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Doob's martingale convergence theorem is known to fail for martingales indexed by directed sets [Die50]. In this note we sharpen Dieudonné's counterexample slightly: instead of non-convergence we obtain that the limit inferior vanishes a.e. although the L^2 limit does not.

We use sequences with particular properties (an example is constructed at the end of the note).

Lemma 1. *There exist sequences $a_k \in (0,1)$, $N_k \in \mathbb{N}$, $k \in \mathbb{N}$, such that*

$$\sum_k a_k^{N_k} < 1 \qquad \text{and} \qquad \prod_k (1 - a_k^{N_k - 1}(1 - a_k))^{N_k} = 0.$$

Let $I = [0,1]$ with Borel σ-algebra $\mathscr{B}$ and Lebesgue measure; denote the trivial σ-algebra by Σ. Consider sequences from Lemma 1 and let $\mathscr{N} = \sqcup_k [1, N_k]$ (disjoint union). Let $\Omega = I^{\mathscr{N}}$ with σ-algebra $\mathscr{B}^{\mathscr{N}}$ and the product measure. We consider the set P consisting of the finite subsets of $\mathscr{N}$ directed by inclusion. For each $p \in P$ we set $\mathscr{B}_p := \mathscr{B}^p \times \Sigma^{\mathscr{N} \setminus p} \subset \mathscr{B}^{\mathscr{N}}$.

Lemma 2. *Let $O = \Omega \setminus \cup_k [0, a_k]^{[1,N_k]} \times I^{\mathscr{N} \setminus [1,N_k]}$. Then $f = 1_O \neq 0$ but $\liminf_{p \in P} \mathbb{E}(f | \mathscr{B}_p) = 0$ a.e.*

Failure of a Wiener-Wintner-type result

Identifying I with a torus $\mathbb{T}$ we turn Ω into a compact abelian group. Choosing rationally independent λ_n, $n \in \mathscr{N}$, we turn it into an ergodic group rotation by T. Now each $\mathscr{B}_p$ is the factor generated by eigenfunctions corresponding to eigenvalues in $\langle \lambda_n \rangle_{n \in p}$.

Note that there exists a full measure set $\Omega' \subset \Omega$ such that for every $\alpha \in \mathbb{T}^d$ the limit

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\|m\alpha\| < \varepsilon} f(T^m x)$$

exists[1] for every $x \in \Omega'$ and equals $\mathbb{E}(f | \mathscr{B}_\alpha)$ a.e., where $\mathscr{B}_\alpha$ is the factor generated by eigenvalues in the group generated by entries of α (in particular, there are only countably many such factors).

Taking f as in Lemma 2 we obtain a full measure set $\Omega'' \subset \Omega'$ such that $\liminf_p \mathbb{E}(f | \mathscr{B}_p)(x) = 0$, $x \in \Omega''$. In particular, for each $x \in \Omega''$ there exists an increasing sequence $p_0 \subset p_1 \subset \dots$ that exhausts $\mathscr{N}$ such that $\lim_i \mathbb{E}(f | \mathscr{B}_{p_i})(x) = 0$. For a corresponding enumeration λ_i of λ_n's we obtain

$$\liminf_{d \to \infty} \lim_{\varepsilon \to 0} \mathbb{E}_{\|m\lambda_1\| < \varepsilon, \dots, \|m\lambda_d\| < \varepsilon} f(T^m x) = 0.$$

Proofs of technical results

Proof of Lemma 1. Let a_k be such that

$$\frac{-\log a_k}{1 - a_k} = 1 + \frac{2 \log \log k}{\log k}$$

and set

$$N_k := \lceil \frac{\log k}{1 - a_k} \rceil.$$

[1] here $\mathbb{E}$ stands for the Cesàro limit; in fact I have only checked corresponding statements for smooth cutoffs

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Then $\frac{1}{2k(\log k)^2} \leq a_k^{N_k} \leq \frac{1}{k(\log k)^2}$, so that (a_k) is summable. On the other hand, the product in the statement of the lemma is zero if and only if

$$\sum_k \log(1 - a_k^{N_k - 1}(1 - a_k))^{N_k} = -\infty.$$

The latter sum is bounded above by

$$-\sum_k N_k a_k^{N_k - 1}(1 - a_k) \leq -\sum_k \log k a_k^{\frac{\log k}{1 - a_k}} \leq -\frac{1}{2} \sum_k \frac{1}{\log k} = -\infty.$$

Finally, it suffices to remove the first few terms of (a_k) to ensure $\sum_k a_k^{N_k} < 1$. □

Proof of Lemma 2. The set O has positive measure by construction.

Let $\varepsilon > 0$ and $p_0 \in P$ be given. We will show that

$$\inf_{p \supset p_0} \mathbb{E}(f \,|\, \mathscr{B}_p) < \varepsilon$$

on a set of full measure. Without loss of generality we may assume $p_0 = \sqcup_{k<K}[1, N_k]$ and $1 - a_k < \varepsilon$ for $k \geq K$.

Let $k \geq K$ and $n \in [1, N_k]$. Consider $p = p_0 \sqcup ([1, N_k] \setminus \{n\}) \supseteq p_0$. By construction we have

$$\mathbb{E}(f \,|\, \mathscr{B}_p) \leq 1 - a_k < \varepsilon \quad \text{on} \quad I^{\mathscr{N} \setminus [1, N_k]} \times [0, a_k]^{[1, N_k] \setminus \{n\}} \times (a_k, 1]^{\{n\}}.$$

Now, for fixed k and $n \in [1, N_k]$ these sets are disjoint and for different k they are pairwise independent. Hence the complement of their union has measure at most

$$\prod_k (1 - a_k^{N_k - 1}(1 - a_k))^{N_k} = 0. \qquad \square$$

References

[Die50] J. Dieudonné, *Sur un théorème de Jessen*, Fund. Math. **37** (1950), 242–248. MR0043176 (13,218d) ↑1