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SS 2016

Seminar: Stable reduction of curves

Tuesdays 16-18h, MZ Room 0.006
Organizational meeting: Tuesday 09.02.2016, 10h(c.t.), MZ Room 0.006.

1) Curves over algebraically closed fields:
Define a curve over an algebraically closed field k as a connected, reduced, proper k-scheme of dimension 1. Define the notion of a non-singular curve and show that non-singular curves are projective [Ha, II.6, Prop. 6.7]. Show that isomorphism classes of non-singular curves are in bijection with field extensions of k of transcendence degree 1 [Ha, I.6, Prop. 6.7, Theorem 6.9]. Define the tangent space of a curve and describe possible singularities, see e.g. [Liu, 4.2.1, Definition 2.1]. Define the notion of an ordinary double point [Liu, 7.5, Definition 5.13] and prove [Liu, 7.5, Proposition 5.15] (in the case $m = 2$). See also [Ha, I.5, Example 5.6.3].

2) Riemann-Roch and projective emebddings:
Define and describe divisors on curves [Ha, II. 6, p. 136 ff.]. Define the arithmetic and geometric genus of a curve (also for singular curves) and prove the Riemann-Roch theorem for non-singular curves [Ha, IV.1] and [Liu, 7.3, Theorem 3.17]. Deduce that a line bundle of positive degree on a smooth curve is ample [Ha, IV, Corollaries 3.2 and 3.3] and [Liu, 7.4, Proposition 4.4]. Apply [Li, Prop. 2.3] to the normalization of a (possibly singular) curve and deduce that any curve is projective.

3) Curves over discrete valuation rings:
A curve over a discrete valuation ring A is a regular scheme X together with a proper and flat morphism $X \to \operatorname{Spec} A$ of relative dimension 1, see also [De, Exemple 1.1] and [Li, Definition 1.7]. Show that the special fiber of X is a divisor on X and define the multiplicities of its irreducible components as in [AW, 1, p. 373]. Define the notion of stable reduction [AW, Introduction] and [BLR, 9.2, Definition 6]. Recall that the Euler characteristic is locally constant [Mu, II.5, Corollary (b)] and deduce that the arithmetic genus of the generic and the special fiber coincide if the special fiber is connected and has a reduced component. Prove the formula for the arithmetic genus of a singular curve [Liu, 4, Proposition 5.4]. Deduce that the special fiber is non-singular (and of genus 0) if the generic fiber is of genus 0. Explain the degenerations in genus 1 and 2, see [Liu, 10.2.1] for $g = 1$ and [Liu, 10, Example 3.6] for $g = 2$.

4) Intersection theory I:
Define the intersection pairing between divisors on X and divisors supported in the special fiber of X. Prove the formulas (a), (b) and (d) of [AW, (1.1)]. See [Li, I.1] and [Ša, Lecture 6], state and prove [Ša, Lemma, p. 92]. See also [Liu, 9.1.2]. A more abstract exposition is given in [De, 1.1-1.9]. Also deduce from the existence of the intersection pairing that the special fiber has a component of multiplicity one,

if $X \to \operatorname{Spec} A$ admits a section (recall that by the valuative criterion $X \to \operatorname{Spec} A$ has a section if and only if its generic fiber has a rational point).

5) Intersection theory II:
Prove that every curve over a discrete valuation ring is projective [Li, I.2]. Introduce the canonical divisor on a relative curve [Liu, 9.1.3, Definition 1.34] and state the relative version of Serre duality [Liu, 6.4.3, Theorem 4.32]. Prove formula (c) of [AW, (1.1)] for the arithmetic genus of an irreducible component of the special fiber:

$$p(C) = 1 + \tfrac{1}{2}(C^2 + C \cdot K),$$

see [Liu, 9.1.3, Theorem 1.37] and [De, 1.10, Proposition 1.11]. If time permits present [De, p. 11, Exercise (b)].

6) Types for the special fiber I:
Define the type of the special fiber of a relative curve [AW, Definition 1.2]. Moreover, define the genus and the graph associated to a type [AW, Definition 1.3, (1.4)]. Prove that there are only finitely many similarity classes of types of genus g [AW, Theorem 1.6, Corollary 1.7]. See also [Liu, 10.1.4].

7) Types for the special fiber II:
Introduce the abelian group associated to the special fiber of a relative curve [AW, (1.10)-(1.19)]. Especially prove the estimate about the number of generators of this group [AW, Theorem 1.16] and [Liu, 10.4, Theorem 4.19].

8) Generalities on group schemes:
Introduce the notion of a group scheme and give the examples of unipotent group schemes, the additive group, the multiplicative group and abelian varieties, see for example [Liu, 7.4.4]. Explain that the multiplication by n is surjective on an abelian variety over an algebraically closed field [Mu, II.4, p. 42 (iv)]. Explain the structure results on connected commutative group schemes [BLR, 9.2, Theorem 1 and Theorem 2].

9) The Picard scheme:
Give some back ground about the Picard scheme [BLR, 8.1, Definition 2] and [BLR, 8.2, Theorem 2]. We always assume that the varieties in question have a rational point, hence sheafification is not necessary, see [FGA, 9.2, Theorem 9.2.5]. Especially explain the identification of the tangent space at the identity is identified with $H^1(X, \mathcal{O}_X)$ [BLR, 8.4]. The Picard scheme of a curve is smooth [BLR, 8.4, Proposition 2]. Show that the Picard scheme of a non-singular curve is an abelian variety and describe the Picard schemes of singular curves, see [BLR, 9.2, Corollary 12] and [Liu, Theorem 5.19]. Moreover, explain that, given a (non-reduced) curve X, the kernel of the map $\operatorname{Pic}^0_X \to \operatorname{Pic}^0_{X_{\mathrm{red}}}$ is a unipotent group [BLR, 9.2. Proposition 5] and [Liu, 7.5. Lemma 5.11].

10) Existence of stable reduction I:
Describe the map from the Picard group of a relative curve to the Picard group of its special fiber [AW, Proposition 2.1]. Prove Proposition 2.3, 2.5 and 2.6 of [AW]. See also [Liu, 4.2] for more details.

11) Existence of stable reduction II:
Finish the proof of the stable reduction theorem, see [AW, Theorem 2.8, Corollary

2.10]. If time permits, give an outlook about the reduction of abelian varieties as in [AW, p. 383, Remark].

References

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