanged by ι, i.e. $\iota(h) = h'$, are called the *edges* $E = E(\Gamma)$ of Γ.

d) The set $L = L(\Gamma) \subset H$ is the set of half-edges fixed by ι (the *legs* of Γ) and $\ell : L \to \{1, \ldots, n\}$ is a bijective map.

e) The graph Γ is *connected*, i.e. any two vertices can be connected by a path consisting of edges[20].

f) We require the *stability condition* that for each vertex $v \in V$ we have

$$2g(v) - 2 + n(v) > 0.$$

An *isomorphism* $\phi : \Gamma \to \Gamma'$ of stable graphs is a collection of bijective maps

$$\varphi_V : V \to V', \varphi_H : H \to H',$$

of their sets of vertices and half-edges which are compatible with the functions g, v, ι, i.e.

$$g'(\varphi_V(v)) = g(v), v'(\varphi_H(h)) = \varphi_V(v(h)), \iota'(\phi_H(h)) = \phi_H(\iota(h)), \ell'(\varphi_H(h)) = \ell(h).$$

Denote by $\mathrm{Aut}(\Gamma)$ the set of isomorphisms $\Gamma \to \Gamma$. This is a group, with group law given by composition. Define the *genus* $g(\Gamma)$ of Γ as the number

$$g(\Gamma) = \sum_{v \in V(\Gamma)} g(v) + 1 + \#E(\Gamma) - \#V(\Gamma), \tag{42}$$

and the *number of legs/markings* $n(\Gamma)$ as $n(\Gamma) = n = \#L(\Gamma)$.

Example 4.6. Let's work out all these things in an example. Consider the stable graph Γ in Figure 15. Its data is given by

$$\begin{aligned}
V(\Gamma) =& \{v_0, v_1, v_2\} \text{ and } g(v_0) = 1, g(v_1) = 1, g(v_2) = 2,\\
H(\Gamma) =& \{h_1, \dots, h_7\} \text{ and } v : H(\Gamma) \to V(\Gamma) \text{ with}\\
& v^{-1}(v_0) = \{h_1\}, v^{-1}(v_1) = \{h_3\}, v^{-1}(v_2) = \{h_2, h_4, h_5, h_6, h_7\},\\
E(\Gamma) =& \{\{h_1, h_2\}, \{h_3, h_4\}, \{h_6, h_7\}\}, \text{ in particular}\\
& \iota(h_1) = h_2, \iota(h_2) = h_1, \dots, \iota(h_7) = h_6, \iota(h_5) = h_5,\\
L(\Gamma) =& \{h_5\} \text{ and } \ell(h_5) = 1.
\end{aligned}$$

Concerning the automorphism group of Γ, there is an automorphism $\tau = (\tau_V, \tau_H) : \Gamma \to \Gamma$ with

$$\tau_V(v_0) = v_1, \tau_V(v_1) = v_0, \tau_V(v_2) = v_2, \tau_H(h_1) = h_3, \tau_H(h_3) = h_1, \tau_H(h_2) = h_4, \tau_H(h_4) = h_2.$$

Similarly, there is an automorphism σ, where σ_V, σ_H fix all vertices and half-edges except for $\tau_H(h_6) = h_7, \tau_H(h_7) = h_6$. You can check that these two commute and generate the automorphism group

$$\mathrm{Aut}(\Gamma) = \langle \tau, \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Finally we see that

$$g(\Gamma) = (1 + 1 + 2) + 1 + 3 - 3 = 5 \text{ and } n(\Gamma) = 1.$$

Exercise 4.7. Define the stable graph associated to a stable curve $(C, p_1, \dots, p_n)$. Check your definition against the picture in Figure 14 and convince yourself that conditions e) and f) of Definition 4.5 are satisfied. We give the definition below for completeness, but it will be less confusing if you first try writing it down yourself.

[20] Mini-Exercise: make precise what this means.

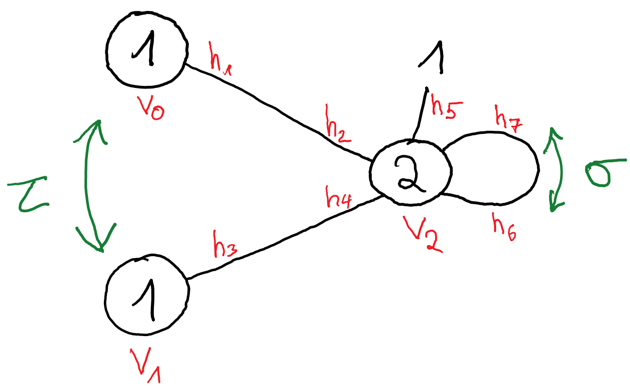


Figure 15: A stable graph Γ with labeled vertices and half-edges (in red) and automorphisms τ, σ generating its automorphism group $\mathrm{Aut}(\Gamma) = \langle \tau, \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

Definition 4.8. Given a stable curve $(C, p_1, \ldots, p_n)$ its associated *dual graph* $\Gamma = \Gamma_C$ is the stable graph defined as follows:

- The vertices $v \in V$ of Γ are in one-to-one correspondence to the irreducible components C_v of C (which canonically correspond to the components $\widetilde{C}_v$ of the normalization $\widetilde{C}$).

  $$V \cong \{C_v : \text{component of } C\} = \{\widetilde{C}_v : \text{component of } \widetilde{C}\}$$

  The map $g : V \to \mathbb{Z}_{\geq 0}$ sends a vertex v to the genus $g(\widetilde{C}_v)$ of the component in the normalization.

- The half-edges $h \in H$ of Γ are in one-to-one correspondence to the union of the preimages $q', q'' \in \widetilde{C}$ of nodes $q \in C$ under the normalization map $\nu : \widetilde{C} \to C$ and the marked points $p_1, \ldots, p_n \in C$.

  $$H \cong \left(\coprod_{\substack{q \text{ node in } C \\ \nu^{-1}(q) = \{q', q''\}}} \{q', q''\} \right) \sqcup \{p_1, \ldots, p_n\}$$

  The map $v : H \to V$ sends half-edges of the form q', q'' to the vertex v for the component $\widetilde{C}_v$ of the normalization containing them, and the half-edges of the form p_i to the vertex v for the component C_v of C containing them. The involution ι exchanges the preimages of nodes ($\iota(q') = q'', \iota(q'') = q'$) and fixes the marked points ($\iota(p_i) = p_i$).

- The legs $L \subset H$ are precisely the marked points

  $$L = \{p_1, \ldots, p_n\}$$

  and the map $\ell : L \to \{1, \ldots, n\}$ sends p_i to i.

Easy exercise 4.9. Check that the genus $g(\Gamma_C)$ of the dual graph of a curve $(C, p_1, \ldots, p_n)$ equals the arithmetic genus of C.

Exercise 4.10. Let $(C, p_1, \ldots, p_n)$ be a stable curve with dual graph Γ and let $(\widetilde{C}_v, (q_h)_{h \in H(v)})_{v \in V(\Gamma)}$ be the components of the normalization of C with marked preimages of nodes and markings. Then there is an exact sequence of groups

$$0 \to \prod_{v \in V(\Gamma)} \mathrm{Aut}(\widetilde{C}_v, (q_h)_{h \in H(v)}) \to \mathrm{Aut}(C, p_1, \ldots, p_n) \to \mathrm{Aut}(\Gamma). \tag{43}$$

Can this sequence in general be extended on the right by 0, i.e. is the map to $\mathrm{Aut}(\Gamma)$ surjective?

Exercise 4.11. a) Show that a stable graph of genus g with n legs has at most $3g-3+n$ edges. (*Note*: This follows from Proposition 4.14 below, and you might look at the proof for inspiration, but there is a purely combinatorial argument for this statement, just using the definition above).

b) Show that given g, n, there are only finitely many stable graphs of genus g with n legs, up to isomorphism.

c) For fixed genus g and number n of legs, convince yourself that there is precisely one stable graph with no edge at all (the *trivial stable graph*). Compute a formula, depending on g, n, for the number of isomorphism classes of stable graphs with exactly one edge.

Remark 4.12. It turns out that stable graphs are incredibly useful when studying problems related to stable curves and their moduli spaces, since they can describe important information about the curves in a purely combinatorial fashion.

However, one issue one encounters is that the number of isomorphism classes of stable graphs grows drastically with g, n. For instance, even for $g = 1, n = 5$ there are already 1576 isomorphism classes of stable graphs. How did I come up with this number? Have I locked myself in my basement for a week, scribbling pages upon pages of stable graphs? No, in fact a few years ago, I locked myself in my basement for 5 months and wrote a computer program to count the graphs for me[21]!

With the help of Jason van Zelm and Vincent Delecroix, this has by now grown into the software package `admcycles` [DSv20] for the open source mathematical software SageMath [S+20]. This package can perform intersection theory on the spaces $\overline{M}_{g,n}$ and as part of this, it can enumerate stable graphs. It can be used online without installation, and you can click on this link to see some example computations, enumerating stable graphs up to isomorphism.

I will occasionally show some examples using this program, but this will be an entirely optional part of the course, and in particular I will not ask anything related to this in the exam.

***Exercise 4.13.** Check your answer to Exercise 4.11 c) for small values of g, n using the software described above. You can also verify that the graph in Figure 15 has precisely 4 automorphisms.

One reason why stable graphs are useful is that the moduli space $\overline{M}_{g,n}$ decomposes as a disjoint union according to possible stable graphs.

[21]If I had just been interested in this particular number of stable graphs, I could also have used an existing computer program written by Aaron Pixton, but I wanted to perform some computations that Aaron's program could not do.

Proposition 4.14. Let $g, n \geq 0$ with $2g-2+n > 0$, then for any stable graph Γ of genus g with n legs, the set

$$M^\Gamma = \{(C, p_1, \ldots, p_n) : \Gamma_C \cong \Gamma\} \subset \overline{M}_{g,n}$$

of curves with stable graph isomorphic to Γ is a nonempty, irreducible, locally closed subset of $\overline{M}_{g,n}$. In particular, the space $\overline{M}_{g,n}$ is the disjoint union

$$\overline{M}_{g,n} = \coprod_\Gamma M^\Gamma,$$

where Γ runs through isomorphism classes of stable graphs. We have

$$\dim M^\Gamma = \sum_{v \in V(\Gamma)} 3g(v) - 3 + n(v) = \dim \overline{M}_{g,n} - \#E(\Gamma). \tag{44}$$

The sets M^Γ are called the *strata* of $\overline{M}_{g,n}$ and the decomposition is called the *stratification according to dual graph.*

To prove the Proposition, let's recall from Fact 3.9 that a nodal curve C is uniquely determined by its normalization together with the data of the pairs of preimages of nodes under the normalization map. Thus the curves $(C, p_1, \ldots, p_n) \in M^\Gamma$ can be uniquely described by specifying the components $\widetilde{C}_v$ of their normalization ($v \in V(\Gamma)$) together with the preimages of nodes and markings p_i under the normalization map $\nu : \widetilde{C} \to C$. This leads to the idea of *gluing morphisms.*

Proposition 4.15. Let Γ be a stable graph of genus g with n legs, then there exists a morphism

$$\xi_\Gamma : \overline{M}_\Gamma = \prod_{v \in V(\Gamma)} \overline{M}_{g(v),n(v)} \to \overline{M}_{g,n} \tag{45}$$

sending a tuple $(C_v, (q_h)_{h \in H(v)})_{v \in V(\Gamma)}$ to the curve $(C, p_1, \ldots, p_n)$ obtained by gluing all pairs $q_h, q_{h'}$ of points corresponding to pairs $\{h, h'\}$ forming edges of Γ and setting $p_i \in C$ to be the image of the marking $q_{\ell^{-1}(i)}$ belonging to the half-edge $\ell^{-1}(i) \in H(\Gamma)$. The morphism ξ_Γ is finite and its image is the closure $\overline{M}^\Gamma$ of M^Γ.

Remark 4.16. The fact that the strata of $\overline{M}_{g,n}$ are parametrized under the maps ξ_Γ by products of smaller-dimensional spaces $\overline{M}_{g(v),n(v)}$ is sometimes called the *recursive boundary structure* of $\overline{M}_{g,n}$. It is one of the most important features of the moduli space of stable curves and the proofs of many results about $\overline{M}_{g,n}$ use it in a very essential way.

Example 4.17. In Figure 16 we see how the gluing map ξ_Γ for a particular stable graph Γ works. It identifies the markings $q_h, q_{h'}$ belonging to half-edges h, h' of Γ forming an edge.

Idea of proof. For the domain of the map ξ_Γ, one can check that $\overline{M}_\Gamma$ is a coarse moduli space for the moduli functor $\overline{\mathcal{M}}_\Gamma : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Sets}$ defined by

$$\overline{\mathcal{M}}_\Gamma(S) = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}(S). \tag{46}$$

Then we can obtain the map ξ_Γ above by constructing a natural transformation

$$\widehat{\xi}_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$$

between moduli functors and then using the properties of coarse moduli spaces.

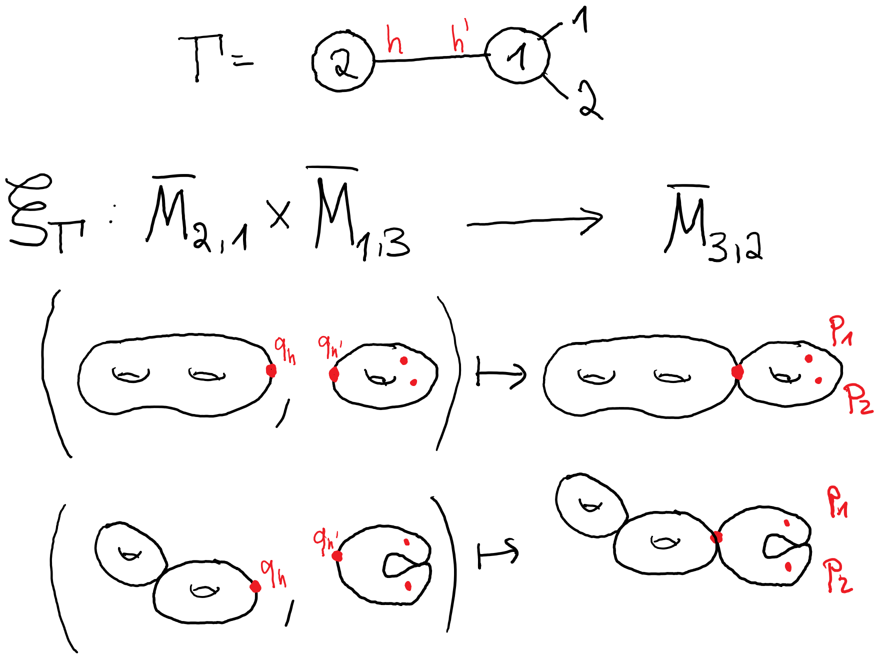


Figure 16: The gluing map associated to a stable graph Γ

Given a scheme S, the natural transformation $\widehat{\xi}_\Gamma$ takes an element of $\overline{\mathcal{M}}_\Gamma(S)$, i.e. a tuple

$$(\pi_v : C_v \to S, (q_h : S \to C_v)_{h \in H(\Gamma)})_{v \in V(\Gamma)}$$

of stable curves over S and glues them to a stable curve by identifying the sections $q_h, q_{h'}$ corresponding to pairs $\{h, h'\}$ forming edges of Γ. The fact that this gluing can be performed in families requires an argument (see e.g. [ACG11, Chapter X, Section 7]). As an illustration, in the simplest case of having two curves $\pi_1 : C_1 \to S$, $\pi_2 : C_2 \to S$ glued along sections $q_1 : S \to C_1$, $q_2 : S \to C_2$, the glued family can be obtained as the union of the images of C_1, C_2 inside the fibre product $C_1 \times_S C_2$ under the maps

$$\begin{tikzcd}
C_1 \arrow[dr, "\mathrm{id}_{C_1} \times (q_2 \circ \pi_1)"] & \\
& C_1 \times_S C_2 \\
C_2 \arrow[ur, "(q_1 \circ \pi_2) \times \mathrm{id}_{C_2}"'] &
\end{tikzcd}$$

Once we construct the natural transformation $\widehat{\xi}_\Gamma$, we obtain the map of coarse moduli spaces by considering the diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_\Gamma \arrow[r] \arrow[d, "\widehat{\xi}_\Gamma"] & h^{\overline{M}_\Gamma} \arrow[d, dashed, "\xi_\Gamma"] \\
\overline{\mathcal{M}}_{g,n} \arrow[r] & h^{\overline{M}_{g,n}}.
\end{tikzcd} \tag{47}$$

Here the horizontal arrows come from the fact that $\overline{M}_\Gamma$ and $\overline{M}_{g,n}$ are coarse moduli spaces of the functors on the right. By definition of a coarse moduli space, the composition

$\overline{\mathcal{M}}_\Gamma \to h^{\overline{M}_{g,n}}$ of the morphisms on the left and bottom must factor through the morphism at the top, giving the map on the right as desired. From the definition of $\widehat{\xi}_\Gamma$ we see that ξ_Γ does what we want on $\mathbb{C}$-points of $\overline{M}_\Gamma$.

As for the properties of ξ_Γ, note that it is proper since its domain is proper and its target is separated ([Vak17, Proposition 10.3.4]). Thus to show that it is finite, it suffices to show that it is quasifinite ([Vak17, Theorem 29.6.2]). Given $(C, p_1, \ldots, p_n)$ in the image of ξ_Γ, what is the preimage? It is the set of tuples of curves $(C_v, (q_h)_{h \in H(v)})_{v \in V(\Gamma)}$ which can be glued together to form $(C, p_1, \ldots, p_n)$.

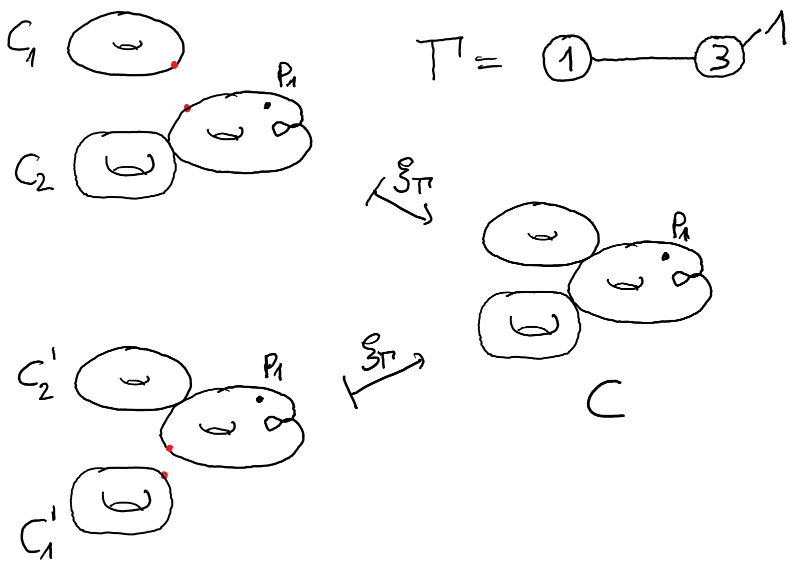


Figure 17: Different choices to obtain a stable curve C by gluing according to the graph Γ, as $C = C_1 \sqcup C_2$ or $C = C_1' \sqcup C_2'$

You can check that such a tuple can be specified by making two finite lists of choices: first you specify the subset Q of the nodes of C which are obtained by gluing markings in the preimage. Given this, you get a (possibly disconnected) curve $\widehat{C}$ by normalizing the nodes in Q[22]. Then you obtain the tuple $(C_v, (q_h)_{h \in H(v)})_{v \in V(\Gamma)}$ of curves by identifying the connected components C_v of $\widehat{C}$ with the vertices $v \in V(\Gamma)$ and the preimages q_h of nodes and markings with the half-edges $h \in H(\Gamma)$. Both the choice of nodes Q and the identification of components and preimages are finite choices, so there are only finitely many possibilities[23].

To show that the image of ξ_Γ is the closure of M^Γ, first note that

$$M^\Gamma = \xi_\Gamma(M_\Gamma), \text{ where } M_\Gamma = \prod_{v \in V(\Gamma)} M_{g(v), n(v)} \subset \overline{M}_\Gamma.$$

Indeed, given a curve in M^Γ we can certainly obtain it by gluing a tuple of smooth curves C_v under the map ξ_Γ (C_v are the components of the normalization of C). Conversely, any

[22] Normally, you cannot choose to "partially normalize" a variety at some points. If you want, you can obtain the normalization $\widehat{C}$ by first normalizing all of C and then gluing back together all nodes *not* in Q.

[23] Note: I am not saying that any choice of Q is allowed: if you choose the wrong set of nodes, you won't get the right number of connected components of $\widehat{C}$ or they won't have the right genus or markings p_i. You can check that if the stable graph of C is isomorphic to Γ, then Q must be the set of *all* nodes of C and the number of choices we have in the second step is exactly the size of the automorphism group $\mathrm{Aut}(\Gamma)$. Thus we expect that the *degree* of ξ_Γ to its image is generically $\#\mathrm{Aut}(\Gamma)$.

such gluing of smooth curves has stable graph Γ. Since ξ_Γ is proper, its image is closed, hence $\overline{M^\Gamma} \subset \xi_\Gamma(\overline{M}_\Gamma)$. On the other hand, since the moduli spaces of curves are irreducible, so is the product $\overline{M}_\Gamma$ and thus the nonempty open subset M_Γ is dense in $\overline{M}_\Gamma$. Finally, this implies

$$\xi_\Gamma(\overline{M}_\Gamma) \subset \overline{\xi_\Gamma(M_\Gamma)} = \overline{M^\Gamma}.$$

□

Exercise 4.18. a) Show (without using Proposition 4.14) that the variety $\overline{M}_\Gamma$ has dimension

$$\dim \overline{M}_\Gamma = \sum_{v \in V(\Gamma)} 3g(v) - 3 + n(v) = \dim \overline{M}_{g,n} - \#E(\Gamma).$$

b) Show that the complement $\partial\overline{M}_{g,n}$ of the locus $M_{g,n} \subset \overline{M}_{g,n}$ of smooth curves is given by the union of the sets $\overline{M}^\Gamma$ for Γ a stable graph with exactly one edge

$$\overline{M}_{g,n} \setminus M_{g,n} = \bigcup_{\Gamma : \#E(\Gamma)=1} \overline{M}^\Gamma.$$

*c) Show that for any stable graph Γ, the set $\overline{M}^\Gamma$ is a union of strata $M^{\Gamma'}$. Give a purely combinatorial description of the Γ' which appear. (*Hint*: A particularly nice way to put the answer to the second part of the question starts with "Consider the category whose objects are stable graphs of genus g with n legs and whose morphisms $\Gamma' \to \Gamma$ are given by …". We are going to see this appear later when discussing the intersection theory of $\overline{M}_{g,n}$.)

By Proposition 4.14, the $\overline{M}^\Gamma$ with $\#E(\Gamma) = 1$ have codimension 1 in $\overline{M}_{g,n}$ and they are called the *boundary divisors* of $\overline{M}_{g,n}$. The stable graphs Γ with $\#E(\Gamma) = 1$ are of one of the two forms in Figure 18 (this is part of the solution of Exercise 4.11 c)).

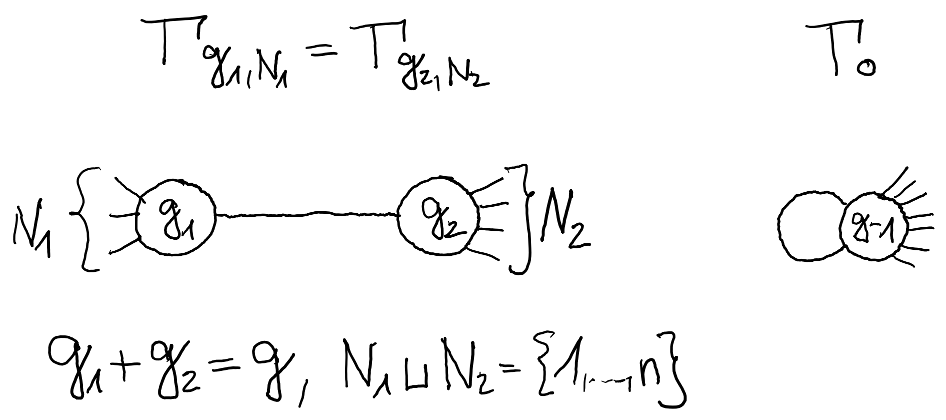


Figure 18: List of stable graphs with precisely one edge; if $g_1 = 0$ we require $n_1 = \#N_1 \geq 2$ because of the stability condition and similarly for $g_2 = 0$ we ask $n_2 = \#N_2 \geq 2$

The corresponding gluing maps take the form

$$\begin{aligned} \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} &\to \overline{M}_{g,n}, \\ \overline{M}_{g-1,n+2} &\to \overline{M}_{g,n}, \end{aligned}$$

and we denote by $\Delta_{g_1,N_1} = \Delta_{g_2,N_2}$ and Δ_0 the images of the respective map. These are precisely the irreducible components of the boundary of $\overline{M}_{g,n}$.

Thus once we prove Proposition 4.14, the exercise above completes the proof of Theorem 3.19 d), stating that the boundary is a Weil divisor (where of course we assumed all the other parts of the Theorem).

Proof of Proposition 4.14. We saw above that M^Γ is the image of $M_\Gamma = \prod_v M_{g(v),n(v)}$ under ξ_Γ. Since M_Γ is nonempty and irreducible, so is M^Γ. Also, the closure $\overline{M}^\Gamma$ is the image of $\overline{M}_\Gamma$ and by a slight extension of the argument above, one can show

$$\xi_\Gamma(\overline{M}_\Gamma \setminus M_\Gamma) = \overline{M}^\Gamma \setminus M^\Gamma.$$

Indeed, a tuple $(C_v, (q_h)_h)_v \in \overline{M}_\Gamma \setminus M_\Gamma$ satisfies that one of the curves C_v has a node, and then the curve $\xi_\Gamma((C_v, (q_h)_h)_v) \in \overline{M}^\Gamma$ has at least $\#E(\Gamma) + 1$ nodes, and thus cannot have dual graph isomorphic to Γ. Since $\overline{M}_\Gamma \setminus M_\Gamma$ is closed and ξ_Γ is proper, the set $\overline{M}^\Gamma \setminus M^\Gamma$ is closed in $\overline{M}^\Gamma$ and thus M^Γ is locally closed inside $\overline{M}_{g,n}$ since it is open in the closed set $\overline{M}^\Gamma$. Finally, we have $\dim \overline{M}^\Gamma = \dim \overline{M}_\Gamma$ since ξ_Γ is finite and thus the formula for $\dim \overline{M}^\Gamma$ follows from Exercise 4.18. □

4.3 Stable curves of genus 0

General results

Before we start looking at examples, let us use the new tool of stable graphs to prove that automorphism groups of genus 0 stable curves are trivial.

Exercise 4.19. Let Γ be a stable graph of genus 0.

a) Show that the *undirected graph* with vertex set $V(\Gamma)$ and edges $\{v(h), v(h')\}$ for $\{h, h'\} \in E(\Gamma)$ is a *tree*. (*Hint:* See e.g. here for the definition of a tree)

b) Show that $\mathrm{Aut}(\Gamma) = \{\mathrm{id}_\Gamma\}$ is trivial.

c) Show that any stable curve $(C, p_1, \ldots, p_n)$ of genus 0 has trivial automorphism group $\mathrm{Aut}(C, p_1, \ldots, p_n) = \{\mathrm{id}_C\}$. (*Hint:* Exercise 4.10)

Corollary 4.20. For $n \geq 3$, the space $\overline{M}_{0,n}$ is a fine moduli space for the functor $\overline{\mathcal{M}}_{0,n}$ and a smooth, irreducible projective variety of dimension $n-3$.

Proof. By Exercise 4.19 we have $\overline{M}^0_{0,n} = \overline{M}_{0,n}$, so the statement follows from Theorem 3.19, in particular part e). □

Example: $n = 3$

Let's turn to some examples. From Exercise 4.11 we know that a stable graph in genus 0 with n legs has at most $3g - 3 + n = n - 3$ edges. Thus for $n = 3$ the only stable graph is the trivial one, which shows $\overline{M}_{0,3} = M_{0,3} = \mathrm{pt}$ is a point.

Example: $n = 4$

For $n = 4$, every nontrivial stable graph has exactly one edge and by Exercise 4.19 above it must have precisely two vertices v_1, v_2 (if it had only one, the edge would be a loop at this vertex, so the graph would not be a tree, if it had more than two vertices, the graph would not be connected). Each vertex is incident to one of the half-edges forming the single edge and by stability we must have $n(v_1), n(v_2) \geq 3$. Since we have precisely four legs to distribute (corresponding to the four marked points), every vertex must get exactly two of them. To make a long story short, here are the possible stable graphs Γ with $(g, n) = (0, 4)$:

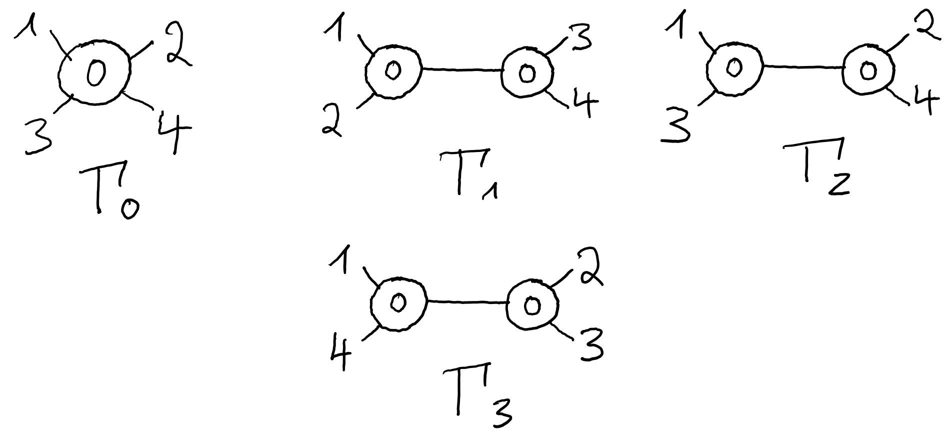


Figure 19: The stable graphs of genus 0 with 4 legs

By Proposition 4.14 we know that

$$\overline{M}_{0,4} = M^{\Gamma_0} \sqcup M^{\Gamma_1} \sqcup M^{\Gamma_2} \sqcup M^{\Gamma_3}, \tag{48}$$

where $M^{\Gamma_0} = M_{0,4} \cong \mathbb{A}^1 \setminus \{0, 1\}$ and the M^{Γ_i} $(i = 1, 2, 3)$ are irreducible, locally closed subset of $\overline{M}_{0,4}$ of dimension 0, i.e. points. Maybe by now you have (correctly) guessed that $\overline{M}_{0,4} \cong \mathbb{P}^1$. Indeed, this follows from Corollary 4.20 and (48) since $\mathbb{P}^1$ is the only smooth, irreducible, projective variety of dimension 1 which contains $M_{0,4} = \mathbb{A}^1 \setminus \{0, 1\}$ as an open subvariety[24]. In Figure 20 you see a picture showing which points in $\mathbb{P}^1$ correspond to which stable curves.

In fact, since $\overline{M}_{0,4}$ is a fine moduli space, we know that the individual curves we drew in Figure 20 actually fit together into a universal curve $\pi : \overline{C}_{0,4} \to \overline{M}_{0,4}$, the universal family of curves for the moduli functor $\overline{\mathcal{M}}_{0,4}$. If you worked on Exercise 4.4, you know that the universal family over $M_{0,4}$ is the trivial family $M_{0,4} \times \mathbb{P}^1 \to M_{0,4}$. The correct way to fill in the missing fibres over $0, 1, \infty \in \overline{M}_{0,4}$ is shown in Figure 21.

Thus the universal curve is given by the composition

$$\pi : \overline{C}_{0,4} = \mathrm{Bl}_{(0,0),(1,1),(\infty,\infty)} \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 = \overline{M}_{0,4} \tag{49}$$

of the blow-up map of $\mathbb{P}^1 \times \mathbb{P}^1$ at three points and the projection from $\mathbb{P}^1 \times \mathbb{P}^1$ to the first factor. The sections $p_1, \ldots, p_4 : \overline{M}_{0,4} \to \overline{C}_{0,4}$ are the strict transforms of the four maps

$$\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1, q \mapsto (q, 0), (q, 1), (q, \infty), (q, q). \tag{50}$$

[24]The varieties $\overline{M}_{0,4}$ and $\mathbb{P}^1$ are both smooth, irreducible projective curves, and such curves are isomorphic if and only if they are birational. And indeed, both contain $\mathbb{A}^1 \setminus \{0, 1\}$.

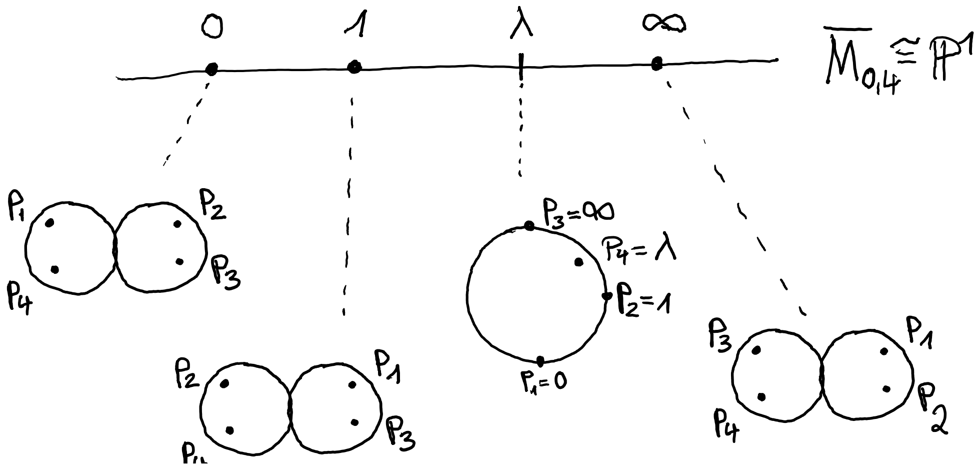


Figure 20: The moduli space $\overline{M}_{0,4} \cong \mathbb{P}^1$ of stable curves

The singular fibres of π over $0, 1, \infty \in \mathbb{P}^1$ are the unions of the strict transform of the fibre of the projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and the exceptional divisor of the blowup, which meet transversally and thus form a nodal curve.

Exercise 4.21. Show that indeed the singular fibres of π over $0, 1, \infty \in \mathbb{P}^1$ are nodal curves according to Definition 3.6.

On the exceptional divisor over $(0,0) \in \mathbb{P}^1 \times \mathbb{P}^1$, you can check that the strict transforms p_1, p_4 of $q \mapsto (q, 0)$ and $q \mapsto (q, q)$ go to *distinct* points, since the maps $(q, 0), (q, q)$ meet at $(0, 0)$ with distinct tangent vectors[25].

The general case $n \geq 5$

In the case $n = 5$ it turns out that

$$\overline{M}_{0,5} \cong \mathrm{Bl}_{(0,0),(1,1),(\infty,\infty)} \mathbb{P}^1 \times \mathbb{P}^1.$$

Sounds familiar? Indeed, we have $\overline{M}_{0,5} \cong \overline{C}_{0,4}$. This is the beginning of a more general story, which we summarize in the following theorem.

Theorem 4.22 ([Knu83a, Kee92])**.** Let $\overline{M}_{0,n}$ be the moduli space of stable curves of genus 0 with n marked points and let $\pi : \overline{C}_{0,n} \to \overline{M}_{0,n}$ be its universal curve.

a) For $n \geq 3$ we have $\overline{M}_{0,n+1} \cong \overline{C}_{0,n}$ and under this identification, the map

$$\pi : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$$

is the so-called *forgetful morphism*[26] *of the marking* $n+1$.

b) The universal curve $\overline{C}_{0,n} \to \overline{M}_{0,n}$ can be obtained from the projection

$$\overline{M}_{0,n} \times \mathbb{P}^1 \to \overline{M}_{0,n}$$

by an iterated blowup along smooth codimension 2 subvarieties (see [Kee92, Section 1] for a precise description).

[25] This uses the identification of the exceptional divisor with $\mathbb{P}(T_{(0,0)}\mathbb{P}^1 \times \mathbb{P}^1)$, see [Vak17, Section 22.3] for details.

[26] See the next section for the general definition of such forgetful morphisms.

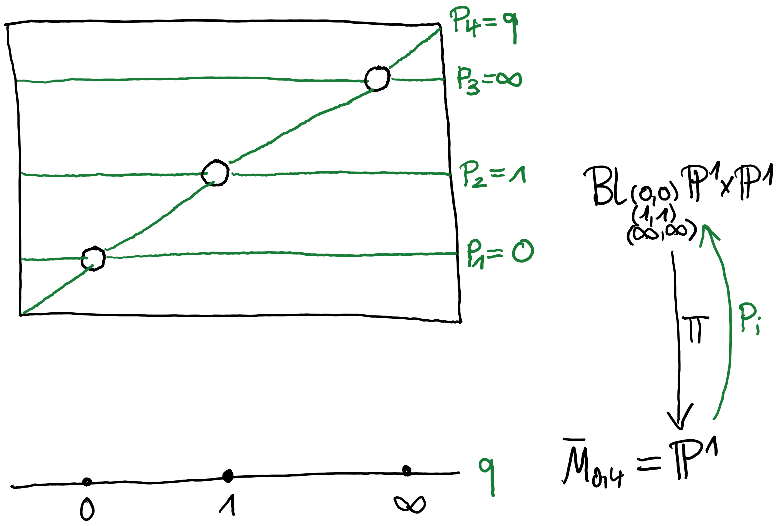


Figure 21: The universal family of curves over the moduli space $\overline{M}_{0,4}$

Remark 4.23. a) Combining the two parts of the above result, we can construct $\overline{M}_{0,n+1}$ from $\overline{M}_{0,n}$ so starting with $\overline{M}_{0,3} = \mathrm{Spec}(\mathbb{C})$ we can find all moduli spaces of stable curves in genus 0 recursively. Alternatively, since we get a factor $\mathbb{P}^1$ every time n increases starting from $n = 3$, we can see that the above procedure will produce $\overline{M}_{0,n}$ as an iterated blowup of the variety $(\mathbb{P}^1)^{n-3}$.

b) If you completed Exercise 4.4, you saw already that the universal curve over $M_{0,n}$ is isomorphic to $M_{0,n} \times \mathbb{P}^1$. This is naturally contained as an open subvariety of the blowup $\overline{C}_{0,n}$ of $\overline{M}_{0,n} \times \mathbb{P}^1$ from above. The universal sections $p_1, \ldots, p_n : \overline{M}_{0,n} \to \overline{C}_{0,n}$ are then the unique extensions of the universal sections $M_{0,n} \to M_{0,n} \times \mathbb{P}^1$ from Exercise 4.4.

The next section will introduce the forgetful morphisms $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ mentioned above for arbitrary g, n. We will see that they *almost*, but not quite, identify $\overline{M}_{g,n+1}$ as the universal curve over $\overline{M}_{g,n}$. Again, our old enemies the automorphisms will ruin everything.

4.4 The forgetful morphism and the universal curve

Since $\overline{M}_{g,n}$ parametrizes curves C together with marked points $p_1, \ldots, p_n$, we might expect that there exist morphisms $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ sending $(C, p_1, \ldots, p_n, p_{n+1})$ to the curve $(C, p_1, \ldots, p_n)$ obtained by forgetting the last marking. Below we will see that this certainly works for smooth curves C, but that the condition that the resulting n-pointed curve is *stable* requires some slight adjustment in our definition.

Easy exercise 4.24. For $2g-2+n > 0$, show that there exists a morphism $M_{g,n+1} \to M_{g,n}$ of the moduli spaces of smooth curves, which on complex points is given by

$$M_{g,n+1}(\mathbb{C}) \to M_{g,n}(\mathbb{C}), (C, p_1, \ldots, p_n, p_{n+1}) \mapsto (C, p_1, \ldots, p_n).$$

What goes wrong when we try to write down the same map for arbitrary stable curves $(C, p_1, \ldots, p_n, p_{n+1})$? Well, it can happen that $(C, p_1, \ldots, p_n)$ is no longer stable. Indeed, the component C_v of C containing the marked point p_{n+1} has one special point less than before in $(C, p_1, \dots, p_n)$. Thus by Proposition 3.13 it can become unstable if $g(C_v) = 0$ and C_v had exactly 3 special points[27].

The left side of Figure 22 shows the various ways in which this situation can happen.

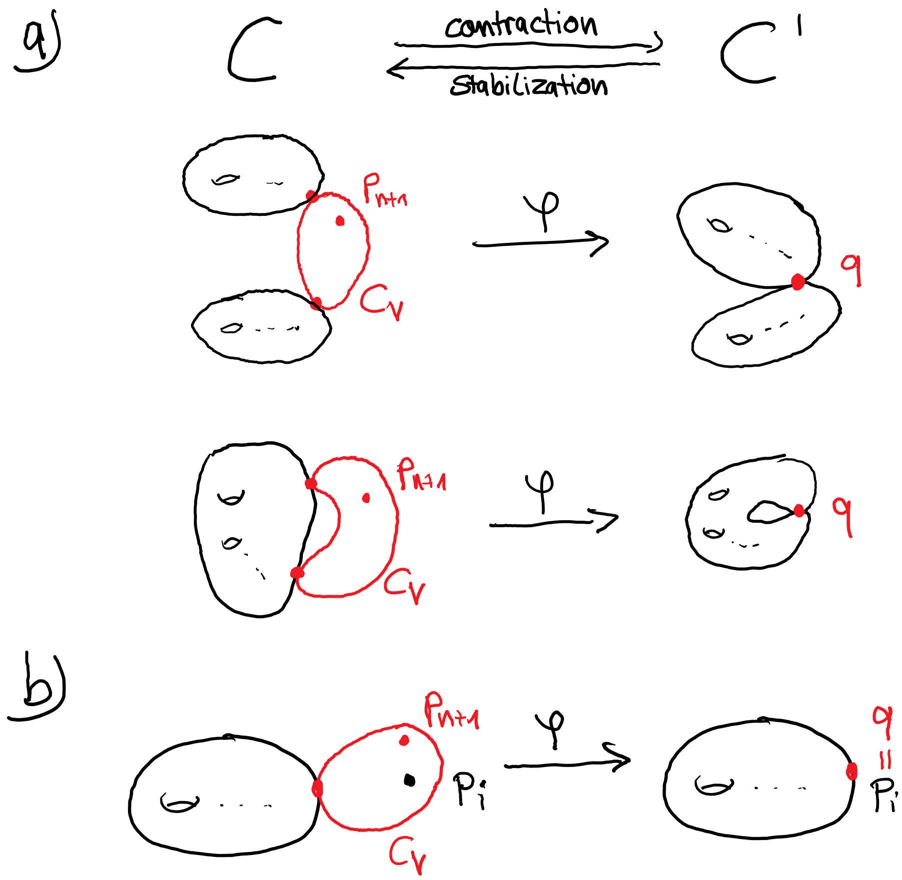


Figure 22: The map φ contracting the component C_v that becomes unstable by forgetting the marking p_{n+1} of C to a point $q \in C'$; the map in the opposite direction (taking (C', q) and inserting a component C_v isomorphic to $\mathbb{P}^1$) is called stabilization and will appear in the proof of Proposition 4.25

Indeed, the other two special points on C_v besides p_{n+1} can be

a) two preimages of nodes, or

b) one preimage of a node and one other marked point p_i.

The right side of Figure 22 shows the solution to our problem of defining the forgetful morphism - we need to construct a morphism $\varphi : C \to C'$ contracting the component C_v to a point $q \in C'$. Then our forgetful morphism $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ is given by

$$\pi((C, p_1, \dots, p_n, p_{n+1})) = \begin{cases} (C, p_1, \dots, p_n) & \text{if } (C, p_1, \dots, p_n) \text{ is stable,} \\ (C', \varphi(p_1), \dots, \varphi(p_n)) & \text{otherwise.} \end{cases} \tag{51}$$

[27]The case of $g(C_v) = 1$ does not cause problems: it would require that p_{n+1} was the only special point of C_v. If $C = C_v$ was smooth, we would be in the case $(g, n+1) = (1, 1)$ which violates $2g - 2 + n > 0$, if C was singular then C_v would have to contain at least one node (otherwise it would be an isolated component of C, so C would not be connected).

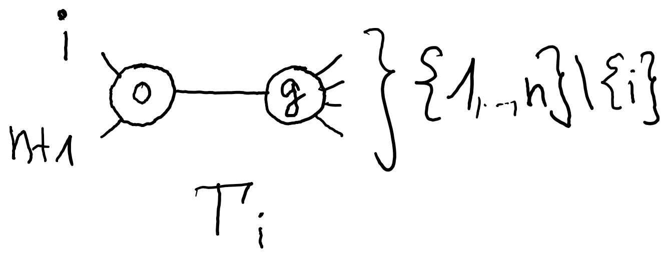
To define the map φ above, we can use Fact 3.9: the curve C' is obtained by taking the normalization $\widetilde{C}$ of C, removing the normalization of the component C_v and gluing the remaining components back together along preimages of nodes according to Figure 22. In case a) above, the two preimages of nodes in C_v are identified, in case b) we simply remove the component C_v. The morphism $\varphi : C \to C'$ is then defined via the second part of Fact 3.9 from the natural map $\widetilde{C} \to C'$, which contracts the normalization of C_v to the point where it was previously attached.

It is not a priori obvious that (51) is the right thing to do, or even that it gives a continuous map. However, the next Proposition tells us that both are the case.

Proposition 4.25 ([Knu83a]). There exists a morphism $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ defined on $\mathbb{C}$-points by (51). Over the locus $\overline{M}^0_{g,n} \subset \overline{M}_{g,n}$ of curves without automorphisms, this map is isomorphic to the universal curve of $\overline{M}^0_{g,n}$. The universal sections $p_1, \ldots, p_n : \overline{M}^0_{g,n} \to \pi^{-1}(\overline{M}^0_{g,n})$ are given by the restriction of the gluing morphisms

$$p_i = \xi_{\Gamma_i} : \overline{M}_{g,n} = \overline{M}_{0,3} \times \overline{M}_{g,n} \to \overline{M}_{g,n+1}$$

associated to the stable graphs



Before saying something about the proof, note that the morphism $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ above is the *unique* extension of the morphism $M_{g,n+1} \to M_{g,n}$ from Easy exercise 4.24 which really just forgets the marking p_{n+1}. The uniqueness follows since the domain $\overline{M}_{g,n+1}$ of π is reduced, the target $\overline{M}_{g,n}$ is separated and since π is determined on the open dense subset $M_{g,n+1} \subset \overline{M}_{g,n+1}$ (see the "Reduced-to-Separated Theorem" [Vak17, Theorem 10.2.2]). Thus you can see the Proposition above as saying that the extension exists and that (51) gives us a *modular interpretation*, i.e. an interpretation what this extension does on the geometric objects (stable curves) that our moduli spaces parametrize.

For the second part of Proposition 4.25 which allows us to interpret the forgetful map π as the universal curve over part of $\overline{M}_{g,n}$ you can have a look at Figure 23, where for chosen points of the fibre of the universal curve we illustrate which $(n+1)$-pointed curves they correspond to.

Below we sketch the formal proof of Proposition 4.25, but even this sketch is rather technical. So if you are happy with the picture and explanation above, feel free to skip it for now.

**Sketch of proof.* As in the proof of Proposition 4.15 we can construct the morphism π by defining a corresponding natural transformation $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ and using that $\overline{M}_{g,n+1}, \overline{M}_{g,n}$ are coarse moduli spaces of the corresponding functors.

In fact, we can do this in two steps: we define a moduli functor $\overline{\mathcal{C}}_{g,n}$ sending a scheme S to the tuples

$$(\pi' : C' \to S; p'_1, \ldots, p'_n, q : S \to C'), \tag{52}$$

such that

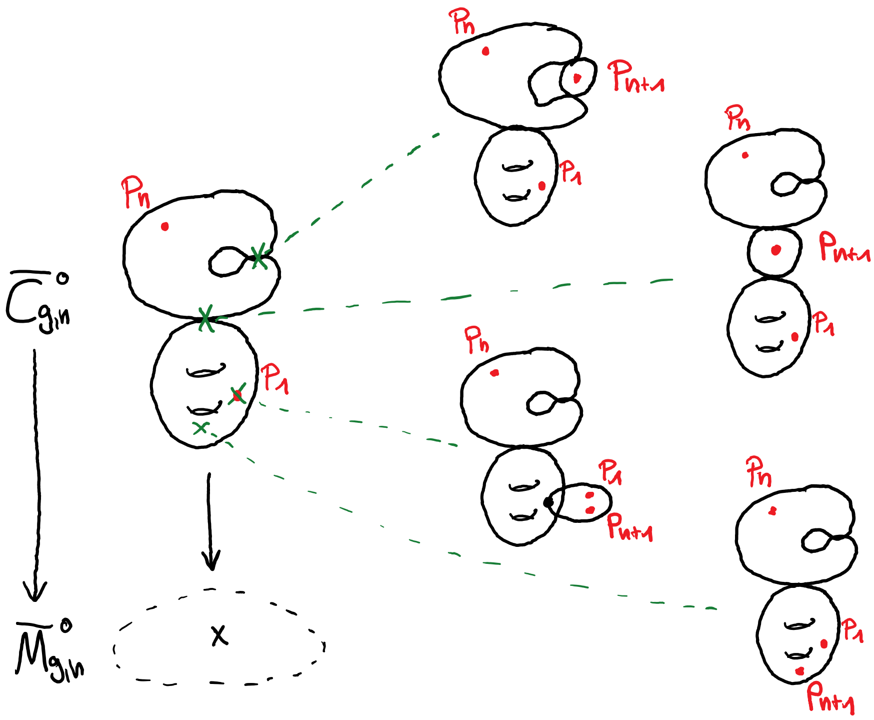


Figure 23: Correspondence of points in the fibre of the universal curve over $\overline{M}^{0}_{g,n}$ with $(n+1)$-pointed stable curves

- $(\pi' : C' \to S; p'_1, \dots, p'_n : S \to C)$ is a family of stable curves of genus g with n marked points,
- $q : S \to C'$ is *any* section of π.

There is a natural transformation $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ which simply forgets the marking q (and does *not* change the curve C). On the other hand, Knudsen [Knu83a] constructs isomorphism of functors

$$\overline{\mathcal{M}}_{g,n+1} \underset{\text{stabilization}}{\overset{\text{contraction}}{\rightleftarrows}} \overline{\mathcal{C}}_{g,n}, \tag{53}$$

and the natural transformation $\overline{\mathcal{M}}_{g,n+1} \cong \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ defines us the morphism $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$. For an excellent explanation of the details of the equivalence (53) see [KV07, Section 1.3] (or look at [Knu83a, Proposition 2.1, Theorem 2.4] for the original formulation and proof).

To summarize the sources above, the contraction map in (53) takes an $(n+1)$-pointed family of curves

$$(\pi : C \to S; p_1, \dots, p_n, p_{n+1} : S \to C), \tag{54}$$

constructs a morphism $\varphi : C \to C'$ which contracts the unstable components of fibres of $C \to S$ as in (51). Then (54) is sent to (52) setting $p'_i = \varphi \circ p_i$ and $q = \varphi \circ p_{n+1}$.

The stabilization functor in (53) does the opposite: starting with the family (52) it modifies the fibres C'_s of $C' \to S$ in which the section q collides with one of the nodes or one of the markings p'_i by inserting an extra component isomorphic to $\mathbb{P}^1$ at the point of intersection. This creates a new family of curves $\pi : C \to S$ and the sections

$p_1, \ldots, p_{n+1} : S \to C$ are obtained from $p_1', \ldots, p_n', q$ as "strict transforms" under the map $C \to C'$.

Finally, we know that the locus $\overline{M}^0_{g,n}$ is a fine moduli space for the functor $\overline{\mathcal{M}}^0_{g,n}$. It is easy to see that the preimage $\pi^{-1}(\overline{M}^0_{g,n}) \subset \overline{M}^0_{g,n+1}$ is also contained in the set $\overline{M}^0_{g,n+1}$ which is a fine moduli space. Defining the functor $\overline{\mathcal{C}}^0_{g,n}$ as the families (52) such that $(C, p_1, \ldots, p_n)$ has only trivial automorphisms, this easily shows that $\pi^{-1}(\overline{M}^0_{g,n})$ is a fine moduli space for $\overline{\mathcal{C}}^0_{g,n}$. On the other hand, one can write down a natural transformation $\overline{\mathcal{C}}^0_{g,n} \to h^{\overline{C}^0_{g,n}}$ making the universal curve $\overline{C}^0_{g,n}$ a fine moduli space of $\overline{\mathcal{C}}^0_{g,n}$. So both $\pi^{-1}(\overline{M}^0_{g,n})$ and $\overline{C}^0_{g,n}$ are fine moduli spaces of the same functor $\overline{\mathcal{C}}^0_{g,n}$, and thus they are isomorphic. From the definition of the stabilization in (53) it follows that the gluing morphisms ξ_{Γ_i} are obtained from the stabilization of the family

$$(\pi : \overline{\mathcal{C}}^0_{g,n} \to \overline{M}^0_{g,n}; p_1, \ldots, p_n, q = p_i : \overline{M}^0_{g,n} \to \overline{\mathcal{C}}^0_{g,n}) \in \overline{\mathcal{C}}_{g,n}(\overline{M}^0_{g,n}).$$

But setting $q = p_i$ exactly corresponds to the i-th section of the universal curve, and this completes the proof that $p_i = \xi_{\Gamma_i}$ over $\overline{M}^0_{g,n}$. □

A final question remains: if $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ is *not* the universal curve outside the locus of curves $(C, p_1, \ldots, p_n) \in \overline{M}_{g,n}$ which have only trivial automorphisms, then what *are* the fibres over curves? For this let $\sigma : C \to C$ be such an automorphism fixing the points $p_1, \ldots, p_n$, then we have for any point $q \in C$ which is smooth and does not coincide with one of the markings $p_1, \ldots, p_n$ that

$$(C, p_1, \ldots, p_n, q) \sim (C, \sigma(p_1), \ldots, \sigma(p_n), \sigma(q)) = (C, p_1, \ldots, p_n, \sigma(q)) \in \overline{M}_{g,n+1}. \quad (55)$$

From this one can check that the closed points of the fibre $\pi^{-1}((C, p_1, \ldots, p_n))$ are in one-to-one correspondence with $\mathrm{Aut}(C, p_1, \ldots, p_n)$-orbits of points in C, and so the fibre is isomorphic to the quotient

$$\pi^{-1}((C, p_1, \ldots, p_n)) \cong C/\mathrm{Aut}(C, p_1, \ldots, p_n). \quad (56)$$

4.5 Genus 1

In comparison to the genus 0 case, the case of genus 1 is much harder: nice properties, such as having a fine moduli space, are missing and there is - to my knowledge - no nice recursive construction as for $\overline{M}_{0,n}$. We content ourselves in looking at one nontrivial example - the case of precisely one marking[28] - and in pointing out the new phenomena that arise from the fact that we can have automorphisms.

The case $n = 1$

The following result makes precise some of the discussion in Section 1.

Proposition 4.26 (see Proposition 4.18 in [Ber13]). The moduli functor $\mathcal{M}_{1,1}$ has as coarse moduli space the affine line $M_{1,1} \cong \mathbb{A}^1$.

**Sketch of proof.* We must show that for any family of elliptic curves over a scheme S we can construct a natural morphism $S \to \mathbb{A}^1$. In the introduction we saw that this should be related to the j-invariant of the elliptic curves. But a priori, this j-invariant only makes sense for elliptic curves given as cubic curves in $\mathbb{P}^1$.

[28]Note that the condition $2g - 2 + n > 0$ implies that $n = 1$ is the simplest possible case for $g = 1$.

To approach this situation, note that given a (smooth) elliptic curve (E, p) one can use Riemann-Roch and Serre Duality to show

$$h^0(E, \mathcal{O}_E(3p)) = 3,\ h^1(E, \mathcal{O}_E(3p)) = 0 \tag{57}$$

and for any basis s_1, s_2, s_3 of sections of $\mathcal{O}_E(3p)$ we obtain an embedding[29]

$$E \hookrightarrow \mathbb{P}^2, q \mapsto [s_1(q) : s_2(q) : s_3(q)]$$

of E as a smooth cubic curve in $\mathbb{P}^2$ (see [Vak17, Section 19.9] for details).

Now we need to do this for families of elliptic curves. Given such a family

$$(\pi : E \to S; p : S \to E) \in \mathcal{M}_{1,1}(S) \tag{58}$$

over some scheme S, we see as in the proof of Proposition 4.2 that (57) implies that the pushforward $\mathcal{V} = \pi_* \mathcal{O}_E(3p)$ is a rank 3 vector bundle over S. Let V be the total space of this vector bundle, then we have an open dense subset

$$U = \{(s_1, s_2, s_3) : s_1, s_2, s_3 \in V_s \text{ form a basis of } V_s = H^0(E_s, \mathcal{O}_{E_s}(3p(s)))\} \subset V \times_S V \times_S V.$$

Let $E_U = E \times_S U$ be the pullback of $E \to S$ under the natural map $U \to S$, then the pullback of V to E_U naturally has three sections s_1, s_2, s_3 (given by the coordinates on U), and they define an embedding $E_U \hookrightarrow \mathbb{P}^2_U = U \times \mathbb{P}^2$.

For this embedding, we want to say that on each fibre there is a cubic equation on $\mathbb{P}^2$, unique up to scaling, that cuts out the image of the elliptic curve in $\mathbb{P}^2$. And indeed, denoting $\mathcal{F}$ the pushforward of $\mathcal{O}_{\mathbb{P}^2_U}(3)$ under $\mathbb{P}^2_U \to U$, there exists a unique $F \in \mathbb{P}(\mathcal{F})$ such that E_U is the vanishing locus of F in $\mathbb{P}^2_U$. Now what data is $F(u)$ at a point $u \in U$? It's just the coefficients of a cubic curve in the corresponding projective space $\mathbb{P}^2_u$. But from these coefficients you can compute the j-invariant of this cubic curve (and the formula is invariant under scaling). This means that we obtain a map $\widehat{j} : U \to \mathbb{A}^1$. But the value of $\widehat{j}$ is constant on the fibres of the morphism $U \to S$ - different points in the fibre just correspond to different ways to embed the same elliptic curve in $\mathbb{P}^2$ and the j-invariant is, well, invariant under this. Using fpqc descent (see [Sta13, Tag 023Q]), we then obtain a morphism $j : S \to \mathbb{A}^1$ as desired.

$$\begin{tikzcd} U \arrow[d] \arrow[dr, "\widehat{j}"] & \\ S \arrow[r, "\exists\, j"'] & \mathbb{A}^1 \end{tikzcd}$$

We'll not go into the details in how to verify that this natural transformation $\mathcal{M}_{1,1} \to h^{\mathbb{A}^1}$ satisfies property a) of a coarse moduli space (being initial among natural transformations $\mathcal{M}_{1,1} \to h^M$ for M a scheme). But concerning part b) of the definition, the above proof showed that every elliptic curve (E, p) can be embedded as a smooth cubic in $\mathbb{P}^2$, so if we accept that the j-invariant classifies those up to isomorphism, it implies that $\mathcal{M}_{1,1} \to h^{\mathbb{A}^1}$ is a bijection on geometric points. □

Corollary 4.27. The moduli space $\overline{M}_{1,1}$ is isomorphic to $\mathbb{P}^1$.

Proof. By Theorem 3.19, the space $\overline{M}_{1,1}$ is a normal, projective variety of dimension 1. Being normal in dimension 1 means it is actually smooth, and as seen in Section 4.3, the only such curve containing an open subset of $\mathbb{A}^1$ is $\mathbb{P}^1$. □

The point $\infty \in \mathbb{P}^1 \cong \overline{M}_{1,1}$ corresponds to the stable curve (E_0, p_0) obtained by starting with $p_0 \in \mathbb{P}^1$ and identifying two points (not equal to p_0) to a node.

[29] Here we use that $\mathbb{P}^2$ is a moduli space of line bundles together with 3 sections not vanishing simultaneously, just what we saw in the proof of Example 2.4!

The case $n = 2$

Exercise 4.28. Figure 24 illustrates the forgetful morphism $\pi : \overline{M}_{1,2} \to \overline{M}_{1,1}$ with the boundary of both spaces marked in red. For each of the points marked in blue, draw their corresponding curves and their dual graphs.

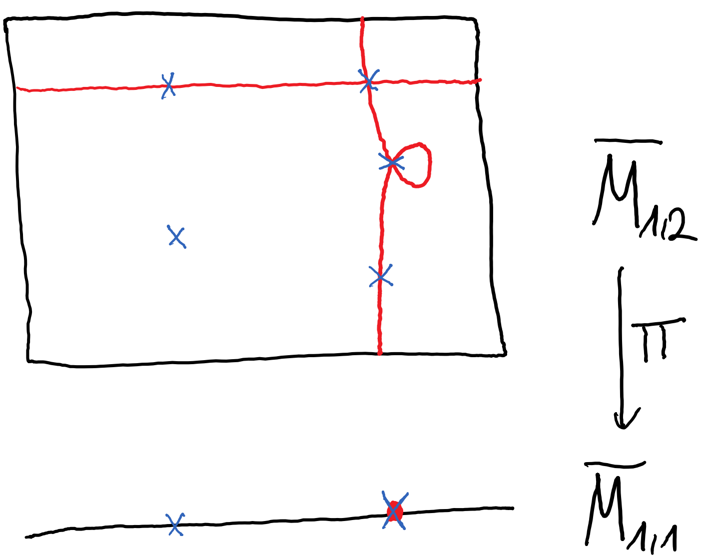


Figure 24: The forgetful morphism $\pi : \overline{M}_{1,2} \to \overline{M}_{1,1}$

A fun construction in $n = 9$

Exercise 4.29. Show that for $Q_1, \ldots, Q_9 \in \mathbb{P}^2$ general points, there exists a unique cubic curve $E_{\mathbf{Q}}$ going through $Q_1, \ldots, Q_9$. Show that this gives rise to a rational map

$$(\mathbb{P}^2)^9 \dashrightarrow \overline{M}_{1,9}, (Q_1, \ldots, Q_9) \mapsto (E_{\mathbf{Q}}, Q_1, \ldots, Q_9). \tag{59}$$

Show that this map is dominant, i.e. that the generic point of $\overline{M}_{1,9}$ is contained in the image. (*Hint:* This last part will involve showing that for any fixed smooth genus 1 curve E, embedded in some way as a cubic $E \hookrightarrow \mathbb{P}^2$ and $Q_1, \ldots, Q_9 \in E$ general points *in* E, the curve E is the unique cubic through $Q_1, \ldots, Q_9$).

Remark 4.30 (*, for people interested in birational geometry)**.** The above exercise shows that $\overline{M}_{1,9}$ is *unirational*, i.e. that it admits a dominant rational map from a projective space. For $g = 0$ we already saw that all spaces $\overline{M}_{0,n}$ have the stronger property of being *rational*, i.e. birational to a projective space. This is the start of an interesting story: for many (small) values of (g, n) there have been rational parametrizations of $\overline{\mathcal{M}}_{g,n}$ as in (59) (see [Ben14, BV05, CF07, Far09, Log03, Ver05]). However, in general the spaces $\overline{M}_{g,n}$ are neither rational nor unirational:

- It turns out that in genus 1, the Hodge number $h^{11,0}(\overline{M}_{1,11})$ is equal to 1, so there exists a nonzero holomorphic 11-form on $\overline{M}_{1,11}$. This implies that there cannot be a dominant rational map $\mathbb{P}^N \dashrightarrow \overline{M}_{1,11}$.

- Classical results by Eisenbud, Harris and Mumford [HM82, Har84, EH87] say that $\overline{M}_g$ is of *general type* for $g \geq 24$ and has positive *Kodaira dimension* for $g = 23$. A variant of this result by Logan [Log03] says that for $g > 3$ the space $\overline{M}_{g,n}$ is of general type for all but finitely many pairs (g, n).

The birational geometry and specifically the Kodaira dimension of the moduli spaces of curves are still an active area of research.

Non-existence of a fine moduli space / universal curve

To show that $\mathcal{M}_{1,1}$ cannot have a fine moduli space, we will construct an explicit example $(\pi : E \to S, p : S \to E)$ of a family of smooth genus 1 curves over a base S, such that the family is Zariski-locally trivial (i.e. a constant family) but not globally trivial. This gives a contradiction: if $M_{1,1}$ was a fine moduli space, the family would be the pullback of the universal family over $M_{1,1}$ under a unique map $S \to M_{1,1}$. The fact that π is Zariski locally trivial would imply that on a Zariski cover of S this map to $M_{1,1}$ is constant (i.e. factors through a point). Since a map is determined by its restriction to a Zariski cover, the map $S \to M_{1,1}$ itself would be constant. But the pullback of a family of curves under a constant map is a trivial family, a contradiction.

For constructing π, since the Zariski topology is very coarse we will need a slightly ugly base S: it is a nodal curve consisting of four rational curves forming a chain, as indicated in Figure 25. To obtain π let (E_0, p_0) be a smooth elliptic curve. By Fact 3.11 there exists a nontrivial automorphism $\sigma : E_0 \to E_0$ which fixes p_0 [30]. We obtain the family $\pi : E \to S$ by gluing the trivial families $U_1 \times E_0$, $U_2 \times E_0$ over the indicated Zariski open cover of S. The intersection $U_1 \cap U_2 \subset S$ has two components, and we glue the families by the identity of E_0 on one component and by σ on the other. The fact that p_0 is invariant under σ shows that the sections $p_0 : U_1 \to U_1 \times E_0$ and $p_0 : U_2 \to U_2 \times E_0$ glue together over the overlaps to a section $p_0 : S \to E$.

Exercise 4.31. Show that the sections $s : S \to E$ of the morphism $\pi : E \to S$ are in bijection with the fixed points of the automorphism σ. Conclude that $\pi : E \to S$ is not isomorphic to the trivial family $S \times E_0 \to S$. (*Hint*: Use that every morphism $\mathbb{P}^1 \to E_0$ is constant. This follows from the more general fact that every morphism $C \to D$ of smooth, projective, irreducible curves with $g(C) < g(D)$ is constant.)

4.6 *Proof of Proposition 4.2

Parts of the script indexed by a * are facultative, so they can be skipped on a first reading and are not part of the exam material. Concerning Proposition 4.2, it is mentioned in [KV07, Section 1.1.1], but proving it in the stated generality is actually quite nontrivial! If you spot gaps or mistakes in the proof below or find better references for the statements I cite, I would be happy if you write me an email.

** Proof of Proposition 4.2.* For part a) note that since $\pi : C \to B$ is proper and the composition $\pi \circ p_1 = \mathrm{id}_B$ is a closed embedding, by the cancellation theorem for properties of morphisms ([Vak17, Theorem 10.1.19]) the map p_1 is a closed embedding. Since the image of p_1 is an effective Cartier divisor when restricted to each fibre, by [Sta13, Tag 062Y] it defines a relative effective Cartier divisor. Let $\mathcal{L} = \mathcal{O}_C(p_1)$ be the associated line bundle on C. We claim that $\mathcal{E} = \pi_* \mathcal{L}$ is a locally free sheaf on B of rank 2. Indeed, for any

[30] If we identify E_0 as the quotient $E_o = \mathbb{C}/\Lambda$ of a $\mathbb{C}$ by a lattice $\Lambda \subset \mathbb{C}$ and choose $p_0 = 0$, one such automorphism is induced by the map $\mathbb{C} \to \mathbb{C}, z \mapsto -z$.

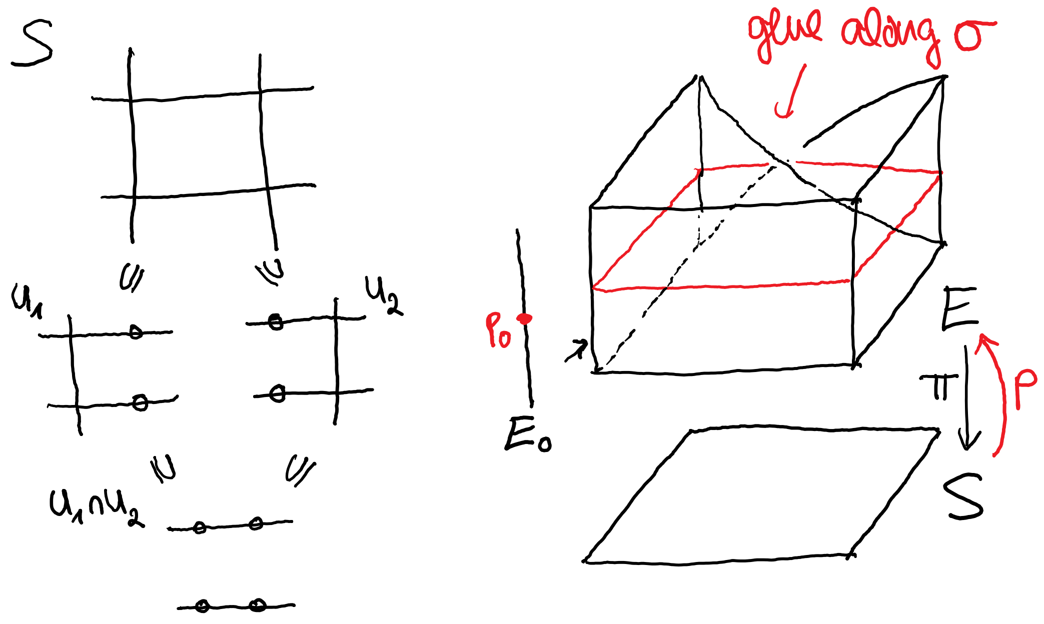


Figure 25: Constructing a nontrivial family over the base S by gluing trivial families on an open cover $S = U_1 \cup U_2$ along a nontrivial automorphism

point $s \in B$ the cohomology group of the fibre $H^1(C_s, \mathcal{L}_s)$ vanishes, since after passing to the algebraic closure $\overline{k(s)}$ of the residue field $k(s)$ (which is flat and thus commutes with formation of H^1) it is isomorphic to

$$H^1(C_{\overline{k(s)}}, \mathcal{L}_{\overline{k(s)}}) = H^1(\mathbb{P}^1_{\overline{k(s)}}, \mathcal{O}(1)) = 0.$$

By the Cohomology and Base Change Theorem (see [Vak17, Theorem 28.1.6., Exercise 28.1.D.] and note that the hypothesis of the base B being locally Noetherian can be removed since π is assumed locally of finite presentation, see [Vak17, Exercise 28.2.M.]), the sheaf $\mathcal{E} = \pi_*\mathcal{L}$ is indeed locally free. By going to a geometric fibre, we see that its rank is

$$h^0(C_{\overline{k(s)}}, \mathcal{L}_{\overline{k(s)}}) = h^0(\mathbb{P}^1_{\overline{k(s)}}, \mathcal{O}(1)) = 2.$$

As in the proof of [Har77, V, Proposition 2.2.] one then shows that $C \cong \mathbb{P}(\mathcal{E})$, completing the proof of a).

Now in case b) assume we have an additional section $p_2 : B \to C$. As in part a) we obtain a line bundle $\mathcal{L}' = \mathcal{O}_C(p_2)$ on C and we claim that the line bundle $\mathcal{L}^\vee \otimes \mathcal{L}$ is a pullback from the base, i.e. there exists $\mathcal{M}$ a line bundle on B with $\mathcal{L}^\vee \otimes \mathcal{L}' = \pi^*\mathcal{M}$. This follows e.g. by [Vak17, Proposition 28.1.11.] (note that we can remove the assumptions on the base being reduced and locally Noetherian with the same arguments used in the previous part). Let $s_0 \in H^0(C, \mathcal{L})$ be the section vanishing along p_1 and $s_\infty \in H^0(C, \mathcal{L}')$ the section vanishing along p_2. Then we have a map of locally free sheaves on B:

$$\Psi : \mathcal{O}_B \oplus \mathcal{M}^\vee \to \mathcal{E}, (a, b) \mapsto a \cdot s_0 + b \cdot s_\infty. \quad (60)$$

Here the section $b \cdot s_\infty$ makes sense since

$$\mathcal{E} = \pi_*\mathcal{L} = \pi_*(\mathcal{L}' \otimes \pi^*\mathcal{M}^\vee) = \mathcal{M}^\vee \otimes \pi_*\mathcal{L}'.$$

On an open cover of B which trivializes $\mathcal{E}$ (so that over the open sets $U \subset B$ the space C_U is isomorphic to $C_U = U \times \mathbb{P}^1$) it is easy to check that Ψ is an isomorphism. This open cover also trivializes the line bundle $\mathcal{M}$ and then the sections s_0, s_1 restrict to a basis of the sections of $\mathcal{L}$ on the fibres of π (since p_1, p_2 are disjoint).

Finally, in case c) we have a third section p_3 and since it is disjoint from p_1, p_2 we have:

$$\mathcal{M} = p_3^* \pi^* \mathcal{M} = p_3^* \mathcal{L}^\vee \otimes \mathcal{L}' = p_3^* \mathcal{O}_C(-p_1 + p_2) = \mathcal{O}_B.$$

Thus by the proof of part b) we can take $\mathcal{E} = \mathcal{O}_B \oplus \mathcal{O}_B$. □

Remark 4.32. A morphism π as in Proposition 4.2 is a special case of a *Brauer-Severi scheme*. By [Gro66, Théorème 8.2] such morphisms are always *étale locally* isomorphic to projective bundles, but not necessarily Zariski locally. An example for a family of smooth genus 0 curves which is not a projective bundle is given by the universal plane conic, defined over an open subset of $\mathbb{P}^5 = \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(2)))$, see [Vak17, Section 18.4.5].

References and further reading

A great introduction to moduli spaces of genus 0 curves is given in [KV07, Chapter 1]. You can also have a look at these lecture notes by Renzo Cavalieri. More material on gluing and forgetful maps (though phrased in the language of stacks that we will see in the next section) can be found in [ACG11, Chapter XII, Section 10].

The moduli problem of elliptic curves (E, p) has a much richer structure than you would expect from the isomorphism $\overline{M}_{1,1} \cong \mathbb{P}^1$ that we give above. You can have a look at the lecture notes [Hai08] for a much more complete picture.

5 Moduli stacks of curves

We have seen in many instances that the existence of automorphisms prevents fine moduli spaces from existing and we mentioned that it causes unpleasant phenomena (e.g. singularities) in the coarse moduli spaces. The solution is that we define a generalization of the notion of a scheme, called an *(algebraic) stack*, such that

- any scheme S can be interpreted as an algebraic stack (similar to the Yoneda embedding allowing us to see schemes as particular examples of moduli functors),
- tools and results from algebraic geometry can be generalized to stacks (i.e. we can define when an algebraic stack $\mathfrak{X}$ is smooth, when a morphism $\mathfrak{X} \to \mathfrak{X}'$ is proper, etc),
- there exists an algebraic stack $\overline{\mathcal{M}}_{g,n}$ serving as a "moduli stack of stable curves" (i.e. morphisms from a scheme S to $\overline{\mathcal{M}}_{g,n}$ are exactly equivalent to families of stable curves over S),
- the algebraic stack $\overline{\mathcal{M}}_{g,n}$ has many nice properties (e.g. $\overline{\mathcal{M}}_{g,n}$ is smooth, the morphism $\overline{\mathcal{M}}_{g,n} \to \mathrm{Spec}(\mathbb{C})$ is proper, etc.).

However, defining stacks and developing the language and results to talk about them requires some serious effort, and goes beyond the scope of this course. So what I will do is to give an outline of the ideas of the definition together with a guide where to learn more. We'll then go on to treat the stacks essentially as a black-box, pretending that they are schemes, with occasional remarks where we need to be more careful.

5.1 An outline of the theory of algebraic stacks

In Lemma 2.1 we saw that we can embed the category of schemes into the category of moduli functors by sending M to the functor $h^M = \mathrm{Mor}(-, M)$. But we saw that the moduli functors $\overline{\mathcal{M}}_{g,n}$ are *not* of the form h^M for some M. The main reason is that given a scheme S and an open cover $S = U_1 \cup U_2$, a morphism $S \to M$ to a scheme M is uniquely determined by its restriction to U_1 and U_2, but a family of curves up to isomorphism is *not* necessarily uniquely determined by its restrictions to U_1, U_2. As we saw in Exercise 4.31, we can obtain a nontrivial family by taking two trivial families of curves over U_1, U_2 and gluing them along a nontrivial automorphism on the overlap $U_1 \cap U_2$.

So we are looking for a definition of a new mathematical object $\overline{\mathcal{M}}_{g,n}$ such that

- it makes sense to speak of a morphism $S \to \overline{\mathcal{M}}_{g,n}$ from a scheme S to $\overline{\mathcal{M}}_{g,n}$, and such morphisms are in bijective correspondence to isomorphism classes of families of stable curves over S,
- given two morphism $f_1, f_2 : S \to \overline{\mathcal{M}}_{g,n}$ corresponding to families $\pi_1 : C_1 \to S$ and $\pi_2 : C_2 \to S$ of curves, we have a notion of isomorphisms $f_1 \to f_2$ corresponding to the set of isomorphisms $C_1 \xrightarrow{\sim} C_2$ of families of curves over S.

Then in the example above, for the scheme $S = U_1 \cup U_2$, a morphism $S \to \overline{\mathcal{M}}_{g,n}$ is determined by its restrictions $f_1 : U_1 \to \overline{\mathcal{M}}_{g,n}$ and $f_2 : U_2 \to \overline{\mathcal{M}}_{g,n}$ *together* with an isomorphism $f_1|_{U_1 \cap U_2} \xrightarrow{\sim} f_2|_{U_1 \cap U_2}$ (telling us how to glue the families on the overlap).

Looking at the requirements above, I claim that we have already seen mathematical objects with such "morphisms between morphisms", namely categories! Given categories $\mathcal{C}_1, \mathcal{C}_2$ we have a notion of a functor $f : \mathcal{C}_1 \to \mathcal{C}_2$, and given two functors $f_1, f_2 : \mathcal{C}_1 \to \mathcal{C}_2$ we have the notion of a natural transformation (or a natural equivalence) $f_1 \to f_2$.

So the idea is that a stack is a category (with some extra data, satisfying suitable properties). In our favorite example, the stack[31] $\overline{\mathcal{M}}_{g,n}$ is the category whose objects are families of stable curves over a scheme:

$$\mathrm{Ob}(\overline{\mathcal{M}}_{g,n}) : (\pi : C \to S; p_1, \ldots, p_n : S \to C) \text{ family of stable genus } g \text{ curves.} \tag{61}$$

The morphisms in the category are a bit peculiar (we'll see in a few lines why this makes sense): they are given by fibre diagrams (or pullbacks) of families of curves[32].

$$\mathrm{Mor}(\pi' : C' \to S', \pi : C \to S) = \left\{ \begin{array}{c} \begin{tikzcd} C' \arrow[r, "\widehat{f}"] \arrow[d, "\pi'"] & C \arrow[d, "\pi"] \\ S' \arrow[r, "f"] & S \end{tikzcd} \end{array} : (\widehat{f}, f) \text{ make } C'/S' \text{ a pullback of } C/S \right\} \tag{62}$$

Note that there is a functor $F : \overline{\mathcal{M}}_{g,n} \to \mathbf{Sch}_{\mathbb{C}}$ to the category of schemes, sending $\pi : C \to S$ to the scheme S and sending the morphism $(\widehat{f}, f)$ in (62) to the morphism $f : S' \to S$. This functor has some very nice properties:

- the preimage of $S \in \mathbf{Sch}_{\mathbb{C}}$ (i.e. the objects of $\overline{\mathcal{M}}_{g,n}$ mapping to S) are precisely the families (61) of stable genus g curves over S,
- given $\pi : C \to S$ and any morphism $f : S' \to S$ of schemes, there exists $\pi' : C' \to S'$ and a morphism $(\widehat{f}, f)$ mapping to the given morphism f under F,
- given the family $\pi : C \to S$, the set of morphisms $(\widehat{f}, f)$ from this family to itself which map to the morphism $f = \mathrm{id}_S : S \to S$ are precisely the automorphisms of the family $\pi : C \to S$ of stable curves. This requires the small check that a map $\widehat{f} : C \to C$ such that $(\widehat{f}, \mathrm{id}_S)$ makes the diagram in (62) into a fibre diagram is an isomorphism. This is the reason why we chose the morphisms in $\overline{\mathcal{M}}_{g,n}$ to be pullback diagrams.

We see that the *fibre* $F^{-1}(S)$ *of* F *over* S, defined as the category whose objects are the objects of $\overline{\mathcal{M}}_{g,n}$ mapping to S and whose morphisms are the morphisms $(\widehat{f}, f)$ sitting over the identity $f = \mathrm{id}_S$, is a *groupoid*, i.e. a category in which all morphisms are isomorphisms. Together with some more technical assumptions (see [Fan01, Section 3]) this makes $(\overline{\mathcal{M}}_{g,n}, F : \overline{\mathcal{M}}_{g,n} \to \mathbf{Sch}_{\mathbb{C}})$ into a *category fibred in groupoids* over $\mathbf{Sch}_{\mathbb{C}}$.

Note that given a scheme M over $\mathbb{C}$, the category $\mathbf{Sch}_M$ of schemes over M is also a category fibred in groupoids: its map $\mathbf{Sch}_M \to \mathbf{Sch}_{\mathbb{C}}$ sends $X \to M$ to X and a morphism $f : X' \to X$ of schemes over M to the morphism $f : X' \to X$ of schemes over $\mathbb{C}$. Note that the fibre of $\mathbf{Sch}_M$ over a scheme $X \in \mathbf{Sch}_{\mathbb{C}}$ is precisely the category of morphisms $X \to M$. This allows us to draw a diagram as follows:

$$\begin{tikzcd} & \text{categories fibred in groupoids} \arrow[d, "(\mathcal{M},F)\mapsto(S\mapsto F^{-1}(S)/\mathrm{iso})"] \\ \mathbf{Sch}_{\mathbb{C}} \arrow[ur, "M\mapsto\mathbf{Sch}_M"] \arrow[r, "M\mapsto h^M"'] & \text{moduli functors} \end{tikzcd} \tag{63}$$

[31]From now on, the symbol $\overline{\mathcal{M}}_{g,n}$ stands for the category we describe here, no longer the moduli functor from before.

[32]Below we omit the sections $p_1, \ldots, p_n$ from the notation since it becomes too complicated otherwise, but they are always part of the data and need to satisfy similar compatibilities.

Here $F^{-1}(S)/\mathrm{iso}$ is the set[33] of objects in the category $F^{-1}(S)$ up to (iso)morphisms of the category. With what we said before, this immediately makes obvious that the diagram (63) commutes. So we see that, extending the Yoneda embedding from Lemma 2.1, the category of schemes can be embedded in the category[34] of categories fibred in groupoids. A *morphism* from a scheme S to a category $(\mathcal{M}, F)$ fibred in groupoids is then a functor f fitting in the diagram

$$\begin{tikzcd} \mathbf{Sch}_S \arrow[rr, "f"] \arrow[dr] & & \mathcal{M} \arrow[dl, "F"] \\ & \mathbf{Sch}_{\mathbb{C}} & \end{tikzcd}$$

and you can check that a morphism $f : S \to \overline{\mathcal{M}}_{g,n}$ is *exactly equivalent* to specifying a family $\pi : C \to S$ of curves over S. So the fundamental idea of fine moduli spaces (that morphisms to them are equivalent to families for the moduli functor) is already baked into the category theory above. But now, we also have a notion of morphisms between morphisms: given $f, f' : \mathbf{Sch}_S \to \overline{\mathcal{M}}_{g,n}$ corresponding to families $\pi : C \to S$ and $\pi' : C' \to S$, we say that an isomorphism from f to f' is a natural equivalence of functors $f' \to f$ making the diagram above commute. This can be seen to be equivalent to giving an isomorphism $C \to C'$ of families of stable curves over S.

To be useful, we need to make sure that we can "do algebraic geometry with the categories above". It turns out that looking at arbitrary categories fibred in groupoids is too general to do this. Therefore, we only look at particular types of such categories, called *(algebraic) stacks*. These are categories fibred in groupoids that satisfy some additional conditions (which allow us to do reasonable algebraic geometry with such categories). The category $\overline{\mathcal{M}}_{g.n}$ above turns out to be an algebraic stack, and instead of giving the definition in full generality, let me just describe them in the specific example of $\overline{\mathcal{M}}_{g,n}$.

$\overline{\mathcal{M}}_{g,n}$ is a stack: Being a *stack* means that $\overline{\mathcal{M}}_{g,n}$ has a "sheaf-like" property. Assume we are given a scheme S and an (étale) cover $(U_i \to S)_i$ together with families $(\pi_i : C_i \to U_i)_i$ of curves. Assume moreover that we have a family $(\varphi_{ij} : C_i|_{U_i \cap U_j} \to C_j|_{U_i \cap U_j})_{ij}$ of isomorphisms of the families of curves on the overlaps $U_i \cap U_j = U_i \times_S U_j$, which are compatible on overlaps (i.e. they satisfy the cocycle condition $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on triple overlaps). Then these glue to a family $\pi : C \to S$ over S and the family is unique up to unique isomorphism[35], see Figure 26.

Given stacks $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ and morphisms $\mathfrak{X} \to \mathfrak{Y}$ and $\mathfrak{Z} \to \mathfrak{Y}$, one can define the *fibre product* $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z}$. Then we say that a morphism $\varphi : \mathfrak{X} \to \mathfrak{Y}$ is *representable* if for every scheme U and morphism $U \to \mathfrak{Y}$, the fibre product $\mathfrak{X} \times_{\mathfrak{Y}} U$ in the diagram

$$\begin{tikzcd} \mathfrak{X} \times_{\mathfrak{Y}} U \arrow[r, "\varphi_U"] \arrow[d] & U \arrow[d] \\ \mathfrak{X} \arrow[r, "\varphi"] & \mathfrak{Y} \end{tikzcd} \tag{64}$$

is isomorphic to a scheme $\mathfrak{X} \times_{\mathfrak{Y}} U \cong S$. Let P be a property of morphisms of schemes invariant under pullback/base change (e.g. being a smooth morphism). Then we say

[33]If you like general nonsense: one probably has to restrict to small categories fibred in groupoids, let's ignore this here.

[34]There is a natural notion of a functor between two categories $(\mathcal{M}_1, F_1), (\mathcal{M}_2, F_2)$ fibred in groupoids: it is a functor $G : \mathcal{M}_1 \to \mathcal{M}_2$ between the underlying categories such that $F_2 \circ G = F_1$.

[35]In the general definition of a stack $\mathcal{M}$, the families of curves over the U_i are replaced by functors $\mathbf{Sch}_{U_i} \to \mathcal{M}$ and the isomorphisms φ_{ij} correspond to natural equivalences between the restrictions of these morphisms to $\mathbf{Sch}_{U_i \cap U_j}$.

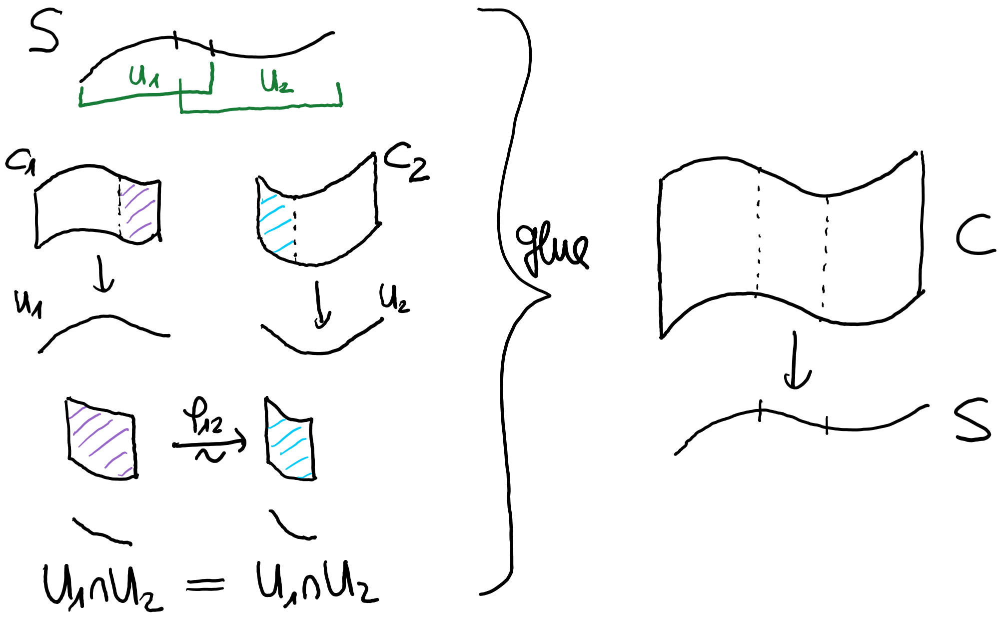


Figure 26: Gluing two families of curves on an open cover $S = U_1 \cup U_2$ along an isomorphism of the restrictions of the families on the overlap $U_1 \cap U_2$

that a representable morphism $\varphi : \mathfrak{X} \to \mathfrak{Y}$ of stacks has property P if for all $U \to \mathfrak{Y}$ the morphism φ_U (of schemes) in (64) has property P.

If you know something about manifolds, you should see the morphism $U \to \mathfrak{Y}$ as a type of chart for $\mathfrak{Y}$ by objects we understand well (we cover a stack using schemes, and a manifold using open subsets of $\mathbb{R}^n$). Then the above says that we can check properties of morphisms between stacks by checking them on all charts (and representable morphisms have the magic property that the preimage of a chart of $\mathfrak{Y}$ is a chart of $\mathfrak{X}$).

$\overline{\mathcal{M}}_{g,n}$ is an algebraic stack: Being an *algebraic stack* means that $\mathfrak{X}$ has a particularly nice chart: there exists a scheme U and a representable, smooth and surjective morphism $U \to \mathfrak{X}$ (smooth and surjective are both properties invariant under pullback, so this statement makes sense). To check this for $\overline{\mathcal{M}}_{g,n}$, note that a morphism $U \to \overline{\mathcal{M}}_{g,n}$ is given by a family $\pi : C \to U$ of stable curves over the scheme U. The fact that this is surjective means that every stable curve appears as a fibre in the morphism π, while smoothness is slightly harder to interpret[36]. Proving the existence of such a $U \to \overline{\mathcal{M}}_{g,n}$ requires some work (see [DM69, Section 5]), but we have seen an example of this at the very start of the course: the family E_t of cubic curves parametrized by $t \in \mathbb{C}$ from (3) gives rise to a family of 1-pointed stable curves of genus 1

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, "\pi"', shift right] & \{([X:Y:Z],t) \in \mathbb{P}^2 \times \mathbb{A}^1 : Y^2Z + X(X-Z)(X-tZ) = 0\} \\ \mathbb{A}^1 \arrow[u, bend right, "p_1"'] & \end{tikzcd} \tag{65}$$

where the section p_1 is given by $p_1(t) = ([0:1:0],t)$. The corresponding map $\mathbb{A}^1 \to \overline{\mathcal{M}}_{1,1}$

[36]If you know the definition of formal smoothness: the property of $U \to \overline{\mathcal{M}}_{g,n}$ being smooth requires that for a ring R and an R-point $R \to U$ with fibre C_R under π and a deformation $C_{R'}$ of C_R over a square-zero extension $R \subset R'$ of R, we can find a morphism $R' \to U$ such that the composition $R' \to U \to \overline{\mathcal{M}}_{g,n}$ is induced by the family $C_{R'}$ of curves.

is a representable, smooth, surjective morphism.

Given an algebraic stack $\mathfrak{X}$ with a representable, smooth, surjective cover $U \to \mathfrak{X}$, we say that $\mathfrak{X}$ is smooth if we can choose the scheme U to be smooth (over $\mathbb{C}$). More generally, given any property Q of schemes that can be checked on a smooth cover, we say that $\mathfrak{X}$ has Q if and only if U has Q.

For $\overline{\mathcal{M}}_{g,n}$ we can indeed choose U to be smooth, so $\overline{\mathcal{M}}_{g,n}$ is a smooth stack. In fact, the cover $U \to \overline{\mathcal{M}}_{g,n}$ can be chosen to be étale (not just smooth). An algebraic stack with this stronger property is called a *Deligne-Mumford stack*. Such stacks are "really close to being a scheme" and for the most part this will allow us to pretend that $\overline{\mathcal{M}}_{g,n}$ behaves just like a scheme which is a fine moduli space for the functor of families of stable curves. One can show that for a Deligne-Mumford stack $\mathcal{M}$ and a point $x : \mathrm{Spec}(\mathbb{C}) \to \mathcal{M}$, the *stabilizer group* Stab_x of x (defined as the set of isomorphisms $x \to x$ from the morphism to itself) is finite. Under mild conditions ($\mathcal{M}$ separated and in characteristic 0) it is conversely true that an algebraic stack $\mathcal{M}$ with finite stabilizer groups at geometric points is Deligne-Mumford. This exactly brings us back to the definition of $\overline{\mathcal{M}}_{g,n}$, since we asked that the automorphism group $\mathrm{Aut}(C, p_1, \ldots, p_n)$ of a stable curve is finite. It was the insight of Deligne and Mumford that this condition exactly ensures that the resulting stack $\overline{\mathcal{M}}_{g,n}$ has nice properties.

Finally, there exists a morphism $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ from the stack $\overline{\mathcal{M}}_{g,n}$ to the coarse moduli space $\overline{M}_{g,n}$ we talked about earlier. This morphism is proper, induces a bijection on geometric points and has the property that any other morphism $\overline{\mathcal{M}}_{g,n} \to M$ to a scheme M must factor through $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ (this is the augmented version of the notion of a coarse moduli space for a moduli functor).

Where to learn more about stacks

Here is a list of resources, ordered in increasing comprehensiveness, which you can use to learn more about stacks:

- the paper "Stacks for Everybody" [Fan01] by Barbara Fantechi (11 pages, a few hours to work through, highly recommended),
- the course on the topic given by Prof. Georg Oberdieck in the Winter semester 2020 (one semester, also highly recommended),
- the book "Algebraic Stacks" (in preparation, by Behrend, Conrad, Edidin, Fantechi, Fulton, Göttsche und Kresch), found on the website of an old course by Andrew Kresch (220 pages, a few months, a great resource for self-study),
- the Stacks project [Sta13] (about 7000 pages, several years of intense study, great to look up results and particular topics, highly non-recommended to read from beginning to end).

5.2 Upgrades of previous results

By using the language of stacks, many results about the moduli spaces of curves that we saw before have a better version (i.e. nicer properties) when talking about the moduli stacks. One caveat: of course, the brief and informal introduction to stacks given in Section 5.1 is not enough to give a precise meaning to all the properties listed below. I still hope you get an idea of their meaning, but you can take those as black boxes for now (we will make them more precise as we need them).

Theorem 5.1. Let $g, n \geq 0$ with $2g - 2 + n > 0$.

a) The categories fibred in groupoids $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ are (algebraic) Deligne-Mumford stacks.

b) They are irreducible, proper and smooth of dimension $3g-3+n$ and there is a natural inclusion $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ as a nonempty, open substack.

c) The boundary $\partial\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is an effective Cartier divisor and even a *normal crossings divisor*[37].

d) The forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ makes $\overline{\mathcal{M}}_{g,n+1}$ the universal curve over $\overline{\mathcal{M}}_{g,n}$. In particular, this morphism is representable, proper, flat and of relative dimension 1.

e) For a stable graph Γ of genus g with n legs, the gluing morphism

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g,n}$$

is representable, finite and a local complete intersection[38]. It has generic degree $\#\mathrm{Aut}(\Gamma)$ onto its image $\overline{\mathcal{M}}^\Gamma$.

Note that e.g. for the forgetful morphism, we originally constructed it by first giving a natural transformation of the corresponding moduli functors (e.g. how to take am $(n+1)$-pointed family of curves and construct an n-pointed family from this). You can check that the *same* construction defines a functor $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ between the corresponding categories. The same discussion applies to the gluing morphisms ξ_Γ.

References and further reading

The origin of the notion of a (Deligne-Mumford) stack is the original paper [DM69] by Deligne and Mumford and Section 4 of this paper gives an introduction to this notion.

[37] Essentially, this means that étale locally the boundary looks like a union of some coordinate hyperplanes in $\mathbb{C}^N$, but see [Sta13, Tag 0CBN] for a formal definition.

[38] This means it can be factored into a regular embedding followed by a smooth morphism.

6 Intersection theory on the moduli of stable curves

As the genus g and the number n of markings increase, the spaces $\overline{M}_{g,n}$ quickly become extremely complicated geometric objects. One way to study such complicated spaces is by computing some *topological invariants* and in particular to study their *singular cohomology groups*. By this last part we mean that you consider the set $\overline{M}_{g,n}(\mathbb{C})$ of complex points of the moduli space, with its complex topology, and study the cohomology groups

$$H^*(\overline{M}_{g,n}) = H^*(\overline{M}_{g,n}(\mathbb{C}), \mathbb{Q}). \tag{66}$$

In Section 6.2 we will see that given a closed algebraic subset $S \subset \overline{M}_{g,n}$ of complex codimension c, we can associate to S a cohomology class $[S] \in H^{2c}(\overline{M}_{g,n})$. For a second algebraic set S' meeting S transversally[39], we then have that the class $[S \cap S']$ associated to the intersection of S, S' is equal to the cup product $[S] \smile [S']$ of their classes $[S], [S']$. We will introduce these notions carefully in Section 6.2, but this is the origin of the word "intersection theory". We will also see a few places where it is more convenient to work with the smooth stacks $\overline{\mathcal{M}}_{g,n}$ instead of the moduli spaces $\overline{M}_{g,n}$, but we'll stick with $\overline{M}_{g,n}$ for the most part and only use the stacks when we need them.

One class of examples of closed subsets $S \subset \overline{M}_{g,n}$ are the closures $S = \overline{M}^{\Gamma}$ of the strata of $\overline{M}_{g,n}$ associated to a given stable graph Γ. So we begin by studying these sets and their intersections in more detail.

6.1 Intersections of strata

In Section 4.2 we saw that the moduli spaces $\overline{M}_{g,n}$ admit a stratification by locally closed subsets

$$\overline{M} = \bigcup_{\Gamma} M^{\Gamma}$$

according to stable graphs Γ. In Proposition 4.15 we showed that the closures $\overline{M}^{\Gamma}$ are parametrized by the gluing maps

$$\xi_\Gamma : \overline{M}_\Gamma = \prod_{v \in V(\Gamma)} \overline{M}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g,n}.$$

In this section we want to answer two natural questions concerning the sets $\overline{M}^{\Gamma}$:

a) Given a stable graph Γ, what are the stable graphs Γ' of curves in the closure $\overline{M}^{\Gamma}$?

b) Given two stable graphs Γ_1, Γ_2, what is the intersection $\overline{M}^{\Gamma_1} \cap \overline{M}^{\Gamma_2} \subset \overline{M}_{g,n}$?

As we saw above, question b) in particular will be related to the computation of the intersection product $[\overline{M}^{\Gamma_1}] \smile [\overline{M}^{\Gamma_2}]$. A good reference for these questions is [GP03, Appendix A], where the answers were worked out in a formal way for the first time.

For question a) we can use that $\overline{M}^{\Gamma} = \xi_\Gamma(\overline{M}_\Gamma)$, so we need to understand how the stable graph of the curve

$$\xi_\Gamma((C_v, p_{1,v}, \ldots, p_{n(v),v})_{v \in V(\Gamma)}) \tag{67}$$

depends on Γ and the stable graphs Γ_v of the curves $(C_v, p_{1,v}, \ldots, p_{n(v),v})$. In Figure 27 you see an example of this.

[39]If S, S' are smooth, this means that at any point $p \in S \cap S'$ we have $T_p\overline{\mathcal{M}}_{g,n} = T_pS + T_pS'$.

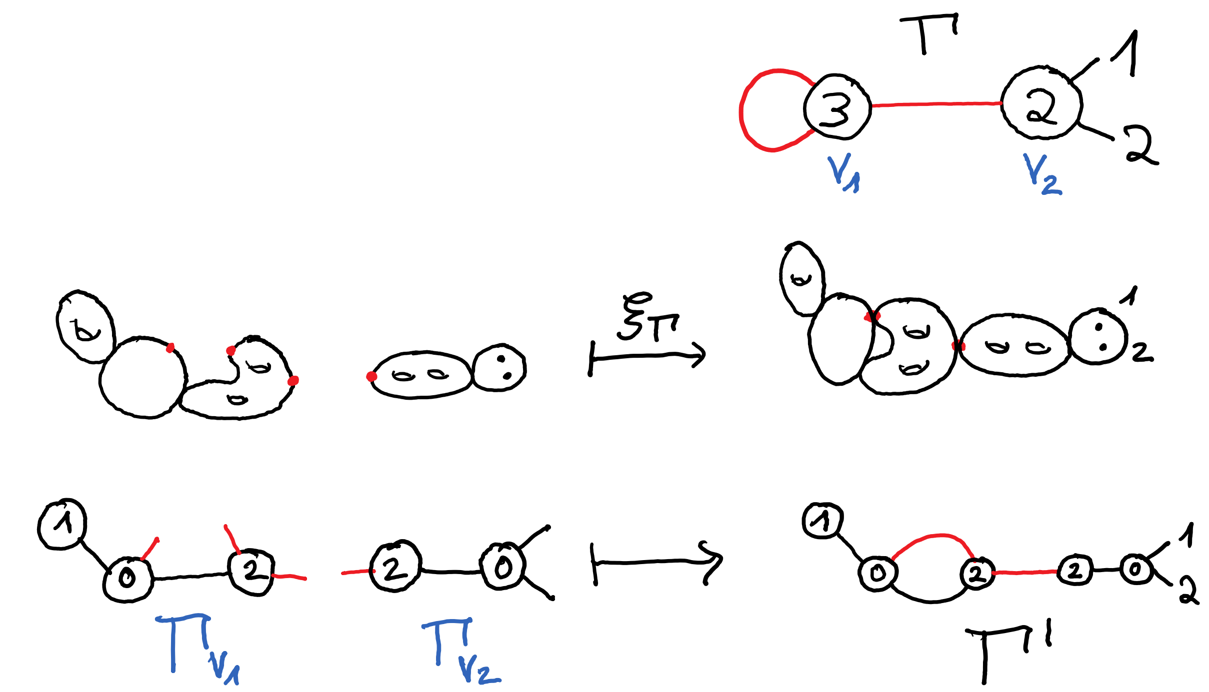


Figure 27: The stable graph Γ' of a curve obtained by gluing curves with stable graphs $\Gamma_{v_1}, \Gamma_{v_2}$ via ξ_Γ is obtained by inserting the graphs $\Gamma_{v_1}, \Gamma_{v_2}$ at the vertices of Γ

Intuitively, it is quite clear what happens: you start with the dual graph Γ and *glue the graphs* Γ_v *into the vertices* v *of* Γ. Making this precise in the formal language of stable graphs is a bit of a headache, which is why I give it as an exercise[40].

Exercise 6.1. Let Γ be a stable graph and for $v \in V(\Gamma)$ let Γ_v be a stable graph of genus $g(v)$ with $n(v)$ legs together with an identification $h : L(\Gamma_v) \xrightarrow{\sim} H(v)$ of the legs of Γ_v with the half-edges $H(v)$ of v in Γ. Define the graph Γ' obtained by gluing the Γ_v into the vertices of Γ and show that it is a stable graph. Convince yourself that for curves $(C_v, p_{1,v}, \ldots, p_{n(v),v})_{v \in V(\Gamma)}$ with stable graphs Γ_v, the dual graph of the curve (67) is equal to Γ'.

Instead of describing the gluing of the graphs Γ_v into Γ explicitly, we define the notion of a morphism $\Gamma' \to \Gamma$ of stable graphs which makes precise the notion that Γ' can be obtained from Γ by gluing *some* graphs Γ_v at the vertices v of Γ.

Definition 6.2. Let Γ, Γ' be stable graphs of genus g with n legs. A *morphism* $\varphi : \Gamma' \to \Gamma$ is defined by two maps[41]

$$\varphi_V : V(\Gamma') \to V(\Gamma), \varphi_H : H(\Gamma) \to H(\Gamma'), \tag{68}$$

satisfying the following conditions:

a) the map φ_H is injective,

---

[40] Bwahaha.

[41] Notice the direction of the maps: φ_V goes from vertices of Γ' to vertices of Γ while φ_H goes from half-edges of Γ to half-edges of Γ'!

b) φ_H sends edges of Γ to edges of Γ'

$$\{h, h'\} \in E(\Gamma) \implies \{\varphi_H(h), \varphi_H(h')\} \in E(\Gamma').$$

Below we will denote by $\varphi_E : E(\Gamma) \to E(\Gamma')$ the corresponding (injective) map of edges.

c) φ_H sends the legs of Γ to the corresponding legs of Γ'

$$\ell_{\Gamma'}(\varphi_H(h)) = \ell_\Gamma(h) \text{ for } h \in L(\Gamma),$$

d) the map φ_V is surjective and compatible with φ_H

$$\varphi_V(v_{\Gamma'}(\varphi_H(h))) = v_\Gamma(h) \text{ for } h \in H(\Gamma),$$

e) given $v_0 \in V(\Gamma)$, the preimage of v_0 under φ_V is a stable graph Γ'_{v_0} of genus $g(v_0)$ with $n(v_0)$ legs. More precisely, the vertices $V_{v_0} = \varphi_V^{-1}(v_0)$ mapping to v_0 under φ_V together with the half-edges $H_{v_0} = v_{\Gamma'}^{-1}(V_{v_0})$ incident to these vertices and all edges $\{h, h'\} \in E(\Gamma') \setminus \varphi_E(E(\Gamma))$ with $h, h' \in H_{v_0}$ form a stable graph Γ_{v_0} and this graph has genus $g(v_0)$ and a number $n(v_0)$ of legs.

We illustrate a morphism $\Gamma' \to \Gamma$ of stable graphs in Figure 28.

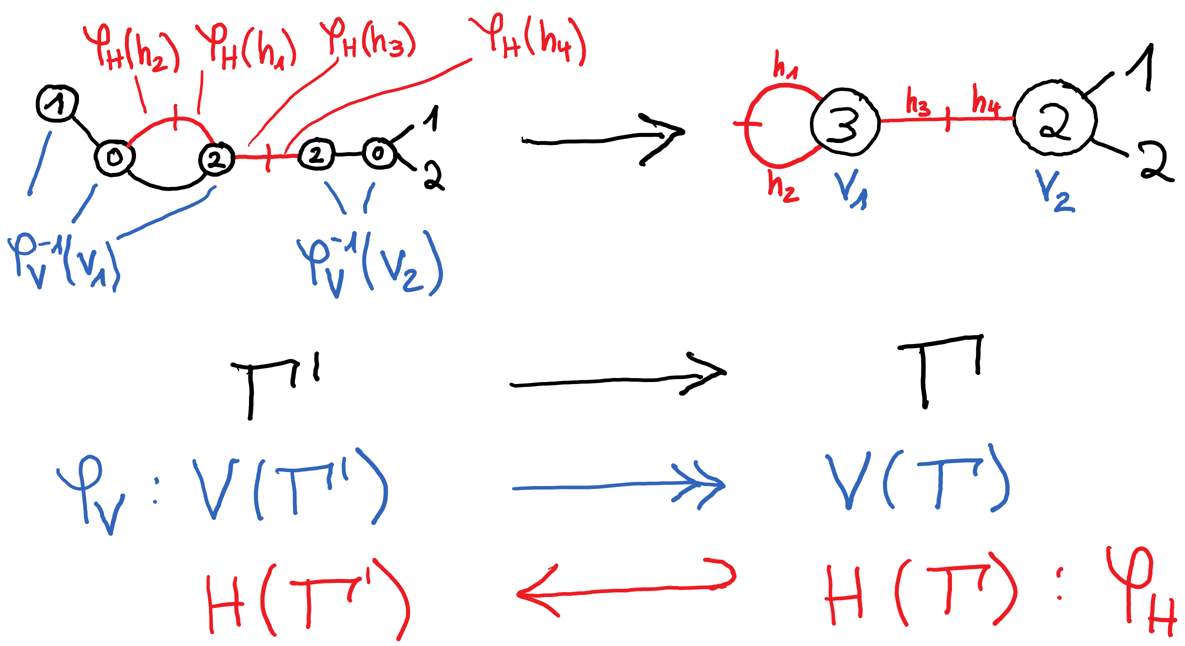


Figure 28: The data of a morphism $\Gamma' \to \Gamma$ illustrated; this is the morphism coming from the gluing in Figure 27

Remark 6.3. a) The existence of a morphism $\Gamma' \to \Gamma$ is precisely equivalent to saying that Γ' can be obtained from Γ by gluing some stable graphs at the vertices of Γ (and these are the stable graphs Γ'_{v_0} appearing in part e) of Definition 6.2).

b) Given a morphism $\varphi : \Gamma' \to \Gamma$, there exists a natural gluing morphism

$$\xi_\varphi : \overline{\mathcal{M}}_{\Gamma'} \to \overline{\mathcal{M}}_\Gamma,$$

where for each $v \in V(\Gamma)$ the component of ξ_φ to the factor $\overline{\mathcal{M}}_{g(v),n(v)}$ of $\overline{\mathcal{M}}_\Gamma$ is given by the gluing map $\xi_{\Gamma'_v}$ (defined on the factors of $\overline{\mathcal{M}}_{\Gamma'}$ associated to vertices of Γ' in $\varphi_V^{-1}(v)$). See Figure 30 for two examples of such gluing morphisms ξ_φ.

c) In the literature, a morphism $\Gamma' \to \Gamma$ is sometimes called a Γ-*structure on* Γ'. Other names you might find are that Γ' *is a specialization of* Γ or that Γ *is a contraction of* Γ'.

d) You can check that there is a category whose objects are stable graphs Γ of genus g with n legs and whose morphisms $\Gamma' \to \Gamma$ are as described in Definition 6.2. In particular there is a natural way to *compose* morphisms φ, φ' of stable graphs, by setting

$$(\varphi \circ \varphi')_V = \varphi_V \circ \varphi'_V, \ (\varphi \circ \varphi')_H = \varphi'_H \circ \varphi_H.$$

You can also check that the notion of an isomorphism of stable graphs defined in Definition 4.5 is equivalent to the notion of an isomorphisms of this category[42]. This makes precise the hint given in Exercise 4.18 *c).

Exercise 6.4. a) Given a stable graph Γ' and a set $E_0 \subset E(\Gamma)$ of edges of Γ, show that there exists a stable graph Γ and a morphism $\varphi : \Gamma' \to \Gamma$ with $\varphi_E(E(\Gamma)) = E_0$. Show that for a second graph $\widetilde{\Gamma}$ and morphism $\widetilde{\varphi} : \Gamma' \to \widetilde{\Gamma}$ with $\widetilde{\varphi}_E(E(\widetilde{\Gamma})) = E_0$, there exists a unique isomorphism $\Gamma \to \widetilde{\Gamma}$ fitting into the diagram

$$\begin{tikzcd} & \Gamma \arrow[dd, "\sim"] \\ \Gamma' \arrow[ur, "\varphi"] \arrow[dr, "\widetilde{\varphi}"'] & \\ & \widetilde{\Gamma} \end{tikzcd}$$

In other words, the map $\Gamma' \to \Gamma$ is unique up to isomorphism. It is called the *contraction of the edges in* $E(\Gamma') \setminus E_0$.

b) Show that every morphism $\Gamma' \to \Gamma$ can be factored as a composition

$$\Gamma' = \Gamma_0 \xrightarrow{\varphi_1} \Gamma_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_d} \Gamma_d = \Gamma$$

of morphisms $\varphi_1, \ldots, \varphi_d$ such that each φ_i contracts a single edge of Γ_{i-1}.

Proposition 6.5. Let Γ be a stable graph in genus g with n legs. Then a curve $(C, p_1, \ldots, p_n) \in \overline{M}_{g,n}$ with stable graph Γ' lies in the closed set $\overline{M}^\Gamma$ if and only if there exists a morphism $\Gamma' \to \Gamma$. In particular, we have

$$\overline{M}^\Gamma = \bigcup_{\Gamma' \to \Gamma} M^{\Gamma'}. \tag{69}$$

[42]Unfortunately, the map φ_H now goes in the opposite direction compared to the convention of Definition 4.5, sorry about that.

Proof. By Proposition 4.15, the set $\overline{M}^{\Gamma}$ is the image of the gluing morphism ξ_Γ. By Exercise 6.1 the stable graphs Γ' of curves in the image of ξ_Γ are precisely those obtained by gluing stable graphs Γ_v into the vertices of Γ and by Remark 6.3 a) this is equivalent to the existence of a morphism $\Gamma' \to \Gamma$. □

Since the closures $\overline{M}^{\Gamma}$ are union of strata $M^{\Gamma'}$, it is now easy to answer question b) above.

Corollary 6.6. Let Γ_1, Γ_2 be two stable graphs of genus g with n legs. Then we have

$$\overline{M}^{\Gamma_1} \cap \overline{M}^{\Gamma_2} = \bigcup_{\substack{\Gamma' \\ \exists\ \Gamma' \to \Gamma_1, \Gamma' \to \Gamma_2}} M^{\Gamma'}, \tag{70}$$

where the union goes over stable graphs Γ' admitting a morphism to Γ_1 *and* Γ_2.

Example 6.7. For $g = 1, n = 2$ we show in Figure 29 all isomorphism classes of stable graphs and which morphisms exist between them. Note the cases where there are two or four morphisms between these graphs (can you write them all down?). You can compare this to the picture of $\overline{M}_{1,2}$ from Figure 24 and check that these morphisms precisely tell you how the closures of the strata intersect. As an example of the statement of Corollary 6.6, we see that

$$\overline{M}^{\Gamma_1} \cap \overline{M}^{\Gamma_2} = M^{\Gamma_3}.$$

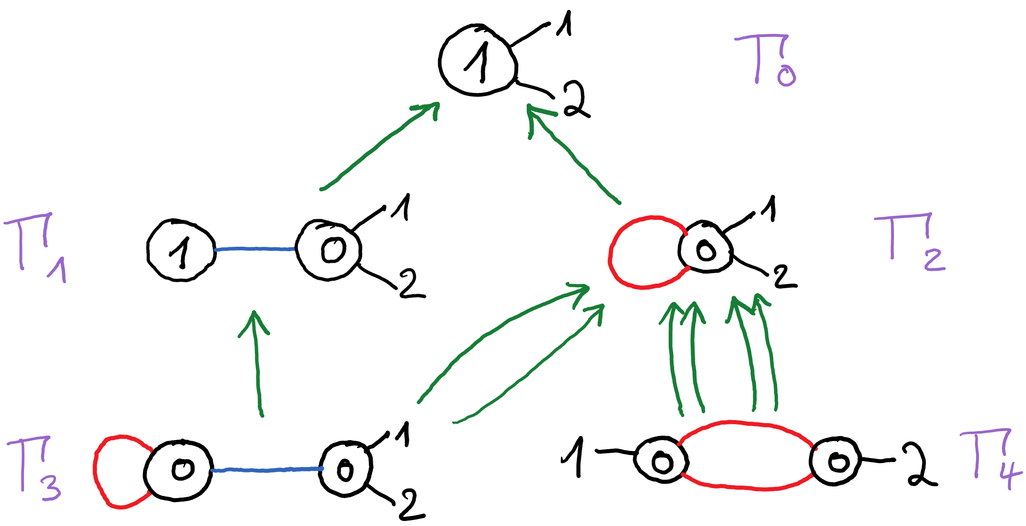


Figure 29: The stable graphs $\Gamma_0, \ldots, \Gamma_4$ in genus $g = 1$ with $n = 2$ legs and morphisms between them. There are four morphisms $\Gamma_4 \to \Gamma_2$ uniquely determined by the image $\phi_H(h)$ of one of the half-edges h of Γ_2, which can map to each of the four half-edges of Γ_4. Note that we did not draw automorphisms of the graphs.

We answered questions a) and b) above to our satisfaction, but in the following sections we will see that it's useful to answer a refined version of question b).

b') Given two stable graphs Γ_1, Γ_2, what is the fibre product

$$\begin{tikzcd} \mathcal{F}_{\Gamma_1,\Gamma_2} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{\Gamma_2} \arrow[d, "\xi_{\Gamma_2}"] \\ \overline{\mathcal{M}}_{\Gamma_1} \arrow[r, "\xi_{\Gamma_1}"'] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

Notice how I sneakily formulated question b') for the moduli stacks instead of the moduli spaces? There is a good reason for this, which we will see below. To answer the question, we will need to consider in more detail the stable graphs Γ' and maps $\Gamma' \to \Gamma_1, \Gamma_2$ above.

Definition 6.8. Given stable graphs $\Gamma_1, \Gamma_2, \Gamma$, a (Γ_1, Γ_2)-*structure on* Γ is a tuple

$$(\Gamma, \varphi_1, \varphi_2) = (\varphi_1 : \Gamma \to \Gamma_1, \varphi_2 : \Gamma \to \Gamma_2) \tag{71}$$

of morphisms from Γ to Γ_1 and Γ_2. The (Γ_1, Γ_2)-structure is called *generic* if

$$E(\Gamma) = \varphi_{1,E}(E(\Gamma_1)) \cup \varphi_{2,E}(E(\Gamma_2)), \tag{72}$$

i.e. every edge $e \in E(\Gamma)$ is the image of an edge in Γ_1 or Γ_2 under the morphisms φ_1, φ_2.

Given a second stable graph Γ' with a (Γ_1, Γ_2)-structure $(\Gamma', \phi'_1, \phi'_2)$ we say that this structure is isomorphic to (71) if there exists an isomorphism $\Gamma \to \Gamma'$ fitting into the diagram

$$\begin{tikzcd} & \Gamma_1 & \\ \Gamma \arrow[ur, "\varphi_1"] \arrow[rr, "\sim"] \arrow[dr, "\varphi_2"] & & \Gamma' \arrow[ul, "\phi'_1"'] \arrow[dl, "\phi'_2"'] \\ & \Gamma_2 & \end{tikzcd}$$

Denote by $\mathfrak{G}_{\Gamma_1,\Gamma_2}$ the set of generic (Γ_1, Γ_2)-structures $(\Gamma, \varphi_1, \varphi_2)$ up to isomorphism.

Example 6.9. In Figure 29, let $\varphi_{3\to1} : \Gamma_3 \to \Gamma_1$ and $\varphi_{3\to2} : \Gamma_3 \to \Gamma_1$ be morphisms as indicated (we have two choices for $\varphi_{3\to2}$). Then

$$(\Gamma_3, \varphi_{3\to1}, \varphi_{3\to2})$$

is a generic (Γ_1, Γ_2)-structure on Γ_3.

Now we can answer question b').

Theorem 6.10. Given stable graphs Γ_1, Γ_2 of genus g with n legs, the fibre product

$$\begin{tikzcd} \mathcal{F}_{\Gamma_1,\Gamma_2} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{\Gamma_2} \arrow[d, "\xi_{\Gamma_2}"] \\ \overline{\mathcal{M}}_{\Gamma_1} \arrow[r, "\xi_{\Gamma_1}"'] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \tag{73}$$

of the gluing morphisms $\xi_{\Gamma_1}, \xi_{\Gamma_2}$ is given by

$$\mathcal{F}_{\Gamma_1,\Gamma_2} = \coprod_{(\Gamma,\varphi_1,\varphi_2)\in\mathfrak{G}_{\Gamma_1,\Gamma_2}} \overline{\mathcal{M}}_\Gamma. \tag{74}$$

The restriction of the diagram (73) to the connected component $\overline{\mathcal{M}}_\Gamma$ of $\mathcal{F}_{\Gamma_1,\Gamma_2}$ associated to $(\Gamma, \varphi_1, \varphi_2) \in \mathfrak{G}_{\Gamma_1,\Gamma_2}$ is given by

$$\begin{array}{ccc} \overline{\mathcal{M}}_\Gamma & \xrightarrow{\xi_{\varphi_2}} & \overline{\mathcal{M}}_{\Gamma_2} \\ {\scriptstyle \xi_{\varphi_1}}\downarrow & & \downarrow{\scriptstyle \xi_{\Gamma_1}} \\ \overline{\mathcal{M}}_{\Gamma_1} & \xrightarrow[\xi_{\Gamma_1}]{} & \overline{\mathcal{M}}_{g,n} \end{array} \tag{75}$$

Proof. We are going to explain how the proof works on the level of $\mathbb{C}$-points. For the more general treatment of families of curves (which you need to define the isomorphism (74)), see e.g. [Sv18, Proposition 2.14].

What is a point in the fibre product (73) above? It is the data of points

$$((C'_v, (q'_h)_{h \in H(v)}))_{v \in V(\Gamma_1)} \in \overline{\mathcal{M}}_{\Gamma_1} \text{ and } ((C''_v, (q''_h)_{h \in H(v)}))_{v \in V(\Gamma_2)} \in \overline{\mathcal{M}}_{\Gamma_2} \tag{76}$$

together with an isomorphism

$$\xi_{\Gamma_1}((C'_V, (q'_h)_h)_v) \xrightarrow{\sim} \xi_{\Gamma_2}((C''_V, (q''_h)_h)_v) = (C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}. \tag{77}$$

Note that here we use that we take the fibre diagram of stacks! If we had written everything with the coarse moduli spaces $\overline{M}_{\Gamma_1}, \overline{M}_{\Gamma_2}$ and $\overline{M}_{g,n}$, the data of a point in the fibre product would be given by two points (76) such that there exists *some* isomorphism (77). For the stacky fibre product, the isomorphism (77) is part of the data! For more on the slightly subtle definition of stacky fibre products, see [Fan01, Section 6.1].

We illustrate the data that we have so far on the left side of Figure 30 (we know so far the collections of curves on the bottom left and top right and how to identify their images under the gluing maps ξ_{Γ_1} and ξ_{Γ_2}).

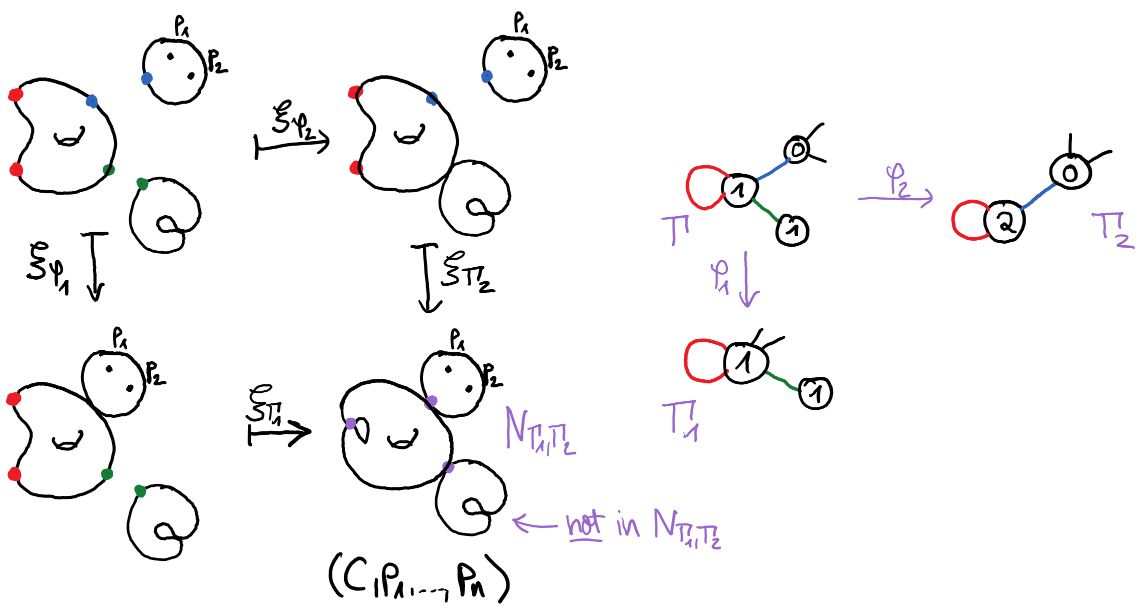


Figure 30: An illustration how starting from curves in $\overline{\mathcal{M}}_{\Gamma_1}$ and $\overline{\mathcal{M}}_{\Gamma_2}$ and an identification of their images under $\xi_{\Gamma_1}, \xi_{\Gamma_2}$ with $(C, p_1, \ldots, p_n)$, we construct a graph Γ (on the right) and an element of $\overline{\mathcal{M}}_\Gamma$ (top left)

Now observe that there is a particular subset N_{Γ_1,Γ_2} of the nodes of the curve C which is the set of those nodes $n_{\{h,h'\}} \in C$ created either by identifying two markings $q'_h, q'_{h'}$ in the map ξ_{Γ_1} (for $\{h,h'\} \in E(\Gamma_1)$) or from markings $q''_h, q''_{h'}$ under ξ_{Γ_2} (for $\{h,h'\} \in E(\Gamma_2)$). These are the purple nodes in Figure 30 (which are the images of the red, green and blue nodes).

We claim that there is a unique stable graph Γ whose edges correspond to the nodes in N_{Γ_1,Γ_2}. It is the graph obtained from the dual graph of C by contracting all edges *not* corresponding to nodes of N_{Γ_1,Γ_2} (according to Exercise 6.4). The vertices v of Γ correspond to the connected components C_v of the partial normalization of C at the nodes in N_{Γ_1,Γ_2} (you see this partial normalization at the top left of Figure 30). The genus $g(v)$ is the arithmetic genus of the nodal curve C_v. The graph Γ has natural morphisms $\varphi_1 : \Gamma \to \Gamma_1$, $\varphi_2 : \Gamma \to \Gamma_2$. For instance, on the level of edges the morphism $\varphi_{1,E} : E(\Gamma_1) \to E(\Gamma)$ sends $\{h,h'\} \in E(\Gamma_1)$ to the edge of Γ corresponding to the node $n_{\{h,h'\}}$ as above. Clearly, the (Γ_1,Γ_2)-structure $(\Gamma,\varphi_1,\varphi_2)$ is generic: the edges of Γ were defined to correspond to nodes of the form $n_{\{h,h'\}}$, so each is either in the image of $\varphi_{1,E}$ or $\varphi_{2,E}$. Moreover, the (Γ_1,Γ_2)-structure $(\Gamma,\varphi_1,\varphi_2)$ we constructed is unique up to isomorphism.

Finally, the curves C_v (for $v \in V(\Gamma)$) together with all marked preimages $\widetilde{q}_{\widetilde{h}}$ of nodes in N_{Γ_1,Γ_2} and markings $p_1,\ldots,p_n$ give an element

$$((C_v, (\widetilde{q}_{\widetilde{h}})_{\widetilde{h}\in H(\Gamma)}))_{v\in V(\Gamma)} \in \overline{\mathcal{M}}_\Gamma. \tag{78}$$

To summarize, what we described above is how to start with the data (76, 77) of a point in $\mathcal{F}_{\Gamma_1,\Gamma_2}$ and use it to construct $(\Gamma,\varphi_1,\varphi_2) \in \mathfrak{G}_{\Gamma_1,\Gamma_2}$ and the point (78) of $\overline{\mathcal{M}}_\Gamma$. This recipe defines you a map from the left-hand side of (74) to the right. On the other hand, the maps $\xi_{\varphi_1}, \xi_{\varphi_2}$ together with the universal property of the fibre diagram $\mathcal{F}_{\Gamma_1,\Gamma_2}$ define you a morphism $\overline{\mathcal{M}}_\Gamma \to \mathcal{F}_{\Gamma_1,\Gamma_2}$. In this way, you define a map from the right to the left side of (74). Using a finite amount of work (which you find in the proof of [Sv18, Proposition 2.14]) you can check that these maps are inverse to each other, finishing the proof. □

6.2 A crash course in intersection theory of complex algebraic varieties

In this section we give an overview of the intersection theory (formulated in the language of singular (co)homology) for algebraic varieties X over the complex numbers. For simplicity, we will formulate things in the setting where X is a smooth, proper variety.

In the end we want to apply this to $X = \overline{\mathcal{M}}_{g,n}$, which is not a variety. Now it is possible to define singular cohomology groups for stacks (see these lecture notes by Behrend) and then one finds that e.g. for $X = \overline{\mathcal{M}}_{g,n}$ the map $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ induces (via pushforward, see below) an isomorphism of cohomology groups $H^*(\overline{\mathcal{M}}_{g,n}) \xrightarrow{\sim} H^*(\overline{M}_{g,n})$. So in the end it does not matter where you do your computations, but many of them are nicer on the smooth stack $\overline{\mathcal{M}}_{g,n}$. However, instead of learning about cohomology groups of stacks properly, we will essentially pretend that they work just like those for schemes, with a few minor adaptions that we will point out. I realize this is not optimal, but it does work surprisingly well and in the references section I'll point out where you can learn how to fill in the missing pieces.

Singular homology and cohomology

Let X be a connected, smooth, proper variety over the complex numbers of complex dimension d. We define its *singular homology*[43] and *singular cohomology groups* as

$$H_*(X) = H_*(X(\mathbb{C}), \mathbb{Q}) \text{ and } H^*(X) = H^*(X(\mathbb{C}), \mathbb{Q}),$$

where the set $X(\mathbb{C})$ of $\mathbb{C}$-points of X is equipped with the complex topology. The fact that X is smooth means that $X(\mathbb{C})$ actually has the structure of a smooth manifold, of dimension $\dim_{\mathbb{R}} X(\mathbb{C}) = 2d$. In particular, we have

$$H_k(X) = 0, \; H^k(X) = 0 \text{ for } k < 0 \text{ or } k > 2d.$$

The fact that X is defined over the complex numbers implies that $X(\mathbb{C})$ has a natural orientation.

Cap product and cup product

Given a homology class $\sigma \in H_k(X)$ and a cohomology class $\alpha \in H^\ell(X)$, we can form their *cap product* $\sigma \frown \alpha \in H_{k-\ell}(X)$ and often say that α acts on σ via this cap product. This gives rise to a perfect pairing

$$H_k(X) \otimes H^k(X) \to H_0(X) \cong \mathbb{Q}, \sigma \otimes \alpha \mapsto \sigma \frown \alpha, \tag{79}$$

allowing us to identify $H^k(X) \cong H_k(X)^\vee$ in a natural way. Here the isomorphism $H_0(X) \cong \mathbb{Q}$ is given by the *degree map*

$$\deg : H_0(X) \to \mathbb{Q}, \; \sum_{i=1}^N a_i[P_i] \mapsto \sum_{i=1}^N a_i, \tag{80}$$

where we use the connectedness of X to conclude that all points $P_i \in X(\mathbb{C})$ are homologous.

On the other hand, the group $H^*(X)$ has a natural ring structure with multiplication given by the cup product $\smile$. This product respects the cohomological grading, i.e. we have

$$H^k(X) \otimes H^\ell(X) \to H^{k+\ell}(X), \; \alpha \otimes \beta \mapsto \alpha \smile \beta. \tag{81}$$

The cap and cup-product satisfy the compatibility

$$\sigma \frown (\alpha \smile \beta) = (\sigma \frown \alpha) \frown \beta.$$

Remark 6.11. For Deligne-Mumford stacks such as $X = \overline{\mathcal{M}}_{g,n}$, the degree map (80) needs to be slightly adapted. It turns out that for a point $P = (C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$, the class $[P]$ of P should have degree

$$\deg([P]) = \frac{1}{\#\mathrm{Aut}(P)} = \frac{1}{\#\mathrm{Aut}(C, p_1, \ldots, p_n)},$$

and thus the degree map (80) becomes

$$\deg : H_0(X) \to \mathbb{Q}, \; \sum_{i=1}^N a_i[P_i] \mapsto \sum_{i=1}^N \frac{a_i}{\#\mathrm{Aut}(P_i)}. \tag{82}$$

Example 6.12. For $n \geq 0$ consider the projective space $X = \mathbb{P}^n$. Its cohomology ring is isomorphic to

$$H^*(\mathbb{P}^n) = \mathbb{Q}[H]/(H^{n+1}), \tag{83}$$

generated by $H \in H^2(\mathbb{P}^n)$. Thus $H^{2j}(\mathbb{P}^n) = \mathbb{Q} \cdot H^j$ for $j = 0, \ldots, n$ and all other cohomology groups vanish. We'll see several interpretations for the generator H below.

[43] When talking about non-compact varieties, it is often more natural to consider the so-called *Borel-Moore homology*, see [Ful84, Chapter 19.1] for a discussion.

Poincaré duality

Our assumptions on X (connected, smooth, proper) imply that the cup-product of cycles of complementary dimension

$$H^k(X) \otimes H^{2d-k}(X) \to H^{2d}(X) \cong \mathbb{Q},\ \alpha \otimes \beta \mapsto \alpha \smile \beta \tag{84}$$

is a perfect pairing. Here the isomorphism $H^{2d}(X) \cong \mathbb{Q}$ is given by sending a $2d$-class α to

$$\int_X \alpha := \deg(\alpha) := \deg([X] \frown \alpha),$$

where $[X]$ is the fundamental class of X (see the next paragraph).

The pairing (84) allows us to identify $H^k(X) \cong H^{2d-k}(X)^\vee$ and combining with the pairing (79), we have a natural isomorphism $H^k(X) \cong H_{2d-k}(X)$. Tracing through the definitions, it is easy to check that this isomorphism is given by

$$H^k(X) \xrightarrow{\sim} H_{2d-k}(X),\ \alpha \mapsto [X] \frown \alpha. \tag{85}$$

Example 6.13. In our example of $X = \mathbb{P}^n$ the pairing (84) is given by

$$(\mathbb{Q} \cdot H^j) \otimes (\mathbb{Q} \cdot H^{n-j}) \to \mathbb{Q} \cdot H^n, (\lambda H^j) \otimes (\mu H^{n-j}) \mapsto \lambda\mu H^n$$

for $k = 2j \in \{0, \ldots, 2n\}$ even (and $H^k(\mathbb{P}^n) = 0$ for k odd).

Fundamental classes of subvarieties

The fact that $X(\mathbb{C})$ is a connected, closed and oriented manifold allows us to define a fundamental class $[X] \in H_{2d}(X)$, which is a generator of the one-dimensional vector space $H_{2d}(X)$.

This can be generalized to (not-necessarily smooth) subvarieties $Z \subset X$. Any such Z admits a finite triangulation in which the singular locus is a subcomplex (see [EH16, Section C.2.1] and references there for details, see Figure 31 for a picture). This can be used to define a fundamental class $[Z] \in H_{2e}(X)$, where $e = \dim_{\mathbb{C}} Z$. Combining this with the Poincaré duality isomorphism (85) we obtain

$$[Z] \in H^{2c}(X),\ \text{for } c = \operatorname{codim}_{\mathbb{C}} Z = d - e. \tag{86}$$

Note that the fundamental class $[X] \in H^0(X)$ is the neutral element for the cup product, i.e.

$$[X] \smile \alpha = \alpha \text{ for all } \alpha \in H^*(X).$$

If $Z \subset X$ is not a subvariety, but a closed subscheme (i.e. possibly nonreduced), we can still define the cycle $[Z]$. For this, let $Z_1, \ldots, Z_r$ be the irreducible components of the reduced scheme Z^{red}, and for $i = 1, \ldots, r$ let

$$m_i = \operatorname{length}_{\mathcal{O}_{Z_i,Z}} \mathcal{O}_{Z_i,Z}$$

be the multiplicity of Z at Z_i (see [Ful84, Appendix A.1] for a definition, for $Z = V(f)$ a hypersurface the number m_i is just the order of vanishing of f at the generic point of Z_i). Then we define

$$[Z] = \sum_{i=1}^r m_i [Z_i] \in H^*(X).$$

Example 6.14. The generator $H \in H^2(\mathbb{P}^n)$ from Example 6.12 is given by the fundamental class $H = [\mathbb{P}^{n-1}]$ of any linear codimension 1 hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. More generally, we have $H^j = [\mathbb{P}^{n-j}]$ for a linear codimension j subspace $\mathbb{P}^{n-j} \subset \mathbb{P}^n$. See Example 6.16 below for an argument why all linear subspaces $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ are homologous.

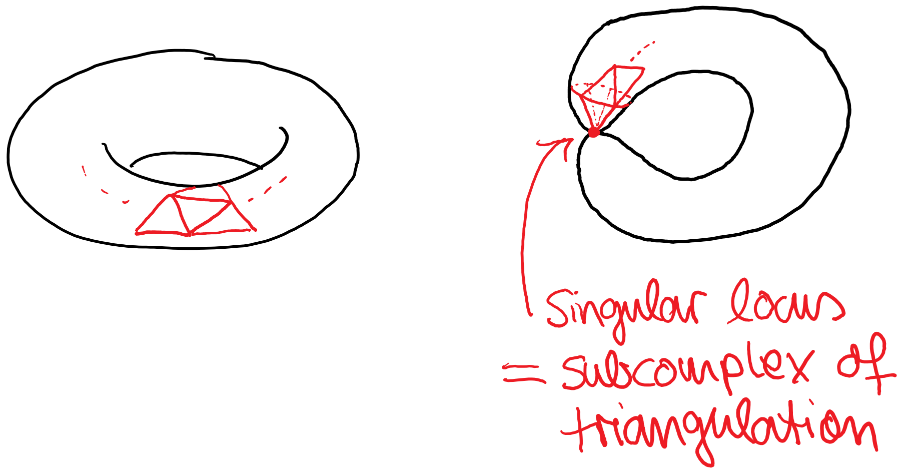


Figure 31: A triangulation of a complex variety X can be used to define a fundamental class as the sum over the singular simplices of the triangulation; the orientation of X tells you how the triangles need to be oriented, if X has singularities you can choose the triangulation in such a way, that the singular locus is a subcomplex

Proper pushforward and flat pullback

Let X, Y be connected, smooth, proper varieties of dimensions d, e and let $f : X \to Y$ be a morphism. Since the induced map $f : X(\mathbb{C}) \to Y(\mathbb{C})$ is continuous, we have

- a pushforward
$$f_* : H_k(X) \to H_k(Y) \tag{87}$$
of homology classes under f and
- a pullback
$$f^* : H^\ell(Y) \to H^\ell(X) \tag{88}$$
of cohomology classes under f. Note that pullback is compatible with the cup product, i.e. $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$.

Using the isomorphism of homology and cohomology groups from (85), we can also see the pushforward as a map
$$f_* : H^\ell(X) \to H^{\ell+2(e-d)}(Y). \tag{89}$$

Note that pushforward and pullback are *functorial*, i.e. for morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have
$$g_*(f_*\alpha) = (g \circ f)_*\alpha, \text{ for } \alpha \in H^*(X),$$
$$f^*(g^*\beta) = (g \circ f)^*\beta, \text{ for } \beta \in H^*(Z).$$

The most important basic compatibility between those operations is the *projection formula*[44]. It says that for $\alpha \in H^*(X)$ and $\beta \in H^*(Y)$ we have
$$f_*(f^*\beta \smile \alpha) = \beta \smile f_*\alpha. \tag{90}$$

[44]Note that for arbitrary maps $f : X \to Y$ of topological spaces with $\sigma \in H_*(X)$ and $\beta \in H^*(Y)$ it is true that $f_*(\alpha \frown f^*\beta) = (f_*\alpha) \frown \beta$ and the projection formula follows from this by inserting $\sigma = [X] \frown \alpha$.

For arbitrary topological spaces X, Y the story would end here. However, in our algebraic setting and assuming some additional properties of f, we can write down more explicitly how the maps f_*, f^* act on fundamental classes $[Z]$ of subvarieties of X, Y. A reference for the statements below is [Ful84, Chapter 1].

Assume[45] that $f : X \to Y$ is *proper*. In this case, given a subvariety $Z \subset X$ the image $Z' = f(Z) \subset Y$ is a subvariety of Y and we have

$$f_*[Z] = \begin{cases} \deg(Z/Z') \cdot [Z'] & \text{if } \dim Z = \dim Z', \\ 0 & \text{otherwise.} \end{cases} \tag{91}$$

Here $\deg(Z/Z')$ is the *degree of* Z *over* Z', which can be defined as the degree of the field extension $\mathbb{C}(Z') \subset \mathbb{C}(Z)$ induced by the restriction of f to Z. It is also the number of preimages in Z of a general point $z' \in Z'$.

On the other hand, for $f : X \to Y$ *flat* and $Z \subset Y$ a closed subvariety, let $f^{-1}(Z) = X \times_Y Z \to X$ be the closed subscheme[46] obtained by pullback via f. Then we have

$$f^*[Z] = [f^{-1}(Z)], \tag{92}$$

where $[f^{-1}(Z)]$ is the fundamental class associated to the subscheme $f^{-1}(Z) \subset X$.

The operations of proper pushforward and flat pullback satisfy the following compatibility condition.

Proposition 6.15 (Proposition 1.7 in [Ful84])**.** Assume we have X, X', Y, Y' connected, smooth, proper and a fibre diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \tag{93}$$

with g flat and f proper. Then g' is flat and f' is proper and for all $\alpha \in H^*(X)$ we have

$$g^* f_* \alpha = (f')_* (g')^* \alpha \in H^*(Y').$$

Example 6.16. Given $F \in \mathbb{C}[X_0, \dots, X_n]_d$ a homogeneous degree d polynomial, let's show that the hypersurface $S = V(F) \subset \mathbb{P}^n$ cut out by F has class $[S] = dH$. For this, consider the universal hypersurface $\mathcal{H}$ over the space $\mathbb{P}^N = \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d)))$:

$$\begin{tikzcd} \mathcal{H} = \{([F], p) \in \mathbb{P}^N \times \mathbb{P}^n : p \in V(F)\} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathbb{P}^n \\ \mathbb{P}^N & \end{tikzcd} \tag{94}$$

The variety $\mathcal{H}$ is smooth, projective and connected (since π_2 is a projective bundle) and π_1, π_2 are both flat and proper. Then we see that the composition $(\pi_2)_*(\pi_1)^*$ sends the class of a point $[F] \in H^{2N}(\mathbb{P}^N)$ to the fundamental class $[V(F)] \in H^2(\mathbb{P}^n)$ of its vanishing locus. Since $\mathbb{P}^N$ is connected, we have $[F] = [X_0^d] \in H^{2N}(\mathbb{P}^N)$ and thus

$$[V(F)] = (\pi_2)_*(\pi_1)^*[F] = (\pi_2)_*(\pi_1)^*[X_0^d] = [V(X_0^d)] = d \cdot [X_0] = d \cdot H \in H^2(\mathbb{P}^n).$$

[45] Given that we assume X, Y to be proper, this is actually automatic! I still write the condition since you can generalize this story to non-proper varieties, see [Ful84, Chapter 1.4].

[46] Note that indeed $f^{-1}(Z)$ can be non-reduced, e.g. for $f : \mathbb{P}^1 \to \mathbb{P}^1, [X_0 : X_1] \mapsto [X_0' : X_1'] = [X_0^2 : X_1^2]$ and $Z = V(X_0')$ we have $f^{-1}(Z) = V(X_0^2)$.

Chern classes of line bundles

Given a line bundle $\mathcal{L}$ on X, we can associate a cohomology class $c_1(\mathcal{L}) \in H^2(X)$ called the first *Chern class* of $\mathcal{L}$. While this makes sense for arbitrary complex line bundles on manifolds (see [BT82, Chapter IV]), in our situation the map is particularly easy.

$$\begin{array}{ccccc} \left\{ \begin{array}{c} \text{line bundles} \\ \mathcal{L} \text{ on } X \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{divisors} \\ D = \sum_i a_i D_i \text{ on } X \end{array} \right\} & \longrightarrow & H^2(X) \\ \mathcal{O}_X(D) & \longleftarrow & D = \sum_i a_i D_i & \longrightarrow & \sum_i a_i [D_i] \end{array} \tag{95}$$

Easy exercise 6.17. Show that given line bundles $\mathcal{L}_1, \mathcal{L}_2$ we have

$$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2).$$

Show that the dual $\mathcal{L}^\vee$ of a line bundle $\mathcal{L}$ on X has first Chern class

$$c_1(\mathcal{L}^\vee) = -c_1(\mathcal{L}).$$

Higher Chern classes of vector bundles

The construction above can be generalized to vector bundles $\mathcal{V}$ of arbitrary ranks r, giving us Chern classes $c_k(\mathcal{V}) \in H^{2k}(X)$ for $k = 1, \ldots, r$ (see [Ful84, Chapter 3]). We will only need the case $k = r$ applied to vector bundles $\mathcal{V}$ which are sums of line bundles $\mathcal{V} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$. In this case, the *top Chern class* is given by the cup product

$$c_{\mathrm{top}}(\mathcal{V}) = c_r(\mathcal{V}) = c_1(\mathcal{L}_1) \smile \cdots \smile c_1(\mathcal{L}_r) \tag{96}$$

of the first Chern classes of the line bundles $\mathcal{L}_i$.

Excess intersection formula

In the next section we will be interested in computing intersection products of cycles of the form $(\xi_\Gamma)_* \alpha$ for Γ a stable graph and $\alpha \in H^*(\overline{\mathcal{M}}_\Gamma)$. The following (general) result will be essential for this. Recall that a *local complete intersection (l.c.i.)* is a morphism that can be factored into a regular embedding followed by a smooth morphism, see [Ful84, Appendix B.7.6]. If the domain of a morphism is smooth, this condition is automatic (in particular, the morphisms ξ_Γ are local complete intersection morphisms (of stacks)).

Proposition 6.18 (Proposition 17.4.1 in [Ful84])**.** Assume we have X, X', Y, Y' connected[47], smooth, proper and a fibre diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array} \tag{97}$$

with g, g' l.c.i. morphisms of codimensions d, d' and f proper. Then for $\alpha \in H^*(X)$ we have

$$g^* f_* \alpha = (f')_* \left(c_{\mathrm{top}}(E) \smile (g')^* \alpha \right) \in H^*(Y'), \tag{98}$$

where E is the rank $d - d'$ bundle on X' given as the quotient

$$E = (f')^* \mathcal{N}_{Y'/Y} / \mathcal{N}_{X'/X} \tag{99}$$

of the pullback of the normal bundle $\mathcal{N}_{Y'/Y}$ of g by the normal bundle $\mathcal{N}_{X'/X}$ of g'.

[47] It's easy to see how you can drop the assumption of X' being connected, and we will use the result in this form.

Concerning the definition of the excess bundle E: we are slightly cheating here, since the definition for general l.c.i. morphisms g, g' is slightly more involved (see [Ful84, Proposition 6.6]). However, if f, g are unramified the above definition makes sense, where the (duals of the) normal bundles $\mathcal{N}_{Y'/Y}$ and $\mathcal{N}_{X'/X}$ can be defined via

$$\mathcal{N}^\vee_{Y'/Y} = g^*\Omega^1_Y/\Omega^1_{Y'} \text{ and } \mathcal{N}^\vee_{X'/X} = (g')^*\Omega^1_X/\Omega^1_{X'}.$$

Example 6.19. Let $i : \mathbb{P}^1 \to \mathbb{P}^2$ be the inclusion of a line in $\mathbb{P}^2$, then we know $i_*[\mathbb{P}^1] = H$ and $H^2 = [\mathrm{pt}]$ is the class of a point. Let's try to find the most complicated way to prove this, by using Proposition 6.18. One sees that the fibre diagram (97) becomes

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \mathbb{P}^1 \arrow[d, "i"] \\ \mathbb{P}^1 \arrow[r, "i"] & \mathbb{P}^2 \end{tikzcd} \tag{100}$$

Then we obtain (using the projection formula (90) and the fact that the fundamental class $[\mathbb{P}^1] \in H^0(\mathbb{P}^1)$ is the neutral element with respect to the cup product on $\mathbb{P}^1$)

$$H^2 = (i_*[\mathbb{P}^1]) \smile (i_*[\mathbb{P}^1]) = i_*(i^*i_*[\mathbb{P}^1] \smile [\mathbb{P}^1]) = i_*(i^*i_*[\mathbb{P}^1]) \tag{101}$$

By the excess intersection formula (98) we compute the term $i^*i_*[\mathbb{P}^1]$ as

$$i^*i_*[\mathbb{P}^1] = (\mathrm{id})_*\left(c_{\mathrm{top}}(E) \smile (\mathrm{id})^*[\mathbb{P}^1]\right) = c_{\mathrm{top}}(E) \smile [\mathbb{P}^1] = c_{\mathrm{top}}(E), \tag{102}$$

with $E = \mathcal{N}_{\mathbb{P}^1/\mathbb{P}^2}/\mathcal{N}_{\mathbb{P}^1/\mathbb{P}^1} = \mathcal{N}_{\mathbb{P}^1/\mathbb{P}^2}$ the excess bundle. It remains to compute this normal bundle, associated to the inclusion $i : \mathbb{P}^1 \to \mathbb{P}^2$. We can make our lives easy and use the fact (see [Ful84, Example 2.5.5]) that for an effective Cartier divisor $D \subset X$ we have $\mathcal{N}_{D/X} = \mathcal{O}_X(D)|_D$, and obtain

$$\mathcal{N}_{\mathbb{P}^1/\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}([\mathbb{P}^1])|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^2}(1)|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathbb{P}^1}(\mathrm{pt}). \tag{103}$$

If you instead like your life to be hard, you can also compute this normal bundle using the Euler sequence of $\mathbb{P}^2$ (see Exercise 6.21 c)). In any case, we see

$$c_{\mathrm{top}}(E) = c_1(\mathcal{O}_{\mathbb{P}^1}(\mathrm{pt})) = [\mathrm{pt}] \in H^2(\mathbb{P}^1)$$

and pushing this forward via $i : \mathbb{P}^1 \to \mathbb{P}^2$ we indeed obtain $[\mathrm{pt}] \in H^4(\mathbb{P}^2)$.

Exercise 6.20. Assume that $X, Y' \subset Y$ are smooth subvarieties of a connected, smooth, proper variety Y. Assume the intersection of X, Y' is *transversal*, i.e. for every $x' \in X \cap Y$ we have

$$T_{x'}Y = T_{x'}X + T_{x'}Y'.$$

Show that the scheme-theoretic intersection $X \cap Y' = X \times_Y Y'$ is reduced and of pure codimension $\mathrm{codim}_Y(X) + \mathrm{codim}_Y(Y')$. Conclude that

$$[X] \smile [Y] = [X \cap Y].$$

In the example of $Y = \mathbb{P}^2$ and $X, Y' \subset Y$ curves of degree d, e meeting transversally, use this to show that X, Y' intersect in precisely $d \cdot e$ points (this is a variant of Bézout's theorem).

Exercise 6.21 (hard, but rewarding)**.** The morphism

$$\mathbb{A}^1 \to \mathbb{A}^2, t \mapsto (t^2 - 1, t^3 - t)$$

extends to the normalization $f : \mathbb{P}^1 \to E_0$ of the nodal cubic curve

$$E_0 = \{Y^2 \cdot Z - X^2(X + Z)\} \subset \mathbb{P}^2,$$

see Figure 32.

a) Show that $f_*[\mathbb{P}^1] = 3 \cdot H \in H^2(\mathbb{P}^2)$ so that $(f_*[\mathbb{P}^1])^2 = 9 \cdot H^2 = 9[\mathrm{pt}] \in H^4(\mathbb{P}^2)$.

b) Now let's show this the hard way. First, compute the fibre product $\mathbb{P}^1 \times_{\mathbb{P}^2} \mathbb{P}^1$ of the map f with itself.

c) Use the Euler sequence of $\mathbb{P}^2$ and the conormal exact sequence for the map f to show that the normal bundle $\mathcal{N}_{\mathbb{P}^1/\mathbb{P}^2}$ of the map f has degree 7 on $\mathbb{P}^1$.

d) Conclude that $(f_*[\mathbb{P}^1])^2 = 9[\mathrm{pt}] \in H^4(\mathbb{P}^2)$.

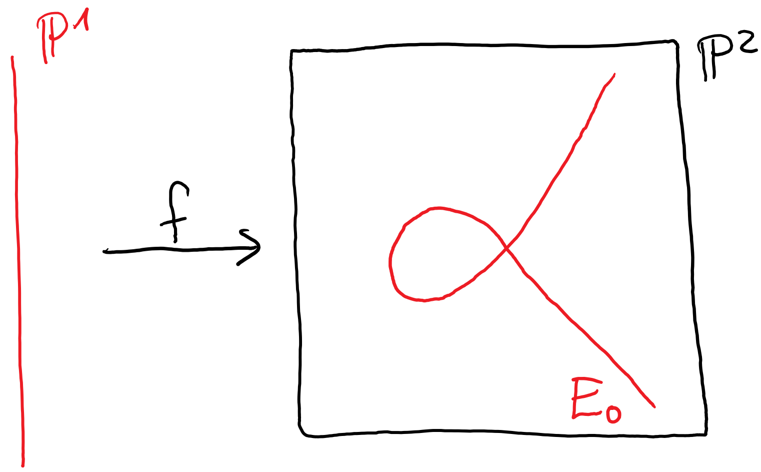


Figure 32: The normalization of the nodal cubic curve E_0

6.3 The tautological ring of the moduli space of stable curves

Now we have all the ingredients we need to start our study of the cohomology groups of the moduli spaces of stable curves. Our first goal here is to write down some interesting cycle classes in $H^*(\overline{\mathcal{M}}_{g,n})$. For this, we have a bunch of tools at our disposal.

- As a modest start, we always have the fundamental class $[\overline{\mathcal{M}}_{g,n}] \in H^0(\overline{\mathcal{M}}_{g,n})$.
- More generally, for every stable graph Γ we have the fundamental class

  $$[\overline{\mathcal{M}}^\Gamma] \in H^{2e}(\overline{\mathcal{M}}_{g,n}),$$

  for $e = \#E(\Gamma)$, of the corresponding closed stratum in $\overline{\mathcal{M}}_{g,n}$.

- If we find some natural line bundle[48] $\mathcal{L}$ on $\overline{\mathcal{M}}_{g,n}$, we can take its Chern class
$$c_1(\mathcal{L}) \in H^2(\overline{\mathcal{M}}_{g,n}).$$
- Once we built a collection of classes α_i on various spaces $\overline{\mathcal{M}}_{g_i,n_i}$, we can obtain even more classes by
  - taking cup products of existing classes,
  - taking pushforwards and pullbacks of (products of) classes α_i under the gluing morphisms ξ_Γ and forgetful morphisms π.

The next definition makes precise what we mean by the system $RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$ of classes that you can obtain using the ingredients above. Its history goes back to the original paper [Mum83] by Mumford, though the formulation presented below was first given in [FP00] by Faber and Pandharipande. Due to the last point in the above list (we can combine classes α_i on different spaces $\overline{\mathcal{M}}_{g_i,n_i}$ via the gluing and forgetful maps), it will be natural to define the sets $RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$ *simultaneously* for all g, n.

Definition 6.22. The *tautological rings* $(RH^*(\overline{\mathcal{M}}_{g,n}))_{g,n}$ are the smallest system of $\mathbb{Q}$-subalgebras
$$RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$$
containing the units $1 = [\overline{\mathcal{M}}_{g,n}] \in H^0(\overline{\mathcal{M}}_{g,n})$ and which are closed under pushforward by all gluing morphisms
$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g,n} \tag{104}$$
and all forgetful morphisms[49]
$$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}. \tag{105}$$
The elements $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n})$ are called *tautological classes.*

Remark 6.23. Let's look more closely at the various parts of the definition and make a couple of comments. In particular, while it might look that the definition omits some of the ingredients we mentioned above (e.g. the classes $[\overline{\mathcal{M}}^\Gamma]$), we'll see that all of those are nonetheless contained in $RH^*(\overline{\mathcal{M}}_{g,n})$.

a) The fancy word "$\mathbb{Q}$-subalgebra" just means that $RH^*(\overline{\mathcal{M}}_{g,n})$ is a $\mathbb{Q}$-subvector space of $H^*(\overline{\mathcal{M}}_{g,n})$ which is invariant under cup product. In other words, given tautological classes α, β and $\lambda \in \mathbb{Q}$, we have that
$$\alpha + \beta,\ \lambda \cdot \alpha \text{ and } \alpha \smile \beta$$
are again tautological.

[48] A similar story works for natural vector bundles $\mathcal{V}$ and their Chern classes $c_\ell(\mathcal{V}) \in H^{2\ell}(\overline{\mathcal{M}}_{g,n})$, though unfortunately we won't have time to see good examples of this.

[49] Note: instead of the classical morphism forgetting marking $n+1$, we must allow morphisms forgetting arbitrary markings i, i.e. defined by $(C, p_1, \ldots, p_{n+1}) \mapsto (C, p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{n+1})$ when C is smooth.

b) Let's also expand what we mean by being "closed under pushforward by all gluing morphisms". For this let Γ be a stable graph and assume we are given some cohomology classes $\alpha_v \in H^*(\overline{\mathcal{M}}_{g(v),n(v)})$ for $v \in V(\Gamma)$. We can then form the class

$$\alpha = \prod_{v \in V} (\pi_v)^* \alpha_v, \text{ for the projections } \pi_v : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g(v),n(v)}. \tag{106}$$

In other words, we pull back the classes α_v from the factor $\overline{\mathcal{M}}_{g(v),n(v)}$ to $\overline{\mathcal{M}}_\Gamma$ and then take their cup product (this is sometimes called the *box product* of the classes α_v). Then we require that if all α_v are tautological, the pushforward

$$(\xi_\Gamma)_* \alpha \in H^*(\overline{\mathcal{M}}_{g,n}) \tag{107}$$

is also tautological. Applying this to the fundamental classes $\alpha_v = [\overline{\mathcal{M}}_{g(v),n(v)}]$, which are tautological by assumption, we have $\alpha = [\overline{\mathcal{M}}_\Gamma]$ is the fundamental class of $\overline{\mathcal{M}}_\Gamma$. Combining the formula (91) for the proper pushforward of fundamental classes with Theorem 5.1 e), stating that ξ_Γ has degree $\#\mathrm{Aut}(\Gamma)$ onto its image $\overline{\mathcal{M}}^\Gamma$, we see that

$$[\overline{\mathcal{M}}^\Gamma] = \frac{1}{\#\mathrm{Aut}(\Gamma)} \xi_* [\overline{\mathcal{M}}_\Gamma] \tag{108}$$

is a tautological class.

The above definition of the tautological ring is short and elegant, but not very explicit (it does not give a convenient way to write down an arbitrary tautological class). Our main goal for the remainder of this section is to give an explicit, finite set of generators of $RH^*(\overline{\mathcal{M}}_{g,n})$ as a $\mathbb{Q}$-vector space. To state the corresponding result, we'll need two more ingredients : the so-called ψ- and κ-classes. Their definition uses in a crucial way the universal curve over $\overline{\mathcal{M}}_{g,n}$.

$$\begin{array}{c} \overline{\mathcal{C}}_{g,n} = \overline{\mathcal{M}}_{g,n+1} \\ {\scriptstyle \pi}\big\downarrow \big\uparrow {\scriptstyle p_i} \\ \overline{\mathcal{M}}_{g,n} \end{array} \tag{109}$$

Definition 6.24. For $i = 1, \ldots, n$, the *i-th cotangent line bundle* $\mathbb{L}_i$ on $\overline{\mathcal{M}}_{g,n}$ is defined as

$$\mathbb{L}_i = p_i^* \Omega_\pi^1, \tag{110}$$

where Ω_π^1 is the sheaf of relative differentials for the morphism π and $p_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n}$ is the section of π corresponding to the i-th marked point. We define the *i-th ψ-class*

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}) \tag{111}$$

to be the first Chern class of this line bundle.

To explain the name, consider Figure 33.

The preimage of $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$ under π is isomorphic to C, and for $q \in C$ a smooth[50] point of C, the sheaf Ω_π^1 has fibre

$$\Omega_\pi^1|_q = T_q^* C.$$

[50] The assumption that q is a smooth point is important: the rank 1 sheaf Ω_π^1 is not even locally free (i.e. a line bundle) at the nodes of C.

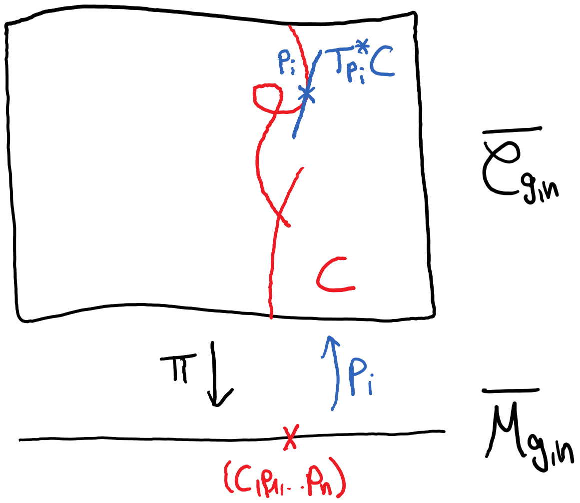


Figure 33: The fibre of the sheaf Ω^1_π at the image of the section p_i is equal to the cotangent space

Thus, pulling back Ω^1_π by setting $q = p_i$, we see that at a point $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$ the line bundle $\mathbb{L}_i$ has fibre

$$\mathbb{L}_i|_{(C,p_1,\ldots,p_n)} = T^*_{p_i}C$$

equal to the cotangent space of C at p_i. This is the kind of natural line bundle $\mathcal{L}$ on $\overline{\mathcal{M}}_{g,n}$ we were talking about at the start of the section. And even though we did not mention the classes ψ_i in the definition of the tautological ring, it turns out that they are nonetheless contained in it.

Proposition 6.25. The ψ-classes ψ_i are tautological.

Proof. Remember from Proposition 4.25 that the morphisms p_i above are actually special cases of gluing morphisms (for the graph Γ_i having two vertices of genus $0, g$ connected by a single edge, with legs $i, n+1$ at the genus 0 vertex and all other legs at the genus g vertex). Thus we have

$$[\Delta_i] = (p_i)_*[\overline{\mathcal{M}}_{g,n}] \in RH^2(\overline{\mathcal{M}}_{g,n+1}). \tag{112}$$

We claim that

$$\psi_i = -\pi_* \left([\Delta_i] \smile [\Delta_i]\right), \tag{113}$$

which would finish the proof (the tautological ring is invariant under cup products and pushforwards by forgetful morphisms). To compute the cup product $[\Delta_i] \smile [\Delta_i]$ we use the excess intersection formula from Proposition 6.18. We start by computing the fibre product of p_i with itself. Since p_i is a section of the separated morphism π, it is a closed embedding (see [Vak17, Exercise 10.1.M]). Thus the fibre product is just given by $\overline{\mathcal{M}}_{g,n}$ itself.

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n} & \xrightarrow{\text{id}} & \overline{\mathcal{M}}_{g,n} \\ \big\downarrow{\scriptstyle \text{id}} & & \big\downarrow{\scriptstyle p_i} \\ \overline{\mathcal{M}}_{g,n} & \xrightarrow{p_i} & \overline{\mathcal{M}}_{g,n+1} \end{array} \tag{114}$$

Alternatively, you can find the diagram (114) as a special case of Theorem 6.10 for $\Gamma_1 = \Gamma_2 = \Gamma_i$. In any case, since the normal bundle $\mathcal{N}_{\overline{\mathcal{M}}_{g,n}, \overline{\mathcal{M}}_{g,n}}$ of the identity is trivial, the excess bundle E is given by the normal bundle $\mathcal{N}_{\overline{\mathcal{M}}_{g,n},\overline{\mathcal{M}}_{g,n+1}}$ of the map p_i. But the dual of this bundle is precisely given by

$$\mathcal{N}^{\vee}_{\overline{\mathcal{M}}_{g,n},\overline{\mathcal{M}}_{g,n+1}} = p_i^*\Omega^1_{\overline{\mathcal{M}}_{g,n+1}}/\Omega^1_{\overline{\mathcal{M}}_{g,n}} = p_i^*\Omega^1_{\pi} = \mathbb{L}_i. \tag{115}$$

Now we're in business and we can first apply the excess intersection formula to obtain

$$(p_i)^*[\Delta_i] = (p_i)^*(p_i)_*[\overline{\mathcal{M}}_{g,n}] = c_1(\mathcal{N}_{\overline{\mathcal{M}}_{g,n},\overline{\mathcal{M}}_{g,n+1}}) = c_1(\mathbb{L}_i^{\vee}) = -c_1(\mathbb{L}_i) = -\psi_i,$$

and then conclude as follows

$$\begin{aligned}\pi_*\left([\Delta_i] \smile [\Delta_i]\right) &= \pi_*\left((p_i)_*[\overline{\mathcal{M}}_{g,n}] \smile [\Delta_i]\right) = \pi_*(p_i)_*\left((p_i)^*[\Delta_i]\right)\\ &= -(\pi \circ p_i)_*\psi_i = -(\mathrm{id}_{\overline{\mathcal{M}}_{g,n}})_*\psi_i = -\psi_i.\end{aligned}$$

□

Definition 6.26. For $a \geq 0$ define the *a-th κ-class* κ_a as the pushforward

$$\kappa_a = \pi_*((\psi_{n+1})^{a+1}) \in RH^{2a}(\overline{\mathcal{M}}_{g,n}). \tag{116}$$

Note that $(\psi_{n+1})^{a+1}$ has cohomological degree $2(a+1)$ on $\overline{\mathcal{M}}_{g,n+1}$ and since

$$\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n+1} - \dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 1,$$

we indeed have κ_a in cohomological degree $2a$. Since ψ_{n+1} is tautological, it follows that κ_a is tautological.

Now that we have ψ- and κ-classes, we combine them with the gluing maps ξ_Γ to obtain the generating set of $RH^*(\overline{\mathcal{M}}_{g,n})$.

Definition 6.27. Let Γ be a stable graph. A *decoration on* Γ is a class $\alpha \in H^*(\overline{\mathcal{M}}_\Gamma)$ which is a product of κ- and ψ-classes pulled back from the factors $\overline{\mathcal{M}}_{g(v),n(v)}$ of $\overline{\mathcal{M}}_\Gamma$. More formally, it is a class

$$\alpha = \prod_{v\in V(\Gamma)} \pi_v^*\alpha_v, \text{ for } \alpha_v = \kappa_{a_{v,1}}^{e_{v,1}} \cdots \kappa_{a_{v,\ell_v}}^{e_{v,\ell_v}} \cdot \psi_1^{f_{v,1}} \cdots \psi_{n(v)}^{f_{v,n(v)}} \in RH^*(\overline{\mathcal{M}}_{g(v),n(v)}). \tag{117}$$

Given Γ and a decoration α on Γ, we define the *decorated stratum class* $[\Gamma, \alpha]$ to be the pushforward

$$[\Gamma, \alpha] = (\xi_\Gamma)_*\alpha \in RH^*(\overline{\mathcal{M}}_{g,n}). \tag{118}$$

Clearly, since all α_v are tautological, the definition of the tautological ring implies that also the $[\Gamma, \alpha]$ are tautological. Note that for dimension reasons we have $H^k(\overline{\mathcal{M}}_{g(v),n(v)}) = 0$ for $k > 2(3g(v) - 3 + n(v))$. This implies that there are only finitely many nonzero classes α_v of the form above, since they vanish unless

$$\sum_j a_{v,j} \cdot e_{v,j} + \sum_i f_{v,i} \leq 3g(v) - 3 + n(v).$$

Note that since ξ_Γ is of relative dimension equal to the number $e = \#E(\Gamma)$ of edges of Γ, we have $[\Gamma, \alpha] \in RH^{2(d+e)}(\overline{\mathcal{M}}_{g,n})$ for $\alpha \in H^{2d}(\overline{\mathcal{M}}_\Gamma)$. We can represent a decorated stratum class by a picture of a stable graph decorated by powers of ψ-classes at half-edges and a monomial in κ-classes at vertices, as illustrated in Figure 34.

Theorem 6.28. The decorated stratum classes $[\Gamma, \alpha]$ form a finite generating set, as a $\mathbb{Q}$-vector space, of the tautological ring $RH^*(\overline{\mathcal{M}}_{g,n})$.

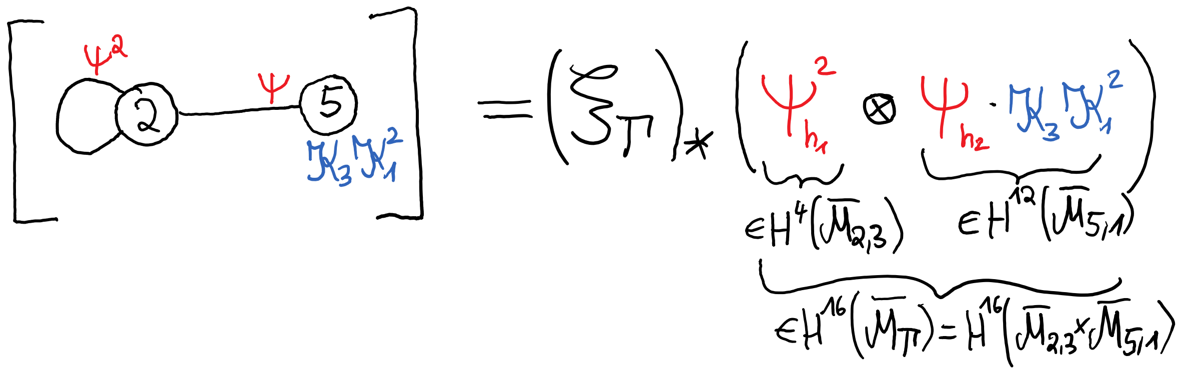


Figure 34: A tautological class in $RH^{20}(\overline{\mathcal{M}}_8)$

Proof. In Definition 6.22 we defined the tautological rings as the minimal system of subspaces of $H^*(\overline{\mathcal{M}}_{g,n})$ satisfying a bunch of properties. So, to conclude we need to show that the $\mathbb{Q}$-vector subspaces $S_{g,n} \subset H^*(\overline{\mathcal{M}}_{g,n})$ spanned by classes $[\Gamma, \alpha]$ have all these properties. Indeed, then they must contain the minimal system $RH^*(\overline{\mathcal{M}}_{g,n})$, but as we saw they are themselves contained in $RH^*(\overline{\mathcal{M}}_{g,n})$, proving equality.

Some parts are straightforward: the $S_{g,n}$ are closed under addition and scalar multiplication with elements of $\mathbb{Q}$, they contain the units $[\overline{\mathcal{M}}_{g,n}]$ (taking Γ the trivial graph and $\alpha = 1$ the trivial product). Also, they are closed under pushforward by gluing maps ξ_Γ: given Γ and decorated classes $[\Gamma_w, \alpha_w] \in S_{g(w),n(w)}$ on the vertices $w \in V(\Gamma)$, we have

$$(\xi_\Gamma)_* \prod_w [\Gamma_w, \alpha_w] = [\Gamma', \alpha], \tag{119}$$

where Γ' is the graph obtained from Γ by gluing in the Γ_w at vertices of Γ (see Exercise 6.1) and the decoration α is obtained by distributing the decorations α_w to the vertices of Γ' (remember that the vertices of Γ' are the union of the vertices of all Γ_w). Thus we see that the class (119) is again a decorated stratum class.

The only parts of Definition 6.22 that require serious work are showing that the spaces $S_{g,n}$ are

a) closed under cup products,

b) closed under pullbacks by forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$.

We will prove part a) in Corollary 6.32 below, and you will show part b) in *Exercise 6.36. Thus, modulo these results, the proof is finished. $\square$

For the proof of Corollary 6.32 we'll need a few more preparations.

Exercise 6.29. Prove that for Γ a stable graph, and $i = 1, \ldots, n$ we have

$$\xi_\Gamma^* \psi_i = (\pi_v)^* \psi_h \in H^*(\overline{\mathcal{M}}_\Gamma), \tag{120}$$

where $h \in H(\Gamma)$ is the half-edge corresponding to the marking i, incident to vertex $v \in V(\Gamma)$. Likewise, for $a \geq 0$ show that

$$\xi_\Gamma^* \kappa_a = \sum_{v \in V(\Gamma)} \pi_v^* \kappa_a \in H^*(\overline{\mathcal{M}}_\Gamma). \tag{121}$$

Hint: In particular for the statement about κ-classes, you should have a look at Proposition 6.34 below.

For the proof of Proposition 6.31 below we will need one more fact, about the normal bundle for the gluing morphisms ξ_Γ. It can be proved using *deformation theory* (we'll discuss this in the optional section about the proof of Theorem 5.1), but for now we'll have to take it as a black box.

Fact 6.30. Given a stable graph Γ, the gluing morphism ξ_Γ is unramified with normal bundle

$$\mathcal{N}_{\xi_\Gamma} = \bigoplus_{\{h,h'\}\in E(\Gamma)} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee \tag{122}$$

being the direct sum over the edges $\{h,h'\}$ of Γ of the tensor products of the *tangent bundles* $\mathbb{L}_h^\vee, \mathbb{L}_{h'}^\vee$ associated to the half-edges h, h'.

You find the proof that the ξ_Γ are unramified in [Knu83a, Corollary 3.9].

Proposition 6.31. Let Γ_1 be a stable graph of genus g with n legs and let $[\Gamma_2,\alpha]$ be a decorated stratum class on $\overline{\mathcal{M}}_{g,n}$. Then the pullback $\xi_{\Gamma_1}^*[\Gamma_2,\alpha]$ is given by

$$\xi_{\Gamma_1}^*[\Gamma_2,\alpha] = \sum_{(\Gamma,\varphi_1,\varphi_2)\in\mathfrak{G}_{\Gamma_1,\Gamma_2}} (\xi_{\varphi_1})_* \left(\xi_{\varphi_2}^*(\alpha)\cdot\gamma_{\mathrm{ex}}\right), \tag{123}$$

where γ_{ex} (depending on $(\Gamma,\varphi_1,\varphi_2)$) is the top Chern class of the excess bundle on $\overline{\mathcal{M}}_\Gamma$ given by

$$\gamma_{\mathrm{ex}} = \prod_{\{h,h'\}\in\varphi_{1,E}(E(\Gamma_1))\cap\varphi_{2,E}(E(\Gamma_2))} (-\psi_h - \psi_{h'}). \tag{124}$$

In particular, the class $\xi_{\Gamma_1}^*[\Gamma_2,\alpha]$ is contained in the *tautological ring of* $\overline{\mathcal{M}}_{\Gamma_1}$, i.e. a sum of terms

$$\prod_{v\in V(\Gamma_1)} \pi_v^*\alpha_v, \text{ for } \alpha_v \in RH^*(\overline{\mathcal{M}}_{g(v),n(v)}).$$

Proof. This result is an application of the excess intersection formula from Proposition 6.18. Indeed, the maps $\xi_{\Gamma_1}, \xi_{\Gamma_2}$ are proper and l.c.i. by Theorem 5.1. We computed their fibre product in Theorem 6.10 to be the disjoint union over spaces $\overline{\mathcal{M}}_\Gamma$ for generic (Γ_1,Γ_2)-structures $(\Gamma,\varphi_1,\varphi_2)$. They fit in diagrams

$$\begin{tikzcd} \overline{\mathcal{M}}_\Gamma \arrow[r, "\xi_{\varphi_2}"] \arrow[d, "\xi_{\varphi_1}"'] & \overline{\mathcal{M}}_{\Gamma_2} \arrow[d, "\xi_{\Gamma_2}"] \\ \overline{\mathcal{M}}_{\Gamma_1} \arrow[r, "\xi_{\Gamma_1}"'] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \tag{125}$$

This explains the corresponding sum in the formula above. Applying Proposition 6.18, the only part of the result left to show is that γ_{ex} is indeed the top Chern class of the excess bundle restricted to $\overline{\mathcal{M}}_\Gamma$.

Using Fact 6.30, we have

$$\mathcal{N}_{\xi_{\Gamma_1}} = \bigoplus_{\{h,h'\}\in E(\Gamma_1)} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee. \tag{126}$$

Using that ξ_{φ_2} is a product of gluing maps (see Remark 6.3 b)), for graphs whose edges correspond to edges in $E(\Gamma)\setminus\varphi_{2,E}(E(\Gamma_2))$, we obtain

$$\mathcal{N}_{\xi_{\varphi_2}} = \bigoplus_{\{h,h'\}\in E(\Gamma)\setminus\varphi_{2,E}(E(\Gamma_2))} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee. \tag{127}$$

Pulling back the normal bundle (126) under the map ξ_{φ_1} and forming the quotient E of this pullback by (127), we obtain

$$E = \xi_{\varphi_1}^* \mathcal{N}_{\xi_{\Gamma_1}} / \mathcal{N}_{\xi_{\varphi_2}} = \bigoplus_{\{h,h'\} \in E(\Gamma_1) \cap E(\Gamma_2)} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee. \tag{128}$$

The fact that γ_{ex} is the top Chern class of this vector bundle follows from Easy exercise 6.17, using that $\psi_h = c_1(\mathbb{L}_h)$ and $\psi_{h'} = c_1(\mathbb{L}_{h'})$. Finally, it is also clear that the expression (123) is contained in the tautological ring of $\overline{\mathcal{M}}_\Gamma$: the map ξ_{φ_1} is a product of gluing maps, and by Exercise 6.29, the term $\xi_{\varphi_2}^*(\alpha) \cdot \gamma_{\text{ex}}$ is a combination of κ- and ψ-classes on the factors of $\overline{\mathcal{M}}_\Gamma$. □

Corollary 6.32. The product of two decorated stratum classes $[\Gamma_1, \alpha_1]$ and $[\Gamma_2, \alpha_2]$ on $\overline{\mathcal{M}}_{g,n}$ is tautological, and given by

$$[\Gamma_1, \alpha_1] \cdot [\Gamma_2, \alpha_2] = \sum_{(\Gamma, \varphi_1, \varphi_2) \in \mathfrak{G}_{\Gamma_1, \Gamma_2}} \left[\Gamma, \xi_{\varphi_1}^*(\alpha_1) \cdot \xi_{\varphi_2}^*(\alpha_2) \cdot \gamma_{\text{ex}}\right], \tag{129}$$

with γ_{ex} as in (124).

Proof. By the projection formula we have

$$[\Gamma_1, \alpha_1] \cdot [\Gamma_2, \alpha_2] = (\xi_{\Gamma_1})_* \alpha_1 \cdot [\Gamma_2, \alpha_2] = (\xi_{\Gamma_1})_* \left((\xi_{\Gamma_1})^* [\Gamma_2, \alpha_2] \alpha_1\right).$$

Thus the result follows by taking the formula from Proposition 6.31, multiplying by α_1, and pushing forward again. In the process we also use the projection formula for ξ_{φ_1} together with the fact that $\xi_{\Gamma_1} \circ \xi_{\varphi_1} = \xi_\Gamma$. □

Exercise 6.33. Verify the computations in the tautological ring of $\overline{\mathcal{M}}_{3,1}$ shown in Figure 35.

The formula from Corollary 6.32 has been implemented in the software package `admcycles` [DSv20]. You can check out some example computations here.

Proposition 6.34. Given a stable graph Γ and $v \in V(\Gamma)$ let $\overline{\mathcal{C}}_v$ be the pullback

$$\begin{tikzcd} \overline{\mathcal{C}}_v \arrow[r] \arrow[d] & \overline{\mathcal{C}}_{g(v),n(v)} \arrow[d] \\ \overline{\mathcal{M}}_\Gamma \arrow[r, "\pi_v"] & \overline{\mathcal{M}}_{g(v),n(v)} \end{tikzcd} \tag{130}$$

of the universal curve on the factor $\overline{\mathcal{M}}_{g(v),n(v)}$ to $\overline{\mathcal{M}}_\Gamma$. Then we have an isomorphism

$$\overline{\mathcal{C}}_v \cong \overline{\mathcal{M}}_{\Gamma(v)} \tag{131}$$

where $\Gamma(v)$ is the stable graph with $n+1$ legs obtained from Γ by adding the leg $n+1$ at vertex v. There exists a commutative diagram

$$\begin{tikzcd} \coprod_{v \in V(\Gamma)} \overline{\mathcal{C}}_v \arrow[r, "\Phi"] \arrow[dr] & \overline{\mathcal{C}}_\Gamma \arrow[r] \arrow[d, "\pi_\Gamma"'] & \overline{\mathcal{C}}_{g,n} \arrow[d, "\pi"'] \\ & \overline{\mathcal{M}}_\Gamma \arrow[r, "\xi_\Gamma"] \arrow[ul, bend left, "p_i'"] & \overline{\mathcal{M}}_{g,n} \arrow[u, bend right, "p_i"'] \end{tikzcd} \tag{132}$$

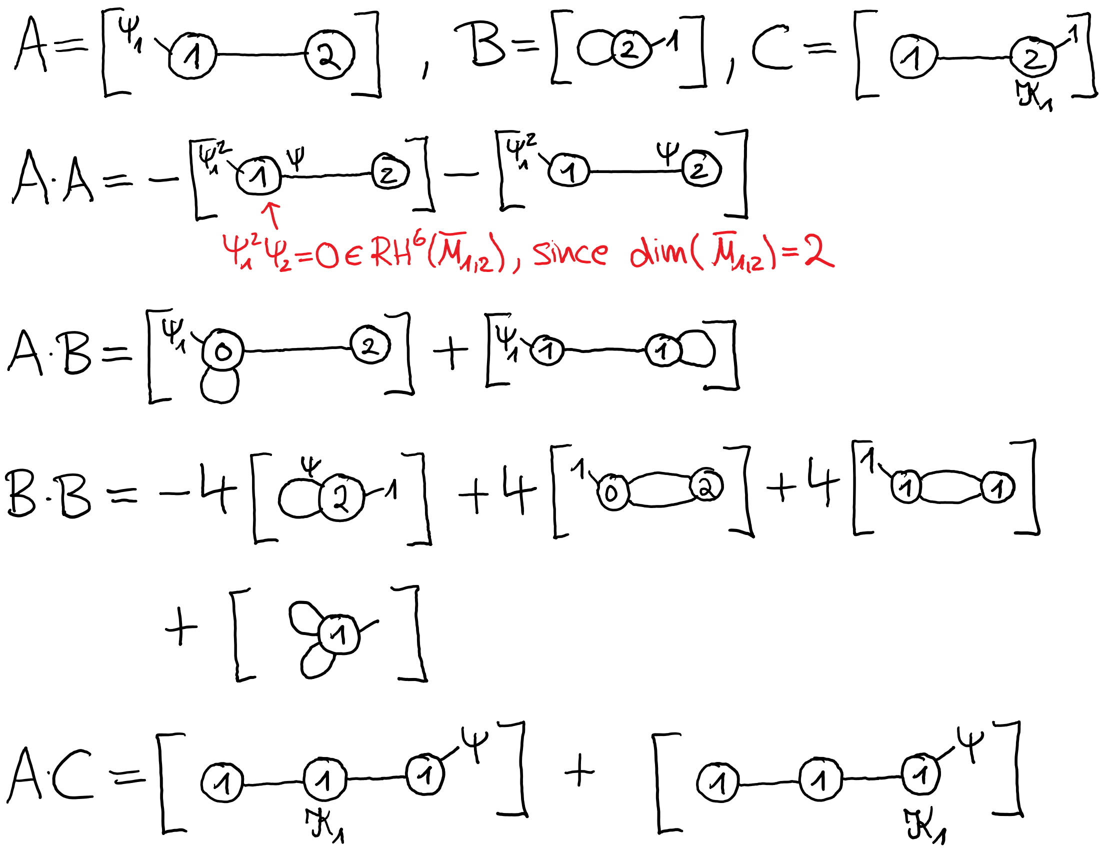


Figure 35: Products in the tautological ring of $\overline{\mathcal{M}}_{3,1}$

where the square on the right is a fibre diagram and the map Φ is the map gluing the families $\overline{\mathcal{C}}_v$ along sections corresponding to half-edges of Γ. In particular, the map Φ is surjective, proper, birational and an isomorphism over the locus in $\overline{\mathcal{C}}_\Gamma$ where the map π_Γ is smooth. In terms of fundamental classes, we have

$$\Phi_* \sum_v [\overline{\mathcal{C}}_v] = [\overline{\mathcal{C}}_\Gamma]\,. \tag{133}$$

Proof. Looking back at the proof (sketch) of Proposition 4.15 and using our new language of stacks, we can see that we *defined* the morphism ξ_Γ by constructing the family $\overline{\mathcal{C}}_\Gamma \to \overline{\mathcal{M}}_\Gamma$ of stable curves over $\overline{\mathcal{M}}_\Gamma$. The fact that $\overline{\mathcal{M}}_{g,n}$ is the moduli stack of stable curves then told us there is a unique map ξ_Γ such that $\overline{\mathcal{C}}_\Gamma$ is the pullback of the universal curve over $\overline{\mathcal{M}}_{g,n}$. This explains the fibre diagram on the right side of (132). But the construction of $\overline{\mathcal{C}}_\Gamma$ precisely started with the disjoint union of curves $\overline{\mathcal{C}}_v$ and glued them together under a map Φ. This map is surjective and an isomorphism away from the locus of nodes in $\overline{\mathcal{C}}_\Gamma$ (in particular, it is birational) and since all $\overline{\mathcal{C}}_v$ are proper, the map Φ is proper as well. For the equality (133) note that the $\overline{\mathcal{C}}_v$ map birationally (in particular of generic degree 1) to the set of irreducible components of $\overline{\mathcal{C}}_\Gamma$, so (133) follows from the definition of proper pushforward[51]. □

See Figure 36 for an illustration of the diagram (132).

[51] Here we are lying a bit more than usual: the stack $\overline{\mathcal{C}}_\Gamma$ is not smooth, so to make full sense of this statement one should use Chow groups (in the sense of [Ful84] or [Vis89]).

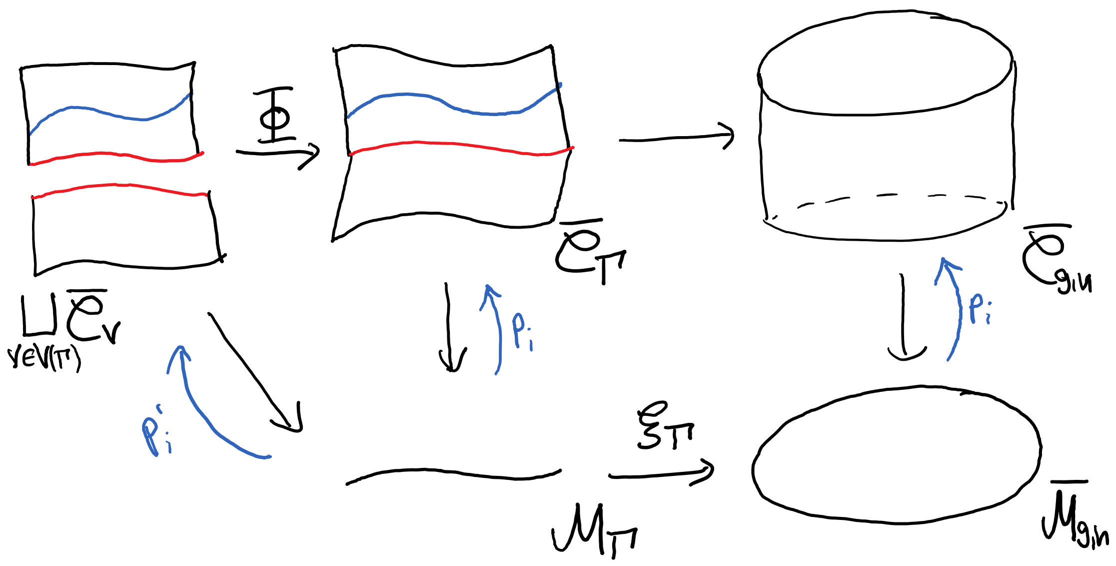


Figure 36: The commutative diagram (132) from Proposition 6.34 illustrated

Proposition 6.35. There exists a commutative diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n+2} \arrow[dr, "G"] \arrow[drr, bend left, "\pi_{n+2}"] \arrow[ddr, bend right, "\pi_{n+1}"'] & & \\
& \overline{\mathcal{M}}_{g,n+1} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n+1} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n+1} \arrow[d, "\pi"'] \\
& \overline{\mathcal{M}}_{g,n+1} \arrow[r, "\pi"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd} \tag{134}$$

where π, π_{n+1} are the forgetful morphisms of marking $n+1$, π_{n+2} is the forgetful morphism of marking $n+2$ and the map G is proper and birational and in particular satisfies

$$G_*[\overline{\mathcal{M}}_{g,n+2}] = \left[\overline{\mathcal{M}}_{g,n+1} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n+1}\right]. \tag{135}$$

Proof. The proof is very similar to the proof of Proposition 6.34, where now we use that the stable curve

$$\overline{\mathcal{M}}_{g,n+1} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n+1}$$

defining the forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ can be constructed by starting with the universal curve $\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g,n+1}$ and contracting some components of its fibres (which become unstable after forgetting the marking $n+1$) under the morphism G, as outlined in the proof of Proposition 4.25. $\square$

***Exercise 6.36.** Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful morphism of the marking $n+1$.

a) Show that

$$\pi^*\psi_i = \psi_i - [\Delta_i] \text{ and } \pi^*\kappa_a = \kappa_a - \psi_{n+1}^a, \tag{136}$$

where $\Delta_i = (p_i)_*[\overline{\mathcal{M}}_{g,n}]$ is again the section of π associated to the i-th marked point. (*Hint*: You can show these formulas using Proposition 6.35 together with the formula (113) for the ψ-class.)

b) Show that the pushforward

$$\pi_* \left(\kappa_{a_1}^{e_1} \cdots \kappa_{a_\ell}^{e_\ell} \cdot \psi_1^{f_1} \cdots \psi_n^{f_n} \right) \tag{137}$$

of a monomial in κ- and ψ-classes is again a polynomial in κ- and ψ-classes on $\overline{\mathcal{M}}_{g,n}$. (*Hint*: Use the projection formula together with part a)).

c) Show that the pullback $\pi^*[\Gamma, \alpha]$ of a decorated stratum class on $\overline{\mathcal{M}}_{g,n}$ is a combination of decorated strata classes on $\overline{\mathcal{M}}_{g,n+1}$.

d) Show that the pushforward $\pi_*[\Gamma, \alpha]$ of a decorated stratum class on $\overline{\mathcal{M}}_{g,n+1}$ is a combination of decorated strata classes on $\overline{\mathcal{M}}_{g,n}$.

This exercise not only finishes the proof of Theorem 6.28, but together with Proposition 6.31 also shows that the tautological rings are invariant under *pullback* (not just pushforward) by gluing and forgetful morphisms.

To summarize the content of this section: we defined the tautological ring $RH^*(\overline{\mathcal{M}}_{g,n})$, a subring of the cohomology $H^*(\overline{\mathcal{M}}_{g,n})$ of $\overline{\mathcal{M}}_{g,n}$. It is genererated by decorated strata classes $[\Gamma, \alpha]$ and we obtained explicit formulas for

- the product $[\Gamma_1, \alpha_1] \smile [\Gamma_2, \alpha_2]$ of two decorated strata classes and
- the pushforward and pullback of decorated strata classes under gluing morphisms ξ_Γ and forgetful morphisms π.

So inside the a priori mysterious cohomology ring $H^*(\overline{\mathcal{M}}_{g,n})$ we have found a subring in which we can do computations, described purely in terms of combinatorics of stable graphs.

6.4 *A panorama of results about the tautological rings

The definition of the tautological ring from the last section opens up a whole box of interesting questions: is every cohomology class on $\overline{\mathcal{M}}_{g,n}$ tautological? Can we give the relations between the generators $[\Gamma, \alpha]$ of $RH^*(\overline{\mathcal{M}}_{g,n})$? While we won't have time to explore these in detail, I wanted to finish by giving you a bit of an overview of the field.

The tautological ring in low genus

For $g = 0$ it is known that every cohomology class on $\overline{\mathcal{M}}_{0,n}$ is tautological. In fact, it is shown in [Kee92] that the vector space $H^*(\overline{\mathcal{M}}_{0,n})$ is generated by *undecorated* strata classes $[\Gamma, 1]$ and that all relations between these classes can be obtained from the *WDVV-relation* in $H^2(\overline{\mathcal{M}}_{0,4})$, illustrated in Figure 37, using forgetful and gluing morphisms.

For $g = 1$ it was shown in [Pet14] that every cohomology class of even degree on $\overline{\mathcal{M}}_{1,n}$ (i.e. those in $H^{2d}(\overline{\mathcal{M}}_{1,n})$ for some d) is tautological, but it was known before that there are nonzero odd cohomology groups, the first one being $H^{11}(\overline{\mathcal{M}}_{1,11}) = \mathbb{Q}$ (see [GP03, Section 4.1]). Since all tautological classes have even degree, this shows that there can be non-tautological cohomology classes on $\overline{\mathcal{M}}_{g,n}$.

Finally, for $g = 2$ the paper [GP03] gives an (explicit) closed algebraic subset $\overline{B} \subset \overline{\mathcal{M}}_{2,20}$ of complex codimension 11, such that its fundamental class $[\overline{B}] \in H^{22}(\overline{\mathcal{M}}_{2,20})$ is *not* tautological. On the other hand, [Pet16] shows that this is the first possible example in genus 2: all even cohomology classes on $\overline{\mathcal{M}}_{2,n}$ are tautological for $n \leq 19$.

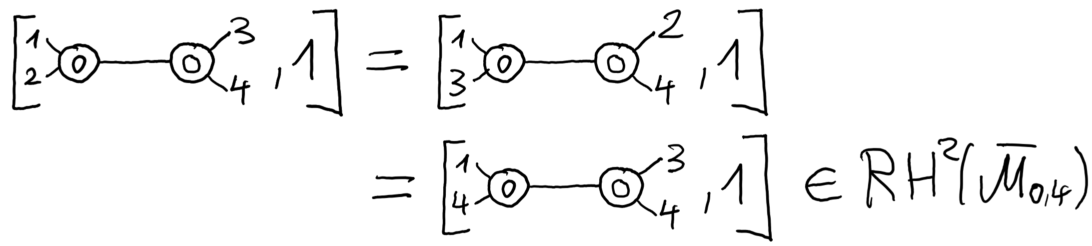


Figure 37: The WDVV relation on $\overline{\mathcal{M}}_{0,4}$; can you see how to prove it?

Remark 6.37. In fact, we know (almost) everything to understand their proof that $[\overline{B}]$ is not tautological: for the boundary divisor gluing map

$$\xi : \overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11} \to \overline{\mathcal{M}}_{2,20}$$

it is shown in [GP03] that the preimage of $\overline{B}$ under ξ is precisely the diagonal $\Delta \subset \overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$. Since this has the correct complex codimension 11, it follows from Proposition 6.18 that $\xi^*([\overline{B}])$ is a nonzero multiple of $[\Delta]$. But it is a general result from cohomology that the class $[\Delta]$ of the diagonal has a formula

$$[\Delta] = \sum_{e_i} e_i \otimes e^i \in H^*(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}),$$

where $(e_i)_i$ and $(e^i)^i$ are Poincaré dual bases of $H^*(\overline{\mathcal{M}}_{1,11})$. Since $H^{11}(\overline{\mathcal{M}}_{1,11}) \neq 0$ by the remark above, we have a nonzero term from $H^{11}(\overline{\mathcal{M}}_{1,11}) \otimes H^{11}(\overline{\mathcal{M}}_{1,11})$ appearing above. This shows that $[\overline{B}]$ is not tautological: otherwise, by Proposition 6.31 the pullback $\xi^*[\overline{B}]$, a multiple of $[\Delta]$, would have to lie in the tautological ring of $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$. But this only has terms coming from $RH^{2d_1}(\overline{\mathcal{M}}_{1,11}) \otimes RH^{2d_2}(\overline{\mathcal{M}}_{1,11})$, so that no term from $H^{11}(\overline{\mathcal{M}}_{1,11}) \otimes H^{11}(\overline{\mathcal{M}}_{1,11})$ could appear.

See [FP13] for more results about tautological and non-tautological cohomology classes on $\overline{\mathcal{M}}_{g,n}$.

Tautological relations

From Theorem 6.28 we know that the decorated strata classes $[\Gamma, \alpha]$ generate the tautological ring $RH^*(\overline{\mathcal{M}}_{g,n})$ as a $\mathbb{Q}$-vector space. However, in general they will not form a basis, but there will be linear relations between them (as we saw in the example of the WDVV-relation in Figure 37). To verify if two tautological classes are equal, we need to understand these.

A first set of relations (for the restrictions of the generators $[\Gamma, \alpha]$ to the moduli space $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ of smooth curves, for $n = 0$) was conjectured by Faber and Zagier and proved by Pandharipande and Pixton (see [PP13]). Later Pixton proposed a generalization of these relations to all $\overline{\mathcal{M}}_{g,n}$ ([Pix12]). These have by now been verified to hold first in cohomology ([PPZ15]) and later in the Chow rings[52] ([Jan17]). It is conjectured that the system of relations proposed by Pixton is complete, e.g. that it contains all tautological relations. However, as of now this conjecture is still quite open!

[52]If you know about Chow rings: analogous to Definition 6.22, the tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$ in Chow can be defined as the smallest system of subrings of the Chow rings of $\overline{\mathcal{M}}_{g,n}$ closed under pushforwards by gluing and forgetful maps. All the proofs we presented can still be carried out in this setting, in particular $R^*(\overline{\mathcal{M}}_{g,n})$ is generated by decorated stratum classes $[\Gamma, \alpha]$. The cycle class map $R^*(\overline{\mathcal{M}}_{g,n}) \to RH^*(\overline{\mathcal{M}}_{g,n})$ is surjective (essentially by definition) and we do not know a single example where it is not injective.

The intersection pairing and integrals of ψ-classes

While we have seen in Corollary 6.32 how to express the intersection products $[\Gamma_1, \alpha_1] \cdot [\Gamma_2, \alpha_2]$ of tautological classes in terms of decorated strata classes $[\Gamma, \alpha]$, we haven't yet seen how to compute the intersection pairing

$$RH^{2d}(\overline{\mathcal{M}}_{g,n}) \otimes RH^{2(3g-3+n-d)}(\overline{\mathcal{M}}_{g,n}) \to \mathbb{Q}, [\Gamma_1, \alpha_1] \otimes [\Gamma_2, \alpha_2] \mapsto \int_{\overline{\mathcal{M}}_{g,n}} [\Gamma_1, \alpha_1] \cdot [\Gamma_2, \alpha_2], \tag{138}$$

since this requires a formula for the degree

$$\int_{\overline{\mathcal{M}}_{g,n}} [\Gamma, \alpha] \in \mathbb{Q}$$

of a decorated stratum class $[\Gamma, \alpha]$. Since $[\Gamma, \alpha]$ is a pushforward of a product of monomials α_v in κ- and ψ-classes on the factors $\overline{\mathcal{M}}_{g(v),n(v)}$ of $\overline{\mathcal{M}}_\Gamma$, it follows that

$$\int_{\overline{\mathcal{M}}_{g,n}} [\Gamma, \alpha] = \prod_{v \in V(\Gamma)} \int_{\overline{\mathcal{M}}_{g(v),n(v)}} \alpha_v.$$

Thus we are reduced to computing integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \kappa_1^{e_1} \cdots \kappa_\ell^{e_\ell} \cdot \psi_1^{f_1} \cdots \psi_n^{f_n}. \tag{139}$$

But κ-classes are essentially forgetful pushforwards of powers of ψ-classes, and using Exercise 6.36 b) you can verify that it is possible to write (139) as a linear combination of integrals

$$\langle \tau_{f_1} \cdots \tau_{f_N} \rangle := \int_{\overline{\mathcal{M}}_{g,N}} \psi_1^{f_1} \cdots \psi_N^{f_N} \tag{140}$$

involving only ψ-classes, for some $N \geq n$. In [Kon92], Kontsevich proved an earlier conjecture of Witten about the intersection numbers (140), essentially giving a way to determine all of them recursively. See [Koc01, Section 3.3] for a nice introduction.

Tautological formulas for interesting cycle classes on $\overline{\mathcal{M}}_{g,n}$

One of the nice things about the tautological ring is that many cohomology classes on $\overline{\mathcal{M}}_{g,n}$ arising from some geometric construction happen to be tautological. In this case, we can find a formula for them, i.e. express them as a linear combination of clases $[\Gamma, \alpha]$. Having such a formula makes it easier to perform computations with them (e.g. compute intersection numbers, compare them with other classes, etc.).

A first example are the so-called *λ-classes*. They are obtained from the *Hodge-bundle* $\mathbb{E}$, a vector bundle of rank g on $\overline{\mathcal{M}}_{g,n}$ whose fibres are given by

$$\mathbb{E}|_{(C,p_1,\ldots,p_n)} = H^0(C, \Omega_C^1), \tag{141}$$

for C smooth[53]. We define the λ-classes as the Chern classes

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,n}), \text{ for } i = 1, \ldots, g.$$

In [Mum83], Mumford uses the *Grothendieck Riemann-Roch formula* (a vast generalization of the classical Riemann-Roch formula we applied to line bundles on curves) to calculate

[53] For a nodal curve C, the cotangent sheaf Ω_C^1 in (141) must be replaced by the *canonical line bundle* ω_C.

an explicit formula for the λ_i, showing that they are indeed contained in the tautological ring.

A second method for obtaining interesting cohomology classes is to find some algebraic subset of $\overline{\mathcal{M}}_{g,n}$ and to take its fundamental class. A particularly nice example are the *hyperelliptic loci* $\overline{\mathrm{Hyp}}_g \subset \overline{\mathcal{M}}_g$. They are defined as the (Zariski) closure of the locus

$$\mathrm{Hyp}_g = \{C : C \text{ is hyperelliptic}\} \subset \mathcal{M}_g \tag{142}$$

of smooth hyperelliptic curves[54]. It turns out that Hyp_g is an irreducible, closed algebraic subset of $\mathcal{M}_g$ of codimension $g-2$, so that we obtain

$$[\overline{\mathrm{Hyp}}_g] \in H^{2(g-2)}(\overline{\mathcal{M}}_g). \tag{143}$$

For $g=2$ we find that every smooth genus 2 curve C is hyperelliptic, so that $[\overline{\mathrm{Hyp}}_2] = [\overline{\mathcal{M}}_2]$. For $g=3$, the cycle $[\overline{\mathrm{Hyp}}_3]$ was first computed by Eisenbud and Harris in [EH87]. Their method was very simple: it was known that $H^*(\overline{\mathcal{M}}_3) = RH^*(\overline{\mathcal{M}}_3)$ and so by Poincaré duality, the intersection pairing

$$RH^2(\overline{\mathcal{M}}_3) \otimes RH^{10}(\overline{\mathcal{M}}_3) \to \mathbb{Q}$$

is perfect. Hence the class $[\overline{\mathrm{Hyp}}_3] \in RH^2(\overline{\mathcal{M}}_{g,n})$ is uniquely determined by its intersection with a generating set of $RH^{10}(\overline{\mathcal{M}}_3)$. Using different techniques, Faber and Pandharipande compute the class $[\overline{\mathrm{Hyp}}_4]$ in [FP05] and in fact show that $[\overline{\mathrm{Hyp}}_g]$ is tautological for all g! Finally, explicit formulas for $[\overline{\mathrm{Hyp}}_5]$ and $[\overline{\mathrm{Hyp}}_6]$ were found in the paper [Sv18] by myself and van Zelm, using essentially the same technique as Eisenbud and Harris, but now aided by our computer program [DSv20].

References and further reading (and rewatching)

A great reference for morphisms of stable graphs, fibre products of gluing maps and the formula for the intersection product of tautological classes is [GP03, Appendix A]. See also Section 2 of my own paper [Sv18] for a slightly more detailed discussion of the same material.

Good references for intersection theory are the classical book [Ful84] by Fulton and a more modern treatment [EH16] by Eisenbud and Harris. Reading them, you will notice that what we do in Section 6.2 is actually just a slightly disguised version of *algebraic intersection theory*, the theory of the Chow groups $A^*(X)$ of an algebraic variety X. It can be formulated purely in terms of algebraic geometric notions (no need for a complex topology). Roughly, for X smooth, the group $A^*(X)$ is generated by classes $[S]$ of subvarieties $S \subset X$ and you divide by an equivalence relation called *rational equivalence*. These groups have an intersection product (defined algebraically) and in the setting of Section 6.2 (X connected, smooth and proper and defined over $\mathbb{C}$) admit a ring homomorphism

$$\mathrm{cl} : A^*(X) \to H^{2*}(X)$$

sending the class $[S] \in A^*(X)$ to the fundamental class $[S] \in H^{2*}(X)$ we discussed above. Many of the operations we introduced (proper pushforward, flat pullback, etc) can be defined already on the level of Chow groups $A^*(X)$.

The theory of Chow groups has also been generalized to algebraic stacks (see [Vis89] for the treatment of Deligne-Mumford stacks and [Kre99] for more general algebraic stacks). This is the appropriate theory to use when studying intersection theory on $\overline{\mathcal{M}}_{g,n}$.

[54] Recall that C is hyperelliptic if it admits a map $C \to \mathbb{P}^1$ which is generically of degree 2.

Starting July 6th 2020 there will be an online reading group "Intersection theory on stacks" organized by Reinier Kramer from the MPI Bonn. If you are interested, you can write an email to Reinier.

For a much more comprehensive overview about the tautological rings, see the survey paper [Pan18] by Pandharipande. Now that you are more familiar with the moduli spaces of curves, you can also consider rewatching his ICM lecture, which I recommended in Section 1.

7 *Ideas of proof for the main theorem

Until now, we have treated the two main theorems about the properties of the moduli spaces $\overline{M}_{g,n}$ and the moduli stacks $\overline{\mathcal{M}}_{g,n}$ (Theorems 3.19 and 5.1) essentially as black boxes, without discussing how to prove them. In this optional section, we'll try to see the main ingredients of the proofs. These require advanced techniques from different areas of modern algebraic geometry, each of which merits their own lecture course. So we'll focus on the global picture and give references which contain the details.

7.1 Construction of the coarse moduli space of curves

Let's start by explaining how to construct the moduli space $\overline{M}_g$ of curves of genus $g \geq 2$ using Geometric Invariant Theory (GIT). A reference for this section is [HM98, Section 4.C].

The basic idea is that we first construct an auxilliary moduli space $\overline{K}_g$, parametrizing a stable curve C together with some additional data, then form a quotient of $\overline{K}_g$ by the action of an algebraic group, dividing out all possible choices of this extra data. We'll explain most of the story starting with the case of smooth curves C, and in the end comment what needs to be done for stable curves. Fixing an integer $k \geq 5$, the auxilliary space for the moduli M_g of smooth curves is

$$K_g = \left\{ (C, \varphi : C \to \mathbb{P}^r) : \begin{array}{c} C \text{ smooth, irreducible curve of genus } g \\ \varphi \text{ non-degenerate embedding}, \varphi^* \mathcal{O}_{\mathbb{P}^r}(1) \cong \omega_C^{\otimes k} \end{array} \right\},$$

where $\omega_C = \Omega_C^1$ is the canonical line bundle of C and

$$r = (2k-1)(g-1) - 1.$$

Also recall that $C \subset \mathbb{P}^r$ is non-degenerate if C is not contained in any hyperplane of $\mathbb{P}^r$.

Now why should we like K_g better than M_g? The reason is that K_g parametrizes subschemes $C \subset \mathbb{P}^r$ of the fixed variety $\mathbb{P}^r$ and Grothendieck ([Gro61]) constructed[55] a fine moduli space $\mathrm{Hilb}(\mathbb{P}^r)$, the *Hilbert scheme of* $\mathbb{P}^r$, parametrizing *all* subschemes of $\mathbb{P}^r$. Then K_g is simply a locally closed subscheme[56] of $\mathrm{Hilb}(\mathbb{P}^r)$ and forms a fine moduli space parametrizing tuples $(C, \varphi : C \to \mathbb{P}^r)$ as above.

Now let's see how to recover M_g from K_g. First we have

$$\deg(\omega_C^{\otimes k}) = 2k(g-1).$$

Since $k \geq 5$, the line bundle $\omega_C^{\otimes k}$ has degree greater than $2g+1$ and thus it is very ample ([Vak17, Section 19.2.11]), its first cohomology vanishes ([Vak17, Section 19.2.5]) and we have

$$h^0(C, \omega_C^{\otimes k}) = 2k(g-1) + 1 - g = r + 1.$$

Thus the morphism $\varphi : C \to \mathbb{P}^r$ is a non-degenerate embedding with $\varphi^* \mathcal{O}_{\mathbb{P}^r}(1) \cong \omega_C^{\otimes k}$ if and only if φ is given by a complete linear system of $\omega_C^{\otimes k}$. In other words (see also Example 2.4), the data of φ is equivalent to the choice of a basis $s_0, \ldots, s_r \in H^0(C, \omega_C^{\otimes k})$

[55] If you are unhappy that we construct the space M_g by using the existence of yet another moduli space $\mathrm{Hilb}(\mathbb{P}^r)$: after a long and technical proof, the existence of the Hilbert scheme follows from the existence of the Grassmannian variety $\mathrm{Gr}(m, n)$, and this can really be constructed by hand, gluing affine spaces along explicit open subschemes.

[56] Roughly, being smooth, irreducible and non-degenerate is an open condition and satisfying $\mathcal{O}_{\mathbb{P}^r}(1)|_C \cong \omega_C^{\otimes k}$ is a closed condition.

up to scaling. The fact that $\omega_C^{\otimes k}$ is very ample automatically implies that any such φ is a closed embedding. Thus we see

$$K_g = \left\{ (C, [s_0 : \ldots : s_r]) : \begin{array}{c} [s_0 : \ldots : s_r] \in \mathbb{P}(H^0(C, \omega_C^{\otimes k})^{\oplus r+1}) \\ s_0, \ldots, s_r \text{ form a basis of} \\ H^0(C, \omega_C^{\otimes k}) \end{array} \right\}.$$

The group PGL_{r+1} acts on K_g by leaving the curve C invariant and acting in a natural way on the tuples $[s_0 : \ldots s_r]$. This action comes from a bigger action $\mathrm{PGL}_{r+1} \curvearrowright \mathrm{Hilb}(\mathbb{P}^r)$ associated to the natural action $\mathrm{PGL}_{r+1} \curvearrowright \mathbb{P}^r$. Inside K_g, for a fixed C this action is simply transitive on the set of all possible choices $[s_0 : \ldots s_r]$ (any two choices of basis are related by a unique base change in PGL_{r+1}). Thus, *if we manage to define a reasonable notion of "quotient" in algebraic geometry*, we should have

$$K_g/\mathrm{PGL}_{r+1} = M_g.$$

The fact that K_g indeed admits such a quotient which is again a scheme can be proved using Geometric Invariant Theory. This is a theory developed by Mumford (see [MFK94]) to prove the existence of such quotients under suitable conditions. Checking these conditions for K_g is the hard technical core of the proof, which we will not discuss further.

To go from the case of smooth curves to the case of stable curves C, we should take ω_C above to be the *canonical sheaf* of C (see [Vak17, Chapter 30]). The canonical sheaf is a coherent sheaf associated to any projective variety. For a nodal, connected projective curve C it turns out to be a line bundle ω_C and it has the amazing property that ω_C is ample if and only if C is stable. Moreover, in this case $\omega_C^{\otimes k}$ is very ample for $n \geq 3$. Then we can define $\overline{K}_g \subset \mathrm{Hilb}(\mathbb{P}^r)$ similar as above and the argument still works.

Several properties of the coarse moduli space $\overline{M}_g$ now follow from this construction. For instance, the fact that each component of the Hilbert scheme is projective together with the machinery of Geometric Invariant Theory implies that the quotient $\overline{M}_g = \overline{K}_g/\mathrm{PGL}_{r+1}$ is in fact also a projective variety.

The above construction is also interesting from the perspective of algebraic stacks: it turns out that $\overline{K}_g$ is smooth (this can be shown using methods from Section 7.2 below, see [HM98, Claim (4.39)]) and the morphism $\overline{K}_g \to \overline{\mathcal{M}}_g$ is representable, smooth and surjective (see [DM69, Section 5]). So the construction of $\overline{K}_g$ is part of the proof that $\overline{\mathcal{M}}_g$ is an *algebraic* stack. Moreover, this atlas $\overline{K}_g \to \overline{\mathcal{M}}_g$ shows that the stack $\overline{\mathcal{M}}_g$ is smooth since $\overline{K}_g$ is smooth.

7.2 Dimension and smoothness

To understand why the stack $\mathcal{M}_{g,n}$ has dimension $3g - 3 + n$, a powerful tool is given by *deformation theory*. This is a theory developed for understanding the local structure (e.g. dimension and smoothness) of moduli spaces or stacks $\mathcal{M}$. Its core idea is to consider the functor of points of $\mathcal{M}$ on schemes of the form $\mathrm{Spec}(A)$ for A a *local Artinian*[57] $\mathbb{C}$*-algebra with residue field* $\mathbb{C}$. The simplest such algebra (apart from $\mathbb{C}$ itself) is given by the *dual numbers*

$$A = \mathbb{C}[\epsilon]/(\epsilon^2).$$

The following exercise shows that for the dual numbers A, understanding the A-points of $\mathcal{M}$ (for $\mathcal{M}$ a scheme) allows us to understand the Zariski tangent spaces at points of $\mathcal{M}$.

[57] Recall that a finitely generated $\mathbb{C}$-algebra is Artinian if it is of finite dimension as a $\mathbb{C}$-vector space.

Exercise 7.1 (see Exercise 12.1.I in [Vak17])**.** Let $\mathcal{M}$ be a complex scheme and let $p : \mathrm{Spec}(\mathbb{C}) \to \mathcal{M}$ be a $\mathbb{C}$-point of $\mathcal{M}$. Show that the elements of the Zariski tangent space

$$T_p\mathcal{M} = (\mathfrak{m}_{\mathcal{M},p}/\mathfrak{m}_{\mathcal{M},p}^2)^\vee, \text{ with } \mathfrak{m}_{\mathcal{M},p} \subset \mathcal{O}_{\mathcal{M},p} \text{ the maximal ideal}$$

are in bijection with the space

$$\mathrm{Mor}_{\mathbb{C}}(\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)), \mathcal{M})_p$$

of morphisms $\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)) \to \mathcal{M}$ such that the composition with

$$\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)) \to \mathrm{Spec}(\mathbb{C}), \epsilon \mapsto 0$$

equals the inclusion $p : \mathrm{Spec}(\mathbb{C}) \to \mathcal{M}$.

So let's try to see why for a smooth curve $C \in \mathcal{M}_g$ (for $g \geq 2$) the tangent space $T_C\mathcal{M}_g$ has dimension $3g-3$. By the exercise, elements of this tangent space correspond to maps $\mathrm{Spec}(A) \to \mathcal{M}_g$ with image $\{C\}$, for $A = \mathbb{C}[\epsilon]/(\epsilon^2)$. Since $\mathcal{M}_g$ is a moduli stack of smooth genus g curves, these are equivalent to families $\mathcal{C} \to \mathrm{Spec}(A)$ of smooth genus g curves such that the pullback $\mathcal{C}_0 \to \mathrm{Spec}(\mathbb{C})$ by the closed embedding $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(A)$ is isomorphic to C.

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, hook] & \mathrm{Spec}(A) \end{tikzcd} \tag{144}$$

Actually, this is a special case of a so-called *first order deformation*, and it makes sense to treat it in some more generality. For this, let X be a complex variety, then a first order deformation of X is a flat morphism $\mathcal{X} \to \mathrm{Spec}(A)$ together with an identification $\mathcal{X}_0 \cong X$ of X with the fibre $\mathcal{X}_0$ of $\mathcal{X}$ over $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(A)$. We illustrate the situation in Figure 38.

To understand the set of first-order deformations of a *smooth* variety X, we'll make a bunch of observations.

a) Since $(\epsilon) \subset A$ is the unique prime ideal, the underlying topological space of $\mathrm{Spec}(A)$ is a single point, and $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(A)$ is an isomorphism on the level of topological spaces. Similarly, since $\mathcal{X}_0 \subset \mathcal{X}$ is cut out by the square-zero ideal (ϵ), it is not difficult to see that the morphism $\mathcal{X}_0 \to \mathcal{X}$ is also an isomorphism of topological spaces. So the Zariski-open subsets U of $\mathcal{C}_0$ are in bijection with the open subsets $\mathcal{U}$ of $\mathcal{C}$. Moreover, it is true that U is affine if and only if $\mathcal{U}$ is affine (see [Ser06, Lemma 1.2.3]).

b) It turns out that first-order deformations $\mathcal{U} \to \mathrm{Spec}(A)$ of smooth, affine schemes U are very simple: each such deformation is isomorphic to the trivial deformation $\mathcal{U} \cong U \times \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ (see [Ser06, Theorem 1.2.4]). Thus if we choose an affine open cover U_i of X, each element of the corresponding affine cover $\mathcal{U}_i$ of $\mathcal{X}$ will be the trivial deformation $\mathcal{U}_i = U_i \times \mathrm{Spec}(A)$ of U_i. Note that this is independent of the particular first-order deformation $\mathcal{X}$: we can first choose the cover U_i of X and know that *every* deformation $\mathcal{X}$ will be trivial on each of the patches $\mathcal{U}_i$ of $\mathcal{X}$.

   Thus all the information of the deformation $\mathcal{X} \to \mathrm{Spec}(A)$ is contained in the data of the isomorphisms

$$\varphi_{ij} : U_{ij} \times \mathrm{Spec}(A) \xrightarrow{\sim} U_{ij} \times \mathrm{Spec}(A) \tag{145}$$

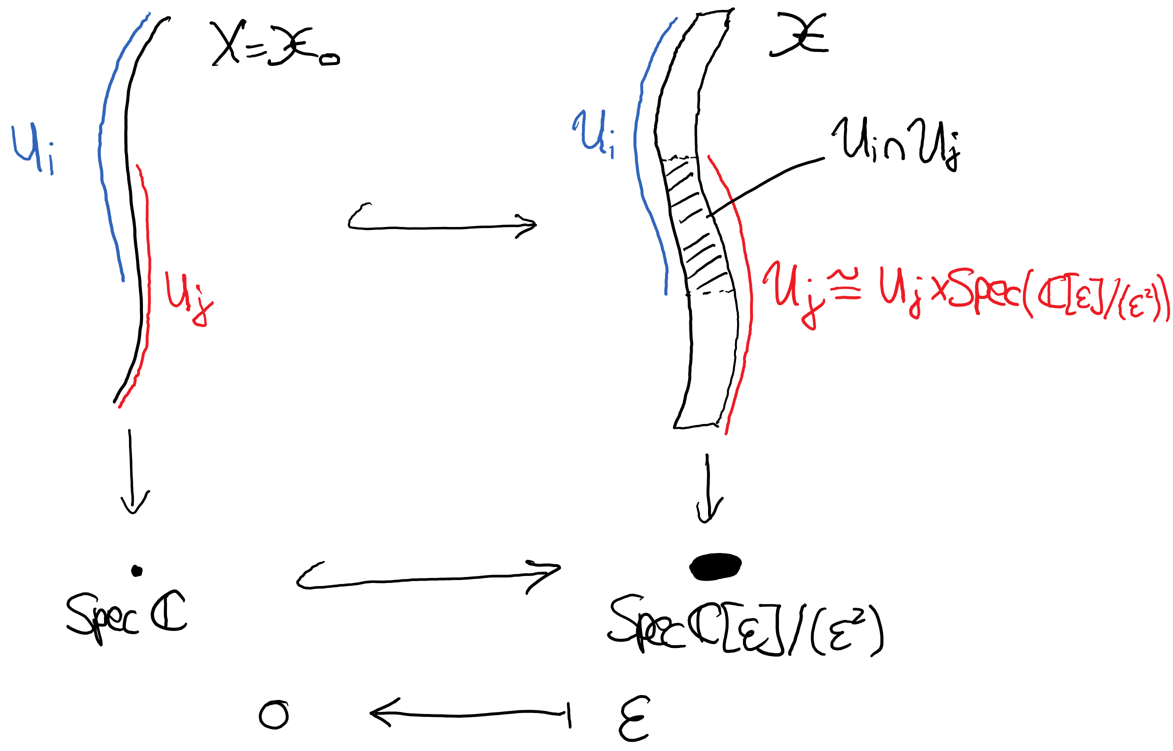


Figure 38: A first order deformation $\mathcal{X}$ of the smooth variety $X/\mathbb{C}$ together with an open, affine cover U_i of X and the induced cover $\mathcal{U}_i = U_i \times \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$ of $\mathcal{X}$

describing how the trivial deformations $U_i \times \mathrm{Spec}(A)$ are glued on the overlaps[58]. Apart from the usual cocycle condition $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$, the morphisms φ_{ij} must also satisfy

$$\varphi_{ij}|_{U_{ij} \times \mathrm{Spec}(\mathbb{C})} = \mathrm{id}_{U_{ij}}$$

since this is how they are glued in the special fibre X over $\mathrm{Spec}(\mathbb{C})$.

c) Since X is separated, the intersection $U_{ij} = U_i \cap U_j$ of the affine opens $U_i, U_j \subseteq X$ is again affine. Then $U_{ij} = \mathrm{Spec}(B)$ and $U_{ij} \times \mathrm{Spec}(A) = \mathrm{Spec}(B[\epsilon]/(\epsilon^2))$. One can check (see [Ser06, Lemma 1.2.6]) that the isomorphisms φ_{ij} above are exactly given by ring morphisms

$$B[\epsilon]/(\epsilon^2) \to B[\epsilon]/(\epsilon^2), (x + \epsilon y) \mapsto (x + \epsilon(y + \eta_{ij}(x)),$$

where $\eta_{ij} \in \mathrm{Der}_{\mathbb{C}}(B, B)$ is a $\mathbb{C}$-linear derivation on B. Such derivations η_{ij} are equivalent to vector fields $v_{ij} \in H^0(U_{ij}, T_{U_{ij}})$.

d) To summarize: for the fixed affine cover U_i of X, we can associate a system

$$(v_{ij} \in H^0(U_{ij}, T_{U_{ij}}))_{ij}$$

of vector fields on the overlaps U_{ij} to any first-order deformation of X. The fact that the gluing maps φ_{ij} satisfy a cocycle condition is equivalent to requiring a

[58] As you see, the idea of obtaining non-trivial objects by taking a collection of trivial objects and gluing them along a compatible system of isomorphisms on overlaps is quite pervasive.

corresponding cocycle condition for the fields v_{ij}. Tracing through the construction, one can check that the data of the first-order deformation $\mathcal{X} \to \mathrm{Spec}(A)$ (up to isomorphism) is equivalent to the corresponding element

$$(v_{ij})_{ij} \in H^1(X, T_X)$$

in the first Čech cohomology of the tangent bundle T_X of X.

Going back to the case $X = C \in \mathcal{M}_g$ we conclude

$$T_C\mathcal{M}_g \cong \left\{ \begin{array}{c} \mathcal{C} \to \mathrm{Spec}(A) \text{ first order} \\ \text{deformation of } C \end{array} \right\} \cong H^1(C, T_C). \tag{146}$$

Finally, by Serre duality we have

$$\begin{aligned} h^1(C, T_C) &= h^1(C, \omega_C^\vee) = h^0(C, (\omega_C^\vee)^\vee \otimes \omega_C) = h^0(C, \omega_C^{\otimes 2}) \\ &= \deg(\omega_C^{\otimes 2}) + 1 - g = 2(2g-2) + 1 - g = 3g - 3\,, \end{aligned}$$

where again we use that the higher cohomology of $\omega_C^{\otimes 2}$ vanishes since it has sufficiently high degree. So that's the magic formula for the dimension of $\mathcal{M}_g$ that we have been looking for!

Let's discuss some extensions of the story above.

Pointed curves

Looking at pointed curves $(C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n}$ is not much more difficult: similar arguments as above (see [HM98, Section 3B]), show that

$$T_{(C,p_1,\ldots,p_n)}\mathcal{M}_{g,n} \cong H^1(C, T_C(-p_1 - \ldots - p_n)),$$

and by an analogous computation to above we get the dimension $3g - 3 + n$.

Stable curves

The deformation theory of stable curves C is slightly more involved: the first-order deformations of C are given by the Ext-group $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$. Let Γ be the stable graph of C and let

$$(C_v, (q_h)_{h \in H(v)})_{v \in V(\Gamma)}$$

be the set of components of the normalization of C with q_h the preimages of the nodes. Then the Ext-group above fits into an exact sequence

$$0 \to \bigoplus_{v \in V(\Gamma)} H^1(C_v, T_{C_v}(-\sum_{h \in H(v)} q_h)) \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) \to \bigoplus_{\{h,h'\} \in E(\Gamma)} T_{q_h}(C_v) \otimes T_{q_{h'}}(C_{v'}) \to 0. \tag{147}$$

Here the subspace space on the left corresponds to *locally trivial* first order deformations of C (which preserve all singularities), whereas for each node $q \in C$ (corresponding to the edge $\{h, h'\} \in E(\Gamma)$) the one-dimensional space $T_{q_h}(C_v) \otimes T_{q_{h'}}(C_{v'})$ detects whether the deformation smoothes the node infinitesimally. Note that the direct sum of the spaces $T_{q_h}(C_v) \otimes T_{q_{h'}}(C_{v'})$ is precisely the fibre of the normal bundle

$$\mathcal{N}_{\xi_\Gamma} = \bigoplus_{\{h,h'\} \in E(\Gamma)} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee$$

of the gluing morphism ξ_Γ that we presented in Fact 6.30. This is no coincidence: in the exact sequence (147), we have that the tangent space of $\overline{\mathcal{M}}_\Gamma$ at the point

$$(C_v, (q_h)_h)_v = (C_v, (q_h)_{h \in H(v)})_{v \in V(\Gamma)} \in \overline{\mathcal{M}}_\Gamma$$

is given by

$$\bigoplus_{v \in V(\Gamma)} H^1(C_v, T_{C_v}(-\sum_{h \in H(v)} q_h)) = T_{(C_v,(q_h)_h)_v} \overline{\mathcal{M}}_\Gamma$$

and

$$\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) = T_C \overline{\mathcal{M}}_g$$

so indeed the direct sum of the $T_{q_h}(C_v) \otimes T_{q_{h'}}(C_{v'})$ is equal to the fibre of the normal bundle of ξ_Γ.

Higher-order deformations and obstructions

We saw that first-order deformations (over $A = \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)))$ allow us to compute the tangent space of a moduli space $\mathcal{M}$. By looking at higher-order deformations (over more general Artinian $\mathbb{C}$-algebras A), we can also say something about its local structure, and for instance detect if $\mathcal{M}$ is smooth.

For this recall that a scheme (or stack) $\mathcal{M}$ of finite type over $\mathbb{C}$ is *formally smooth* if for every surjective map $A' \to A$ of Artinian local $\mathbb{C}$-algebras with kernel I of $\mathbb{C}$-dimension 1 and every map $\mathrm{Spec}(A) \to \mathcal{M}$ there exists a map $\mathrm{Spec}(A') \to \mathcal{M}$ such that the diagram

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[rr, hook] \arrow[dr] & & \mathrm{Spec}(A') \arrow[dl, dashed, "\exists"] \\ & \mathcal{M} & \end{tikzcd}$$

commutes (see [Sta13, Tag 02HY]). Applying this to $\mathcal{M} = \mathcal{M}_g$, we need to show that every family of smooth curves over $\mathrm{Spec}(A)$ can be extended to $\mathrm{Spec}(A')$.

In this case, we can use a criterion which is valid in a more general situation: if X is a smooth variety such that $H^2(X, T_X) = 0$, then any deformation of X over $\mathrm{Spec}(A)$ can be extended to $\mathrm{Spec}(A')$, for A, A' as above (see [Ser06, Section 1.2.5]). One says that deformations of X are *unobstructed* in this case. For $X = C$ a smooth curve, we clearly have the vanishing $H^2(C, T_C)$ for dimension reasons, which shows that $\mathcal{M}_g$ is formally smooth.

Infinitesimal automorphisms

Now we saw that given a smooth, projective variety X, the first and second cohomology group of X told us important things about deformations of X. But it seems we skipped a case here: what about the *zeroth* cohomology? Indeed, it tells us something about (infinitesimal) automorphisms of X!

To make this precise, we note that the group $\mathrm{Aut}(X)$ of automorphisms of X can be given a natural scheme structure. This works by identifying an automorphism $\varphi : X \to X$ with its graph $\Gamma_\varphi \subset X \times X$ and then realizing $\mathrm{Aut}(X)$ as an open subscheme of the Hilbert scheme $\mathrm{Hilb}(X \times X)$ parametrizing all subschemes $\Gamma \subset X \times X$. Then it turns out that at the identity $\mathrm{id}_X \in \mathrm{Aut}(X)$ we have

$$T_{\mathrm{id}_X} \mathrm{Aut}(X) = H^0(X, T_X).$$

Looking at the example of $X = C$ a smooth curve of genus g, we check

$$\dim H^0(C, T_C) = \begin{cases} 3 & \text{for } g = 0, \\ 1 & \text{for } g = 1, \\ 0 & \text{for } g \geq 2. \end{cases}$$

Comparing with Fact 3.11, this corresponds to the statements that $\mathrm{PGL}_2 = \mathrm{Aut}(\mathbb{P}^1)$ has dimension 3, the automorphism group of a genus 1 curve E had dimension 1 (containing E itself), and the automorphism group of a curve of genus at least 2 is discrete.

Summarizing again, for X a smooth projective variety, we saw that there exist correspondences

$$\begin{aligned} H^0(X, T_X) &\longleftrightarrow \text{infinitesimal } \textbf{automorphisms} \text{ of } X, \\ H^1(X, T_X) &\longleftrightarrow \text{first-order } \textbf{deformations} \text{ of } X, \\ H^2(X, T_X) &\longleftrightarrow \textbf{obstructions} \text{ to extending deformations of } X. \end{aligned}$$

To conclude this section, here are two fun exercises which show that deformation theory can also be applied to other moduli spaces that we have already seen before.

Exercise 7.2 (Dimension of the Jacobian)**.** Fix a smooth genus g curve C and consider the Jacobian $\mathrm{Jac}(C) = \mathbf{Pic}^0_{C/\mathbb{C}}$, the moduli space of degree 0 line bundles on C. In analogy with (146) above, show that there exists a correspondence

$$T_{\mathcal{O}_C}\mathrm{Jac}(C) \cong \left\{ \begin{array}{c} \mathcal{L} \text{ first order deformation} \\ \text{of } \mathcal{O}_C \text{ on } C \times \mathrm{Spec}(A) \end{array} \right\} \cong H^1(C, \mathcal{O}_C). \tag{148}$$

Note that you need to define what a first order deformation of $\mathcal{O}_C$ on the trivial family $C \times \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ means. Conclude that the tangent space of $\mathrm{Jac}(C)$ at $\mathcal{O}_C$ is g-dimensional. Since $\mathrm{Jac}(C)$ is a group scheme and we are in characteristic zero, it is automatically smooth (see [Sta13, Tag 047N]), so this actually implies that $\mathrm{Jac}(C)$ is smooth of dimension g.

Exercise 7.3. Show that the tangent space $T_p\mathbb{P}^n$ of $\mathbb{P}^n$ at any closed point p is n-dimensional, by using the description of $\mathbb{P}^n$ as a moduli space together with Exercise 7.1.

7.3 Density of the locus of smooth curves and local structure of the boundary

Let's start with something basic that we never checked carefully: that the set $\mathcal{M}_{g,n}$ of smooth curves is open in $\overline{\mathcal{M}}_{g,n}$.

Proposition 7.4. Let $\pi : \mathcal{C} \to B$ be a family of stable curves. Then the locus $B_0 \subseteq B$ of b such that the fibre $\mathcal{C}_b$ is smooth is an open subset of B.

Proof. By definition, the locus $\mathcal{C}^{\mathrm{sm}} \subseteq \mathcal{C}$ of points where the morphism π is smooth is open in $\mathcal{C}$. Its complement, the set of nodes in $\mathcal{C}$, is closed and since π is proper by assumption, the locus

$$B^{\mathrm{sing}} = \pi(\mathcal{C} \setminus \mathcal{C}^{\mathrm{sm}}) \subseteq B$$

is also closed. But then $B_0 = B \setminus B^{\mathrm{sing}}$ is indeed open. $\square$

Corollary 7.5. The inclusion $i : \mathcal{M}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is representable and an open embedding.

Proof. For any $B \to \overline{\mathcal{M}}_{g,n}$, corresponding to a family $\pi : \mathcal{C} \to B$ of stable curves over B, the pullback of i under $B \to \overline{\mathcal{M}}_{g,n}$ is precisely the open embedding $B_0 \subseteq B$ from Proposition 7.4. Thus i is representable (since B_0 is a scheme) and an open embedding (since this property can be checked on a smooth cover of $\overline{\mathcal{M}}_{g,n}$ by schemes). □

Another fact we haven't discussed yet is that $\mathcal{M}_{g,n}$ is nonempty for every g, n with $2g - 2 + n > 0$. For this it suffices to show that for every $g \geq 0$ there exists a smooth curve C of genus g, since then (by Easy Exercise 3.12) the curve $(C, p_1, \ldots, p_n)$ has finite automorphisms for $2g - 2 + n$ for any distinct $p_1, \ldots, p_n \in C$.

To construct a curve of any genus $g \geq 0$ let

$$\lambda_1, \ldots, \lambda_{2g+2} \in \mathbb{C}$$

be $2g + 2$ distinct values, then the unique smooth projective curve C containing the affine curve

$$C^0 = \{(x, y) : y^2 = (x - \lambda_1) \cdots (x - \lambda_{2g+2})\} \subset \mathbb{C}^2$$

is smooth of genus g (see [Vak17, Section 19.5]). In fact, the morphism $C^0 \to \mathbb{C}, (x, y) \mapsto x$ extends to a double cover $C \to \mathbb{P}^1$, making C a *hyperelliptic* curve.

To summarize, we have now seen that $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ is nonempty and open. If we also knew that $\overline{\mathcal{M}}_{g,n}$ is irreducible, this clearly would imply that $\mathcal{M}_{g,n}$ is dense in $\overline{\mathcal{M}}_{g,n}$. On the other hand, it would be nice to see explicitly how, given a stable curve C_0, we can "approximate" it by smooth curves. In other words, we would like to find a family $\mathcal{C} \to B$ over a (small) irreducible base B such that $C_0 = \mathcal{C}_{b_0}$ appears as the fibre of some $b_0 \in B$ and such that the set

$$U = \{b \in B : \mathcal{C}_b \text{ is smooth}\} \subseteq B$$

is nonempty and dense in B. Indeed, such a family would correspond to a morphism $\varphi : B \to \overline{\mathcal{M}}_{g,n}$ and we have

$$C_0 = \varphi(b_0) \in \varphi(B) = \varphi(\overline{U}) \subseteq \underbrace{\overline{\varphi(U)}}_{\subseteq \mathcal{M}_{g,n}} \subseteq \overline{\mathcal{M}_{g,n}}.$$

Such a family $\mathcal{C} \to B$ was constructed in [DM69, Section 1] over a base of the form

$$B = \operatorname{Spec} \mathbb{C}[[t_1, \ldots, t_{3g-3+n}]],$$

with C_0 appearing as the fibre over the maximal ideal $b_0 = (x_1, \ldots, x_{3g-3+n})$ and the general fibre over B being smooth. The construction is based on ideas of (higher-order) deformation theory and in fact it gives a small, formal neighborhood of C_0 in $\overline{\mathcal{M}}_{g,n}$.

Instead of working with these complete local rings and formal schemes, let's explain what happens in the related language of complex-analytic spaces. Assume first that C_0 has a single node q, then by the definition of a node we can find a small (complex) open neighbourhood W of q of the form

$$W = \{(x, y) \in B_\epsilon(0) : xy = 0\} \subset \mathbb{C}^2.$$

But the singularity $(xy = 0)$ can be smoothed in a 1-parameter family. Let $\Delta \subset \mathbb{C}$ be a small disc around 0, then we have a family

$$\mathcal{W} = \{(x, y, t) \in B_\epsilon(0) \times \Delta : xy = t\} \to \Delta, (x, y, t) \mapsto t$$

such that all fibres $(xy = t)$ for $t \neq 0$ are smooth.

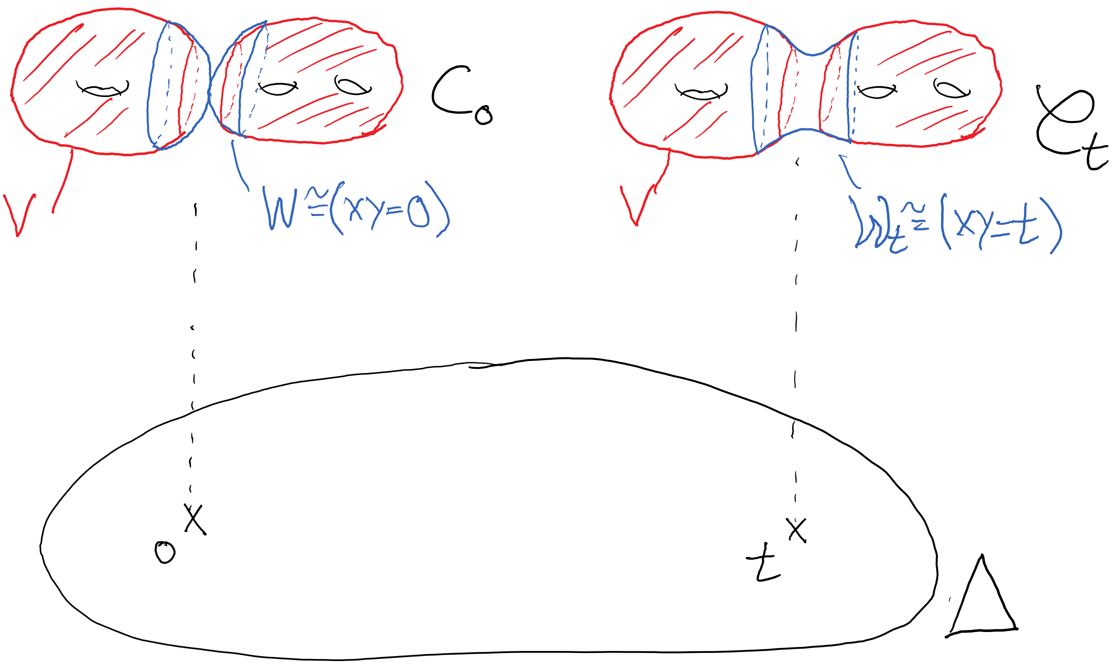


Figure 39: A deformation $\mathcal{C} \to \Delta$ of the curve C_0 over the disc Δ obtained by gluing the scheme $\mathcal{W}$ cut out by $xy = t$ (in blue) and the trivial family $V \times \Delta$ (in red)

To get a global deformation of C_0, let $V \subset C_0$ be an open subset not containing a small ball around the node q, such that $W \cup V = C_0$ and such that W, V overlap in two small annuli, as illustrated in Figure 39.

Then, in the complex-analytic world, we can glue the family $\mathcal{W}$ smoothing the node q to the trivial family $V \times \Delta \to \Delta$, obtaining a family $\mathcal{C} \to \Delta$ with smooth fibres away from $\{t = 0\} \subset \Delta$.

The story for curves with multiple nodes is similar: you get a coordinate t_e for every edge $e \in \Gamma(C_0)$ in the dual graph and a family $\mathcal{C} \to \Delta^{E(\Gamma)}$ such that locally around the node corresponding to e the curve has an equation $(x_e y_e = t_e)$.

The above family $\mathcal{C} \to \Delta^{E(\Gamma)}$ defines a map $\Delta^{E(\Gamma)} \to \overline{\mathcal{M}}_{g,n}$ whose image meets the boundary of $\overline{\mathcal{M}}_{g,n}$ transversally in the point C_0. This can be upgraded to get a full set of complex-analytic coordinates[59] of $\overline{\mathcal{M}}_{g,n}$ in a small neighborhood of C_0. For this, we need to choose additional coordinates parametrizing deformations of the normalization of C_0 (with the marked preimages of nodes). Such deformations can be made (painfully) explicit using *Schiffer variations* (see [HM98, Section 3.B]). For now, you should just think of them as coming from a local, holomorphic chart of the corresponding spaces $\overline{\mathcal{M}}_{g(v),n(v)}$ parametrizing the components of the normalization. We illustrate the situation in Figure 40.

After having described this in the language of complex analysis, let us finish this section by returning to the algebraic world and giving a well-defined statement there. For this, we define the *completed local ring* $\widehat{\mathcal{O}}_{C_0,\overline{\mathcal{M}}_{g,n}}$ of $\overline{\mathcal{M}}_{g,n}$ at a point C_0 to be the completed local ring $\widehat{\mathcal{O}}_{C_0',U}$ of any étale cover $U \to \overline{\mathcal{M}}_{g,n}$ at a preimage point $C_0' \in U$ of C_0. The fact that étale maps induce isomorphisms on completed local rings implies that this is independent

[59] Here we need to be slightly careful: $\overline{\mathcal{M}}_{g,n}$ is of course still a stack and C_0 might have automorphisms. What we describe in the following can be seen as a local complex-analytic chart of an étale cover of $\overline{\mathcal{M}}_{g,n}$ by a scheme (which exists since $\overline{\mathcal{M}}_{g,n}$ is a Deligne-Mumford stack.

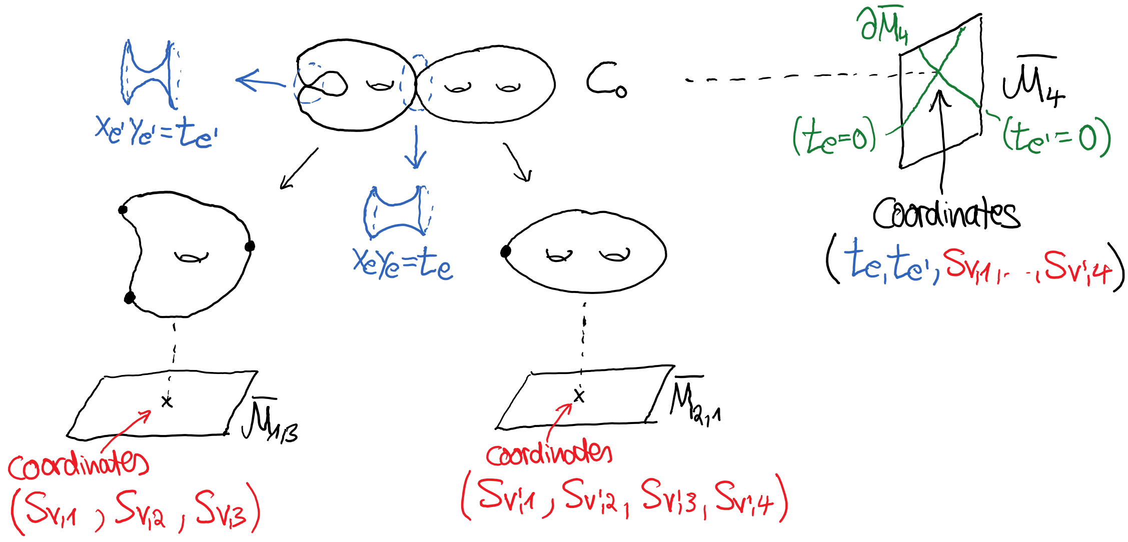


Figure 40: Holomorphic coordinates around the point $C_0 \in \overline{\mathcal{M}}_4$ arising from coordinates $t_e, t_{e'}$ smoothing the two nodes of C_0 and coordinates $s_{v,j}$ deforming the two components of the normalization of C_0. Note that the boundary $\partial\overline{\mathcal{M}}_4$ is cut out by the equations $t_e = 0$ and $t_{e'} = 0$ around C_0

of choices.

Theorem 7.6. Let $C_0 \in \overline{\mathcal{M}}_{g,n}$ be a closed point, let $\Gamma = \Gamma(C_0)$ be the stable graph associated to C_0 and let $(C_v, (q_h)_{h\in H(v)})_{v\in V(\Gamma)}$ be a preimage of C_0 under the gluing map ξ_Γ. Then we have an isomorphism

$$\widehat{\mathcal{O}}_{C_0,\overline{\mathcal{M}}_{g,n}} \cong \mathbb{C}[[t_e, s_{v,j} : e \in E(\Gamma), v \in V(\Gamma), j = 1, \ldots, 3g(v) - 3 + n(v)]]. \tag{149}$$

Here the coordinates t_e cut out the preimage of the boundary $\partial\overline{\mathcal{M}}_{g,n}$ under the natural map $\mathrm{Spec}(\widehat{\mathcal{O}}_{C_0,\overline{\mathcal{M}}_{g,n}}) \to \overline{\mathcal{M}}_{g,n}$, so that

$$\partial\overline{\mathcal{M}}_{g,n}|_{\mathrm{Spec}(\widehat{\mathcal{O}}_{C_0,\overline{\mathcal{M}}_{g,n}})} = \bigcup_{e\in E(\Gamma)} V(t_e). \tag{150}$$

On the other hand, for the boundary gluing map ξ_Γ, the functions $s_{v,j}$ pull back to generators of the completed local rings $\widehat{\mathcal{O}}_{C_v,\overline{\mathcal{M}}_{g(v),n(v)}}$ of the moduli spaces $\overline{\mathcal{M}}_{g(v),n(v)}$ occuring as factors of $\overline{\mathcal{M}}_\Gamma$ at the points $(C_v, (q_h)_h)_v$.

The fact that the boundary $\partial\overline{\mathcal{M}}_{g,n}$ pulls back to a union of coordinate hyperplanes 150 exactly shows that the boundary is indeed a normal crossings divisor on $\overline{\mathcal{M}}_{g,n}$. On the other hand, it is also clear that the preimage of $\mathcal{M}_{g,n}$, which is the complement of these hyperplanes, is dense in $\mathrm{Spec}(\widehat{\mathcal{O}}_{C_0,\overline{\mathcal{M}}_{g,n}})$.

References and further reading

A very comprehensive references for deformation theory is the book [Ser06] by Sernesi. You can also have a look at the book [Har10] by Hartshorne, in particular Section 27, where he discusses the moduli space of curves.

There are two more properties of $\overline{\mathcal{M}}_{g,n}$ that we have not discussed yet: irreducibility and properness.

For irreducibility, you can use the fact that $\overline{\mathcal{M}}_{g,n}$ is smooth to conclude that it is enough to show that the stack is *connected*. For this, one strategy is to start with any curve and degenerate it to the boundary of $\overline{\mathcal{M}}_{g,n}$ in a connected family. Then it suffices to show that the boundary is connected, which follows by an induction argument (the boundary is covered by lower-dimensional spaces). This strategy is described in [HM98, Section 6.A]. A different strategy is to use *Teichmüller theory* to study the moduli space $\mathcal{M}_{g,n}$ of smooth curves (see [ACG11, Chapter XV]) and show its irreducibility. Since we described an independent proof that $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ is dense in the last section, the irreducibility of $\mathcal{M}_{g,n}$ implies the irreducibility of $\overline{\mathcal{M}}_{g,n}$.

For properness, you can use the *valuative criterion of properness* (see [Sta13, Tag 0CL9]). Essentially you have to show that for a valuation ring Λ with fraction field K and a morphism $\mathrm{Spec}(K) \to \overline{\mathcal{M}}_{g,n}$ there is a unique extension of this morphism to $\mathrm{Spec}(\Lambda)$. This translates to saying that you have a family of stable curves $\mathcal{C}_K \to \mathrm{Spec}(K)$ and want to complete it (uniquely) to a family $\mathcal{C}_\Lambda \to \mathrm{Spec}(\Lambda)$. This construction is known as *stable reduction*, and described in [HM98, Section 3.C] or [ACG11, Chapter X.4].

8 Solutions to selected exercises

Solution 8.1 (Solution of Exercise 2.20). a) The morphism $s : U \to \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$ is given by $s = ([s_{1,1} : s_{1,2}], [s_{2,1}, s_{2,2}])$, for line bundles $\mathcal{L}_1, \mathcal{L}_2$ together with sections $s_{1,1}, s_{2,1}$ of $\mathcal{L}_1$ and $s_{1,2}, s_{2,2}$ of $\mathcal{L}_2$. The fact that s factors through the complement of Δ implies that the matrix

$$A = \begin{pmatrix} s_{1,1} & s_{1,2} \\ s_{2,1} & s_{2,2} \end{pmatrix} \tag{151}$$

is invertible at every point of U (i.e. its determinant, which is a section of $\mathcal{L}_1 \otimes \mathcal{L}_2$ is nowhere zero on U).

On an open cover $U = \bigcup_i U_i$ trivializing both line bundles, we can identify all sections above with actual functions, so that the formula A gives a well-defined map $U_i \to \mathrm{PGL}_2$ (here we use the non-vanishing of the determinant). Then s and s_0 are equivalent since $A \cdot s_0 = s$.

b) The restriction of $s_{\mathbb{P}^1 \times \mathbb{P}^1}$ to $U = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$ satisfies the conditions of part a), so that on an open cover U_i of U it is equivalent to the constant morphism s_0. But $s_0 \in h(U_i)$ is itself a pullback of $i \in h(\mathrm{pt})$ under the map $U_i \to \mathrm{pt}$. Thus we have a diagram of schemes and elements of h on these schemes which are pullbacks under the morphisms

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^1 & U \arrow[l] & \coprod_i U_i \arrow[l] \arrow[r] & \mathrm{pt} \\
s_{\mathbb{P}^1 \times \mathbb{P}^1} & s_{\mathbb{P}^1 \times \mathbb{P}^1}|_U & \coprod_i s_{\mathbb{P}^1 \times \mathbb{P}^1}|_{U_i} & i
\end{tikzcd}$$

Applying the natural transformation Φ' we obtain a diagram of morphisms to M' as follows:

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[drr, "\Phi'(s_{\mathbb{P}^1 \times \mathbb{P}^1})"'] & U \arrow[l] \arrow[dr] & \coprod_i U_i \arrow[l] \arrow[r] \arrow[d] & \mathrm{pt} \arrow[dl, "\psi"] \\
& & M' &
\end{tikzcd}$$

As all maps $U_i \to M'$ factor through $\mathrm{pt} \to M'$ and the U_i form a Zariski cover, also $U \to M'$ factors through pt. Since the image of $\mathrm{pt} \to M'$ is closed and the map $\Phi'(s_{\mathbb{P}^1 \times \mathbb{P}^1})$ is continuous, it must factor (as a set map) through the image point of pt. We claim that it also factors as a scheme morphism. Indeed, it must factor through any affine open neighborhood of the image point of pt, but since $\mathbb{P}^1 \times \mathbb{P}^1$ is projective, such a map is constant as a map of schemes and thus factors through pt.

c) The map $f = s$ has the desired property for the first part of the exercise. Given such X and $s \in h(X)$, the natural transformation Φ' sends s to a morphism $\Phi'(s) : X \to M'$. We show that Φ' factors via Φ and the natural transformation $\Psi : h^{\mathrm{pt}} \to h^{M'}$ associated to $\psi : \mathrm{pt} \to M'$ above. This is equivalent to showing that $\Phi'(s)$ factors through ψ. But we have a diagram

$$\begin{tikzcd}
h(\mathbb{P}^1 \times \mathbb{P}^1) \arrow[r, "h(f)"] \arrow[d, "\Phi'"] & h(X) \arrow[d, "\Phi'"] \\
\mathrm{Mor}(\mathbb{P}^1 \times \mathbb{P}^1, M') \arrow[r, "\circ f"] & \mathrm{Mor}(X, M')
\end{tikzcd}$$

For the element $s_{\mathbb{P}^1 \times \mathbb{P}^1}$ in the upper left, going down and right sends it to the morphism

$$X \xrightarrow{f} \mathbb{P}^1 \times \mathbb{P}^1 \to \mathrm{pt} \xrightarrow{\psi} M'$$

by part b). On the other hand, the map $h(f)$ sends $s_{\mathbb{P}^1 \times \mathbb{P}^1}$ to s by the choice of f and going down to $\mathrm{Mor}(X, M')$ we obtain $\Phi'(s)$. Thus this morphism factors through ψ as desired.

d) The element

$$(i' : \mathrm{pt} \xrightarrow{([1:0],[1:0])} \mathbb{P}^1 \times \mathbb{P}^2) \in h(\mathrm{pt}).$$

is not equivalent to $i \in h(\mathrm{pt})$ and in fact $h(\mathrm{pt}) = \{i, i'\}$ has two elements, whereas $\mathrm{Mor}(\mathrm{pt}, \mathrm{pt})$ has exactly one element.

9 Questions

Question 9.1 (see Exercise 2.8)**.** Let $\pi_X : X \to Z$, $\pi_Y : Y \to Z$ be morphisms of schemes. For any scheme S define

$$h(S) = (h^X \times_{h^Z} h^Y)(S) = \left\{ (\sigma_X, \sigma_Y) : \begin{array}{c} \sigma_X : S \to X, \sigma_Y : S \to Y \\ \text{such that } \pi_X \circ \sigma_X = \pi_Y \circ \sigma_Y \end{array} \right\}$$

a) Show that h defines a moduli functor.

b) Prove that the fibre product $X \times_Z Y$ is a fine moduli space for h (you can use standard properties of the fibre product). What is its universal family?

Question 9.2 (see Exercise 2.17)**.** a) Show that every fine moduli space is also a coarse moduli space (in particular, make precise what this statement means).

b) Show that given a moduli functor h having a coarse moduli space (M, Φ), this space is unique up to isomorphism.

Question 9.3. Let $E \subset \mathbb{P}^2$ be a smooth, irreducible cubic curve.

a) Compute the geometric and arithmetic genus of E.

b) Let $L \subset \mathbb{P}^2$ be a line in general position and consider the curve $C = E \cup L$. You can use without proof that C is a nodal curve. Is C stable? If so, draw its dual graph and compute its arithmetic and geometric genus.

Question 9.4 (see Exercise 3.12)**.** Let C be a smooth, complex, irreducible projective curve of genus g and $p_1, \ldots, p_n \in C$ be distinct points.

a) Show that $\mathrm{Aut}(C, p_1, \ldots, p_n)$ is finite if and only if $2g - 2 + n > 0$.

b) For $C = \mathbb{P}^1$ and $n = 3$, compute the orders of the groups $\mathrm{Aut}(\mathbb{P}^1, p_1, p_2, p_3)$ and

$$\mathrm{Aut}(\mathbb{P}^1, \{p_1, p_2, p_3\}) = \{\varphi \in \mathrm{Aut}(\mathbb{P}^1) : \varphi(\{p_1, p_2, p_3\}) = \{p_1, p_2, p_3\}\}.$$

Question 9.5 (see Exercise 4.4)**.** Explain the isomorphism

$$M_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta \tag{152}$$

that we discussed in the lecture. In particular, for $n = 4$ compute which point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is associated to the point

$$(\mathbb{P}^1, \infty, 42, 0, \pi) \in M_{0,4}.$$

What is the universal family over $M_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta$?

Question 9.6 (see Exercise 4.11)**.** a) Show that a stable graph of genus g with n legs has at most $3g - 3 + n$ edges.

b) Compute the number of isomorphism classes of stable graphs with exactly one edge for $g = 5, n = 4$.

Question 9.7 (see Exercise 4.19). a) Show that the graph Γ from Figure 41 has trivial automorphism group.

b) Compute the order of the automorphism group $\mathrm{Aut}(\Gamma')$ of Γ'. Let (C, p_1) be a stable curve with dual graph Γ'. Does the automorphism group $\mathrm{Aut}(C, p_1)$ have the same order as $\mathrm{Aut}(\Gamma')$?

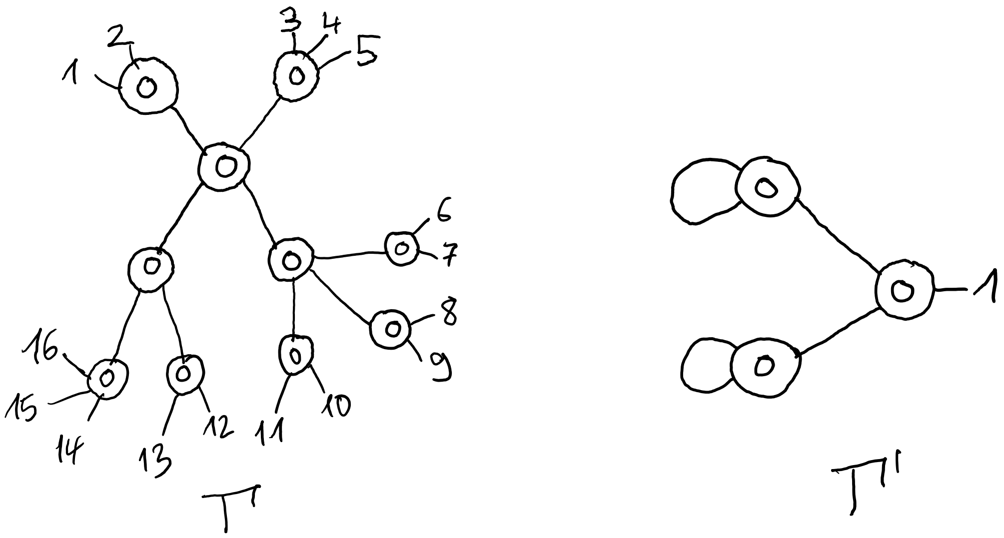


Figure 41: Stable graphs Γ and Γ'

Question 9.8 (see Exercise 4.28). Figure 42 illustrates the forgetful morphism $\pi : \overline{M}_{1,2} \to \overline{M}_{1,1}$ with the boundary of both spaces marked in red. For each of the points marked in blue, draw their corresponding curves and their dual graphs.

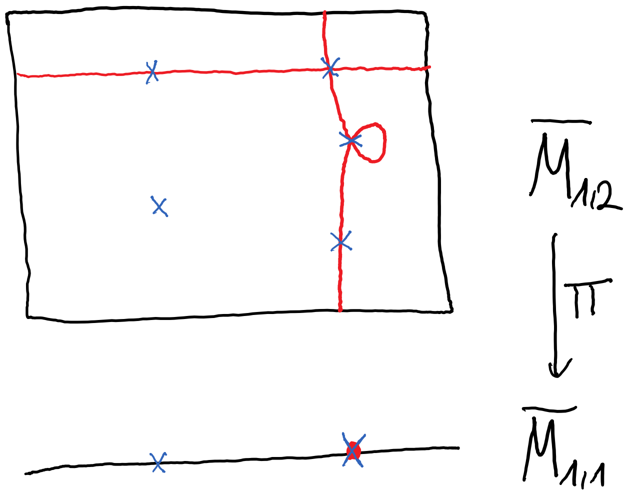


Figure 42: The forgetful morphism $\pi : \overline{M}_{1,2} \to \overline{M}_{1,1}$

Question 9.9. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism of degree d. Compute

$$f_* : H^*(\mathbb{P}^1) \to H^*(\mathbb{P}^1) \text{ and } f^* : H^*(\mathbb{P}^1) \to H^*(\mathbb{P}^1)$$

on the basis $1, H$ of $H^*(\mathbb{P}^1)$.

Question 9.10. Consider the stable graphs Γ_1, Γ_2 in Figure 43.

a) What is the genus g and number of legs n of these graphs. What are the cohomological degrees $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ such that the decorated stratum classes $[\Gamma_i, 1]$ (with $\alpha = 1 \in H^0(\overline{\mathcal{M}}_{\Gamma_i})$) are contained in $H^{k_i}(\overline{\mathcal{M}}_{g,n})$?

b) The set $\mathfrak{G}_{\Gamma_1,\Gamma_2}$ of generic (Γ_1, Γ_2)-structures $(\Gamma, \varphi_1, \varphi_2)$ has precisely 3 elements. Draw the three possible graphs Γ that appear. You don't have to prove that these are the only ones.

c) Compute the cup product $[\Gamma_1, 1] \smile [\Gamma_2, 1]$ as a sum of decorated stratum classes.

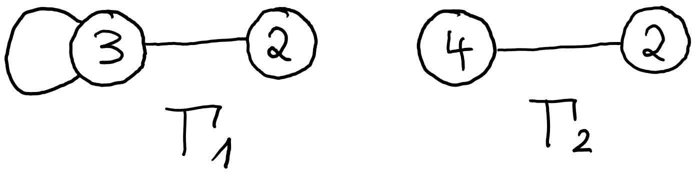


Figure 43: Two stable graphs

References

[ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[Ben14] L. Benzo. Uniruledness of some moduli spaces of stable pointed curves. *J. Pure Appl. Algebra*, 218(3):395–404, 2014.

[Ber13] José Bertin. Algebraic stacks with a view toward moduli stacks of covers. In Pierre Dèbes, Michel Emsalem, Matthieu Romagny, and A. Muhammed Uludağ, editors, *Arithmetic and Geometry Around Galois Theory*, pages 1–148, Basel, 2013. Springer Basel.

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[BV05] Andrea Bruno and Alessandro Verra. $\overline{M}_{15}$ is rationally connected. In *Projective varieties with unexpected properties*, pages 51–65. Walter de Gruyter, Berlin, 2005.

[CF07] G. Casnati and C. Fontanari. On the rationality of moduli spaces of pointed curves. *J. Lond. Math. Soc. (2)*, 75(3):582–596, 2007.

[DM69] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[DSv20] Vincent Delecroix, Johannes Schmitt, and Jason van Zelm. admcycles – a Sage package for calculations in the tautological ring of the moduli space of stable curves. *arXiv e-prints*, page arXiv:2002.01709, February 2020.

[EH87] D. Eisenbud and J. Harris. The kodaira dimension of the moduli space of curves of genus ... 23. *Inventiones mathematicae*, 90:359–388, 1987.

[EH16] David Eisenbud and Joe Harris. *3264 and All That: A Second Course in Algebraic Geometry*. Cambridge University Press, 2016.

[Fan01] Barbara Fantechi. Stacks for Everybody. In Carles Casacuberta, Rosa Maria Miró-Roig, Joan Verdera, and Sebastià Xambó-Descamps, editors, *European Congress of Mathematics*, pages 349–359, Basel, 2001. Birkhäuser Basel.

[Far09] Gavril Farkas. The global geometry of the moduli space of curves. In *Algebraic geometry—Seattle 2005. Part 1*, volume 80 of *Proc. Sympos. Pure Math.*, pages 125–147. Amer. Math. Soc., Providence, RI, 2009.

[FP00] C. Faber and R. Pandharipande. Logarithmic series and Hodge integrals in the tautological ring. *Michigan Math. J.*, 48:215–252, 2000. With an appendix by Don Zagier, Dedicated to William Fulton on the occasion of his 60th birthday.

[FP05] Carel Faber and Rahul Pandharipande. Relative maps and tautological classes. *JEMS*, 7:13–49, 2005.

[FP13] C. Faber and R. Pandharipande. Tautological and non-tautological cohomology of the moduli space of curves. In *Handbook of moduli. Vol. I*, volume 24 of *Adv. Lect. Math. (ALM)*, pages 293–330. Int. Press, Somerville, MA, 2013.

[Ful84] William Fulton. *Intersection theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1984.

[GP03] Tom Graber and Rahul Pandharipande. Constructions of nontautological classes on moduli spaces of curves. *Michigan Math. J.*, 51(1):93–109, 2003.

[Gro61] Alexander Grothendieck. Techniques de construction et théorèmes d'existence en géométrie algébrique iv : les schémas de hilbert. In *Séminaire Bourbaki : années 1960/61, exposés 205-222*, number 6 in Séminaire Bourbaki, pages 249–276. Société mathématique de France, 1961. talk:221.

[Gro66] Alexander Grothendieck. Le groupe de brauer : I. algèbres d'azumaya et interprétations diverses. In *Séminaire Bourbaki : années 1964/65 1965/66, exposés 277-312*, number 9 in Séminaire Bourbaki, pages 199–219. Société mathématique de France, 1966. talk:290.

[Hai08] Richard Hain. Lectures on Moduli Spaces of Elliptic Curves, 2008.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Har84] J. Harris. On the kodaira dimension of the moduli space of curves, ii. the even-genus case. *Inventiones mathematicae*, 75:437–466, 1984.

[Har10] Robin Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics*. Springer, New York, 2010.

[HM82] Joe Harris and David Mumford. On the kodaira dimension of the moduli space of curves. *Inventiones mathematicae*, 67:23–86, 1982.

[HM98] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[Jan17] Felix Janda. Relations on $\overline{M}_{g,n}$ via equivariant Gromov-Witten theory of $\mathbb{P}^1$. *Algebr. Geom.*, 4(3):311–336, 2017.

[Kee92] Sean Keel. Intersection theory of moduli space of stable n-pointed curves of genus zero. *Trans. Amer. Math. Soc.*, 330(2):545–574, 1992.

[Kle05] Steven L. Kleiman. The Picard scheme. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 235–321. Amer. Math. Soc., Providence, RI, 2005.

[Knu83a] Finn F. Knudsen. The projectivity of the moduli space of stable curves, ii: The stacks $m_{g,n}$. *Mathematica Scandinavica*, 52:161–199, Dec. 1983.

[Knu83b] Finn F. Knudsen. The projectivity of the moduli space of stable curves, iii: The line bundles on $m_{g,n}$, and a proof of the projectivity of $m_{g,n}$ in characteristic 0. *Mathematica Scandinavica*, 52:200–212, 1983.

[Koc01] Joachim Kock. *Notes on psi classes*. 2001.

[Kon92] Maxim Kontsevich. Intersection theory on the moduli space of curves and the matrix Airy function. *Comm. Math. Phys.*, 147(1):1–23, 1992.

[Kre99] Andrew Kresch. Cycle groups for Artin stacks. *Invent. Math.*, 138(3):495–536, 1999.

[KV07] Joachim Kock and Israel Vainsencher. *An invitation to quantum cohomology*, volume 249 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2007. Kontsevich's formula for rational plane curves.

[Liu] Qing Liu. Homotopy invariance of the Picard group. Mathematics Stack Exchange. (version: 2013-07-03).

[Log03] Adam Logan. The Kodaira dimension of moduli spaces of curves with marked points. *Amer. J. Math.*, 125(1):105–138, 2003.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[Mum83] David Mumford. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 271–328. Birkhäuser Boston, Boston, MA, 1983.

[Pan18] Rahul Pandharipande. A calculus for the moduli space of curves. In *Algebraic geometry: Salt Lake City 2015*, volume 97 of *Proc. Sympos. Pure Math.*, pages 459–487. Amer. Math. Soc., Providence, RI, 2018.

[Pet14] Dan Petersen. The structure of the tautological ring in genus one. *Duke Math. J.*, 163(4):777–793, 2014.

[Pet16] Dan Petersen. Tautological rings of spaces of pointed genus two curves of compact type. *Compos. Math.*, 152(7):1398–1420, 2016.

[Pix12] A. Pixton. Conjectural relations in the tautological ring of $\overline{M}_{g,n}$. *ArXiv e-prints*, July 2012.

[PP13] R. Pandharipande and A. Pixton. Relations in the tautological ring of the moduli space of curves. *ArXiv e-prints*, January 2013.

[PPZ15] Rahul Pandharipande, Aaron Pixton, and Dimitri Zvonkine. Relations on $\overline{M}_{g,n}$ via 3-spin structures. *J. Amer. Math. Soc.*, 28(1):279–309, 2015.

[S$^+$20] W. A. Stein et al. *Sage Mathematics Software (Version 9.0)*. The Sage Development Team, 2020. `http://www.sagemath.org`.

[Sch17] Peter Scholze. Algebraic Geometry I, 2017. Lecture notes.

[Ser06] Edoardo Sernesi. *Deformations of algebraic schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[Sta13] The Stacks Project Authors. Stacks project. `http://stacks.math.columbia.edu`, 2013.

[Sv18] Johannes Schmitt and Jason van Zelm. Intersections of loci of admissible covers with tautological classes. *arXiv e-prints*, page arXiv:1808.05817, August 2018.

[Vak17] Ravi Vakil. The rising sea - Foundations of Algebraic Geometry, 2017. Lecture notes.

[Ver05] Alessandro Verra. The unirationality of the moduli spaces of curves of genus 14 or lower. *Compos. Math.*, 141(6):1425–1444, 2005.

[Vis89] Angelo Vistoli. Intersection theory on algebraic stacks and on their moduli spaces. *Invent. Math.*, 97(3):613–670, 1989.