8 Questions

Question 8.1 (see Exercise 2.8)**.** Let $\pi_X : X \to Z$, $\pi_Y : Y \to Z$ be morphisms of schemes. For any scheme S define

$$h(S) = (h^X \times_{h^Z} h^Y)(S) = \left\{ (\sigma_X, \sigma_Y) : \begin{array}{c} \sigma_X : S \to X, \sigma_Y : S \to Y \\ \text{such that } \pi_X \circ \sigma_X = \pi_Y \circ \sigma_Y \end{array} \right\}$$

a) Show that h defines a moduli functor.

b) Prove that the fibre product $X \times_Z Y$ is a fine moduli space for h (you can use standard properties of the fibre product). What is its universal family?

Question 8.2 (see Exercise 2.17)**.** a) Show that every fine moduli space is also a coarse moduli space (in particular, make precise what this statement means).

b) Show that given a moduli functor h having a coarse moduli space (M, Φ), this space is unique up to isomorphism.

Question 8.3. Let $E \subset \mathbb{P}^2$ be a smooth, irreducible cubic curve.

a) Compute the geometric and arithmetic genus of E.

b) Let $L \subset \mathbb{P}^2$ be a line in general position and consider the curve $C = E \cup L$. You can use without proof that C is a nodal curve. Is C stable? If so, draw its dual graph and compute its arithmetic and geometric genus.

Question 8.4 (see Exercise 3.12)**.** Let C be a smooth, complex, irreducible projective curve of genus g and $p_1, \ldots, p_n \in C$ be distinct points.

a) Show that $\mathrm{Aut}(C, p_1, \ldots, p_n)$ is finite if and only if $2g - 2 + n > 0$.

b) For $C = \mathbb{P}^1$ and $n = 3$, compute the orders of the groups $\mathrm{Aut}(\mathbb{P}^1, p_1, p_2, p_3)$ and

$$\mathrm{Aut}(\mathbb{P}^1, \{p_1, p_2, p_3\}) = \{\varphi \in \mathrm{Aut}(\mathbb{P}^1) : \varphi(\{p_1, p_2, p_3\}) = \{p_1, p_2, p_3\}\}.$$

Question 8.5 (see Exercise 4.4)**.** Explain the isomorphism

$$M_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta \tag{135}$$

that we discussed in the lecture. In particular, for $n = 4$ compute which point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is associated to the point

$$(\mathbb{P}^1, \infty, 42, 0, \pi) \in M_{0,4}.$$

What is the universal family over $M_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta$?

Question 8.6 (see Exercise 4.11)**.** a) Show that a stable graph of genus g with n legs has at most $3g - 3 + n$ edges.

b) Compute the number of isomorphism classes of stable graphs with exactly one edge for $g = 5, n = 4$.

Question 8.7 (see Exercise 4.19). a) Show that the graph Γ from Figure 35 has trivial automorphism group.

b) Compute the order of the automorphism group $\mathrm{Aut}(\Gamma')$ of Γ'. Let (C, p_1) be a stable curve with dual graph Γ'. Does the automorphism group $\mathrm{Aut}(C, p_1)$ have the same order as $\mathrm{Aut}(\Gamma')$?

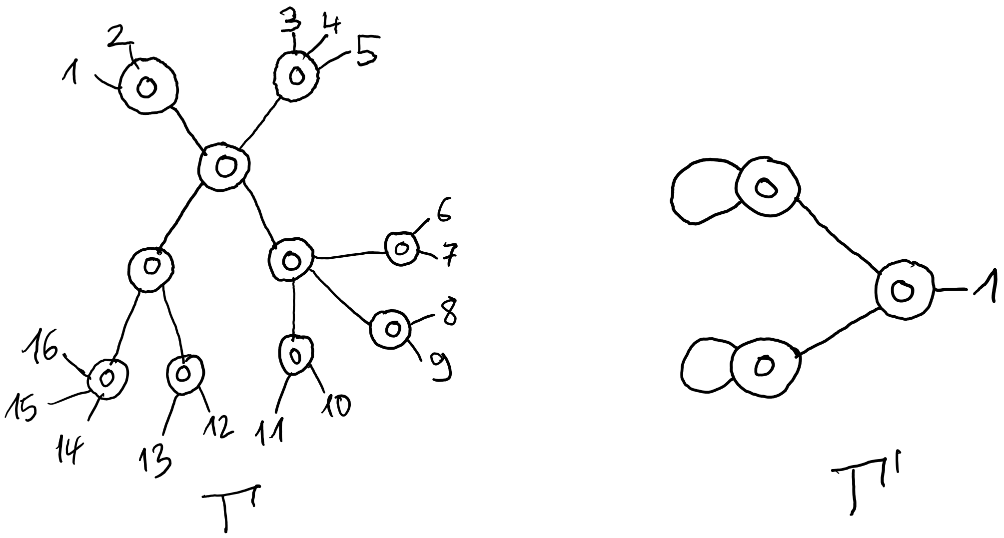

Figure 35: Stable graphs Γ and Γ'

Question 8.8 (see Exercise 4.28). Figure 36 illustrates the forgetful morphism $\pi : \overline{M}_{1,2} \to \overline{M}_{1,1}$ with the boundary of both spaces marked in red. For each of the points marked in blue, draw their corresponding curves and their dual graphs.

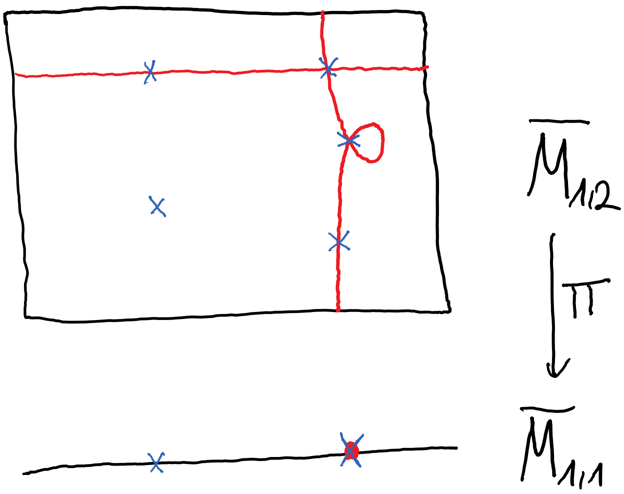

Figure 36: The forgetful morphism $\pi : \overline{M}_{1,2} \to \overline{M}_{1,1}$

Question 8.9. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism of degree d. Compute

$$f_* : H^*(\mathbb{P}^1) \to H^*(\mathbb{P}^1) \text{ and } f^* : H^*(\mathbb{P}^1) \to H^*(\mathbb{P}^1)$$

on the basis $1, H$ of $H^*(\mathbb{P}^1)$.

Question 8.10. Consider the stable graphs Γ_1, Γ_2 in Figure 37.

a) What is the genus g and number of legs n of these graphs. What are the cohomological degrees $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ such that the decorated stratum classes $[\Gamma_i, 1]$ (with $\alpha = 1 \in H^0(\overline{\mathcal{M}}_{\Gamma_i})$) are contained in $H^{k_i}(\overline{\mathcal{M}}_{g,n})$?

b) The set $\mathcal{G}_{\Gamma_1,\Gamma_2}$ of generic (Γ_1, Γ_2)-structures $(\Gamma, \varphi_1, \varphi_2)$ has precisely 3 elements. Draw the three possible graphs Γ that appear. You don't have to prove that these are the only ones.

c) Compute the cup product $[\Gamma_1, 1] \smile [\Gamma_2, 1]$ as a sum of decorated stratum classes.

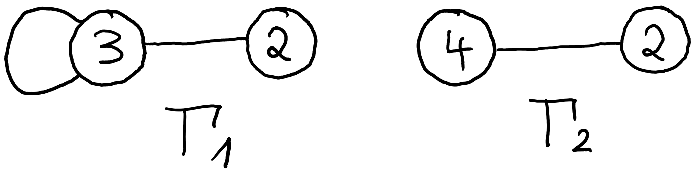

Figure 37: Two stable graphs