GRADUATE SEMINAR: MODULAR FORMS AND GALOIS REPRESENTATIONS UNIVERSITY OF BONN, SOMMERSEMESTER 2017

Modular forms and Galois representations are fundamental objects in modern number theory. Apparently, they look quite different: Modular forms are certain holomorphic functions on the Poincaré upper half plan, while Galois representations are of arithmetic nature. However, Eichler-Shimura, Deligne, Deligne-Serre proved in 1960s-1970s, the following amazing theorem:

Let $N \geq 1$ be an integer, and $\Gamma_1(N)$ be the usual congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of matrices congruent to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ modulo N. Then, to each cuspidal eigenform f of level $\Gamma_1(N)$ and weight $k \geq 1$, one can attach a compatible system of ℓ-adic Galois representations $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ which are unramified at all primes not dividing $N\ell$. Here, ρ_f and f are related in the sense that, for any rational prime $p \nmid N\ell$, the trace of the image of Frobenius elements at p under ρ_f is equal to the p-th coefficient of the q-expansion of f.

This important and deep theorem was the starting point of many further developments of modern number theory, especially of the so-called Langlands program. The aim of this Seminar is to guide the students to understanding Deligne's construction of the Galois representation ρ_f when $k \geq 2$, and Deligne-Serre's proof when $k = 1$.

The preliminaries for the Seminar are the following

- Algebraic number theory, including ramification theory of extensions of number fields, structure of local Galois group, Chebotarev density theorem.
- Algebraic geometry: the language of schemes, sheaves, line bundles, algebraic correspondences, algebraic curves.
- Etale cohomology: basic definition of usual cohomology groups, cohomology with compact support, the proper base change theorem, the comparison of étale cohomology with usual singular cohomology of varieties over $\mathbb{C}$.

The contents of the talks are as follows:

Lecture 1: The complex theory of modular forms. (April 26th) Define modular forms and cusp forms of level $\Gamma_1(N)$ and weight $k \geq 1$. Introduce the complex modular curve $Y_1(N)_{\mathbb{C}} = \Gamma_1(N)\backslash\mathfrak{H}$, and its canonical compactification $X_1(N)_{\mathbb{C}}$. Introduce the modular line bundle ω on $X_1(N)_{\mathbb{C}}$, and explain the Kodaira-Spencer isomorphism $\omega^2 \cong \Omega^1_{X_1(N)_{\mathbb{C}}}(\log C)$, where $C = X_1(N)_{\mathbb{C}} - Y_1(N)_{\mathbb{C}}$. Explain that the space of modular forms of weight k and level $\Gamma_1(N)$ is isomorphic to $H^0(X_1(N)_{\mathbb{C}}, \omega^k)$ and the space of cusp forms is isomorphic to $H^0(X_1(N)_{\mathbb{C}}, \omega^k(-C))$. Use Riemann-Roch to show the dimension formula for modular and cusp forms. Reference is [DS05, Chap. 3].

Lecture 2: Hecke operators. (May 3rd) Define Hecke operators T_n on the space of modular forms for each $n \geq 1$. Show that Hecke operators T_n for $(n, N) = 1$ are self-adjoint for the Peterson inner product on cusp forms, and deduce that the Hecke algebra $\mathbb{T}$ generated by the endomorphisms T_p with $(p, N) = 1$ on the space of cusp forms is semi-simple. Introduce old forms and new forms, and state the strong multiplicity one theorem for newforms. (Reference: [DS05, Chap. 5].)

Lecture 3: Arithmetic theory of modular curves. (May 10th) Show that $Y_1(N)_{\mathbb{C}}$ is in bijection with elliptic curves over $\mathbb{C}$ together with a point of order N. Define the moduli problem $\mathcal{M}_1(N)$ over $\mathbb{Z}[1/N]$ of elliptic curves together with a $\Gamma_1(N)$-level structure, and prove that $\mathcal{M}_1(N)$ is representable by an affine smooth scheme $Y_1(N)$ over $\mathbb{Z}[1/N]$ for $N \geq 4$. Introduce the canonical compactification $X_1(N)$ as a proper smooth scheme over $\mathbb{Z}[1/N]$ by considering Tate curves. Reference is [KM85, Chap. 4] and [DR72, Chap. VII].

Lecture 4: Arithmetic theory of modular forms and Hecke operators (May 17th). Give the moduli interpretation of modular forms and cusp forms, and explain that the q-expansions of modular forms are equivalent to evaluating the modular form at Tate curves. Prove the q-expansion principle. (Reference: [Ka79, §1.6] and [DS74, Section 2].) Discuss some generalities on algebraic correspondences and induced morphisms on cohomology groups. Introduce the modular curve $X(\Gamma_1(N) \cap \Gamma_0(p))$. Prove that the Hecke operator T_p on modular forms of level $\Gamma_1(N)$ coincides with the action of the algebraic correspondence given by $X(\Gamma_1(N) \cap \Gamma_0(p))$. Prove that if $f = \sum_{n=1}^{\infty} a_n q^n$ is an eigenform for all Hecke operators, then $\mathbb{Q}(a_n : n \geq 1)$ is a number field. (Reference: [Ka79, §1.11])

$\star$ **Lecture 5: Eichler-Shimura isomorphism. (May 24th)** Let $a : \mathcal{E} \to Y_1(N)$ be the universal elliptic curve, and $\mathbb{L} = R^1 a_*(\mathbb{Z})$ viewed as a $\mathbb{Z}$-local system on $Y_1(N)_{\mathbb{C}}$. For any integer $k \geq 2$, put $\mathbb{L}_{k-2} := \mathrm{Sym}^{k-2}(\mathbb{L})$. Prove that there exist Eichler-Shimura isomorphisms

$$\begin{aligned}
H^1(Y_1(N)_{\mathbb{C}}, \mathbb{L}_{k-2}) \otimes_{\mathbb{Z}} \mathbb{C} &\cong M_k(\Gamma_1(N)) \oplus \overline{S_k(\Gamma_1(N))},\\
H^1_c(Y_1(N)_{\mathbb{C}}, \mathbb{L}_{k-2}) \otimes_{\mathbb{Z}} \mathbb{C} &\cong S_k(\Gamma_1(N)) \oplus \overline{M_k(\Gamma_1(N))}\\
\widetilde{H}^1(Y_1(N)_{\mathbb{C}}, \mathbb{L}_{k-2}) \otimes \mathbb{C} &\cong S_k(\Gamma_1(N)) \oplus \overline{S_k(\Gamma_1(N))}
\end{aligned}$$

where $H^1(Y_1(N)_{\mathbb{C}}, \mathbb{L}_{k-2})$ (resp. $H^1_c(Y_1(N)_{\mathbb{C}}, \mathbb{L}_{k-2})$) denotes the singular cohomology (resp. the compact supported singular cohomology) of the complex manifold $Y_1(N)_{\mathbb{C}}$, and $\widetilde{H}^1(Y_1(N)_{\mathbb{C}}, \mathbb{L}_{k-2})$ denotes the image of $H^1_c(Y_1(N)_{\mathbb{C}}, \mathbb{L}_{k-2})$ in $H^1(Y_1(N)_{\mathbb{C}}, \mathbb{L}_{k-2})$. Reference: [Ve61] and [De69, 2.10]. (This is a slightly difficult lecture, since the language in [Ve61] seems a little old. The lecturer should translate it into modern cohomological language.)

Lecture 6: Étale cohomology. (May 31st) Fix a rational prime ℓ. Define similarly $\mathbb{L}_\ell := R^1 a_*(\mathbb{Z}_\ell)$ and $\mathbb{L}_{k-2,\ell} = \mathrm{Sym}^{k-2}(\mathbb{L}_\ell)$ viewed as $\mathbb{Z}_\ell$-local system on $Y_1(N)$. Let $f : Y_1(N) \to \mathrm{Spec}(\mathbb{Z}[1/N])$ be the natural morphism. Prove that $R^1 f_*(\mathbb{L}_{k-2,\ell})$ is a $\mathbb{Z}_\ell$-local system on $\mathrm{Spec}(\mathbb{Z}[1/N])$ commuting with all base change.

Put $V_\ell := R^1 f_*(\mathbb{L}_{k-2,\ell})_{\overline{\eta}} \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell$, where $\bar{\eta}$ is the geometric point of $\mathrm{Spec}(\mathbb{Z}[1/N])$ given by $\overline{\mathbb{Q}}$. Deduce that that V_ℓ is a $\overline{\mathbb{Q}}_\ell$-vector space equipped with commuting actions by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the Hecke algebra $\mathbb{T} = \overline{\mathbb{Q}}_\ell[T_n : n \geq 1]$; moreover, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on V_ℓ is unramified outside $N\ell$. Reference: [De69, Section 3 and Corollary 4.2]

Lecture 7: Integral model of $X(\Gamma_1(N) \cap \Gamma_0(p))$. (June 14th) Let p be a prime coprime to N. Introduce the moduli problem that defines an integral model of $X(\Gamma_1(N) \cap \Gamma_0(p))$ over $\mathbb{Z}[1/N]$. Show that the fiber of $X(\Gamma_1(N) \cap \Gamma_0(p))$ over $\mathbb{F}_p$ is isomorphic to two copies of $X_1(N)_{\mathbb{F}_p}$ cutting transversally at supersingular locus. Reference: [De69, Section 4], [DR72, Chap. VI, Section 6].

Lecture 8: Finish of the construction in weight $k \geq 2$. (June 21st) Let f be a newform of level $\Gamma_1(N)$ and weight $k \geq 2$. Let V_ℓ be the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{T}$-module studied in Lecture 6. Denote by $V_{\ell,f}$ be the f-isotypical component of V_ℓ for the Hecke action. Deduce from the Eichler-Shimura isomorphism that $V_{\ell,f}$ has dimension 2 over $\overline{\mathbb{Q}}_\ell$. Deduce from Lecture 7 the Eichler-Shimura congruence relation, hence show that the *contragredient* of $V_{\ell,f}$ is the Galois representation attached to f. Follow Ribet's argument to show that $V_{\ell,f}$ is irreducible. Combine with Weil II to give the application to Ramanujan's conjecture for holomorphic cusp forms. Reference: [De69, Section 5] and [Ri76].

Lecture 9: Results in weight one and applications (June 28th) State the results for weight one cusp forms. Give applications to the Fourier coefficients of weight one modular forms, [DS74, Section 9]. Discuss the general statement of Artin's holomorphy conjecture for complex representations of the absolute Galois group of a number field, and some general reductions. Prove the functional equation of L-functions of weight one modular forms, and deduce the applications to Artin conjecture for 2-dimensional representations coming from modular forms of weight one. Reference: [DS74, Section 4].

Lecture 10: Rankin-Selberg L-function and applications to weight one forms (July 5th) Introduce Rankin-Selberg L-function of two cusp forms, prove its analytic properties (Analytic continuation, functional equations, poles). Then explain the first estimation of the Fourier coefficients of weight one modular forms proved in [DS74, Section 5].

Lecture 11: Construction of Galois representations modulo ℓ. (July 19th) The aim of this lecture is to prove the following statement, which is the first step of Deligne-Serre's construction of Galois representation attached to a weight one modular form: *Let f be a modular form of weight $k \geq 1$ level $\Gamma_1(N)$ and nebentypus ϵ with coefficients in a number field K. Let λ be a finite place of K, and k_λ be the residue field of characteristic ℓ. Assume that f is a eigenvector for all T_p modulo λ with $p \nmid N\ell$ of eigenvalue $a_p \in k_\lambda$. Let $k_f \subseteq k_\lambda$ be the subfield generated by a_p and $\epsilon(p)$. Then there exists a semi-simple Galois representation*

$$\rho_\lambda\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(k_\lambda)$$

such that

$$\operatorname{Tr}(\rho(\operatorname{Frob}_p)) = a_p, \quad \det(\rho(\operatorname{Frob}_p)) = \epsilon(p)p^{k-1}.$$

Reference: [DS74, Théorème 6.7].

$\star$ **Lecture 12: End of the proof in weight one. (July 26th)** Follow [DS74, Section 7, 8] to finish the proof of Deligne-Serre's construction of the Galois representation attached to a weight one eigenform f. By Lecture 11, for each rational prime ℓ that is completely split in the coefficient field K of f, one can attach to f a mod ℓ Galois representation $\rho_\ell : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\mathbb{F}_\ell)$. The key point is to show that $\operatorname{Im}(\rho_\ell)$ is bounded independent of ℓ.

References

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