Dr. E. Hellmann SS 2014

Algebraic Geometry II

Exercise Sheet 7

Due Date: 16.06.2014

Exercise 1:

Let k be a field and let X be an integral k-scheme of finite type. Let $K = k(X) = \mathcal{O}_{X,\eta}$ denote its function field (here $\eta \in X$ is the generic point). Show that the following conditions are equivalent:

(i) The field extension K over k is separable.

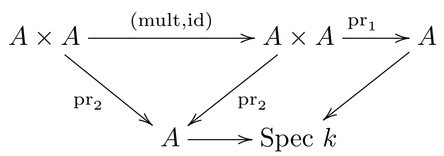
(ii) There exists a dense open subset $U \subset X$ such that U is smooth over k.

Exercise 2:

(i) Let X and Y be S-schemes and let $\mathrm{pr}_X : X \times_S Y \to X$ resp. $\mathrm{pr}_Y : X \times_S Y \to Y$ denote the canonical projections. Show that $\Omega^1_{X\times_S Y/S} \cong \mathrm{pr}_X^*\Omega^1_{X/S} \oplus \mathrm{pr}_Y^*\Omega^1_{Y/S}$.

(ii) Compute the cotangent bundle of the projective space: Let A be a ring and $\mathbb{P}^n_A \to \mathrm{Spec}\, A$ be the projective space of dimension n over A. Show that the kernel of the universal quotient $\mathcal{O}^{n+1}_{\mathbb{P}^n_A} \to \mathcal{O}(1)$ is given by $\Omega_{\mathbb{P}^n_A/A}(1)$. Especially show that $\Omega^1_{\mathbb{P}^1_A/A} \cong \mathcal{O}(-2)$.

(iii) Let $p : G \to \mathrm{Spec}\, k$ be a group scheme with unit $e : \mathrm{Spec}\, k \to G$. Let $\Omega^1_{G/k}(e)$ denote the fiber of $\Omega^1_{G/k}$ at e. Show that $\Omega^1_{G/k} \cong \mathcal{O}_G \otimes_k \Omega^1_{G/k}(e) = p^*e^*\Omega^1_{G/k}$. Especially $\Omega^1_{G/k}$ is free.
Hint: consider the commutative diagram



Exercise 3:

Let k be a field and let X be a k-scheme. Let $Y_1, Y_2 \subset X$ be closed subschemes and let $Y_1 \cap Y_2$ be their scheme-theoretic intersection. Let x be a k-valued point of $Y_1 \cap Y_2$ such that X, Y_1 and Y_2 are smooth over Spec k at x of relative dimension d, resp. $d - c_1$, resp. $d - c_2$. Show that the following conditions are equivalent:

(a) $Y_1 \cap Y_2$ is smooth at x of relative dimension $d - c_1 - c_2$.

(b) $T_xY_1 + T_xY_2 = T_xX$.

Exercise 4:

(i) Let k be a field of characteristic p and let $t \in k$ be an element that is not a p-th power. Show that the curve $X = \text{Spec } k[X,Y]/(Y^2 - X^p + t)$ is regular (i.e. all local rings are regular) but not smooth.

(ii) Determine the set of points at which the following morphisms are smooth.

(a) $\text{Spec } \mathbb{Z}[T]/(T^2+1) \to \text{Spec } \mathbb{Z}$.

(b) $\text{Spec } A[T_1, \dots, T_n]/(T_1 T_2 \cdots T_n - \varpi) \to \text{Spec } A$, where A is a discrete valuation ring and $\varpi \in A$ is a uniformizer.

(iii) Let k be a field of characteristic char $k \neq 2$ and let $X = \text{Spec } k[T_1, T_2]/(T_2^2(1-T_1^2)-1) \subset \mathbb{A}_k^2$. Compute the set of all points where dT_2 (resp. dT_1) generates Ω_X^1/k, i.e. the set of point where T_2 (resp. T_1) is a uniformizing parameter. This is the set of points where the morphism $g : X \to \mathbb{A}_k^1$ defined by T_2 (resp. T_1) is étale. Compute the sheaf of differentials of g.

(iv) For $X = \text{Spec } k[T_1, T_2]/(T_2^2 - T_1^2(T_1+1)) \subset \mathbb{A}_k^2$ (resp. $X = \text{Spec } k[T_1^2, T_2^2]/(T_2^2 - T_1^3)$) let $g : \tilde{X} \to X$ denote the blow-up at $(0,0) \in X$. Show that $\tilde{X}$ is smooth over k and compute the tangent space $T_{(0,0)}X$, the morphism on tangent spaces induced by g and the differentials $\Omega^1_{\tilde{X}/X}$.

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeomII`