Ajtai–Szemerédi theorem over non-commutative groups

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Corners theorem

Theorem (Ajtai and Szemerédi, 1974)

Let $E \subset \mathbb{Z} \times \mathbb{Z}$ be a set with positive upper density

$$\overline{d}(E) = \limsup_{N \to \infty} \frac{|E \cap \{1, \dots, N\}^2|}{N^2} > 0.$$

Then it contains (∞ many) corners, that is, subsets of the form

$$\{(a, b), (a + n, b), (a, b + n)\}, \quad n \neq 0.$$

Theorem (Bergelson, McCutcheon, and Zhang, 1997)

Let G be a countable amenable group and $E \subset G \times G$ be a set with positive upper density. Then it contains (∞ many) corners, that is, subsets of the form

$$\{(a, b), (ga, b), (ga, gb)\} \quad g \neq \mathrm{id}_G.$$

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Ergodic formulation

$$\exists \text{ corner } \{(a,b),(ga,b),(ga,gb)\} \subset E \iff 1_E \cdot T_1^g 1_E \cdot T_1^g T_2^g 1_E \neq 0,$$

$$\text{where } T_1^g f(a,b) = f(ga,b), \quad T_2^g f(a,b) = f(a,gb).$$

Gelfand–Naimark theorem appplied to a suitable subalgebra $1_E \in \mathfrak{A} \subset \ell^\infty(G^2)$ gives a corresponding ergodic measure-preserving action on a regular Borel probability space

$$T_1 \times T_2 : G \times G \curvearrowright (X,\mu), \quad C(X) \cong \mathfrak{A}, \quad \int 1_E d\mu = \overline{d}(E).$$

Theorem (BMZ, 1997)

Let $f \in L^\infty(X,\mu)$ with $f \geq 0$ and $f \not\equiv 0$. Then

$$\liminf_{N\to\infty} \mathbb{E}_{g\in F_N} \int f \cdot T_1^g f \cdot T_1^g T_2^g f d\mu > 0.$$

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Our result

Theorem (Chu, Z.-K., 2013)

Let $T_1 \times T_2 : G \times G \curvearrowright (X, \mu)$ be a measure-preserving action and $f \in L^\infty(X, \mu)$ with $f \geq 0$. Then for every $\varepsilon > 0$ there exists an almost periodic function $\chi : G \to \mathbb{R}$ such that

$$\lim_{N\to\infty} \frac{\mathbb{E}_{g\in F_N}\chi(g) \int f \cdot T_1^g f \cdot T_1^g T_2^g f d\mu}{\mathbb{E}_{g\in F_N}\chi(g)} \geq \Big(\int f^{3/4} d\mu \Big)^4 - \varepsilon.$$

Corollary

Let $E \subset G \times G$. Then for every $\varepsilon > 0$ the set

$$R_\varepsilon = \{g \in G : \overline{d}(E \cap (g^{-1}, \mathrm{id})E \cap (g^{-1}, g^{-1})E) \geq \overline{d}(E)^4 - \varepsilon\}$$

is both left and right syndetic, that is, for some finite set $F \subset G$,

$$FR_\varepsilon = R_\varepsilon F = G.$$

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Corollary

Let $E \subset G \times G$*. Then for every* $\varepsilon > 0$ *the set*

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$$F R_\varepsilon = R_\varepsilon F = G.$$

Characteristic factors

Every measure-preserving system admits an extension that we call *magic*, i.e. such that

$$A(X|I_2, T_1) = I_1 \vee I_2,$$

where I_i is the T_i-invariant factor (subalgebra of $L^\infty(X,\mu)$).
For a magic system

$$\lim_{N\to\infty} \mathbb{E}_{g\in F_N} \chi(g) \int f \cdot T_1^g f \cdot T_1^g T_2^g f \mathrm{d}\mu$$

$$= \lim_{N\to\infty} \mathbb{E}_{g\in F_N} \chi(g) \underbrace{\int \mathbb{E}(f|I_1 \vee K_{1,2}) \cdot T_1^g \mathbb{E}(f|I_1 \vee I_2) \cdot T_1^g T_2^g \mathbb{E}(f|I_2 \vee K_{1,2}) \mathrm{d}\mu}_{=:c(g)}$$

with $c(g)$ almost periodic. Choose almost periodic χ supported on $\{g : c(g) > c(\mathrm{id}) - \varepsilon\}$.
Remains to obtain a lower bound for

$$c(\mathrm{id}) = \int \mathbb{E}(f|B_0) \cdot \mathbb{E}(f|B_1) \cdot \mathbb{E}(f|B_2).$$

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Product of expectations lemma

Lemma

Let $f_0, \dots, f_n \geq 0$ *be functions on a probability space* (X, B, μ) *and let* $B_1, \dots, B_n \subset B$ *be arbitrary sub-*σ*-algebras. Then*

$$\int f_0 \prod_{i=1}^{n} \mathbb{E}(f_i|B_i) \geq \Big(\int \prod_{i=0}^{n} f_i^{\frac{1}{n+1}} \Big)^{n+1}.$$

Proof.

By Hölder's inequality

$$\int \prod_{i=0}^{n} f_i^{\frac{1}{n+1}} = \int \big(f_0^{\frac{1}{n+1}} \prod_{i=1}^{n} \mathbb{E}(f_i|B_i)^{\frac{1}{n+1}}\big) \cdot \prod_{i=1}^{n} \Big(\frac{1_{\{\mathbb{E}(f_i|B_i)>0\}} f_i}{\mathbb{E}(f_i|B_i)} \Big)^{\frac{1}{n+1}}$$

$$\leq \Big(\int f_0 \prod_{i=1}^{n} \mathbb{E}(f_i|B_i) \Big)^{\frac{1}{n+1}} \cdot \prod_{i=1}^{n} \Big(\underbrace{\int \frac{1_{\{\mathbb{E}(f_i|B_i)>0\}}}{\mathbb{E}(f_i|B_i)} f_i}_{=|\{\mathbb{E}(f_i|B_i)>0\}|\leq 1} \Big)^{\frac{1}{n+1}}. \qquad \square$$

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Let $f_0, \dots, f_n \geq 0$ be functions on a probability space (X, B, μ) and let $B_1, \dots, B_n \subset B$ be arbitrary sub-σ-algebras. Then

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Proof.

By Hölder's inequality

$$\begin{aligned}
\int \prod_{i=0}^{n} f_i^{\frac{1}{n+1}} &= \int \big(f_0^{\frac{1}{n+1}} \prod_{i=1}^{n} \mathbb{E}(f_i|B_i)^{\frac{1}{n+1}}\big) \cdot \prod_{i=1}^{n} \Big(\frac{1_{\{\mathbb{E}(f_i|B_i)>0\}} f_i}{\mathbb{E}(f_i|B_i)} \Big)^{\frac{1}{n+1}} \\
&\leq \Big(\int f_0 \prod_{i=1}^{n} \mathbb{E}(f_i|B_i) \Big)^{\frac{1}{n+1}} \cdot \prod_{i=1}^{n} \Big(\underbrace{\int \frac{1_{\{\mathbb{E}(f_i|B_i)>0\}}}{\mathbb{E}(f_i|B_i)} f_i}_{=|\{\mathbb{E}(f_i|B_i)>0\}| \leq 1} \Big)^{\frac{1}{n+1}}. \qquad \square
\end{aligned}$$