ALGEBRAIC AND GEOMETRIC STABILITY

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Abstract. These notes are based on three lectures given at the Introductory School on Moduli Spaces at the MPIM in October 2015. We discuss some general ideas surrounding the construction of moduli spaces of objects (of fixed additive invariants) in (quasi-)abelian categories over fields. In this context, there are two (a priori unrelated) concepts of stability, one algebraic and one from Geometric Invariant Theory. The goal is to compare them in two elementary examples: filtered $\mathbb{F}_q$-vector spaces and quiver representations over an algebraically closed field.

Contents

Outline

In this note, we explain two notions of stability. Firstly, in a (quasi-)abelian category $\mathcal{C}$ over a field, in which every object has a finite rank attached to it, we introduce so-called degree (or equivalently, slope) functions. This yields a functorial filtration of the category with semistable graded pieces, which means that their subobjects behave well with respect to the slope function.

On the other hand, we present the concept of semistable points from Geometric Invariant Theory. These are defined with respect to a linearization of the action of a reductive group G on an algebraic variety X, namely as points where the action is particularly nice, allowing passage to a good quotient.

In case X is attached to $\mathcal{C}$, but classifies surplus data, and the G-action parametrizes the excess, the resulting quotient represents the intended moduli problem for objects in $\mathcal{C}$. If the linearization corresponds to the degree function, the notion of semistable objects in $\mathcal{C}$ might be an algebraic description of the semistable points in X, so that the quotient will be a well-behaved moduli space for a suitable (also well-behaved) full subcategory of $\mathcal{C}$ of semistable objects.

We present here two elementary examples where this connection is visible, namely filtered vector spaces (and their flag varieties) over $\mathbb{F}_q$ as well as quiver representations (and their representation varieties). For the case of vector bundles, where a lot of the theory originally comes from, there is a summary in [6], App. 5, C.

The first talk §1 mostly contains material from the survey paper [1], discussing the algebraic notion of stability. The second §2 and third §3 parts, where we give an overview of the relevant results from Geometric Invariant Theory and present the case of flag varieties, respectively, are largely based on the book [3]. Finally, Section §4 on quiver representations is taken from the paper [5].

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1. Slope filtrations

Let k be a field, and $\mathcal{C}$ an (essentially small) exact k-linear category, and $\mathrm{sk}(\mathcal{C})$ its set of isomorphism classes. Our categories come equipped with a rank function

$$\mathrm{rk}\colon \mathrm{sk}(\mathcal{C}) \to \mathbb{N},$$

which is zero only on $0 \in \mathcal{C}$ and extends to a morphism $\mathrm{rk}\colon K_0(\mathcal{C}) \to \mathbb{Z}$ on the Grothendieck group. Note that this is in particular a finiteness condition, as the rank bounds the length of filtrations in $\mathcal{C}$. Let us present some examples of moduli problems in this context.

In each case, we will need to fix further additive invariants (for example, but not necessarily, the Grothendieck class itself).

Example 1.1. (a) Let X be a smooth projective (connected) curve over k, and $\mathcal{C} = \mathrm{Bun}_X$ the category of vector bundles on X. For $M \in \mathcal{C}$, of course $\mathrm{rk}(M) = \dim_{k(x)}(M \otimes k(x))$ means the dimension of the fibre at a point $x \in X$. The moduli problem is of the form

$$\mathrm{Sch}\,/k \ni S \mapsto \{S\text{-families of vector bundles on } X \text{ of rank } n\},$$

where an S-family just means a vector bundle on $X \times_{\mathrm{Spec}\, k} S$.
(b) Let $Q = (Q_0, Q_1,\ s, t\colon Q_1 \to Q_0)$ be a (finite, connected, acyclic) quiver, and $\mathcal{C} = \mathrm{rep}_k(Q)$ the category of finite dimensional representations of Q over k, that is, functors $Q \to \mathrm{vect}_k$ to the category of finite dimensional vector spaces over k. The rank function we consider in this case is $\mathrm{rk}(M) = \sum_{i \in Q_0} \dim_k(M_i)$. We would like to study the functor

$$\mathrm{Sch}\,/k \ni S \mapsto \{\text{functors } Q \to \mathrm{Bun}_S \text{ of constant rank } n \text{ on } S\}.$$

(c) Finally, let $\mathcal{C} = \mathbb{Z}\text{-}\mathrm{fil}_k$ be the category of Z-filtered finite dimensional k-vector spaces, where Z is a totally ordered set (here, Z will be either $\mathbb{Z}$ or finite). More precisely, $(V, F^\bullet) \in \mathcal{C}$ means $V \in \mathrm{vect}_k$, together with a (decreasing) separated exhaustive filtration

$$F^\bullet\colon Z^{\mathrm{op}} \to (\{\text{subspaces of } V\}, \subseteq).$$

Rather than fixing the rank $\mathrm{rk}(V, F^\bullet) = \dim V$, we remember a fixed $V \in \mathrm{vect}_k$, and consider

$$\mathrm{Sch}\,/k \ni S \mapsto \{F^\bullet\colon Z^{\mathrm{op}} \to (\{\text{locally on } S \text{ direct summands of } V \otimes \mathcal{O}_S\}, \subseteq)\},$$

where our filtrations $F^\bullet$ are always supposed to be exhaustive and separated.

Arbitrary exact categories are too general a framework for our purposes, but two of the above categories are not abelian (cf. Example 1.5). We now introduce an appropriate context to work in, immediately followed by a useful, more intuitive characterization via torsion pairs.

Definition 1.2. A quasi-abelian category is an additive category $\mathcal{C}$ with (co-)kernels such that $\mathrm{Ext}(-,-)$ is a bifunctor.

Proposition 1.3 ([2], Prop. B.3)**.** *An additive category $\mathcal{C}$ with (co-)kernels is quasi-abelian if and only if there is a fully faithful embedding $\mathcal{C} \hookrightarrow \mathcal{A}$ into an abelian category, and a full subcategory $\mathcal{T} \subseteq \mathcal{A}$, such that $\mathrm{Hom}(\mathcal{T}, \mathcal{C}) = 0$, and every $A \in \mathcal{A}$ sits in a short exact sequence*

$$0 \to T \to A \to M \to 0, \tag{1.1}$$

with $T \in \mathcal{T},\ M \in \mathcal{C}$. In this case, 1.1 is unique, and $A \mapsto T =: A_{\mathrm{tors}}$ resp. $A \mapsto A/A_{\mathrm{tors}}$, are right resp. left adjoint to the respective embeddings.

Remark 1.4. It follows from the construction that $K_0(\mathcal{A}) = K_0(\mathcal{C})$ in the above situation. Hence, if $\mathcal{C}$ is equipped with a rank function, it extends to $\mathcal{A}$ (though as no longer a rank function in this sense), and one would expect $\mathcal{T} \overset{(*)}{=} \{T \in \mathcal{A} \mid \mathrm{rk}(T) = 0\}$. This is the case in the examples below, but not in general.

Example 1.5. (a) For a smooth projective curve X, let $\mathcal{C} = \mathrm{Bun}_X$. Then $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_X(1)$ has (co-)kernel $= 0$, but is not an isomorphism (since $\mathrm{Hom}(\mathcal{O}_X(1), \mathcal{O}_X) = 0$), therefore $\mathcal{C}$ is not abelian. In fact, its abelian envelope $\mathcal{A}$ as in Proposition 1.3 is given by $\mathcal{A} = \mathrm{Coh}_X$, the category of coherent sheaves on X, and $\mathcal{T}$ is the full subcategory of torsion sheaves.

(c) In $\mathcal{C} = n\text{-}\operatorname{fil}_k$, the morphism $(V,\ V \supseteq 0 \supseteq \ldots \supseteq 0) \xrightarrow{\mathrm{id}_V} (V,\ V \supseteq V \supseteq 0 \supseteq \ldots \supseteq 0)$ again has (co-)kernel $= 0$, but its "inverse" does not respect the filtrations, hence it is not an isomorphism. Here, $\mathcal{C}$ embeds into $\mathcal{A} = \operatorname{rep}_k(A_n)$ as $(V, F^\bullet) \mapsto (F^n V \hookrightarrow \ldots \hookrightarrow F^1 V)$, and the torsion subcategory is given by $\mathcal{T} = \{V_1 \to \ldots \to V_{n-1} \to 0\}$.

We now introduce the other type of additive invariant we will consider.

Definition 1.6. A degree function on $\mathcal{C}$ is a group homomorphism $\deg\colon K_0(\mathcal{C}) \to \mathbb{Z}$ such that $\deg(M) \leq \deg(N)$ for any $M \to N$ with (co-)kernel $= 0$.

Example 1.7. (a) Let $\mathcal{C} = \operatorname{Bun}_X$ as above. The usual notion of degree of a divisor induces $\deg\colon \operatorname{Pic}(X) \to \mathbb{Z}$, which in turn extends to $K_0(\mathcal{C})$ via $\deg(M) := \deg(\det M)$. Note that e.g. $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_X(1)$ resp. $\mathcal{O}_X(1) \not\twoheadrightarrow \mathcal{O}_X$ are consistent with the definition of a degree function.
(b) For $\mathcal{C} = \operatorname{rep}_k(Q)$, recall $K_0(\mathcal{C}) \xrightarrow{\sim} \mathbb{Z}^{Q_0},\ M \mapsto \underline{\dim}(M) = (\dim(M_i))_{i \in Q_0}$. Therefore, the degree functions on $\mathcal{C}$ are precisely given by $\operatorname{Hom}(K_0(\mathcal{C}), \mathbb{Z}) \cong \mathbb{Z}^{Q_0},\ \deg_\theta \mapsto \theta$, where

$$\deg_\theta(M) = \sum_{i \in Q_0} \theta_i \dim(M_i).$$

Of course, since $\mathcal{C}$ is abelian, all of these indeed satisfy the property of a degree function.
(c) By Example 1.5, we can try to descend (b) to $n\text{-}\operatorname{fil}_k$. It turns out that the cone of degree functions in $\operatorname{Hom}(K_0(n\text{-}\operatorname{fil}_k), \mathbb{Z})$ is given by $\{\deg_\theta \mid \theta \in \mathbb{N}^{n-1} \times \mathbb{Z}\}$. The choice of θ is then essentially equivalent to an embedding into $\mathcal{C} = \mathbb{Z}\text{-}\operatorname{fil}_k$, with its "universal" degree function

$$\deg_\bullet\colon K_0(\mathcal{C}) \to \mathbb{Z},\ (V, F^\bullet) \mapsto \sum_{\lambda \in \mathbb{Z}} \lambda \dim(\operatorname{gr}^\lambda V) = \sum_{i=1}^{r} \lambda_i \dim(\operatorname{gr}^{\lambda_i} V),$$

where $\operatorname{gr}^\lambda V = F^\lambda V / F^{\lambda+1} V$ are the graded pieces and λ_i the jumps of the filtration. Indeed, under the other natural identification $K_0(n\text{-}\operatorname{fil}_k) \cong \mathbb{Z}^n$, namely via $\underline{\dim}(\operatorname{gr}^\bullet)$, we get

$$\mathbb{N}^{n-1} \times \mathbb{Z} \xrightarrow{\sim} \{\eta \in \mathbb{Z}^n \mid \eta_1 \geq \ldots \geq \eta_n\},\ \theta \mapsto (\theta_1 + \ldots + \theta_n, \ldots, \theta_{n-1} + \theta_n, \theta_n). \tag{1.2}$$

For later use in the geometric situation, note that $\deg_\bullet$ extends to (the "K-points") $\mathbb{Z}\text{-}\operatorname{fil}_{K|k}$, the category of pairs $(V \in \operatorname{vect}_k,\ F^\bullet\colon \mathbb{Z}^{\mathrm{op}} \to (\{\text{subspaces of } V \otimes_k K\}, \subseteq)$.

For the rest of the section, we will introduce the resulting algebraic notion of stability on our categories, as well as its most important properties.

Definition 1.8. The slope of an object $0 \neq M \in \mathcal{C}$ is defined to be $\mu(M) = \frac{\deg(M)}{\operatorname{rk}(M)} \in \mathbb{Q}$. It is called (semi-)stable, if $\mu(M') \underset{(\leq)}{<} \mu(M)$ for all non-trivial subobjects $0 \subsetneq M' \subsetneq M$.

Remark 1.9. Unlike the rank function, the target of degree map being $\mathbb{Z}$ here is not essential. In fact, it could be replaced by any totally ordered abelian group Λ. Then μ is a map

$$\mu\colon \operatorname{sk}(\mathcal{C}) \longrightarrow \mathbb{Z}^{-1}\Lambda := ((\mathbb{Z} \setminus \{0\})^{-1}\Lambda) \amalg \{\infty\} = \Lambda_{\mathbb{Q}} \amalg \{\infty\},\ \text{where } \Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q},$$

with total ordering induced by $\frac{\lambda}{n} \leq \frac{\mu}{m} \Leftrightarrow m\lambda \leq n\mu$ for $m, n \in \mathbb{N}$, and of course $\infty > \Lambda_{\mathbb{Q}}$. Note that Λ is automatically torsion-free, and $\Lambda_{\mathbb{Q}}$ is uniquely divisible.

Proposition 1.10 ([1], Thm. 1.4.7)**.** *There exists a (unique, functorial) filtration*

$$F^\bullet\colon \mathbb{Q}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C},\ (\lambda, M) \mapsto F^\lambda M,$$

such that for each $M \in \mathcal{C}$, the associated flag

$$0 \subseteq F^{\lambda_1} M \subseteq \ldots \subseteq F^{\lambda_n} M = M \tag{1.3}$$

is uniquely determined by having successive quotients semistable of decreasing slopes

$$\mu(\operatorname{gr}^{\lambda_1} M) = \lambda_1 > \ldots > \lambda_n = \mu(\operatorname{gr}^{\lambda_n} M).$$

Conversely, $F^\bullet$ uniquely determines the degree function via $\deg(M) = \sum_{\lambda \in \mathbb{Q}} \lambda \operatorname{rk}(\operatorname{gr}^\lambda M)$.

This result was first obtained by Harder-Narasimhan [4] in the context of vector bundles. Accordingly, the flag 1.3 is called the Harder-Narasimhan (or HN-)filtration.

Remark 1.11. Keeping in mind Remark 1.4 (and assuming $(*)$ there), we can extend this to the abelian envelope $\mathcal{A}$. The HN-filtration of $A \in \mathcal{A}$ is then simply $0 \subseteq A_{\mathrm{tors}} \subseteq F^\bullet(A/A_{\mathrm{tors}})$, which respects the decreasing slope condition, since $\mu(A_{\mathrm{tors}}) = \infty$, by definition.

Example 1.12. It follows that the HN-filtration on $\mathcal{C} = \mathbb{Z}\text{-}\mathrm{fil}_k$ is rather tautological. We can say equivalently that an object $(V, F^\bullet) \in \mathcal{C}$ is semistable if and only if $F^\bullet$ has precisely one jump. However, for $K \neq k$, the slope filtration on $\mathbb{Z}\text{-}\mathrm{fil}_{K|k}$ is certainly not trivial.

Remark 1.13. Labeling the filtration by the slopes (rather than settling for the flags (1.3)) is responsible for the functoriality of $F^\bullet$.

In most cases, the HN-filtration is related to the Jordan-Hölder filtration, as follows. Consider the full subcategory $\mathcal{C}_\lambda = \{0\} \amalg \{M \in \mathcal{C} \mid M \text{ is semistable of slope } \mu(M) = \lambda\}$. Usually (cf. [1], Cor. 1.4.10, Prop. 2.2.11), $\mathcal{C}_\lambda$ will be abelian, noetherian and artinian, with simple objects precisely the stable objects in $\mathcal{C}$ of slope λ. Thus the Jordan-Hölder filtration on $\mathcal{C}_\lambda$ yields a refinement of the HN-filtration on $\mathcal{C}$. We have $\mathrm{Hom}(\mathcal{C}_\lambda, \mathcal{C}_\mu) = 0$ for $\lambda > \mu$.

Finally, we present an example where $\mathcal{C}_\lambda$ can be considered more interesting than $\mathcal{C}$ itself.

Example 1.14. Let K be a complete discretely valued field of characteristic 0 with perfect residue field k of characteristic p, and $L|K$ a finite (totally ramified) extension. The easiest example is $K = \mathbb{Q}_p$. Let σ be a lift of the Frobenius on k to K.

A filtered isocrystal over $L|K$ is an element $(V, F^\bullet) \in \mathbb{Z}\text{-}\mathrm{fil}_{L|K}$, together with a σ-semilinear automorphism φ of V, that is, $\varphi\colon V \otimes_{K,\sigma} K \xrightarrow{\sim} V$. Let $\mathcal{C}$ denote their category, and define

$$\deg_\sigma(V, F^\bullet, \varphi) := -\nu_p(\det \varphi).$$

We are interested in the degree function $\deg = \deg_\bullet + \deg_\sigma$ on $\mathcal{C}$ (recall Example 1.7 (c)). Colmez and Fontaine have shown that there is an equivalence of categories

$$\mathcal{C}_0 \xrightarrow{\sim} \mathrm{rep}_K^{\mathrm{cris}}(G_L),$$

where $\mathrm{rep}_K^{\mathrm{cris}}(G_L)$ denotes the category of certain well-behaved continuous representations of the absolute Galois group G_L in K-vector spaces (the realm of the local Langlands program).

Filtered isocrystals arise in p-adic Hodge Theory as ("crystalline") cohomology groups (with natural Frobenius action; they become K-vector spaces after inverting p) of smooth projective varieties over L, whose reductions mod p are smooth projective varieties over k, and which carry a Hodge filtration (over L) by comparison with de Rham cohomology.

2. Geometric invariant theory

Let G be a (linearly) reductive group over k. For our purposes, it is enough to know about the general linear group $G = \mathrm{GL}_d$. This is the affine group scheme

$$\mathrm{GL}_d = \mathrm{Spec}\left(k[x_{ij}, t]_{1 \le i,j \le d}/(\det(x_{ij})t - 1)\right),$$

which represents the functor on k-algebras $R \mapsto \mathrm{GL}_d(R)$. This defines the group structure on G, in particular the multiplication $G \times_k G \to G$, via the Yoneda embedding. We also consider for $V \in \mathrm{vect}_k$ the linear group $\mathrm{GL}_V : R \mapsto \mathrm{Aut}(V \otimes_k R)$, which of course can be identified with $\mathrm{GL}_{\dim V}$. A G-action on a scheme X over k is a morphism $G \times_k X \to X$ such that the obvious diagrams commute. In particular, this yields an action on the global sections via $g.f(x) = f(g^{-1}x)$ for $g \in G(R),\ f \in \Gamma(X, \mathcal{O}_X) \otimes_k R$.

Example 2.1. Let $V \in \mathrm{vect}_k$, and let $\rho\colon G \to \mathrm{GL}_V$ be a representation. This defines what is called a linear G-action on $X = \mathrm{Spec}(\mathrm{Sym}(V^*)) \cong \mathbb{A}_k^{\dim V}$, namely on R-points via

$$G(R) \times (V \otimes_k R) \to (V \otimes_k R),\ (g, v) \mapsto \rho(R)(g)v.$$

A case of particular interest for us is when X is the so-called representation variety $\mathcal{R} = \mathcal{R}_{\underline{d},Q}$ of dimension vector $\underline{d} \in \mathbb{N}^{Q_0} \subseteq K_0(\mathrm{rep}_k(Q))$ of a quiver Q, which arises as above from

$$V = V_{\underline{d}} = \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_k(k^{d_{s(\alpha)}}, k^{d_{t(\alpha)}}),\ \text{and } G = \mathrm{GL}_{\underline{d}} = \prod_{i \in Q_0} \mathrm{GL}_{d_i}.$$

The action is by change of basis, via $(g_i)_{i \in Q_0}.(\varphi_\alpha)_{\alpha \in Q_1} = (g_{t(\alpha)}\varphi_\alpha g_{s(\alpha)}^{-1})_{\alpha \in Q_1}$ on k-points.

Remark 2.2. Intuitively, if X is the representation variety from above, the quotient $X/\operatorname{GL}_{\underline{d}}$ should be a moduli space for representations of Q over k of dimension vector $\underline{d}$. However, whenever G acts on an affine scheme $X = \operatorname{Spec}(A)$ linearly, then $A^G \cong k$, i.e., the classical invariant theory quotient $\operatorname{Spec}(A^G)$ is trivial.

The idea of taking $\operatorname{Spec}(A^G)$ as the quotient is to separate orbits by invariant functions, cf. Example 2.3. Note that it is not obvious that A^G is in fact a k-algebra of finite type (here, the assumption on G is crucial!). It turns out that $\operatorname{Spec}(A) \to \operatorname{Spec}(A^G)$ is always a *good* quotient (cf. Theorem 2.6), but in case of the linear action above, clearly not a suitable quiver moduli space (it only detects *semi-simple* representations).

Geometric Invariant Theory identifies the locus of points of which we can take a good, projective quotient, which will yield a moduli space for (semi-)stable quiver representations.

Example 2.3. Let $G = \operatorname{GL}_d$ act on $\mathrm{M}_{d\times d} = \operatorname{Spec}(A) = \mathbb{A}_k^{d^2}$, with $A = k[x_{ij}]_{1\leq i,j\leq d}$, by conjugation. For example, for $d = 2$ (and $k = \overline{k}$), we have the invariant functions $\det, \operatorname{tr} \in A^G$. The fibres of $(\det, \operatorname{tr})\colon \mathrm{M}_{2\times 2}(k) \to k^2$ form either a 2-dimensional closed orbit, if $\lambda_1 \neq \lambda_2$ are distinct eigenvalues, or the orbit $G.\begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix}$, $t \neq 0$, which does not contain the closed point $\lim\limits_{t\to 0}\begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, which of course forms its own orbit.

The fact that $A^G \cong k[\det, \operatorname{tr}]$ is essentially the same statement as that the above is a complete description of the closed orbits, parametrized by the quotient $\operatorname{Spec}(A^G) \cong \mathbb{A}_k^2$. The same holds true for any d, where $A^G \cong k[e_1, \ldots, e_d]$ is generated by the coefficients of the characteristic polynomial (of the coordinates of $\mathrm{M}_{d\times d}$),

$$\det(\lambda \cdot \operatorname{id} - (x_{ij})) = \sum_{i=0}^{d} (-1)^{d-i} e_{d-i} \lambda^i,$$

which are by definition the elementary symmetric polynomials (in the eigenvalues). Their fundamental theorem therefore says that $\mathrm{M}_{d\times d} / \operatorname{GL}_d = \operatorname{Spec}(A^G) \cong \mathbb{A}_k^d$.

Now let X be a projective variety with G-action over k (or, more generally, a proper scheme). A linearization of the G-action is an embedding $X \hookrightarrow \mathbb{P}_k^n$ such that X is a closed G-invariant subvariety, where G acts on $\mathbb{P}_k^n$ by a *linear* action on $\mathbb{A}_k^{n+1} \hookleftarrow \widehat{X}$.

Equivalently, such an embedding is given by a G-equivariant (very ample) line bundle $\mathcal{L}$ on X (i.e., lifting the G-action on X to the geometric line bundle associated to $\mathcal{L}$). We will simply say $\mathcal{L} \in \operatorname{Pic}^G(X)$ is a linearization.

If $X \cong \operatorname{Proj}(A) \subseteq \mathbb{P}_k^n$, where $A = k[x_0, \ldots, x_n]/I$, following Remark 2.2, we might guess that $\operatorname{Proj}(A^G)$ is a good quotient (note that since the grading on A is preserved by G, it descends to A^G). However, in order for the quotient map

$$\pi\colon \operatorname{Proj}(A) \twoheadrightarrow \operatorname{Proj}(A^G),\ \mathfrak{p} \mapsto \mathfrak{p} \cap A^G,$$

to be well-defined, of course we need $(\mathfrak{p} \cap A^G) \neq A_+^G = \bigoplus_{i>0} A_i^G$. Equivalently,

$$\exists f \in A_i^G,\ i > 0:\ f(\mathfrak{p}) \neq 0,$$

i.e. $\mathfrak{p} \in X_f = \{\mathfrak{q} \in X \mid f \notin \mathfrak{q}\}$. Note that this depends on the linearization, and that in this case $A_i = \Gamma(X, \mathcal{L}^{\otimes i})$. This motivates the definition of the geometric notion of stability.

Definition 2.4. Let X be a (proper) projective variety with G-action over k, and $\mathcal{L}$ a linearization. Then $x \in X$ is called semistable, if ($X_f \ni x$ is an affine open neighbourhood)

$$\exists f \in \Gamma(X, \mathcal{L}^{\otimes i})^G,\ i > 0 : f(x) \neq 0.$$

If moreover the G-action on X_f is closed and $\dim(Gx) = \dim(G)$, then x is called stable.

Remark 2.5. Note that $x \in X \subseteq \mathbb{P}_k^n$, with lift $\widehat{x} \in \mathbb{A}_k^{n+1}$, is semistable if and only if $0 \notin \overline{G\widehat{x}}$. Indeed, any G-equivariant section $f \in \Gamma(X, \mathcal{L}^{\otimes i})^G,\ i > 0$, which does not vanish on x, yields the function $f - f(\widehat{x})$, separating 0 and $G\widehat{x}$ (hence $\overline{G\widehat{x}}$ by continuity).

Theorem 2.6 ([6]). *The (semi-)stable loci $X^{\mathrm{s}}, X^{\mathrm{ss}} \subseteq X$ are G-invariant open subvarieties. There exists a good quotient $\pi\colon X^{\mathrm{ss}} \twoheadrightarrow Y$, which is projective over k. In particular,*

$$\pi(x_1) = \pi(x_2) \Leftrightarrow \overline{Gx_1} \cap \overline{Gx_2} \neq \emptyset,$$

and the topology on Y is the quotient topology. It satisfies the universal property

$$\mathrm{Hom}_G(X^{\mathrm{ss}}, Z) \cong \mathrm{Hom}(Y, Z),$$

for Z with trivial G-action. Moreover, there is an open subvariety $Y^{\mathrm{s}} \subseteq Y$ such that

$$\pi|_{X^{\mathrm{s}}}\colon X^{\mathrm{s}} = \pi^{-1}(Y^{\mathrm{s}}) \twoheadrightarrow Y^{\mathrm{s}}$$

is a good quotient whose fibres $\pi^{-1}(y)$ over $y \in Y^{\mathrm{s}}$ are precisely the (closed) G-orbits.

Remark 2.7. If $X \hookrightarrow \mathbb{P}^n_k$ corresponds to $\mathcal{L} \in \mathrm{Pic}^G(X)$, and $\widehat{X} \subseteq \mathbb{A}^{n+1}_k$ is the affine cone of X, then G acts on $\Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}}) \cong \bigoplus_{i\geq 0} \Gamma(X, \mathcal{L}^{\otimes i}) =: B$. Then $Y = \mathrm{Proj}(B^G)$, and

$$\pi\colon X^{\mathrm{ss}} = \bigcup_{f\in B^G} X_f \twoheadrightarrow Y$$

is glued together from affine quotients, which we already know to be good. For a proper definition of good quotients, see [6], Definition 0.6.

In order to see any relation to the algebraic notion of stability, we need a numerical characterization of (semi-)stable points. This we will discuss for the rest of the section.

Remark 2.8. Let $\lambda\colon \mathbb{G}_m := \mathrm{GL}_1 \longrightarrow G$ be a group morphism (also called a cocharacter, one-parameter subgroup, or 1-PS for short), and $x \in X$. Then the induced orbit map

$$\mathbb{G}_m = \mathbb{A}^1_k \smallsetminus \{0\} \longrightarrow X,\ t \mapsto \lambda(t)x,$$

extends to $\mathbb{A}^1_k \to X$, by the valuative criterion for properness (in particular, for X projective). Then clearly, $x_0 = \lim_{t\to 0} \lambda(t)x \in X$ is fixed under the induced $\mathbb{G}_m$-action on X through λ. Hence $\mathbb{G}_m$ acts on the fibre $\mathcal{L}_{x_0}$ at x_0, necessarily via some character $\chi\colon \mathbb{G}_m \to \mathbb{G}_m$ of $\mathbb{G}_m$. But those are given by $\mathrm{End}(\mathbb{G}_m) \cong \mathbb{Z}$, $(t \mapsto t^a) \mapsto a$, and $\chi \mapsto \mu \in \mathbb{Z}$ is the number we were looking for.

Let us first give an equivalent, more explicit definition, for $X \subseteq \mathbb{P}_{V,k} \cong \mathbb{P}^n_k$, with G-action linearized as $\rho\colon G \to \mathrm{GL}_V$. In this case, the composition $\rho \circ \lambda\colon \mathbb{G}_m \to \mathrm{GL}_V$ is precisely the same as a grading into eigenspaces $V = \bigoplus_{i\in\mathbb{Z}} V_i$, where

$$V_i := \{v \in V \mid (\rho\circ\lambda)(t)v = t^i v \text{ for all } t \in \mathbb{G}_m\}. \tag{2.1}$$

Then $\mu = \min\{i \in \mathbb{Z} \mid \widetilde{x}_i \neq 0\}$, where $x \leftrightarrow \widetilde{x} \in \mathbb{P}_V(k)$ decomposes as $\widetilde{x} = \sum_{i\in\mathbb{Z}} \widetilde{x}_i \in \bigoplus_{i\in\mathbb{Z}} V_i$.

Finally, we can also characterize μ in terms of ρ (and orbit closures, cf. Remark 2.11) by

$$\lim_{t\to 0} t^{-\mu}(\rho\circ\lambda)(t)x \text{ exists and is } \neq 0.$$

Definition 2.9. In the notation of Remark 2.8, the (GIT-)slope of x (wrt $\mathcal{L}, \lambda$) is defined by

$$\mu^{\mathcal{L}}(x,\lambda) = \mu(x, \rho\circ\lambda) := -\mu.$$

The sign in Definition 2.9 is purely for aesthetic reasons, in regards to the following result.

Theorem 2.10 (Hilbert-Mumford Criterion). *Let $G, X, \mathcal{L}$ be as before, and let $x \in X$. Then*

$$x \textit{ is semistable wrt } \mathcal{L} \Leftrightarrow \mu^{\mathcal{L}}(x,\lambda) \geq 0 \textit{ for all (non-trivial) } \lambda\colon \mathbb{G}_m \to G,$$

$$x \textit{ is stable wrt } \mathcal{L} \Leftrightarrow \mu^{\mathcal{L}}(x,\lambda) > 0 \textit{ for all non-trivial } \lambda\colon \mathbb{G}_m \to G.$$

Remark 2.11. Assume $G = \mathbb{G}_m$ and let $X \ni x$ and $\rho\colon G \to \mathrm{GL}_V$ be as in Remark 2.8. Then x is semistable $\Leftrightarrow 0 \notin \overline{G\widehat{x}}$, as we saw in Remark 2.5, which is to say

$$\lim_{t\to 0} \rho(t)\widehat{x} \neq 0 \text{ and } \lim_{t\to\infty} \rho(t)\widehat{x} \neq 0, \text{ whenever they exist.}$$

Here, $\lim\limits_{t\to\infty}$ means in $\mathbb{P}^1_k = \mathbb{A}^1_k \cup \{\infty\}$, to which we can further extend the orbit map. But using the last characterization of the slope in Remark 2.8, this is equivalent to

$$\mu(x,\rho) \geq 0 \text{ and } \mu(x,\rho^{-1}) \geq 0,$$

where ρ^{-1} means $\rho \circ \lambda$ for $\lambda \leftrightarrow -1 \in \mathbb{Z}$. This suffices since $\mu^{\mathcal{L}}(x,\lambda^r) = r\mu^{\mathcal{L}}(x,\lambda)$ for $r \in \mathbb{N}$.

3. Period domains over $\mathbb{F}_q$

Keeping in mind the above results, it mainly goes to find the correct linearization to match the additive data we fix in our examples. We start with filtered vector spaces.

Fix $V \in \mathrm{vect}_k$, and let $\underline{d} = (d_1 \geq \ldots \geq d_n) \in \mathbb{N}^n$ with $d_1 = \dim V$. The corresponding flag variety $\mathcal{F} = \mathcal{F}_{\underline{d},V}$ is the functor on k-algebras

$$\mathcal{F}\colon R \mapsto \{V \otimes_k R \supseteq \ldots \supseteq F^n(V \otimes_k R) \text{ locally direct summands of } \mathrm{rk}_R(F^i(V \otimes_k R)) = d_i\}$$

with the obvious action of $G = \mathrm{GL}_V$. It is represented by the projective variety $G/P_{\underline{d}}$ (via the embedding (3.3)), where $P_{\underline{d}} \subseteq G$ is the standard parabolic subgroup of type $\underline{d}$, the stabilizer of flags of type $\underline{d}$ in V, i.e. upper triangular block matrices of the appropriate sizes. Then the G-action becomes just left multiplication.

Remark 3.1. Let $K|k$ be any field extension. Fix a degree function in $\mathrm{Hom}(K_0(n\text{-}\mathrm{fil}_{K|k}),\mathbb{Z})$, noting that $K_0(n\text{-}\mathrm{fil}_{K|k})$ does not depend on K. Recall from Example 1.7 (*c*) that the degree is given by $\theta \in \mathbb{Z}^n$ with descending entries (under (1.2)), and we can correspondingly embed

$$\theta\colon n\text{-}\mathrm{fil}_{K|k} \hookrightarrow \mathbb{Z}\text{-}\mathrm{fil}_{K|k}\,. \tag{3.1}$$

Let $\mu_\theta = \theta^*\mu_\bullet$ be the corresponding slope function. We use the suggestive notation

$$\mathcal{F}(K)^{\mathrm{ss}} = \{[F^\bullet] \in \mathcal{F}(K) \mid (V,F^\bullet) \in n\text{-}\mathrm{fil}_{K|k} \text{ is a semistable object}\}.$$

Then by definition, we can write these "semistable points" as the complement

$$\mathcal{F}(K)^{\mathrm{ss}} = \mathcal{F}(K) \smallsetminus \bigcup_{U \subsetneq V} Z_U(K), \tag{3.2}$$

where $Z_U \subseteq \mathcal{F}$ are in fact *closed* subvarieties, defined by setting

$$Z_U(K) := \{[F^\bullet] \in \mathcal{F}(K) \mid \mu_\theta(U, F^\bullet \cap U) > \mu_\theta(V,F^\bullet)\},$$

which follows from linear algebra, namely upper semi-continuity of the map

$$\mathcal{F}(K) \to \mathbb{Q},\ [F^\bullet] \mapsto \mu_\theta(U, F^\bullet \cap U).$$

This means that $\mathcal{F}(K)^{\mathrm{ss}}$ can only be ensured to be the K-points of a k-variety, if the union is finite, meaning $k = \mathbb{F}_q$ ought to be a finite field. We will thus assume this from now on.

Note that here, we can see the importance of not confusing a k-scheme with its k-points, recalling from Example 1.12 that $\mathcal{F}(k)^{\mathrm{ss}}$ is usually empty.

In the same situation for Example 1.14, the left-hand side of (3.2) has an interpretation as the K-points of a variety in the sense of p-adic (rigid-analytic) geometry.

Now, we make use of the following embedding,

$$\mathcal{F} = \mathcal{F}_{\underline{d},V} \hookrightarrow \prod_{i=1}^n \mathrm{Gr}_{d_i,V} \hookrightarrow \prod_{i=1}^n \mathbb{P}_{\bigwedge^{d_i} V} \hookrightarrow \mathbb{P}^N_k, \tag{3.3}$$

where the first arrow is the embedding $[F^\bullet] \mapsto ([F^iV])_{i=1,\ldots,n}$ into Grassmannians $\mathrm{Gr}_{d_i,V}$, the second map is the product of Plücker embeddings $[U] \mapsto [\det U]$, and finally, we apply the Segre embedding. This is not quite the embedding which defines our linearization, but rather we weight it by θ, as follows.

Let $\mathcal{L}_i$ denote the pullback to $\mathrm{Gr}_{d_i,V}$ of the canonical line bundle on $\mathbb{P}_{\bigwedge^{d_i} V}$. Then we set

$$\mathcal{L}_\theta := \boxtimes_{i=1}^n \mathcal{L}_i^{\otimes(\theta_i - \theta_{i+1})}|_{\mathcal{F}} \in \mathrm{Pic}^G(\mathcal{F}).$$

Definition 3.2. The scalar product of two $\mathbb{Z}$-filtrations $F_1^\bullet, F_2^\bullet$ of V over $K|k$ is defined by

$$\langle F_1^\bullet, F_2^\bullet\rangle = \sum_{i,j\in\mathbb{Z}} ij\cdot \dim_K(\operatorname{gr}^i_{F_1^\bullet}\operatorname{gr}^j_{F_2^\bullet} V).$$

Remark 3.3. Here, it really is essential that we are working with $\mathbb{Z}$-filtrations and consider the embedding (3.1), since the scalar product compares exactly *where* the jumps occur.

Lemma 3.4 ([3], Lemma 2.2.1)**.** *Let $\lambda\colon \mathbb{G}_m \to G$ and let $[F^\bullet] \in \mathcal{F}(K)$ be a fixed point of the induced $\mathbb{G}_m$-action through λ on $\mathcal{F}$. Then*

$$\mu^{\mathcal{L}_\theta}([F^\bullet],\lambda) = -\langle F^\bullet, F^\bullet_\lambda\rangle, \tag{3.4}$$

where $F^\bullet_\lambda$ is the filtration of V by the eigenspaces from (2.1) *as $F^j_\lambda V = \bigoplus_{i\geq j} V_i$.*

Remark 3.5. The key point is that the fibre over $[F^\bullet]$ of $\mathcal{L}_\theta$ is precisely

$$\mathcal{L}_{\theta,[F^\bullet]} \cong \bigotimes_{i=1}^n \det(F^{\theta_i}V)^{\otimes(\theta_i-\theta_{i+1})} \cong \bigotimes_{i=1}^n \det(\operatorname{gr}^{\theta_i} V)^{\otimes\theta_i},$$

where on each factor $\mathbb{G}_m$ turns out to act through λ precisely appropriately.

The fixed point condition is overcome by replacing $[F^\bullet]$ by $[F_0^\bullet] = \lim_{t\to 0}\lambda(t)[F^\bullet]$, which is allowed, because it changes neither side of (3.4).

Finally, $(V,F^\bullet) \in n\text{-}\operatorname{fil}_{K|k} \subseteq \mathbb{Z}\text{-}\operatorname{fil}_{K|k}$ is semistable if and only if $\langle F^\bullet, F^\bullet_{\mathrm{rat}}\rangle \leq 0$ for all *k-rational* $\mathbb{Z}$-filtrations $F^\bullet_{\mathrm{rat}}$ of V. These are exactly the *split* filtrations, i.e. those given by *gradings* of V, which as we know correspond to morphisms $\lambda\colon \mathbb{G}_m \to G$ as $F^\bullet_\lambda$.

Theorem 3.6 ([3], Theorem 2.2.3)**.** *Let G, V, θ be as before, and let $(V,F^\bullet) \in n\text{-}\operatorname{fil}_{K|k}$. Then*

$$(V,F^\bullet) \textit{ is } \mu_\theta\textit{-semistable } \Leftrightarrow \mu^{\mathcal{L}_\theta}([F^\bullet],\lambda)\geq 0 \textit{ for all (non-trivial) } \lambda\colon \mathbb{G}_m\to G.$$

Remark 3.7. Recall that the Hilbert-Mumford criterion 2.10 only applies in the case of an algebraically closed field k (at least in positive characteristic), where we do not have to worry about whether points are defined over the base field k. One can of course analyze how both notions of stability behave under base change and Galois descent.

4. Quiver varieties

Let Q be a quiver. Assume for simplicity $k = \overline{k}$ (in regards to Remark 3.7, cf. [7], §4). Fix a dimension vector $\underline{d} \in \mathbb{N}^{Q_0}$, and let $\mathcal{R} = \mathcal{R}_{\underline{d},Q} = \operatorname{Spec}(A)$, with $A = \operatorname{Sym}(V_{\underline{d}}^*)$, be the representation variety from Example 2.1, with its linear action of $\mathrm{GL}_{\underline{d}}$.

As above, we would like to choose a linearization corresponding to a fixed degree θ. Now, the only line bundle on $\mathcal{R}$ is the trivial one, on which we have to specify a lift of the $\mathrm{GL}_{\underline{d}}$-action. But that is necessarily given by an action on the fibre via some character

$$\chi\colon \mathrm{GL}_{\underline{d}} \to \mathbb{G}_m,$$

so $\mathrm{GL}_{\underline{d}}$ acts by $g(x,z) = (gx,\chi(g)z)$ on $\mathcal{R}\times_k \mathbb{A}^1_k$. Hence, linearizations are parametrized by

$$\operatorname{Pic}^{\mathrm{GL}_{\underline{d}}}(\mathcal{R}) \cong \operatorname{Hom}(\mathrm{GL}_{\underline{d}},\mathbb{G}_m) \cong \prod_{i\in Q_0}\operatorname{Hom}(\mathrm{GL}_{d_i},\mathbb{G}_m) \cong \mathbb{Z}^{Q_0}, \tag{4.1}$$

where $\theta\in\mathbb{Z}^{Q_0}$ corresponds to the character $\chi_\theta\colon \mathrm{GL}_{\underline{d}} \to \mathbb{G}_m,\ g\mapsto \prod_{i\in Q_0}\det(g_i)^{\theta_i}$.

Remark 4.1. The representation $\mathrm{GL}_{\underline{d}} \to \mathrm{GL}_{V_{\underline{d}}}$ is not faithful. Indeed, the scalar action

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\ \delta\ } \mathrm{GL}_{\underline{d}} \longrightarrow \mathrm{GL}_{V_{\underline{d}}} \text{ via } \delta(t) = \begin{pmatrix} t & & \\ & \ddots & \\ & & t\end{pmatrix},$$

is precisely its kernel. Therefore, we really act with the group $G := \mathrm{PGL}_{\underline{d}} = \mathrm{GL}_{\underline{d}}/\mathbb{G}_m$.

Thus, our characters χ_θ from (4.1) have to satisfy exactly the condition

$$\chi_\theta(t) = \prod_{i\in Q_0} \det(t\cdot \mathrm{id})^{\theta_i} = 1 \ \forall t \in \mathbb{G}_m, \text{ that is, } \sum_{i\in Q_0} \theta_i d_i = 0. \tag{4.2}$$

In this case, the G-equivariant sections of powers of a linearization χ take the form

$$A^{G,\chi^n} := \{f \in A \mid f(gx) = \chi(g)^n f(x) \text{ for all } x\}.$$

Accordingly, the candidate for our quiver moduli space is the projective quotient scheme

$$\mathrm{Proj}\Big(\bigoplus_{n\geq 0} A^{G,\chi^n}\Big) \longrightarrow \mathrm{Spec}(k). \tag{4.3}$$

Furthermore, the GIT-slope $\mu^\chi(x,\lambda)$ for $\lambda\colon \mathbb{G}_m \to G$ of course does not depend on x, and in fact the induced $\mathbb{G}_m$-action on the fibre is given by $\chi\circ\lambda \in \mathrm{End}(\mathbb{G}_m) \cong \mathbb{Z}$, meaning that

$$\mu^\chi(x,\lambda) = -\mu, \text{ when } \chi(\lambda(t)) = t^\mu \ \forall t\in\mathbb{G}_m,$$

cf. Remark 2.8. The Hilbert-Mumford criterion 2.10 now simply says that

$$x\in\mathcal{R} \text{ is } \chi\text{-semistable} \Leftrightarrow \mu^\chi(x,\lambda)\geq 0 \text{ for all } \lambda\colon \mathbb{G}_m\to G \text{ such that } \lim_{t\to 0}\lambda(t)x \text{ exists.}$$

Let $[M]\in\mathcal{R}$, so $M\in \mathrm{rep}_k(Q)$ with $\underline{\dim}(M) = \underline{d}$. Consider the filtrations $F^\bullet_\lambda M_i$ from (3.4), for each $i\in Q_0$. Denote the arrows in M by $\varphi_\alpha\colon M_{s(\alpha)}\to M_{t(\alpha)}$. Then

$$\lambda(t)\varphi_\alpha \equiv t^{m-n}\varphi_\alpha \text{ on } \mathrm{gr}^n_\lambda M_{s(\alpha)} \to \mathrm{gr}^m_\lambda M_{t(\alpha)},$$

since by definition, $\mathrm{gr}^n_\lambda M_{s(\alpha)} = M_{s(\alpha),n}$ is the eigenspace for the $\mathbb{G}_m$-action. Therefore,

$$\lim_{t\to 0}\lambda(t)\varphi_\alpha \text{ exists} \Leftrightarrow \varphi_\alpha\equiv 0 \text{ on } \mathrm{gr}^n_\lambda M_{s(\alpha)}\to \mathrm{gr}^m_\lambda M_{t(\alpha)} \text{ for all } m<n.$$

This means in turn that φ_α respects the filtration, so that $F^n_\lambda M\subseteq M$ are subrepresentations, defining a filtration of M. The limit is the associated graded representation

$$\lim_{t\to 0}\lambda(t)[M] = \Big[\bigoplus_{n\in\mathbb{Z}} \mathrm{gr}^n_\lambda M\Big] \in \mathcal{R}.$$

Then we can express the GIT-slope pairing (exactly when $\lim_{t\to 0}\lambda(t)[M]$ exists) as

$$-\mu^{\chi_\theta}([M],\lambda) = \sum_{i\in Q_0}\theta_i\sum_{n\in\mathbb{Z}} n\cdot\dim(\mathrm{gr}^n_\lambda M_i) = \sum_{n\in\mathbb{Z}} n\cdot\deg_\theta(\mathrm{gr}^n_\lambda M) = \sum_{n\in\mathbb{Z}}\deg_\theta(F^n_\lambda M). \tag{4.4}$$

Note that the sum is well-defined, because (4.2) tells us that for all $n\ll 0$, we have

$$\deg_\theta(F^n_\lambda M) = \deg_\theta(M) = \sum_{i\in Q_0}\theta_i d_i = 0.$$

Theorem 4.2. *Let $\eta\in\mathbb{Z}^{Q_0}$, and $\theta = \mathrm{rk}(\underline{d})\eta - \deg_\eta(\underline{d})$. Then χ_θ is a character of $\mathrm{PGL}_{\underline{d}}$, and M is μ_η-semistable $\Leftrightarrow$ M is μ_θ-semistable $\Leftrightarrow$ $[M]\in\mathcal{R}_{\underline{d},Q}$ is χ_θ-semistable.*

Proof. Of course, θ is defined precisely such that it satisfies (4.2). Namely,

$$\sum_{i\in Q_0}\theta_i d_i = \deg_\theta(\underline{d}) = \mathrm{rk}(\underline{d})\deg_\eta(\underline{d}) - \deg_\eta(\underline{d})\,\mathrm{rk}(\underline{d}) = 0.$$

Now assume $[M]$ is semistable. Any $0\subsetneq M'\subsetneq M$ defines a two-step filtration, corresponding to some $\lambda\colon\mathbb{G}_m\to G$ such that $\lim_{t\to 0}\lambda(t)[M]$ exists. Then by (4.4) and Hilbert-Mumford,

$$\deg_\theta(M') = -\mu^{\chi_\theta}([M],\lambda)\leq 0.$$

Hence, $(\mu_\theta(M) =)\ 0 \geq \mu_\theta(M') = \mathrm{rk}(\underline{d})\mu_\eta(M') - \deg_\eta(\underline{d})$, and so $\mu_\eta(M')\leq \frac{\deg_\eta(\underline{d})}{\mathrm{rk}(\underline{d})} = \mu_\eta(M)$. The converse follows by the same argument, but applied to an arbitrary λ as in (4.4). Indeed, whether M is semistable with respect to μ_θ or μ_η, we always have

$$\mu_\theta(F^n_\lambda M) = \mathrm{rk}(\underline{d})\mu_\eta(F^n_\lambda M) - \deg_\eta(\underline{d}) \leq \mathrm{rk}(\underline{d})\mu_\eta(M) - \deg_\eta(\underline{d}) = \mu_\theta(M) = 0,$$

whence we obtain that $\deg_\theta(F^n_\lambda M)\leq 0$ for all n,λ. ☐

Remark 4.3. It is shown in [5], §5, that (4.3) indeed yields a coarse moduli space for semistable quiver representations of dimension vector $\underline{d}$, and moreover, that if $\underline{d}$ is relatively prime, the corresponding moduli space of stable representations is fine.

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