V4A2 - Algebraic Geometry II (Sommersemester 2017)

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Introduction

This course was taught in Bonn, Germany over the Sommersemester 2017, by Prof. Dr. Peter Scholze.

We started by looking at properties of flat maps between schemes, as a way of parametrising nice families of schemes over the base scheme. This naturally lead us into the study of smooth, unramified, and étale morphisms, as well as the sheaf of Kähler differentials. Whilst proving that smoothness is local on source, we defined the first cohomology group $H^1(X, \mathcal{F})$ of an abelian sheaf on a scheme $\mathcal{F}$ using $\mathcal{F}$-torsors. This pushed us to define sheaf cohomology in general, and explore some of the associated homological algebra surrounding it. After proving some technical statements about the cohomology of coherent sheaves, and various base change properites, we had all the fire-power we needed to state and prove the Riemann-Roch Theorem and Serre Duality. We wrapped up the course with the theory of formal functions, Zariski's main theorem and Stein factorisation.

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1 Overview of the Course 20/04/2017

Recall that algebraic geometry is the study of geometric objects that are locally defined as the solution set of a system of polynomial equations,

$$p_1(x_1, \dots, x_n) = \dots = p_m(x_1, \dots, x_n) = 0,$$

over some field k. This is encoded algebraically in the k-algebra $A = k[x_1, \dots, x_n]/(p_1, \dots, p_m)$, and geometrically as the affine scheme $\operatorname{Spec} A$, the set of all prime ideals of A, with the Zariski topology, and a sheaf of k-algebras on it. By gluing together these affine schemes we obtain general schemes (in this case only schemes of finite type over a field k). This semester we will explore two main topics:

I Families of schemes; the notations of flatness and smoothness.

II Cohomology of (quasi-coherent) sheaves.

This lecture we'll just see a preview of both of these topics.

I - Families of Schemes

Definition 1.1. *A family of schemes (parametrised by the base scheme S) is a morphism $f : X \to S$.*

The intuition here should be that for all points $s \in S$ we have a scheme $\operatorname{Spec} k(s)$ and an inclusion $\operatorname{Spec} k(s) \hookrightarrow S$. For all point $s \in S$ we can then define the fibre of f at s simply as the pullback,

$$X_s := X \times_S \operatorname{Spec} k(s),$$

which is a scheme over $k(s)$. In this way we can move from morphisms $f : X \to S$ to a family of schemes X_s over $k(s)$ parametrised by $s \in S$. This process forgets some information, so knowing the morphism f is important. However, this definition is mostly useless in this generality, as the following stupid example ilustrates.

Example 1.2. Start with a scheme S, and choose any collection of schemes $X(s)$ such that each $X(s)$ is a scheme over $k(s)$, with no assumed compatibility. Then we let $X = \coprod_{s \in S} X(s)$, which is a scheme, and we obtain a map,

$$f : X = \coprod_{s \in S} X(s) \longrightarrow \coprod_{s \in S} \operatorname{Spec} k(s) \longrightarrow S.$$

For $s' \in S$, we then have,

$$X_{s'} = X \times_S \operatorname{Spec} k(s') = \coprod_{s \in S} X(s) \times_S \operatorname{Spec} k(s') = X(s'),$$

since the only time the fibre $X(s) \times_S \operatorname{Spec} k(s')$ is non-empty is when $s = s'$. This is a silly family of schemes, since over each point $s \in S$, the fibres $X_s = X(s)$ need to have no relations amongst each other.

We need to somehow find a condition which encodes the idea of a continuous family of schemes. This leads us to the notions of flatness and smoothness.

Recall 1.3. Let A be a ring and M be an A-module, then we have a functor $- \otimes_A M$ from the category of A-modules onto itself. This functor is always right exact, so if

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0,$$

is an exact sequence of A-modules, then

$$N' \otimes_A M \longrightarrow N \otimes_A M \longrightarrow N'' \otimes_A M \longrightarrow 0,$$

is also an exact sequence of A-modules. However, the map $N' \otimes_A M \to N \otimes_A M$ might not be injective.

Definition 1.4. *An A-module M is flat if the functor $-\otimes_A M$ is exact.*

This is equivalent to the following statement: if $N' \hookrightarrow N$ is an injective map of A-modules, then $N \otimes_A M \hookrightarrow N \otimes_A M$ is an injective map of A-modules.

Definition 1.5. *A morphism of schemes $f : X \to S$ is flat if for all open affine $U = \operatorname{Spec} A \subseteq X$ mapping to $\operatorname{Spec} R \subseteq S$, the corresponding map $R \to A$ makes A a flat R-module.*

Remark 1.6. This definition and the interpretation of this as a good definition for a continuous family of schemes was suggested by Serre. It can be tricky to obtain a good geometric understanding of this.

Example 1.7 (Non-flat ring map)**.** Let A be a local ring, and $f \in \mathfrak{m}_A \subseteq A$ be a nonzero divisor of A. Then A/fA is not a flat A-module. More generally, if B is any nonzero A/fA-algebra, then B is not a flat A-module. The object $\operatorname{Spec} B$ is somehow vertical over $\operatorname{Spec} A$, so there are non-empty fibres of $\operatorname{Spec} B \to \operatorname{Spec} A$ over $V(f)$, but the fibres away from $V(f)$ are empty. Visually, we might view $\operatorname{Spec} B$ as hanging vertically over $\operatorname{Spec} A$. To see this, note that we have an injective map $fA \hookrightarrow A$. But its tensor, $fA \otimes_A B \to B$ is not injective unless $fA \otimes_A B = 0$ (since its image must be contained in fB, which is 0). But since f is a nonzero divisor, $fA \cong A$, so $fA \otimes_A B \cong A \otimes_A B \cong B \neq 0$. Hence B is not flat.

More generally we have the following statement

Proposition 1.8. *If A is a ring and M is a flat A-module, then the set*

$$Z_M = \{x \in X = \operatorname{Spec} A \,|\, M \otimes_A k(x) \neq 0\},$$

is generalising, i.e. if $x \in Z_M$ and $x \in \overline{\{y\}}$ for some $y \in \operatorname{Spec} A$, then y is also in Z_M.

Proof. Choose an $x \in Z_M$, and a generalisation of x, say $y \in \operatorname{Spec} A$. Then we have prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of A corresponding to x and y respectively. We need to show $M \otimes k(y) = (M/\mathfrak{q})_{\mathfrak{q}} \neq 0$. We can replace A by $A/\mathfrak{q}$, and M by $M/\mathfrak{q} = M \otimes_A A/\mathfrak{q}$, so A is a local integral domain, and M is a non-zero flat A-module. This works since flatness is preserved by base change (see Lemma 2.7). Setting $K = \operatorname{Frac} A = A_{\mathfrak{q}}$ we then have an inclusion $A \hookrightarrow K$, and after tensoring with M over A we obtain an inclusion $M \hookrightarrow M \otimes_A K = M_{\mathfrak{q}}$. Since $M \neq 0$, this implies $M_{\mathfrak{q}} \neq 0$. □

Example 1.9. If $A = \mathbb{Z}$ then $M = \mathbb{Z}/p\mathbb{Z}$ is not flat. Indeed, we have that

$$Z_M = \{(p)\}.$$

But $\mathbb{Z}$ is an integral domain, so it has generic a point $\eta = (0)$ which is not in Z_M. On the other hand, if $M = \mathbb{Q}$, then we see that $Z_{\mathbb{Q}} = \{\eta\}$, which is closed under generalisations.

If $f : X \to S$ is a flat map, then our definition of a continuous family of schemes will make sense. Note that if $S = \operatorname{Spec} k$ for some field k, then all morphisms f as above are flat. This is algebraically clear, since all modules over a field are free (vector spaces), and also geometrically clear since $\operatorname{Spec} k$ is a single point.

There is a stronger condition than flatness, called smoothness. We will see that this condition is interesting even when $S = \operatorname{Spec} k$. In fact, we have a theorem which states that f is smooth if and only if f is flat and for all $s \in S$, the morphism $f_s : X_s \to \operatorname{Spec} k(s)$ is smooth (see Theorem 6.12). This tells us than in order to know that a morphism is smooth, we only need to know that it is smooth over a collection of fields.

Example 1.10. Consider the family of curves parametrised by $y^2 = x^3 + x^2 + t$, for varying t, over a field k of characteristic not equal to 2. Different values for t parametrize different curves. In particular, when $t > 0$ the respective curve is smooth, but for $t = 0$ we obtain a nodal singularity (the nodal cubic). Hence this family of curves is not smooth, but it is flat since the deformations are continuous. If we remove the point $t = 0$ from the fibre and all other points that create singularities, we obtain a smooth family of curves.

There are other adjectives we can use here too. A proper flat map gives us a continuous family of compact (possibly singular) spaces, and a proper smooth map over $\mathbb{C}$ will be a continuous family of compact complex manifolds. We will add more adjectives like unramifed and étale as the semester carries on.

II - Sheaf Cohomology

This machinery will be used to prove the Riemann-Roch theorem.

Recall 1.11. If k is an algebraically closed field, and C is a projective smooth (or equivalently proper normal) curve over k, then the genus g is defined as,

$$g = \dim_k \Gamma(C, \Omega^1_{C/k}),$$

where $\Omega^1_{C/k}$ is canonical line bundle on C, as defined in lecture 29 of [7].

Theorem 1.12 (Riemann-Roch)**.** *For all line bundles $\mathcal{L}$ on C we have,*

$$\dim_k \Gamma(C, \mathcal{L}) - \dim_k \Gamma(C, \Omega^1_{C/k} \otimes \mathcal{L}^\vee) = \deg \mathcal{L} + 1 - g.$$

Our proof of the Riemann-Roch theorem will have three major steps:

Step 1 For any (quasi-coherent) sheaf $\mathcal{F}$ on a scheme X, and for $i \geq 0$, we define the cohomology groups $H^i(X, \mathcal{F})$ (the ith cohomology group of X with coefficients in $\mathcal{F}$), with $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ in such a way that, given an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

of sheaves, we obtain a long exact sequence on sheaf cohomology,

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to H^1(X, \mathcal{F}) \to \cdots.$$

Step 2 We prove that for any line bundle $\mathcal{L}$ on a proper, normal curve C, the equality

$$\dim_k H^0(C, \mathcal{L}) - \dim_k H^1(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g$$

holds. We note that various finiteness properties must hold here too, such as $\dim_k H^i(C, \mathcal{L})$ is finite, which crutially uses properness.

Step 3 Finally we prove that given a proper, smooth scheme X over a field k of dimension d, and a vector bundle $\mathcal{E}$ over X , we have an isomorphism

$$H^i(X, \mathcal{F}) \cong \mathrm{Hom}(H^{d-i}(X, \Omega^d_{X/k} \otimes \mathcal{F}^\vee), k).$$

This is called the Serre duality. In particular, if $X = C$, then $d = 1$ and we see that $H^1(C, \mathcal{F})$ is dual to $H^0(C, \Omega^1_{C/k} \otimes \mathcal{F}^\vee)$.

Remark 1.13. If $\mathcal{E}$ is a coherent sheaf over a projective smooth scheme X/k with $d = \dim X$, then we can define its Euler characteristic,

$$\chi(X, \mathcal{E}) = \sum_{i=1}^{d} (-1)^i \dim_k H^i(X, \mathcal{E}),$$

assuming that each $H^i(X, \mathcal{E})$ is a finite dimensional k-vector space, and $H^i(X, \mathcal{E}) = 0$ for large values of i. Famously, Hirzebruch expressed this as an explicit formula in terms of the Chern classes of $\mathcal{E}$, which are somehow generalisations of the degree of a line bundle. This is called the Riemann-Roch-Hirzebruch theorem, and it can be seen written in the entrance of the Max Plank Institute for Mathematics in Bonn.

2 Flatness 24/04/2017

We start by recalling some definitions.

Definition 2.1. 1. *Given a ring A, and an A-module M, then we say that M is flat (as an A-module) if the functor $- \otimes_A M$ is exact.*

2. *If $\phi : A \to B$ is a map of rings, then ϕ is flat if B a flat A-module through ϕ.*

3. *If $\phi : A \to B$ is a flat map of rings such that the induced morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, then we call ϕ faithfully flat (treuflach auf Deutsch).*

We now give a range of examples of flat modules and morphisms.

Example 2.2 (Localisations)**.** If $S \subseteq A$ is a multiplicatively closed subset of A, then $A \to A[S^{-1}]$ is flat. Indeed, in this case we can see that $- \otimes_A A[S^{-1}]$ is given by $M \mapsto M[S^{-1}]$ on any A-module M, which we already know to be exact from exercise 6.3(i) from last semester.

Example 2.3 (Filtered Colimits)**.** We already know that filtered colimits of A-modules are exact. Using the fact that the tensor product of filtered colimits is the filtered colimit of the tensor product (which we can see by adjunction) allows us to conclude that filtered colimits of flat modules are flat from exercise 3.4 from last semester.

We can relate the previous two examples by realising the localisation of A at S as a filtered colimit,

$$A[S^{-1}] = \operatorname{colim}_{s \in S} A[s^{-1}].$$

For example, we saw last semester that $A[f^{-1}]$ for any $f \in A$ is simply the colimit of the diagram

$$A \xrightarrow{\cdot f} A \xrightarrow{\cdot f} A \xrightarrow{\cdot f} \cdots,$$

where all the maps are multiplication by f.

Theorem 2.4 (Lazard)**.** *An A-module is flat if and only if it is a filtered colimit of finite free A-modules.*

Example 2.5 (Completions)**.** Let A be a noetherian ring with an ideal $I \subseteq A$, and

$$\widehat{A} = \varprojlim A/I^n,$$

the I-adic completion of A. Then the morphism $A \to \widehat{A}$ is flat. To see this, if M is a finitely generated A-module, we know that $M \otimes_A \widehat{A} \to \widehat{M}$ is an isomorphism, and that $M \mapsto \widehat{M}$ is exact (see Lemma 10.95.1 part (3) in [8]). But M is a flat A-module if and only if for all finitely generated ideals $I \subseteq A$, the induced map

$$I \otimes_A M \longrightarrow M$$

is injective. Applying this to $M = \widehat{A}$, we get that it is enough to show that $\widehat{I} \to \widehat{A}$ is injective. But this is just the exactness of completions of finitely generated A-modules mentioned above.

Example 2.6. Let $X = \operatorname{Spec} A = \bigcup_{i=1}^{n} D(f_i)$ for some collection of $f_i \in A$. Notice that the map,

$$A \longrightarrow B = \prod_{i=1}^{n} A[f_i^{-1}],$$

is faithfully flat. Indeed, B is a finite direct sum[1] of flat modules so it is flat, and the induced map

$$\operatorname{Spec} B \longrightarrow \operatorname{Spec} A = X,$$

is surjective since the $D(f_i)$'s form a cover of X.

[1]Recall finite direct sums are isomorphic to finite direct products in an abelian category (see Definition 9.2).

Lemma 2.7. *Let $\phi : A \to B$ be a map of rings, M be an A-module and N be a B-module.*

1. *If M is a flat A-module, then $M \otimes_A B$ is a flat B-module.*
2. *If ϕ is flat and N is a flat B-module, then N is also flat as an A-module.*

Proof. To prove 1, notice we have the following commutative diagram (up to canonical isomorphism) of functors.

$$\begin{array}{ccc} B\text{−Mod} & \xrightarrow{-\otimes_B (B \otimes_A M)} & B\text{−Mod} \\ \downarrow{\scriptstyle \text{forg.}} & & \downarrow{\scriptstyle \text{forg.}} \\ A\text{−Mod} & \xrightarrow{-\otimes_A M} & A\text{−Mod} \end{array}$$

The two vertical functors are exact. Note that we can check the exactness of B-modules in the category of abelian groups, so it suffices to check exactness after forgetting to the category of A-modules. The bottom functor is exact by assumption, hence the top functor is also exact. For Part 2 we use a similar argument. Consider the following commutative diagram (up to canonical isomorphism) of functors.

$$\begin{array}{ccc} A\text{−Mod} & \xrightarrow{-\otimes_A B} & B\text{−Mod} \\ {\scriptstyle -\otimes_A N = (-\otimes_A B)\otimes_B N}\downarrow & & \downarrow{\scriptstyle -\otimes_B N} \\ A\text{−Mod} & \xleftarrow[\text{forg.}]{} & B\text{−Mod} \end{array}$$

The top and right funtors are exact by assumption and the bottom functor is always exact, hence the left functor must also be exact. □

The important proposition we would like to prove this lecture is the following.

Proposition 2.8 (Flatness Descent). *If $\phi : A \to B$ is faithfully flat and M is an A-module, then M is flat if and only if $M \otimes_A B$ is flat.*

If we apply this to Example 2.6 we see that it implies that M is flat if and only if $M[f_i^{-1}]$ is a flat $A[f_i^{-1}]$-module for $i = 1, \dots, n$. In particular, we only need to check flatness on localisations at elements which generate the unit ideal of A.

To prove this proposition we need a lemma:

Lemma 2.9. *Let $\phi : A \to B$ be a flat map, and $C^\bullet$ be an integer graded complex (so we have maps $d : C^i \to C^{i+1}$ for all $i \in \mathbb{Z}$, such that $d^2 = 0$) of A-modules. Then $C^\bullet$ is exact implies that $C^\bullet \otimes_A B$ is exact. In fact, we have*

$$H^*(C^\bullet \otimes_A B) \cong H^*(C^\bullet) \otimes_A B.$$

Conversely, if ϕ is faithfully flat, then exactness of $C^\bullet \otimes_A B$ implies exactness of $C^\bullet$.

It might seem strange that we have to work with complexes, but we will need this added generality later on. For now, we recall some definitions and facts from homological algebra.

Recall 2.10. If $C^\bullet$ is a complex of A-modules, then we define submodules $B^i \subseteq Z^i \subseteq C^i$ as follows: B^i is the image of $d : C^{i-1} \to C^i$ and Z^i is the kernel of $d : C^i \to C^{i+1}$. Notice that $B^i \subseteq Z^i$ since $d^2 = 0$. We define the cohomology of $C^\bullet$ as

$$H^i(C^\bullet) = Z^i / B^i.$$

We say that $C^\bullet$ is an exact complex (also called acyclic) if its defining sequence is exact, i.e. $\ker d = \operatorname{im} d$ for all d. Then we have that $Z^i = B^i$ for all $i \in \mathbb{Z}$, and thus $H^i(C^\bullet) = 0$ for all $i \in \mathbb{Z}$.

Proof of Lemma 2.9. The first part of the lemma is a formal consequence of the following two short exact sequences,

$$0 \longrightarrow Z^i \longrightarrow C^i \longrightarrow B^{i+1} \longrightarrow 0,$$
$$0 \longrightarrow B^i \longrightarrow Z^i \longrightarrow H^i \longrightarrow 0,$$

which, after tensoring with the flat A-module B, gives us short exact sequences,

$$0 \longrightarrow Z^i \otimes_A B \longrightarrow C^i \otimes_A B \longrightarrow B^{i+1} \otimes_A B \longrightarrow 0, \tag{2.11}$$

$$0 \longrightarrow B^i \otimes_A B \longrightarrow Z^i \otimes_A B \longrightarrow H^i \otimes_A B \longrightarrow 0. \tag{2.12}$$

Sequence 2.11 tells us that $B^{i+1} \otimes_A B$ is the image of the map $d \otimes_A B$, which has kernel $Z^i \otimes_A B$. Sequence 2.12 tells us that, the $H^i(C^\bullet) \otimes_A B$ is isomorphic to the quotient of $Z^i \otimes_A B$ by $B^i \otimes_A B$, which is by definition $H^i(C^\bullet \otimes_A B)$. For the converse, note that $H^i(C^\bullet \otimes_A B) \cong H^i(C^\bullet) \otimes_A B$, so if we setting $M = H^i(C^\bullet)$ we see that it suffices to check that $M \otimes_A B = 0$ implies $M = 0$. Assume not, and take an element $0 \neq x \in M$. Let $I \subsetneq A$ be the annihilator of x. Then we have an inclusion $A/I \hookrightarrow M$, from which by flatness we obtain another injection, $B/IB \hookrightarrow M \otimes_A B$. Then observe the following pullback square.

$$\begin{array}{ccc} \mathrm{Spec}(B/IB) & \longrightarrow & \mathrm{Spec}(A/I) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

We know the bottom map is surjective, and that the right map is a closed immersion, so the left map must also be closed and the top map is surjective. This implies that $B/IB \neq 0$, which contradicts our assumption since we have an injection $B/BI \hookrightarrow M \otimes_A B = 0$. □

Proof of Proposition 2.8. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of A-modules. We have to show that the sequence, after tensoring with $- \otimes_A M$, is still exact. The map ϕ is faithfully flat, so by Lemma 2.9, it suffices to check exactness after tensoring the sequence with B. We are reduced to showing the exactness of,

$$0 \longrightarrow N' \otimes_A M \otimes_A B \longrightarrow N \otimes_A M \otimes_A B \longrightarrow N'' \otimes_A M \otimes_A B \longrightarrow 0.$$

However, we know $M \otimes_A B$ is flat as a B-module, so flatness of ϕ implies that $M \otimes_A B$ is flat as an A-module by Lemma 2.7. The result follows. □

We now generalize these results in scheme-theoretic language.

Proposition 2.13 (Definition/Proposition)**.** *Let X be a scheme and $\mathcal{M}$ a quasi-coherent sheaf (of $\mathcal{O}_X$-modules) on X. Then $\mathcal{M}$ is flat if one of the following equivalent conditions hold:*

1. *For all open affines $U = \operatorname{Spec} A \subseteq X$, $\mathcal{M}(U)$ is a flat $\mathcal{O}_X(U)$-module.*
2. *There is a cover of X by open affines $U = \operatorname{Spec} A \subseteq X$ such that $\mathcal{M}(U)$ is a flat A-module.*

We must show that the conditions are equivalent.

Proof. The first implication is trivial. For the converse, suppose that $X = \bigcup U_i$ for some $U_i = \mathrm{Spec}(A_i)$ is a cover such that $\mathcal{M}(U_i)$ are flat A_i-modules . For simplicity, we will consider one of those U_i and denote it by $U = \mathrm{Spec}(A)$. So $\mathcal{M}(U)$ is a flat A-module. Let $f \in A$ such that $V = \operatorname{Spec} A[f^{-1}] = D_U(f) \subseteq U$. Clearly, we have that $\mathcal{M}(V) = \mathcal{M}(U) \otimes_A A[f^{-1}]$, since $\mathcal{M}$ is quasi-coherent, so $\mathcal{M}(V)$ is a flat $A[f^{-1}] = \mathcal{O}_X(V)$-module. Such V's form a basis for the topology on U, and thus on X. Therefore, if $\operatorname{Spec} A' = U' \subseteq X$ is any open affine, we can find $f'_1 \ldots, f'_n \in A'$ such that $D_{U'}(f'_i) = D_{U_i}(f_i) \subseteq U_i$ for some U_i, f_i as above. Thus we see that

$$\mathcal{M}(D_{U'}(f'_i)) = \mathcal{M}(U') \otimes_{A'} A'[f_i'^{-1}]$$

is a flat $A'[f_i'^{-1}]$-module, for $i = 1, \dots, n$. As flatness descends along the faithfully flat map,

$$A' \longrightarrow \prod_{i=1}^{n} A'[f_i'^{-1}],$$

we conclude that $\mathcal{M}(U')$ is a flat A'-module. $\square$

Now we need to define flat morphisms of schemes.

Proposition 2.14 (Definition/Proposition)**.** *A morphism of schemes $f : Y \to X$ is said to be flat if one of the following equivalent conditions hold:*

1. *For all open affine $V = \operatorname{Spec} B \subseteq Y$ mapped to an open affine $U = \operatorname{Spec} A \subseteq X$, the map $A \to B$ is flat.*
2. *There is a cover of Y by open affines $V = \operatorname{Spec} B \subseteq Y$ mapping to open affines $U = \operatorname{Spec} A \subseteq X$ such that $A \to B$ is flat.*

Proof. Again, one of the implications is trivial. For the other implication, we start by proving that if we shrink U and V the map is still flat. Then we show that if Y, X are affine the result holds, and the result follows by gluing.

Indeed, since localisations are flat, we can restrict to open subsets of U, and since the composition of flat maps is flat we can restrict to open subsets of V. For the second part, let $f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ be a map of affine schemes, and assume that there exists a cover by distinguished open sets on X and Y such that the restricted maps are flat. So, there is a cover $Y = \bigcup_{i=1}^{n} D(g_i)$ such that $A \to B[g_i^{-1}]$ is flat for $i = 1, \dots, n$.[2] Now we argue as in the proof of flat descent. Given a short exact sequence of A-modules,

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

we want to conclude that the sequence after applying $- \otimes_A B$ is still exact. However, the map $B \to \prod_{i=1}^{n} B[g_i^{-1}]$ is faithfully flat, so we can check after tensoring with the latter product ring. This follows from flat descent, and the flatness of $B[g_i^{-1}]$ over A. $\square$

Remark 2.15. In [5], Grothendieck considers $f : Y \to X$ and a quasi-coherent sheaf $\mathcal{N}$ on Y. Then he defines "flatness of $\mathcal{N}$ over $\mathcal{O}_X$". If $Y = X$, this definition recovers flatness of quasi-coherent sheaves and if $\mathcal{N} = \mathcal{O}_Y$ it recovers flatness of the map f. We will consider this more general approach in Definition 16.1 when it is needed.

We need the following result, but we will not prove it.

Proposition 2.16. *Let P be the collection of all flat morphisms of schemes. Then P is closed under composition, and satisfies the base change, product, and the local on the source and on the target properties.*

Proof. This is Proposition 14.3 in [2]. $\square$

Notice that this is the first class of morphisms that we have defined (so far) that is local on the source. This essentially follows by definition. The fact that these morphisms are also local on the target means we can prove every other property stated above on affine schemes.

We now begin a detour into the subject of faithfully flat descent. Recall the following proposition from last semester (see [7, Cor. 11.10]).

[2] A priori we only get that flatness of $A[f_i^{-1}] \to B[g_i^{-1}]$, but $A \to A[f_i^{-1}]$ is also flat.

Proposition 2.17. *The functor $M \mapsto M_i = M[f_i^{-1}]$ from the category of A-modules to the category of collections of $A[f^{-1}]$-modules M_i and isomorphisms $\alpha_{ij} : M_i[f_j^{-1}] \to M_j[f_i^{-1}]$ which satisfy the cocycle condition, is an equivalence of categories.*

The idea in what follows is to see the above statement as a special case of a much more general statement. The general statement deals with faithfully flat maps, while the above only deals with the faithfully flat map,

$$A \longrightarrow \prod_{i=1}^{n} A[f_i^{-1}].$$

Theorem 2.18. *Let $\phi : A \to B$ be a faithfully flat map. Then the category of A-modules is equivalent to the category of B-modules, N, together with an isomorphism $\alpha : N \otimes_A B \cong B \otimes_A N$ of $B \otimes_A B$-modules such that the following diagram commutes,*

$$\begin{array}{ccc} N \otimes_A B \otimes_A B & \xrightarrow{\alpha \otimes_A \mathrm{id}_B} & B \otimes_A N \otimes_A B \\ & \searrow^{\alpha} & \downarrow{\scriptstyle \mathrm{id}_B \otimes_A \alpha} \\ & & B \otimes_A B \otimes_A N \end{array},$$

where the diagonal α is the map switching the tensor factors N and $B \otimes_A B$.

The resulting functor F sends an A-module M to $(N = B \otimes_A M, \alpha_{\mathrm{can}})$ where the isomorphism is the canonical isomorphism,

$$\alpha_{\mathrm{can}} : N \otimes_A B = B \otimes_A M \otimes_A B \cong B \otimes_A B \otimes_A M = B \otimes_A N.$$

Remark 2.19. This functor has a right adjoint. We will use it to prove the above equivalence. It is defined as

$$(N, \alpha) \longmapsto \mathrm{eq}\left(N \rightrightarrows B \otimes_A N \right) = \{ n \in N \mid \alpha(n \otimes 1) = 1 \otimes n \},$$

where the top map is $n \mapsto 1 \otimes n$ and the bottom one is $n \mapsto \alpha(n \otimes 1)$. Indeed, given any M and (N, α) we can easily check the natural isomorphism of hom-sets required for an adjunction:

$$\mathrm{Hom}((M \otimes_A B, \alpha_{\mathrm{can}}), (N, \alpha)) = \left\{ f \in \mathrm{Hom}_B(M \otimes_A B, N) \;\middle|\; \begin{array}{ccc} M \otimes_A B \otimes_A B & \xrightarrow{f \otimes_A \mathrm{id}_B} & N \otimes_A B \\ \downarrow{\scriptstyle \alpha_{\mathrm{can}}} & & \downarrow{\scriptstyle \alpha} \\ B \otimes_A M \otimes_A B & \xrightarrow{\mathrm{id}_B \otimes_A f} & B \otimes_A N \end{array} \right\}$$

$$= \left\{ f_0 \in \mathrm{Hom}(M, N) \;\middle|\; \begin{array}{ccccc} M & \xrightarrow{f_0} & N & \longrightarrow & N \otimes_A B \\ \downarrow{\scriptstyle =} & & & & \downarrow{\scriptstyle \alpha} \\ M & \xrightarrow{f_0} & N & \xrightarrow{n \mapsto 1 \otimes n} & B \otimes_A N \end{array} \right\} = \mathrm{Hom}_A\Big(M, \mathrm{eq}\big(N \rightrightarrows B \otimes_A N \big)\Big).$$

To prove Theorem 2.18 we will need to show the unit and counit of the above adjunction are equivalences, i.e. we need to see that

(unit) for all A-modules M, the map $M \to \mathrm{eq}\left(B \otimes_A M \rightrightarrows B \otimes_A B \otimes_A M \right)$ is an isomorphism, and

(counit) for all (N, α) satifying the cocycle condition, if $M = \mathrm{eq}\left(N \rightrightarrows B \otimes_A N \right)$ then $M \otimes_A B \to N$ is an isomorphism.

3 Faithfully Flat Descent 27/04/2017

This lecture we shall prove Theorem 2.18 seen last time.

Remark 3.1 (Geometric Interpretation)**.** Let $f : Y = \operatorname{Spec} B \to \operatorname{Spec} A = X$ be a faithfully flat map, and let $\mathcal{M}$ be a quasi-coherent sheaf on X. Then $\mathcal{N} = f^*\mathcal{M}$ is a quasi-coherent sheaf on Y. Note that we have an isomorphism.

$$\alpha : p_1^*\mathcal{N} \cong p_2^*\mathcal{N},$$

where $p_i : Y \times_X Y \to Y$ are the canonical projections. Indeed, considering the usual diagram

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{p_2} & Y \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{f} & X \end{array},$$

we immediatelly see that

$$p_1^*\mathcal{N} = p_1^*f^*\mathcal{M} = (p_1 \circ f)^*\mathcal{M} = (p_2 \circ f)^*\mathcal{M} = p_2^*f^*\mathcal{M} = p_2^*\mathcal{N}.$$

Furthermore, this isomorphism α satisfies the cocycle condition on $Y \times_X Y \times_X Y$. Actually, these two conditions encode the fact that $\mathcal{N}$ comes from a quasi-coherent module on X, meaning that given a quasi-coherent sheaf $\mathcal{N}$ on Y, it is the pullback of a quasi-coherent sheaf on X if and only if we have an isomorphism α as above satisfying the cocycle condition. This is called descent because we want to descend from a cover of Y down to X. We can descend modules by Theorem 2.18, but with more case we can descend schemes, morphisms, and properties (such as the descent of flatness from Proposition 2.8).

For the proof of Theorem 2.18 we recall Remark 2.19 which says F has a right adjoint, G, sending,

$$(N, \alpha) \longmapsto \operatorname{eq}\Big(\; N \rightrightarrows B \otimes_A N \; \Big).$$

In geometric terms, using the same notation as in Remark 3.1, this equaliser can be simply written as

$$\{s \in H^0(Y, \mathcal{N}) = N \mid p_1^*(s) = p_2^*(s) \in H^0(Y \times_X Y, p_1^*\mathcal{N}) \cong_\alpha H^0(Y \times_X Y, p_2^*\mathcal{N})\}.$$

Note that the definition of G does not use the cocycle condition. Since we have an adjunction between the categories in Theorem 2.18, we simply need to show that the unit and counit are isomorphisms. We'll see that proving that the unit is an isomorphism does not make use of the cocycle conditions.[3]

Proposition 3.2. *Let $\phi : A \to B$ be a faithfully flat map. Then the sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow B \otimes_A B,$$

where the last map sends $b \mapsto b \otimes 1 - 1 \otimes b$ is exact.

Proof. Suppose that $\phi : A \to B$ has a section, so a ring map $\sigma : B \to A$ such that $\sigma \circ \phi = \mathrm{id}_A$.[4] Then it is clear that ϕ must be injective. Let $b \in B$ such that $b \otimes 1 = 1 \otimes b \in B \otimes_A B$. The map $\sigma \otimes_A \mathrm{id}_B : B \otimes_A B \to B$ sends $b \otimes 1 \mapsto \sigma(b)$ and $1 \otimes b \mapsto b$, so $b = \sigma(b) \in A$, hence $b \in A$ (note that we didn't even need faithful flatness in this case).

[3]We remark this now, since the occurance of the cocycle conditions in the proof that the counit is an isomorphism can be considered as a little subtle.

[4]This would usually be called a retraction of ϕ, but when we consider the maps on spectra the map induced by σ is really a section.

For the general case, recall that flat descent (Proposition 2.8) states that we can check the exactness of our sequence above after applying the functor $-\otimes_A B$. If we let $A' = B$, $B' = B \otimes_A B$, then our sequence becomes,

$$0 \longrightarrow A' \longrightarrow B' \longrightarrow B' \otimes_{A'} B',$$

where last map is still $b' \mapsto b' \otimes 1 - 1 \otimes b'$. But in this case we have a trivial section $m : B' = B \otimes_A B \to B = A'$ of ϕ, which is simply multiplication of B as an A-algebra. The result follows. □

The following is not a corollary of the above proposition, but it is a corollary of the proof as we will see.

Corollary 3.3. *Let A, B, ϕ as above. For any A-module, M, the natural map*

$$M \longrightarrow FG(M) = \mathrm{eq}\left(B \otimes_A M \rightrightarrows B \otimes_A B \otimes_A M \right),$$

(where the two maps on the equalizer are the same as Remark 2.19) is an isomorphism, i.e. the sequence,

$$0 \longrightarrow M \longrightarrow M \otimes_A B \longrightarrow M \otimes_A B \otimes_A B,$$

is exact, where the maps are $m \mapsto m \otimes 1$ and $m \otimes b \mapsto m \otimes b \otimes 1 - m \otimes 1 \otimes b$.

Proof. Instinctively, we would tensor Proposition 3.2 with M, but tensoring is not left exact, so we cannot naïvely do this. We shall use a similar argument as the one in the proof above . It is enough to prove this after tensoring with $-\otimes_A B$, so we can again assume that $\phi : A \to B$ has a section σ. Then $M \to M \otimes_A B$ has a section, and is therefore injective. To see exactness in the middle, take $\sum_i m_i \otimes b_i \in M \otimes_A B$ satisfying,

$$\sum_i m_i \otimes b_i \otimes 1 = \sum_i m_i \otimes 1 \otimes b_i \in M \otimes_A B \otimes_A B.$$

Then the map,

$$\mathrm{id}_M \otimes_A \sigma \otimes_A \mathrm{id}_B : M \otimes_A B \otimes_A B \longrightarrow M \otimes_A A \otimes_A B = M \otimes_A B,$$

sends $\sum_i m_i \otimes b_i \otimes 1 \mapsto \sum_i m_i \otimes \sigma(b_i)$, and $\sum_i m_i \otimes 1 \otimes b_i \mapsto \sum_i m_i \otimes b_i$. Hence,

$$\sim_i m_i \otimes b_i = \sum_i m_i \otimes \sigma(b_i) = \sum_i m_i \sigma(b_i) \otimes 1 \in M.$$

□

This gives us the unit case, and we will now procced to show the counit is an isomorphism. Let N be a B-module, and $\alpha : N \otimes_A B \cong B \otimes_A N$ a given isomorphism.

Proposition 3.4. *If the pair (N, α) satisfies the cocycle condition, then the map*

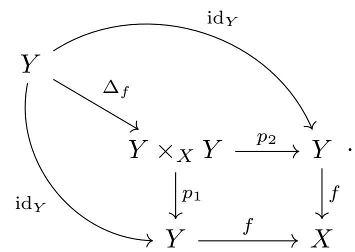
$$\psi : N = (N \otimes_B B \otimes_A B) \otimes_{B \otimes_A B} B = (N \otimes_A B) \otimes_{B \otimes_A B} B \xrightarrow{\alpha \otimes_{B \otimes_A B} \mathrm{id}_B} (B \otimes_A N) \otimes_{B \otimes_A B} B = N,$$

is the identity.

Remark 3.5 (Geometric Interpretation)**.** Recall the set up of Remark 3.1. We have $\alpha : p_1^* \mathcal{N} \cong p_2^* \mathcal{N}$. Then we can see that the map ψ can be translated to the map,

$$\Delta_f^*(\alpha) : \Delta_f^* p_1^* \mathcal{N} \xrightarrow{\cong} \Delta_f^* p_2^* \mathcal{N},$$

where Δ_f is the diagonal map, defined by the commutative diagram,



But note that $\Delta_f^* p_1^* \mathcal{N} = \mathcal{N} = \Delta_f^* p_2^* \mathcal{N}$, so the above proposition is just saying that $\Delta_f^*(\alpha) = \mathrm{id}_{\mathcal{N}}$.

Proof. The idea is to use the fact that ψ is an automorphism of N. In particular, this tells us that it suffices to show that $\psi \circ \psi = \psi$. This is precisely where the cocycle condition is used. Let $n \in N$, and write $\alpha(n \otimes 1) = \sum_i b_i \otimes x_i$ for some $b_i \in B$ and $x_i \in N$. Then we can write,

$$\psi(n) = \sum_i b_i x_i \in N,$$

straight from the definitions. We now write $\alpha(x_i \otimes 1) = \sum_j b_{ij} \otimes y_{ij}$ for some $b_{ij} \in B$ and $y_{ij} \in N$. The cocycle condition applied to $n \otimes 1 \otimes 1$ then tells us that,

$$\sum_{i,j} b_i \otimes b_{ij} \otimes y_{ij} = \sum_i b_i \otimes 1 \otimes x_i \in B \otimes_A B \otimes_A N.$$

By applying the B-module action on N twice , we get

$$\sum_{i,j} b_i b_{ij} y_{ij} = \sum_i b_i x_i \in N.$$

Then we obtain

$$\psi(\psi(n)) = \sum_i b_i \psi(x_i) = \sum_{i,j} b_i b_{ij} y_{ij} = \sum_i b_i x_i = \psi(n).$$

□

We can finally prove the counit result.

Proposition 3.6. *Let (N, α) be a pair as above satisfying the cocycle condition, and let*

$$M = \mathrm{eq}\Big(N \rightrightarrows B \otimes_A N \Big).$$

Then the natural map $M \otimes_A B \to N$ is an isomorphism, i.e. $FG(N, \alpha) \cong (N, \alpha)$, since the α's are automatically compatible.

Proof. It is enough to check this after $- \otimes_A B$. Note that the equaliser commutes with $- \otimes_A B$, by flatness of ϕ. Hence we may assume that ϕ has a section $\sigma : B \to A$. To show injectivity, let $m_i \in M$ and $b_i \in B$ be such that $\sum_i b_i \otimes m_i$ is in the kernel of $M \otimes_A B \to N$, i.e. $\sum_i b_i m_i = 0$. Recall that $\alpha(m_i \otimes 1) = 1 \otimes m_i$, so $0 = \alpha(\sum_i b_i m_i \otimes 1) = \sum_i b_i \otimes m_i \in B \otimes_A M$.

For surjectivity, let $n \in N$, and write $\alpha(n \otimes 1) = \sum_i b_i \otimes x_i \in B \otimes_A N$. Since $\psi(n) = n$, we have the equality $n = \sum_i b_i x_i$. We also write $\alpha(x_i \otimes 1) = \sum_{ij} b_{ij} \otimes y_{ij}$ as before, and by the cocycle condition we get,

$$\sum_{i,j} b_i \otimes b_{ij} \otimes y_{ij} = \sum_i b_i \otimes 1 \otimes x_i \in B \otimes_A B \otimes_A N$$

as above. Applying the map $\sigma \otimes_A \mathrm{id}_B \otimes_A \mathrm{id}_N$, and using the equality above and the fact that α is linear, we obtain the equality

$$\alpha\left(\sum_i \sigma(b_i)x_i \otimes 1\right) = \sum_i \sigma(b)\alpha(x_i\otimes 1) = \sum_i \sigma(b_i)\left(\sum_j b_{ij} \otimes y_{ij}\right) = \sum_i \sigma(b_i)\otimes x_i = 1\otimes\left(\sum_i \sigma(b_i)x_i\right).$$

This implies that $\sum_i \sigma(b_i)x_i \in M \subseteq N$, since $\sigma(b_i) \in A$. We now specialise along

$$B \otimes_A B \otimes_A N \xrightarrow{\mathrm{id}_B \otimes_A \sigma \otimes_A \mathrm{id}_N} B \otimes_A N \xrightarrow{\text{act.}} N\ ,$$

and consider

$$\sum_i b_i \left(\sum_j \sigma(b_{ij})y_{ij}\right). \tag{3.7}$$

Applying the previous argument to the x_i's, we see that the term inside the parentheses in 3.7 is in M. Since we have $\sum_{i,j} b_i \otimes b_{ij} \otimes y_{ij} = \sum_i b_i \otimes 1 \otimes x_i$, we then notice the expression 3.7 is simply equal to

$$\sum_i b_i x_i = n.$$

This implies that M generates N as a B-module, as required. □

This immediatelly implies Theorem 2.18.

Proof of Theorem 2.18. This follows from Remark 2.19, Corollary 3.3, and Proposition 3.6. □

Remark 3.8. This can be generalised to a categorical context by the Barr-Beck Monadicity Theorem. It can be generalised further in algebra too. Given a map of rings $\phi : A \to B$ we can ask when the functor F of Theorem 2.18 is fully faithful or essentially surjective. There is an obvious necessary condition for fully faithfulness (in fact even for faithfulness):

For any A-module M, $M \cong \mathrm{Hom}_A(A, M) \to \mathrm{Hom}_B(B, B \otimes_A M) \cong B \otimes_A M$ must be injective, or equivalently[5], for all ideals $I \subseteq A$ we must have $A/I \hookrightarrow B/IB$ is injective. This condition on ϕ is sometimes called universally injectivity. It turns out this is in fact sufficient too.

Theorem 3.9. *If ϕ is universally injective, then F is an equivalence of categories.*

This is satisfied, for example, if $\phi : A \to B$ has a splitting as a map of A-modules. There is also a related conjecture of Hochster from 1973.

Conjecture 3.10 (Direct Summand Conjecture)**.** *If A is regular, and $\phi : A \to B$ is finite injective, then it splits as a map of A-modules.*

This was proved to be true in 2016 by Yves André.

Remark 3.11. It is easy to obtain this theorem if $\mathbb{Q} \subseteq A$, since in that case we have a trace map tr: $B \to A$ and the composition of ϕ with tr is simply the degree of B/A, which is some $d \in A^\times$. In this case, $d^{-1}\mathrm{tr}$ is a splitting. If $\mathbb{F}_p \subseteq A$, then the conjecture was proved shortly after Hochster's conjecture from 1973, but this requires lots of theory developed by Hochster. There was almost no progress on the mixed characteristic case until André.

[5]Peter puts a little astrix here, in case this isn't 100% true.

4 Smoothness 04/05/2017

Today we are going to talk about smoothness, and some of our intuition of smoothness should come from the study of tangent spaces. We saw in exercise 11.4(i) last semester, that the tangent space of a scheme over a field k is naturally isomorphic to the set of maps from $\operatorname{Spec} k[\epsilon]/\epsilon^2$. This is an example of a first order thickening of $\operatorname{Spec} k$, in the following sense.

Definition 4.1. *A closed immersion $i : S_0 \to S$ of schemes is a first order thickening (resp. nth order thickening) of S_0, if the corresponding ideal sheaf $\mathcal{I} = \ker(i^\flat : \mathcal{O}_S \to i_*\mathcal{O}_{S_0})$ satisfies $\mathcal{I}^2 = 0$ (resp. $\mathcal{I}^{n+1} = 0$).*

Locally we have $\operatorname{Spec} A_0 \subseteq \operatorname{Spec} A$, where $A_0 = A/I$, with $I^2 = 0$ (resp. $I^{n+1} = 0$). In commutative algebra the first case is called a square zero extension. The map i is said to be split if there is a section $s : S \to S_0$ such that $s \circ i = \mathrm{id}_{S_0}$.

Remark 4.2. There is a bijection of sets between the set of split first order thickenings of S_0 and the set of quasi-coherent $\mathcal{O}_{S_0}$-modules. The map in one direction is given by sending a thickening S to its ideal subsheaf, and for the converse direction by sending a quasi-coherent module $\mathcal{M}$ to the relative spectrum $\underline{\operatorname{Spec}}(\mathcal{O}_{S_0}[\mathcal{M}])$. Notice that if $\mathcal{M}$ is any quasi-coherent $\mathcal{O}_{S_0}$-module, then $\mathcal{O}_{S_0}[\mathcal{M}]$, defined as $\mathcal{O}_{S_0} \oplus \mathcal{M}$, is a quasi-coherent $\mathcal{O}_{S_0}$-algebra through the map,

$$(f, m) \cdot (g, n) = (fg, fn + gm).$$

We should think of this as $(f + m\epsilon)(g + n\epsilon)$ with $\epsilon^2 = 0$.

Notice as well that there are non-split square zero extensions, e.g. $\mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$.

Definition 4.3. *A morphism $f : X \to S$ of schemes is called formally smooth (resp. formally étale, resp. formally unramified) if for all commutative diagrams,*

$$\begin{array}{ccc} T_0 & \xrightarrow{u_0} & X \\ {\scriptstyle i}\downarrow & \overset{u}{\nearrow} & \downarrow{\scriptstyle f} \\ T & \longrightarrow & S \end{array},$$

where $i : T_0 \subseteq T$ is a first order thickening of affine schemes, there is a lift $u : T \to X$ such that the whole diagram commutes (resp. there is exactly one such lift u, resp. there is at most one such u).

Formally smoothness essentially says that there is no obstruction to extending maps to first order thickenings.

Remark 4.4. Equivalently, we could have defined formal smoothness (étaleness, unramified) using a nth order thickening instead of a first order thickening. Indeed, since a nth order thickening is a composition of n first order thickenings, the equivalence is clear. Hence this definition is really that of a smooth map, not just a one time differentiable map.

Clearly, formally étale is equivalent to formally smooth plus formally unramified.

Example 4.5. We now give examples of some maps that lie within the classes of maps we just defined.

- Open immersions are formally étale. Indeed, if $f : X \to S$ is an open immersion, then we know that a morphism $T \to S$ factors over X if and only if $f(|T|) \subseteq |X|$. Since the underlying topological spaces of T and T_0 for a first order thickening are identical, the result is clear.
- Closed immersions are formally unramified. To see this just note that $f : X \to S$ being a closed immersion implies that $X(T) \to S(T)$ is injective on T-valued points for any scheme T, hence formally unramified. We can also come up with explicit examples of when a closed immersion is not

formally étale, i.e. not also formally smooth. Consider the closed immersion $\operatorname{Spec}\mathbb{Z}/2 \to \operatorname{Spec}\mathbb{Z}$, then there is no dotted arrow in the following diagram,

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}/4 \\ \downarrow & \nearrow & \downarrow \\ \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \end{array},$$

where all the maps are the canonical quotients or the identity.

- The natural map $X = \mathbb{A}^n_S \to S$ is formally smooth. To check this we may assume that $S = \operatorname{Spec}\mathbb{Z}$, since we will see that these classes of maps are all stable under base change. Given $T_0 = \operatorname{Spec} A_0$, $T = \operatorname{Spec} A$ and a surjection $A \twoheadrightarrow A_0$, we see that the map $u_0 : T_0 \to \mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[X_1, \ldots, X_n]$ corresponds exactly to some choice of elements $x_0, \ldots, x_n \in A_0$. We define the map $u : T \to X$ by sending $X_1, ..., X_n$ to some $\widetilde{x}_1, \ldots, \widetilde{x}_n \in A$ lifting $x_1, \ldots, x_n \in A_0$, which is always possible by the surjectivity of $A \twoheadrightarrow A_0$. u might not be unique since $A \twoheadrightarrow A_0$ is not necessarily injective.
- If R is a perfect $\mathbb{F}_p$-algebra[6] then $\operatorname{Spec} R \to \operatorname{Spec}\mathbb{F}_p$ is formally étale. This is exercise 4.3.
- Let A be a ring, $a \in A^\times$, and $n \in \mathbb{Z}$ such that n is invertible in A. Then the canonical map

$$\operatorname{Spec} A[X]/(X^n - a) \longrightarrow \operatorname{Spec} A,$$

is finite and formally étale. The is exercise 5.1.

Proposition 4.6. *The classes of formally smooth/étale/unramified maps satisfy BC, COMP, PROD, LOCS and LOCT. Furthermore, given a general morphism of schemes $g : Y \to X$ and a formally unramified map $f : X \to S$, if $f \circ g$ is formally smooth (resp. formally étale, resp. formally unramified) then g is formally smooth (resp. formally étale, resp. formally unramified).*

Proof. Showing that these classes of maps satisfy BC, COMP and PROD is straightforward from the definitions. To show formally étale and formally unramified maps are LOCS and LOCT is also an elementary exercise. On the other hand, to show that (non-finitely presented) formally smooth maps are LOCS and LOCT is actually *very* difficult. In this latter case we need a different characterisation of formally smooth and a big theorem by Raynaud and Gruson (see Theorem 7.3).

For the last part of the proposition, we'll prove the case of formally smoothness, and the others follow by similar arguments. Assume $f \circ g$ is formally smooth. We want to see that this implies that g also is formally smooth. Suppose that we have a commutative diagram as above:

$$\begin{array}{ccc} T_0 & \xrightarrow{u_0} & Y \\ {\scriptstyle i}\downarrow & \overset{u}{\nearrow} & \downarrow{\scriptstyle g} \\ T & \longrightarrow & X \end{array}.$$

We know the corresponding diagram involving $f \circ g$ has a lift $u : T \to Y$ since $f \circ g$ is formally smooth, so we obtain,

$$\begin{array}{ccc} T_0 & \xrightarrow{u_0} & Y \\ {\scriptstyle i}\downarrow & \overset{u}{\nearrow} & \downarrow{\scriptstyle f\circ g} \\ T & \xrightarrow{v} & S \end{array}.$$

[6] An $\mathbb{F}_p$-algebra R is perfect if the Frobenius map $R \to R$ sending $x \mapsto x^p$ is an isomorphism.

We fix such a $u : T \to Y$ and claim that this is a desired lifting for the first diagram. We only have to check the commutivity of our original lifting problem using this u. This is solved by considering the following diagram,

$$\begin{array}{ccc} T_0 & \xrightarrow{u_0} & X \\ {\scriptstyle i}\big\downarrow & \overset{v}{\nearrow}{\scriptstyle g\circ u} & \big\downarrow{\scriptstyle f} \\ T & \longrightarrow & S \end{array} .$$

Since f is formally unramified, there can only exist one lift $T \to X$ making the diagram commute. But we have two such maps, v and $g \circ u$, hence they must be equal. Commutativity, and hence the result follow. □

Definition 4.7. *A morphism $f : X \to S$ is smooth (resp. étale, resp. unramified) if it is formally smooth (resp. formally étale, resp. formally unramified) and f is locally of finite presentation (resp. locally of finite presentation, resp. locally of finite type).*

Remark 4.8. In [5] unramified maps are also asked to be locally of finite presentation, but this excludes some closed immersions for which the ideal sheaf if not locally finitely generated. We are following the convention of [8].

Recall the following definitions.

Definition 4.9. *A morphism $f : X \to S$ is locally of finite type (resp. finite presentation) if one of the following equivalent conditions hold:*

1. *For any open affine $U = \operatorname{Spec} A \subseteq X$ mapping to $\operatorname{Spec} R \subseteq S$, A is a finitely generated (resp. finitely presented) R-algebra.*
2. *There exists an affine cover of $X = \bigcup U_i = \bigcup Spec(A_i)$, each mapping to $\operatorname{Spec} R_i \subseteq S$, such that A_i are finitely generated (resp. finitely presented) R_i-algebras.*

Remark 4.10. Note that if we assume in addition that f is quasi-compact (resp. qcqs[7]), then f is actually of finite type (resp. finite presentation). Both of these classes of maps satisfy BC, COMP, PROD, LOCS and LOCT.

Corollary 4.11. *The classes of smooth maps, étale maps, and unramified maps all satisfy BC, COMP, PROD, LOCS and LOCT.*

The finiteness conditions imposed above facilitate the proof of these statements.

Example 4.12. Let k be a field (or a ring) and set $T_0 = \operatorname{Spec} k$. Let X be a k-scheme, $f : X \to \operatorname{Spec} k$, and $T_0 \subseteq T = \operatorname{Spec} k[\epsilon]/\epsilon^2$ (note that we have a map $k[\epsilon]/\epsilon^2 \to k$, sending ϵ to 0, which induces the desired inclusion on spectra). Fix a $k-$point, $u_0 : \operatorname{Spec} k \to X \in X(k)$. If the scheme X is formally smooth (i.e. the base morphism f is smooth) then we have a map $u : T \to X$ making the following diagram commute,

$$\begin{array}{ccc} T_0 & \xrightarrow{u_0 = x} & X \\ \big\downarrow & \overset{u}{\nearrow} & \big\downarrow \\ T & \longrightarrow & \operatorname{Spec} k \end{array} .$$

The lift u can be thought of associating a tangent direction to the point $u_0 : \operatorname{Spec} k \to X$. If we assume furthermore that f is of finite presentation, then it is actually smooth.

In this example $T_0 \to T$ is split, so we always have a lift, but the set of maps here is something we will study in more detail now.

[7] qcqs=quasi-compact and quasi-separated. We will use this as an adjective for schemes, and maps of schemes.

Definition 4.13. *Let k, X, T_0, T as above, and $x \in X(k)$ be a k-point of X. The tangent space of X at x is defined as $T_xX = \{\text{maps } u : T \to X \mid \text{above diagram commutes}\}$.*

Let's compute this. Without loss of generality, assume $X = \operatorname{Spec} A$ for a k-algebra A. The point x corresponds to a map $\phi : A \to k$, and u to a map $\widetilde{\phi} : A \to k[\epsilon]/\epsilon^2 \cong k \oplus k\epsilon$ lifting ϕ as maps of k-algebras. From this, we deduce that we can write $\widetilde{\phi}$ as

$$\widetilde{\phi}(a) = \phi(a) + d(a)\epsilon,$$

for some map $d : A \to k$. Then, using the fact that $\widetilde{\phi}$ is a map of k-algebras, we can deduce the following properties for d.

1. For all $\lambda \in k, a \in A$,the equality $d(\lambda a) = \lambda d(a)$ holds.
2. For all $a, b \in A$, the equality $d(a + b) = d(a) + d(b)$ holds.
3. For all $a, b \in A$, the equality $d(ab) = \phi(a)d(b) + \phi(b)d(a)$.

Properties 1 and 2 above are simply k-linearity, and property 3 is often called the Leibniz rule (à la Calculus 1).

Definition 4.14. *Let $R \to A$ be a map of rings and M be an A-module. A derivation of A over R with values in M is a R-linear map $d : A \to M$ (*not *a map of A-modules) such that for all $a, b \in A$ we have $d(ab) = ad(b) + bd(a)$ (i.e. d satisfies the Leibniz rule).*

In fact there exists a universal derivation, and hence a universal A-module of differentials.

Proposition 4.15. *Let $R \to A$ be a map of rings. Then there exists an A-module $\Omega^1_{A/R}$ and a universal derivation $d : A \to \Omega^1_{A/R}$ of A/R, i.e. any R-derivation $d' : A \to M$ factors uniquely through $\Omega^1_{A/R}$. In particular, we have a functorial isomorphism $\operatorname{Der}_R(A, M) \cong \operatorname{Hom}_A(\Omega^1_{A/R}, M)$. We call $\Omega^1_{A/R}$ the A-module of Kähler differentials.*

Proof. This proof is purely formal. We define $\Omega^1_{A/R}$ to be the free A-module generated by the symbols $d(a)$ for all $a \in A$ subject to the relations $d(\lambda a) = \lambda d(a)$, $d(a + b) = d(a) + d(b)$ and the Leibniz rule for all $a, b \in A$ and $\lambda \in R$. □

We get the following corollary:

Corollary 4.16. *In the context of Example 4.12,*

$$T_xX = \operatorname{Der}_k(A, k) \cong \operatorname{Hom}_A(\Omega^1_{A/k}, k) \cong \operatorname{Hom}_k(\Omega^1_{A/k} \otimes_{A,\phi} k, k) = \left(\Omega^1_{A/k} \otimes_{A,\phi} k\right)^\vee .$$

Example 4.17. Let $Z = \operatorname{Spec} k[X, Y]/XY$, which is simply the union of the X and Y axes, and let $x = (0, 0)$. What is then T_xZ? Note that the map $\widetilde{\phi} : k[X, Y]/XY \to k[\epsilon]/\epsilon^2$ must send $X \mapsto a\epsilon$ and $Y \mapsto b\epsilon$ for $a, b \in k$. We can then see that T_xZ is 2-dimensional. In fact it is isomorphic to $T_x\mathbb{A}^2_k$, which is seemingly counterintuitive. However, let us now consider $u_0 : \operatorname{Spec} k[\epsilon]/\epsilon^2 \to Z$ given by $X, Y \mapsto \epsilon$ and try to extend this to a map

$$u : \operatorname{Spec} k[\epsilon]/\epsilon^3 \to \operatorname{Spec} k[\epsilon]/\epsilon^2 \to Z.$$

The latter would be given by a map of rings $k[X, Y]/XY \to k[\epsilon]/\epsilon^3$ sending $X \mapsto \epsilon + a\epsilon^2$ and $Y \mapsto \epsilon + b\epsilon^2$. Then $0 = XY \mapsto \epsilon^2 \neq 0$, which is a contradiction. Hence there is no such lift of u_0. This show Z is not smooth over k.

Now we will try to compute some modules of Kähler differentials.

Proposition 4.18. *Let $R \to A$, $R \to S$ and $A \to B$ be maps of rings. Then the following properties are satisfied:*

1. $\Omega^1_{A \otimes_R S/S} \cong \Omega^1_{A/R} \otimes_R S$.
2. *The sequence,*
$$\Omega^1_{A/R} \otimes_A B \longrightarrow \Omega^1_{B/R} \longrightarrow \Omega^1_{B/A} \longrightarrow 0,$$
is exact (but the first map is not necessarily injective).
3. *If in addition $A \to B$ is surjective, and $I = \ker(A \to B)$, then the sequence,*
$$I \otimes_A B = I/I^2 \xrightarrow{d} \Omega^1_{A/R} \otimes_A B \longrightarrow \Omega^1_{B/R} \longrightarrow \Omega^1_{B/A} = 0,$$
is exact.
4. *If $A = R[X_i | i \in I]$ (which we will write as $R[X_i]$ for now) is a free polynomial algebra indexed by some set I, then,*
$$\Phi : \bigoplus_i A \cdot dX_i \longrightarrow \Omega^1_{A/R},$$
is an isomorphism.

Corollary 4.19. *If $A = R[X_i]/(f_j)$ then,*
$$\Omega^1_{A/R} = \left(\bigoplus_i A \cdot dX_i \right) / (d(f_j)).$$

Proof. This is simply part 3 and 4 of the proposition above. □

Proof of Proposition 4.18. Part 1 follows quickly from universal properties. Parts 2 and 3 use the definition of Ω^1, i.e. the presentation given in the proof of Proposition 4.15. For part 4, we first observe the map Φ in question is surjective. We need to see that for all $f = f(X_i) \in A = R[X_i]$, the element df lies in the image, but,
$$f = \sum_{\text{finite},(n_i) \in \mathbb{N}^I} r_{(n_i)} \prod_{i \in I} X_i^{n_i},$$
and when we apply the derivation d and the Leibniz rule multiple times, we obtain,
$$df = \sum r_{(n_i)} d \left(\prod_{i \in I} X_i^{n_i} \right) = \sum_{i \in I} n_i X_i^{n_i - 1} \prod_{j \neq i} X_j^{n_j} dX_i.$$
This is in the image of Φ. For injectivity, we construct a derivation $d : A \to \bigoplus_{i \in I} A \cdot dX_i$ which maps,
$$f \longmapsto \sum_{i \in I} \frac{\partial f}{\partial X_i} \cdot dX_i.$$
Checking this is actually a derivation is basically Calculus 1. Hence we obtain a map
$$\Omega^1_{A/R} \longrightarrow \bigoplus_{i \in I} A \cdot dX_i,$$
which is a spliting of our original map Φ. Thus Φ is injective. □

5 Kähler Differentials 08/05/2017

Recall the definitions of the module of Kähler differentials from last lecture (Proposition 4.15 and what followed). Notice that assigning to a map $R \to A$ to $\Omega^1_{A/R}$ is a functorial construction. In other words, given the following commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array},$$

then we obtain a canonical morphism $A' \otimes_A \Omega^1_{A/R} \to \Omega^1_{A'/R'}$ which sends $a' \otimes da \mapsto a' df(a)$. This can be shown to be an isomorphism if the square above is a pushout (part 1 of Proposition 4.18). The following lemma compares liftings of a certain kind with derivations.

Lemma 5.1 (Derivations and Liftings)**.** *Given the following commutative diagram of ring homomorphisms,*

$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow & \swarrow & \downarrow f \\ B & \xrightarrow{\pi} & B/I \end{array},$$

with $I^2 = 0$ (so I is naturally a B/I-module).[8]

1. *Given $\phi_1, \phi_2 : A \to B$ are two lifts of f (i.e. $\pi \circ \phi_i = f$), then $\delta : \phi_1 - \phi_2 : A \to I$ is an R-linear derivation.*

2. *Given $\phi : A \to B$ a lift of f, and $\delta : A \to I$ an R-linear derivation, then $\phi + \delta : A \to B$ is another lift of f.*

Notice that together, parts 1 and 2 imply that the set of R-linear derivation from A to I act freely and transitively on the set of liftings of f, given a lifting exists.

Proof. Both proofs require some equations and some calculating. For part 1 let $a, b \in A$, then we have

$$\delta(ab) = \phi_1(ab) - \phi_2(ab) = \phi_1(a)\phi_1(b) - \phi_1(a)\phi_2(b) + \phi_1(a)\phi_2(b) - \phi_2(a)\phi_2(b) = \phi_1(a)\delta(b) + \phi_2(b)\delta(a).$$

The A-module action on I comes through either ϕ_1 or ϕ_2, they give the same action, so the equality above becomes,

$$\delta(ab) = a\delta(b) + b\delta(a),$$

and δ is a derivation since it is clearly R-linear. For the second part we consider if the map $\phi + \delta$ is really a map of algebras, since then it is clearly a lift. For $a, b \in A$ we have,

$$(\phi + \delta)(ab) = \phi(ab) + \delta(ab) = \phi(a)\phi(b) + a\delta(b) + b\delta(a) = \phi(a)\phi(b) + \phi(a)\delta(b) + \phi(b)\delta(a) + \delta(a)\delta(b).$$

The last equality comes from the fact that $\delta(a)\delta(b) = 0$ as $I^2 = 0$. From here we immediately obtain,

$$(\phi + \delta)(ab) = (\phi + \delta)(a)(\phi + \delta)(b),$$

which finishes our proof. □

There are some corollaries that we quickly obtain from this lemma.

[8] Notice that this diagram becomes our lifting diagrams in our definition of formally smooth etc. See Definition 4.3.

Corollary 5.2. *We have an isomorphism between the R-linear derivations from A to an A-module M and morphisms ϕ such that ϕ is an R-linear section $A \to A[M]$ of the projection map $A[M] \to A$.*[9]

Corollary 5.3. *A map* $\operatorname{Spec} A \to \operatorname{Spec} R$ *of affine schemes is formally unramified if and only if* $\Omega^1_{A/R} = 0$.

The proof of the latter corollary comes exactly from considering the image of the diagram in Lemma 5.1 under the functor Spec, and then applying Definition 4.3. Now we would like to localise our modules of Kähler differentials, which would enable us to discuss quasi-coherent sheaves of Kähler differentials in the long run.

Lemma 5.4. *Given $S \subseteq A$ a multiplicative subset, then*

$$\Omega^1_{A/R} \otimes_A A[S^{-1}] \cong \Omega^1_{A[S^{-1}]/R}.$$

Proof. We could try to prove this by defining a bijection $da \otimes \frac{a'}{s} \mapsto \frac{a'}{s} d\left(\frac{a}{1}\right)$ with inverse determined by the quotient rule,

$$d\left(\frac{a}{s}\right) \longmapsto da \otimes \frac{1}{s} + ds \otimes \frac{a}{s^2}.$$

We then would have to check this is well defined etc. Alternatively, we can try to obtain an alternative definition for the module of Kähler differentials $\Omega^1_{A/R}$.

Proposition 5.5. *Let $X = \operatorname{Spec} A$, $S = \operatorname{Spec} R$, a map $X \to S$, and $\Delta_{X/S} : X \to X \times_S X$ be diagonal map, i.e. the closed embedding given by multiplication of A as an R-algebra, with corresponding ideal $I \subseteq A$. Then we can identify the module of Kähler differentials of A/R as,*

$$\Omega^1_{A/R} \cong I/I^2 \cong \Delta^*_{X/S}(\widetilde{I})(X).$$

Using this proposition to finish the proof of Lemma 5.4, we see,

$$I \otimes_{A \otimes_R A} \left(A[S^{-1}] \otimes_R A[S^{-1}]\right) \cong \ker\left(A[S^{-1}] \otimes_R A[S^{-1}] \longrightarrow A[S^{-1}]\right) =: J.$$

This gives us the following chain of isomorphisms,

$$\Omega^1_{A/R} \otimes_A A[S^{-1}] \cong I/I^2 \otimes_A A[S^{-1}] \cong J/J^2 \cong \Omega^1_{A[S^{-1}]/R}.$$

□

Let us prove this proposition now.

Proof of Proposition 5.5. Given the hypotheses of Proposition 5.5, we consider the R-linear derivation $\delta : A \to I/I^2$ defined by $a \mapsto 1 \otimes a - a \otimes 1$. We need to check this is a derivation first,

$$\delta(ab) = 1 \otimes ab - ab \otimes 1 = 1 \otimes ab - a \otimes b + a \otimes b - ab \otimes 1 = b\delta(a) + a\delta(b),$$

where the A-module structure on I/I^2 is $a\bar{i} = (a \otimes 1)i = (1 \otimes a)i$. From this we obtain a map $\Omega^1_{A/R} \to I/I^2$ which explicitly sends $da \mapsto 1 \otimes a - a \otimes 1$. To obtain a map in the opposite direction, consider the universal derivation $d : A \to \Omega^1_{A/R}$, and from this we define maps,

$$\phi_1, \phi_2 : A \longrightarrow A[\Omega^1_{A/R}], \qquad \phi_1(a) = (a, 0), \qquad \phi_2(a) = (a, da).$$

From the universal property of the tensor product we now obtain a map,

$$A \otimes_R A \longrightarrow A[\Omega^1_{A/R}], \qquad a \otimes b \longmapsto a(b + db),$$

[9] Here we are using the notation $A[M]$ for the ring $A \oplus \epsilon M$, with multiplication $(a, m)(a', m') = (aa', am' + a'm)$. This means we have $M \to A[M] \to A$, where the latter map is the projection onto A, is a square zero extension since $M^2 = 0$.

which in turn gives us,

$$I/I^2 \longrightarrow \Omega^1_{A/R}, \qquad \sum_i a_i \otimes b_i \longmapsto \sum_i a_i(b_i + db_i) = \sum_i a_i db_i.$$

It is then a quick exercise to check these maps are mutual inverses. □

Remark 5.6. There is geometric motivation for this, which essentially notices $(I/I^2)^\vee$ as the normal bundle of the diagonal, and $(\Omega^1_{A/R})^\vee$ as the tangent bundle of X, so the isomorphism between them is simply a projetion of the diagonal onto one coordinate $X \times_S X \to X$.

We will now use the localisation of Lemma 5.4 to globalise Kähler differentials.

Corollary 5.7. *The assignment $D(f) \mapsto \Omega^1_{A[f^{-1}]/R} \cong \Omega^1_{A/R} \otimes_A A[f^{-1}]$ on principle open sets of $X = \operatorname{Spec} A$ over $S = \operatorname{Spec} R$, determines a quasi-coherent sheaf $\Omega^1_{X/S}$ on X. Moreover, $\Omega^1_{X/S} \cong \Delta^*_X(\widetilde{I})$ where I is the ideal corresponding to the closed embedding $\Delta : X \to X \times_S X$.*

This motivates a general definition as well.

Definition 5.8. *Let $f : X \to S$ be any morphism of schemes, then we have $\Delta_{X/S} : X \to X \times_S X$ is a locally closed immersion, and hence can be factored as a closed immersion $\Delta^U_{X/S} : X \to U$ and an open immersion $U \subseteq X \times_S X$. We then define,*

$$\Omega^1_{X/S} = \left(\Delta^U_{X/S}\right)^* (\mathcal{I}),$$

where as usual, $\mathcal{I}$ is the ideal subsheaf defining X inside U. If X is separated, then we can take $U = X \times_S X$. Notice this statement is independent of U, and if we restrict $\Omega^1_{X/S}$ to some open affine $U = \operatorname{Spec} A$ we recover,

$$\Omega^1_{X/S}\Big|_U \cong \widetilde{\Omega^1_{A/R}}.$$

The following proposition generalises exercise 4.1, (c.f. Proposition 4.18 as well).

Proposition 5.9. *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes.*

1. *There exists a canonical derivation $d : \mathcal{O}_X \to \Omega^1_{X/S}$ which restricts to the universal derivation in the affine situation.*
2. *The sheaf $\Omega^1_{X/S}$ commutes with base change with respect to S.*
3. *The sequence,*

$$f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0,$$

   where the first map sends $da \mapsto df^\#(a)$ for some $a \in \mathcal{O}_Y$, is exact.
4. *If f is a closed immersion with ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$, then the sequence,*

$$\mathcal{I}/\mathcal{I}^2 := f^*\mathcal{I} \xrightarrow{d} f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow 0,$$

   is exact.

We obtain a useful generalisation of Corollary 5.3.

Corollary 5.10. *Given any morphism $f : X \to S$ of schemes, then the following are equivalent.*

1. *The morphism f is formally unramifed.*
2. *The sheaf $\Omega^1_{X/S} = 0$.*

Moreover, if f is locally of finite type, then in addition we have the equivalent condition,

3. *The diagonal $\Delta_{X/S} : X \to X \times_S X$ is an open immersion (recalling that locally closed immersions are unramified).*

Proof. Everything can be proved locally, so we have $X = \operatorname{Spec} A$, $S = \operatorname{Spec} R$ and I is the kernel of $A \otimes_R A \to A$, the multiplication map. We have then seen the equivalence between 1 and 2, and the fact that 3 implies 2 comes from the fact that $I = 0$, since $i^*\mathcal{O}_{X\times_X X} \to \mathcal{O}_X$ is an isomorphism. We now assume Part 2, then f is locally of finite type and $\Delta_{X/S}$ is locally of finite presentation. Indeed, for R-algebra generators a_i, the elements $1 \otimes a_i - a_i \otimes 1$ generate I. Assume that $I/I^2 = 0$, then Nakayama implies $I_x = 0$ for all $x \in \Delta_{X/S}(\operatorname{Spec} A)$, which implies $\Delta_{X/S}$ is flat, and exercise 2.3[10] shows $\Delta_{X/S}$ is then an open immersion. $\square$

Proposition 5.11. *Given maps $f : X \to Y$ and $g : Y \to S$ of schemes. Then the sequence,*

$$f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0,$$

is exact.

1. *If f is formally smooth, then the sequence above is exact on the left (so the first map is injective) and locally split.*

2. *If $g \circ f$ is formally smooth and the above exact sequence is locally split, then f is formally smooth.*

Proof. All of these conditions are local, so we can set $X = \operatorname{Spec} C$, $Y = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$. Then we have to show,

$$0 \longrightarrow C \otimes_B \Omega^1_{B/A} \longrightarrow \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \longrightarrow 0,$$

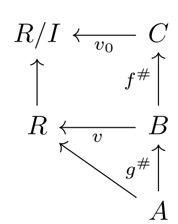
is exact and locally split when f is formally smooth. Specifically, we need some retraction ρ of the first map which is defined as $c \otimes db \mapsto cdf^{\#}(b)$. Since f is formally smooth, we obtain the lifting $\eta : C \to C[C \otimes_B \Omega^1_{B/A}]$ defined by $\eta = (\psi, \delta)$ where δ in particular is an A-linear derivation $C \to C \otimes_B \Omega^1_{B/A}$. Recall the codomain of η as a module is simply $C \oplus (C \otimes_B \Omega^1_{B/A})$. The universal properties of $\Omega^1_{C/A}$ gives us a map,

$$\rho : \Omega^1_{C/A} \longrightarrow C \otimes_B \Omega^1_{B/A},$$

which is given by $dc \mapsto \delta(c)$. To check that ρ is a retraction we see that,

$$\rho(cdf^{\#}(b)) = c\delta(f^{\#}(b)) = cdf^{\#}(b),$$

which finishes part 1. For the second part, let $g \circ f$ be formally smooth, and let the exact sequence in question be locally split. To show f is formally smooth, we consider the following lifting problem,



,

where $I^2 = 0$. Since $g \circ f$ is formally smooth, then we have $u' : C \to R$ such that $u' \circ f^{\#} \circ g^{\#} = v \circ g^{\#}$. We need to find some A-linear derivation $\delta : C \to I$, such that $(u' + \delta) \circ f^{\#} = v$, but this is equivalent to $v - u' \circ f^{\#} = \delta \circ f^{\#}$ as an A-linear derivation from B to I. However, we have identified the

[10] Exercise 2.3 reads as follows. Given a closed immersion $f : X \to S$. Show that f is flat and locally of finite presentation if and only if f is an open immersion.

A-linear derivations from C and I as $\mathrm{Hom}_C(\Omega^1_{C/A}, I)$ and the A-linear derivations from B to I as $\mathrm{Hom}_B(\Omega^1_{B/A}, I) \cong \mathrm{Hom}_C(C \otimes_B \Omega^1_{B/A}, I)$, and the map,

$$\gamma : \mathrm{Hom}_C(\Omega^1_{C/A}, I) \cong \mathrm{Der}_A(C, I) \longrightarrow \mathrm{Der}_A(B, I) \cong \mathrm{Hom}_C(C \otimes_B \Omega^1_{B/A}, I),$$

induced by $C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A}$ identifies withe restructing derivations from C to B. We have a retraction of this map however, some

$$\rho : \mathrm{Hom}_C(C \otimes_B \Omega^1_{B/A}, I) \longrightarrow \mathrm{Hom}_C(\Omega^1_{C/A}, I),$$

hence the map γ is an epimorphism and we can find our desired δ. □

The following is a useful corollary, since next lecture we use it to talk about the Jacobi criterion.

Corollary 5.12 (Uniformising Parameters)**.** *Given a map of schemes $g : X \to S$. Then g is smooth if and only if for each $x \in X$ there is an open neighbourhood $x \in U \subseteq X$ and sections $f_1, \dots, f_n \in \Gamma(U, \mathcal{O}_U)$ such that in the following diagram,*

$$\begin{array}{ccc} U & \xrightarrow{f=(f_1,\dots,f_n)} & \mathbb{A}^n_S \\ & {\scriptstyle g|_U}\searrow & \downarrow \\ & & S \end{array},$$

the map f is étale and,

$$\Omega^1_{X/S}|_U \cong \bigoplus_{i=1}^{n} \mathcal{O}_U \cdot df_i.$$

Proof. One direction is simple, since if we work locally we notice that if f is étale, then g is automatically smooth as $\mathbb{A}^n_S$ is smooth over S. Conversely, assume g is smooth, then we will see next lecture that $\Omega^1_{X/S}$ is projective and finitely presented, implies $\Omega^1_{X/S}$ is finite locally free. For each $x \in X$ we may choose $x \in U \subseteq X$ and $V \subseteq S$ small enough, so we can assume the restricted map $g|_U : U \to V$ has domain and codomain affine schemes, say $U = \operatorname{Spec} B$ and $V = \operatorname{Spec} R$, such that on U the sheaf $\Omega^1_{X/S}$ is finite free. Possibly after shrinking further we have an isomorphism,

$$\bigoplus_{i=1}^{r} B \cdot df_i \cong \Omega^1_{B/R},$$

where $f_i \in B$. In fact, $\Omega^1_{B/R}$ is generated by df_i for $f_i \in B$ and hence, locally around x, we can find a basis of $\Omega^1_{B/R}$ among the df_i's. These f_i's give us a map $\operatorname{Spec} B \to \operatorname{Spec} A$, where $A = R[X_1, \dots, X_n]$, by sending $X_i \mapsto f_i$, which factor through the diagram,

$$\begin{array}{ccc} U & \xrightarrow{f=(f_1,\dots,f_n)} & \mathbb{A}^n_R \\ & {\scriptstyle g|_U}\searrow & \downarrow \\ & & \operatorname{Spec} R \end{array}.$$

Examining the exact sequence of Proposition 5.11,

$$0 \longrightarrow \Omega^1_{A/R} \otimes_A B \xrightarrow{\cong} \Omega^1_{B/R} \longrightarrow \Omega^1_{B/A} \longrightarrow 0, \tag{5.13}$$

where the first map sends $dX_i \mapsto df_i$, is exact on the left, since this first map is clearly an isomorphism. This sequence is also split, again from explicit isomorphism and a decomposition into direct sums. Proposition 5.11 then tells us f is smooth (as it is clearly finitely presented). Sequence 5.13 and Corollary 5.3 tell us f is also unramified, hence étale. □

Proposition 5.14. *Given the following diagram of schemes,*

$$\begin{array}{ccc} Z & \overset{i}{\hookrightarrow} & X \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array},$$

with i a closed immersion defined by the ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, then we consider the following sequence,

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} i^*\Omega^1_{X/S} \longrightarrow \Omega^1_{Z/S} \longrightarrow 0.$$

1. *If f is formally smooth, then our sequence above is exact, the map d is injective, and this sequence is locally split.*

2. *If g is formally smooth, our sequence is exact, locally split, and the map d is injective, then f is formally smooth.*

Proof. Just like any proof, we work locally, so let $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$ and $Z = \operatorname{Spec} B/J$ for some ideal $J \subseteq B$. For part 1 we need a retraction of d, so we set up the following lifting problem,

$$\begin{array}{ccccc} B/J & \xleftarrow{\text{id}} & B/J & \xleftarrow{\pi} & B \\ \uparrow & {\scriptstyle u}\swarrow & \uparrow{\scriptstyle v} & \swarrow & \\ B/J^2 & \longleftarrow & A & & \end{array}.$$

Since f is formally smooth, we have the dotted arrow u, but then we have two lifts of $B \to B/J^2$, the canonical quotient map $\pi' : B \to B/J^2$ sending $b \mapsto \bar{b}$, and the composite $u \circ \pi$. Hence we obtain a derivation $\delta = \pi' - u \circ \pi : B \to J/J^2$. This in turn gives us a map $\rho' : \Omega^1_{B/A} \to J/J^2$ of B-modules sending $db \mapsto \delta(b)$, and then a map $\rho : B/J \otimes_B \Omega^1_{B/A} \to J/J^2$ of B/J-modules, which maps $a \otimes db \mapsto a\delta(b)$. If $b \in J$, then $\rho(db) = b$ modulo J, since $\pi(b) = 0$, and hence we have a retraction of d. For the final part, we need to solve a lifting problem of the form,

$$\begin{array}{ccccc} R/I & \longleftarrow & B/J & \longleftarrow & B \\ \uparrow & \swarrow & \uparrow{\scriptstyle v} & \nearrow{\scriptstyle g^{\#}} & \\ R & \longleftarrow & A & & \end{array},$$

for some ideal $I \subseteq R$ such that $I^2 = 0$. We have assumed $g^{\#} : A \to B$ is formally smooth, hence we obtain the lifting $v : B \to R$, and we then procede to modify this such that $v|_J = 0$, which would give us our desired lift $B/J \to R$. A different choice for v is of the form $v + \delta$ for some $\delta \in \operatorname{Der}_A(B, I)$, hence we need to find a derivation $\delta \in \operatorname{Der}_A(B, I)$ such that $\delta|_J = -v|_J$. However, we have the map,

$$\operatorname{Hom}_{B/J}(B/J \otimes_A \Omega^1_{B/A}, I) \cong \operatorname{Hom}_B(\Omega^1_{B/A}, I) \cong \operatorname{Der}_A(B, I) \longrightarrow \operatorname{Hom}_B(J/J^2, I),$$

which is an epimorphism as our sequence in the hypotheses is locally split. Hence we can lift the map $-v|_J \in \operatorname{Hom}_B(J/J^2, I)$ to some $\delta \in \operatorname{Der}_A(B, I)$. $\square$

6 Smoothness and the Jacobi Criterion 11/05/2017

We begin with another important proposition about modules of Kähler differentials which we have forgotten until now.

Proposition 6.1. *If $f : \operatorname{Spec} A \to \operatorname{Spec} R$ is formally smooth, then $\Omega^1_{A/R}$ is projective.*

An immediate corollary of this is $\Omega^1_{A/R}$ is finite locally free if f is smooth, since finitely presented projective modules are in particular finite projective, which is equivalent to finite locally free.

Proof. Take $M \to M'$ to be a surjection of A-modules, then we have to show the induced map,

$$\operatorname{Der}_R(A, M) \cong \operatorname{Hom}_A(\Omega^1_{A/R}, M) \longrightarrow \operatorname{Hom}_A(\Omega^1_{A/R}, M') \cong \operatorname{Der}_R(A, M'),$$

is surjective. In other words, given $\phi' : A \to A[M'] = A \oplus \epsilon M'$ (with $\epsilon^2 = 0$) such that $a \mapsto a + \epsilon\delta(a)$ we would like a $\phi : A \to A[M]$ which is mapped to ϕ' (recall that derivations $\delta \in \operatorname{Der}_R(A, M)$ identify with sections $\phi : A \to A[M]$ by Corollary 5.2. We then set up the following lifting problem,

$$\begin{array}{ccc} A[M'] & \xleftarrow{\ \phi'\ } & A \\ \uparrow & \swarrow_{\phi} & \uparrow \\ A[M] & \longleftarrow & R \end{array},$$

and since $R \to A$ is formally smooth we have the map ϕ which lifts ϕ', and hence $\phi = \operatorname{id}_A + \epsilon\delta$, where $\delta \mapsto \delta'$, and we're done. ☐

Recall Proposition 5.14. We want to view a concrete example of this. Let $S = \operatorname{Spec} R$, $X = \mathbb{A}^n_S = \operatorname{Spec} B$ where $B = R[X_1, \ldots, X_n]$ and $Z = \operatorname{Spec} A = V(I)$, where $I = (f_1, \ldots, f_r)$ is an ideal, and $A \cong B/I$. Then we have the following commutative diagram with exact rows,

$$\begin{array}{ccccccc} \bigoplus_{i=1}^r A \cdot e_i & \xrightarrow{\ \mathcal{J}\ } & \bigoplus_{i=1}^n A \cdot dX_i & & & & \\ \downarrow & & \downarrow \cong & & & & \\ I/I^2 & \xrightarrow{\ d\ } & A \otimes_B \Omega^1_{B/R} & \longrightarrow & \Omega^1_{A/R} & \longrightarrow & 0 \end{array}, \tag{6.2}$$

where the first vertical map sends the generators $e_i \mapsto \overline{f}_i \in I/I^2$. We now stare at this diagram, and recognise the matrix $\mathcal{J}$ as the Jacobi matrix,

$$\mathcal{J} = \left(\frac{\partial f_i}{\partial X_j}\right)_{i,j}.$$

Choose some $z \in Z$ with corresponding prime ideal $\mathfrak{p} \subseteq A$, and assume $\overline{f}_1, \ldots, \overline{f}_r \in I/I^2$ form a basis of $I/I^2 \otimes k(z)$. Proposition 5.14 implies that Z is smooth in a neighbourhood of z if and only if the bottom row of Sequence 6.2 is exact on the left (so the first map is injective) and locally split after we tensor everything $- \otimes_A A_{\mathfrak{p}}$. Lemma 6.3 to come will then show us this is equivalent to Sequence 6.2 being exact on the left after tensoring with $- \otimes_A k(z)$, which is equivalent to $\mathcal{J}(z)$ is injective, thus $\mathcal{J}(z)$ has maximal rank r.

Lemma 6.3. *Given a local ring A and $M : A^r \to A^n$ an A-linear matrix. Then M is injective and split if and only if $M \otimes_A k$ is injective, where k is the residue field of A.*

Proof. If M is injective and split, then $M \otimes_A k$ is injective. Assume the converse now, so $r \leq n$ and without loss of generality we may take M to be of the form,

$$M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix},$$

where M_1 is an $r \times r$ matrix with entries in A such that the determinant of M_1 in k is nonzero. As A is local this implies M_1 is invertible, since the determinant of M_1 lies in $A^\times$, which implies we have the following commutative diagram,

$$\begin{array}{ccc} A^r & \xrightarrow{M} & A^n \\ {\scriptstyle M_1^{-1}}\big\uparrow\cong & \nearrow_{N} & \\ A^r & & \end{array} , \qquad N = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ * & * & * \end{pmatrix}$$

and N clearly has a splitting. ☐

The above Lemma and Diagram 6.2 lead us to a classical smoothness criterion.

Theorem 6.4 (Jacobian Criterion)**.** *Given the following diagram of schemes,*

$$\begin{array}{ccc} Z & \overset{i}{\hookrightarrow} & \mathbb{A}^n_R \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle j} & \\ & \operatorname{Spec} R & \end{array} ,$$

where i is a closed immersion which is locally of finite presentation, then for any $z \in Z$, f is smooth at z if and only if $Z = V(f_1, \ldots, f_r)$ locally around z and the rank of the Jacobi matrix $\mathcal{J}(z) = \left(\frac{\partial f_i(z)}{\partial x_j}\right)$ is r.

Proof. [11] Lemma 6.3 and the discussion proceeding it tells us that if f is smooth then the Jacobian matrix has this particular property. For the other direction, we write $Z = V(f_1, \ldots, f_r)$, set $I = (f_1, \ldots, f_r) \subseteq R[X_1, \ldots, X_n] = B$, $S = \operatorname{Spec} R$ and $A = B/I$. From this we obtain (locally) the sequence with exact rows,

$$\begin{array}{ccccccc} 0 \longrightarrow & \bigoplus_{i=1}^r A \cdot f_i & \xrightarrow{\mathcal{J}} & \bigoplus_{i=1}^n A \cdot dX_i & & & \\ & \big\downarrow & & \big\downarrow\cong & & & \\ & I/I^2 & \xrightarrow{d} & A \otimes_B \Omega^1_{B/R} & \longrightarrow & \Omega^1_{A/R} & \longrightarrow 0 \end{array} ,$$

where $\mathcal{J} = \left(\frac{\partial f_i}{\partial x_j}\right)$ is the Jacobian matrix. Our point $x \in X$ corresponds to a prime ideal $\mathfrak{p} \subseteq A'$, and our hypotheses imply that $\mathcal{J} \otimes_{A'} k(\mathfrak{p})$ is injective, which implies that the left vertical arrow is an isomorphism and also that d is injective and locally split at x. Spreading out this splitting, we then use Proposition 5.14 part 2 to finish the proof. ☐

Notice that in the above proof we really used the projectivity to obtain splittings to make our arguments. We now wish to state a corollary of Proposition 5.11.

Corollary 6.5. *Given a map $f : X \to Y$ in the category of schemes over S. If f is formally étale then $f^*\Omega_{Y/S} \to \Omega^1_{X/S}$ which maps $da \mapsto df^\#(a)$ is an isomorphism.*

Proof. The map f is formally étale, so it is formally smooth and formally unramified. From the former condition we use Proposition 5.11 to obtain that the desired map is injective, and the fact f is formally unramified implies the last term in the exact sequence of Proposition 5.11 is zero using Corollary 5.10. ☐

[11] This proof, and following lemma, and it's proof were proved in lecture 7, as an amendment, but they obviously fit here in the notes.

Lemma 6.6. *Given $f : X \to S$ which is formally étale, $x \in X$ and $s = f(x)$ such that $k = k(s) \cong k(x)$. Then,*

$$\widehat{\mathcal{O}}_{S,s} = \lim_n \mathcal{O}_{S,s}/\mathfrak{m}_{S,s}^n \longrightarrow \widehat{\mathcal{O}}_{X,x} = \lim_n \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^n,$$

is an isomorphism.

Proof. Consider the category $\mathcal{C}_{\leq n}$ who has objects are local rings $(A, \mathfrak{m}_A)$ such that $\mathfrak{m}_A^n = 0$ with an isomorphism $\iota : k \overset{\cong}{\to} A/\mathfrak{m}_A$, and where the morphisms are local ring homomorphisms of local rings commuting with these ι's. For example, both $\mathcal{O}_{S,s}/\mathfrak{m}_{S,s}^n$ and $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^n$ are both naturally elements of $\mathcal{C}_{\leq n}$. We have representable functors,

$$F_n = \mathrm{Hom}_{\mathcal{C}_{\leq n}}(\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^n, -), \qquad G_n = \mathrm{Hom}_{\mathcal{C}_{\leq n}}(\mathcal{O}_{S,s}/\mathfrak{m}_{S,s}^n, -).$$

We also have a natural transformation $\eta : F_n \Rightarrow G_n$ sending $\phi \mapsto \phi \circ f^\#$. We want to see this is an isomorphism for every n, so it suffices to show η is a bijection by induction, using the obvious fact that $\mathcal{C}_{\leq n}$ is a full subcategory of $\mathcal{C}_{\leq n}$. When $n = 1$ this is simply the fact that $k(s) \cong k(x)$, so we proceed to the inductive step. Let $A \in \mathcal{C}_{\leq n}$ and let $A' = A/\mathfrak{m}_A^{n-1} \in \mathcal{C}_{\leq n-1}$. We now look at $\eta : F_n(A') \to G_n(A')$, which we know is a bijection from our inductive hypothesis as $F_n(A') = F_{n-1}(A')$ and $G_n(A') = G_{n-1}(A')$. However we have the following isomorphism,

$$F_n(A) \cong F_n(A') \times_{G_n(A')} G_n(A),$$

whose justification comes from the following lifting problem, which is solved uniquely using the fact f is formally étale,

$$\begin{array}{ccc} A' & \longleftarrow & \mathcal{O}_{X,x} \\ \uparrow & \swarrow & \uparrow \\ A & \longleftarrow & \mathcal{O}_{S,s} \end{array}.$$

This tells us η is a natural isomorphism, and hence the objects representing these functors are isomorphic. Since these objects are the elements of the diagram defining the completion, we see the completions of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{S,s}$ at their respective maximal ideals are isomorphic. ☐

Now we will start to think about smooth schemes over fields. The following proposition is exercise 5.2.

Proposition 6.7. *Given a field k and $f : X \to \mathrm{Spec}\, k$ then the following are equivalent.*

1. *The map f is étale.*
2. *The map f is unramified.*
3. *The map f is smooth and locally quasi finite*[12].
4. *The scheme X is simply a disjoint union,*

$$X = \coprod \mathrm{Spec}\, l,$$

where the l are finite separable field extensions of k.

We now come to a theorem we will prove today.

Theorem 6.8. *Let k be a field, and $f : X \to \mathrm{Spec}\, k$, then the following are equivalent.*

1. *The map f is smooth.*

[12] A map $g : Y \to S$ is locally quasi finite if for all $s \in S$, the fibre $Y_s = Y \times_S \mathrm{Spec}\, k(s)$ is a discrete topological space and g is locally of finite type. Essentially we would like the fibres to be finite, but this condition is not closed under base change, so we come to this definition.

2. *The map f is geometrically regular*[13].

3. *For one algebraically closed field extension K over k we have X_K is a regular scheme.*

To prove this theorem, we need some notes from commutative algebra.

Proposition 6.9 (Commutative Algebra Facts)**.** *Let A be a local noetherian ring.*

1. *The ring A is regular if and only if the completion $\widehat{A} = \lim_n A/\mathfrak{m}_A^n$ is regular.*

2. *If a map $A \to B$ is local and flat, and B is regular, then A is regular.*

3. *Given A is regular then $A_\mathfrak{p}$ is regular for all prime ideals $\mathfrak{p} \subseteq A$.*

Proof of Theorem 6.8. To see part 1 implies part 2, we can take k to be algebraically closed since we can just take a change of base, this means locally $g : X \to \mathbb{A}_k^n$ is étale (from Corollary 5.12). Pick some $x \in X$, then without loss of generality we can take $x \in X$ to be closed by part 3 of Proposition 6.9. We now have $k \cong k(x) \cong k(g(x))$ which implies by étaleness of our map g and Lemma 6.7 that

$$\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{\mathbb{A}_k^n, g(x)} \cong k[|t_1, \dots, t_n|],$$

which we know is clearly regular. Part 1 of Proposition 6.9 then implies $\mathcal{O}_{X,x}$ is regular. The implication that part 2 implies part 3 is clear, so now we assume part 3 is true. We claim in this case that f is smooth if and only if $f_K : X_K \to \operatorname{Spec} K$ is smooth. Without loss of generality we can take $X = \operatorname{Spec} A$ with $A_K = A \otimes_R K$, which implies,

$$\Omega^1_{A/R} \otimes_R K \cong \Omega^1_{A_K/K}.$$

In particular, $\Omega^1_{A/k}$ is locally free of finite rank if and only if $\Omega^1_{A_K/K}$ is locally free of finite rank. We need a quick lemma for this now.

Lemma 6.10. *Let $A \to B$ be faithfully flat, and M be a A-module.*

1. *The module M is of finite type (resp. of finite presentation) if and only if $M \otimes_A B$ is of finite type over C (resp. of finite presentation).*

2. *The module M is flat over A if and only if $M \otimes_A B$ is flat over B.*

Proof of Lemma 6.10. For part 1, one direction is obvious. Conversely, we write $M = \operatorname{colim} N$ where $N \subseteq M$ are the finitely generated submodules of M, then we have,

$$M \otimes_A B \cong \operatorname{colim}(N \otimes_A B) = N' \otimes_A B,$$

since directed colimits commute with the tensor product. We notice N' is finitely generated, then, using the fact that $A \to B$ is faithfully flat, we see that $M = N'$. One direction of part 2 is also obvious, so assume $M \otimes_A -$ is exact. Then, $M \otimes_A -$ is exact if and only if $B \otimes_A M \otimes_A -$ is exact by faithful flatness, but we can rewrite this as,

$$B \otimes_A M \otimes_A - \cong (B \otimes_A M) \otimes_B (B \otimes_A -),$$

where $B \otimes_A M$ is flat over B and $B \otimes_a -$ is also exact. ∎

[13] A map $f : X \to \operatorname{Spec} k$ is said to be geometrically regular if for all algebraically closed field extensions K over k we have $X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$ is a regular scheme, so it's stalks are all regular rings.

Let us continue the proof of Theorem 6.8. We want to show f is smooth if and only if f_K is smooth. Clearly if f is smooth then f_K is smooth. Thus assume that f_K is smooth, so then $\Omega^1_{A/K}$ is locally free of finite rank. Let $g : X \to \mathbb{A}^n_k$ be a closed immersion with corresponding ideal I. Then f is smooth if and only if the following sequence,

$$0 \longrightarrow I/I^2 \xrightarrow{d} g^*\Omega^1_{k[X_1,\dots,X_n]/k} \longrightarrow \Omega^1_{A/k} \longrightarrow 0,$$

is exact and locally split, but since $\Omega^1_{A/k}$ is locally free this is equivalent to asking the map d to be injective. This however is equivalent to the map

$$d_K : I_K/I_K^2 \longrightarrow g_K^*\Omega^1_{K[X_1,\dots,X_n]/K}$$

being injective. This is true since f_K is smooth, so we're done. □

We now have another general theorem we will appeal to.

Theorem 6.11. *Let k be a field, and X a scheme over k which is locally of finite type, and take some closed $x \in X$ such that $k(x)$ is a separable field extension of k and $\mathcal{O}_{X,x}$ is regular, then X is smooth in a neighbourhood of x.*

Proof of Theorem 6.11. The field extension $k(x)$ over k is separable, so this implies the map $\operatorname{Spec} k(x) \to \operatorname{Spec} k$ is étale, which gives us the following exact sequence (by Proposition 5.14).

$$0 \longrightarrow \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} \longrightarrow k(x) \otimes_{\mathcal{O}_{X,x}} (\Omega^1_{X/k})_x \longrightarrow \Omega^1_{k(x)/k} \longrightarrow 0.$$

We note that $\Omega^1_{k(x)/k)} = 0$ since this field extension is separable (using Corollary 5.3 and Proposition 6.7), so we have and isomorphism

$$\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} \cong k(x) \otimes_{\mathcal{O}_{X,x}} (\Omega^1_{X/k})_x.$$

Pick some closed immersion $g : X \hookrightarrow \mathbb{A}^n_k$ with corresponding ideal I. Then we obtain the exact sequence,

$$I/I^2 \longrightarrow g^*\Omega^1_{\mathbb{A}^n_k/k} \longrightarrow \Omega^1_{X/k} \longrightarrow 0.$$

Let $d = \dim \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} = \dim \mathcal{O}_{X,x}$, where the latter equality comes from the fact that $\mathcal{O}_{X,x}$ is a regular ring. Take $f_1, \dots, f_{n-d} \in I$ such that $\overline{df}_i$ are linearly independent after tensoring with $k(x)$. Then we define $X_0 = V(f_1, \dots, f_{n-d})$. Clearly we have the containment $X \subseteq X_0$, and we see that X_0 is smooth and of dimension d from the linear independence of the basis we just chose (using the Jacobian criterion). Then $\mathcal{O}_{X_0,x}$ is regular of dimension d by what we have shown and we have a surjection $\mathcal{O}_{X_0,x} \twoheadrightarrow \mathcal{O}_{X,x}$ of local regular rings of dimension d, hence an isomorphism. This recognises $X = X_0$ in a neighbourhood around x. □

There is one more theorem for today, which is a criterion for smoothness we have promised for a little while now.

Theorem 6.12. *Given a map $f : X \to S$ of locally finite presentation, then the following are equivalent,*

1. *The map f is smooth.*
2. *The map f is flat and has smooth fibres.*
3. *The map f is flat and has smooth geometric fibres*[14]*.*

Again, we are going to need a quick local criterion for flatness, so let us state it quickly.

[14] A map $X \to S$ has smooth geometric fibres if for all algebraically closed fields k and maps $\operatorname{Spec} k \to S$, we have $X \times_S \operatorname{Spec} k \to \operatorname{Spec} k$ is smooth.

Theorem 6.13 (Local Criterion for Flatness)**.** *Given a local noetherian ring $(C, \mathfrak{m})$ and a C-module M, then M if flat if and only if the multiplication map $\mathfrak{m} \otimes_C M \to M$ is injective.*

Proof of Theorem 6.12. The fact that part 3 implies part 2 is simply the descent of smoothness. To show part 1 is equivalent to part 2 we can first reduce to the local case, so let $S = \operatorname{Spec} R$, $X = \operatorname{Spec} A$ and $A = B/I$ for $B = R[X_1, \ldots, X_n]$ and some finitely generated ideal $I \subseteq B$. We can even reduce to the case when all our rings are noetherian, by recalling the fact that if a map $A \to R$ is a finitely presented map of rings, then there is a noetherian R' and a finitely presented map $A' \to R'$ of rings such that $A' = A \otimes_{R'} R$.

Now we assume part 1 is true, so locally we have $X = V(f_1, \ldots, f_r) \subseteq V(f_1, \ldots, f_i) =: X_i \subseteq \mathbb{A}^n_S$ for $i \leq r$, both closed subschemes which are smooth over S by the Jacobian criterion. We use this as our inductive step, so it suffices to show $f : X \to S$ is flat with smooth fibres. However, f is already smooth and hence flat, so we only really need to show the smooth fibres condition. If we take $x \in X$ with X flat and smooth over S and $t \in \Gamma(X, \mathcal{O}_X)$ such that $dt \neq 0$ inside $\Omega^1_{X/S} \otimes k(s)$ and $t(x) = 0$, then $Z = V(f) \to S$ is flat and smooth. Let $s = f(x)$, then we have an exact sequence,

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Z,z} \longrightarrow 0.$$

We have to show $\mathfrak{m}_{S,s} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Z,z} \to \mathcal{O}_{Z,z}$ is injective. However, we know $\mathcal{O}_{X,x}/\mathcal{O}_{S,s}$ is flat, so then the snake lemma and the local criterion for flatness mentioned above (Theorem 6.13) imply it suffices to show the morphism on the cokernel is injective. In other words, the map,

$$K/\mathfrak{m}_{S,s}K \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_{S,s}\mathcal{O}_{X,x},$$

is injective. This map fits into the following commutative diagram however,

$$\begin{array}{ccccc}
 & & K/\mathfrak{m}_{S,s}K & & \\
 & \nearrow\!\!\!\nearrow & & \searrow & \\
\mathcal{O}_{X,x}/\mathfrak{m}_{S,s}\mathcal{O}_{X,x} & & \xrightarrow{\quad \bar{t} \quad} & & \mathcal{O}_{X,x}/\mathfrak{m}_{S,s}\mathcal{O}_{X,x} \\
\downarrow \cong & & & & \downarrow \cong \\
\mathcal{O}_{X_s,x} & & \xrightarrow{\quad \bar{t} \quad} & & \mathcal{O}_{X_s,x}
\end{array},$$

but $\bar{t}$ is injective since $\bar{t} \neq 0$ and $\mathcal{O}_{X_S,x}$ is regular and hence integral. Now assume part 2, with A, B and R as they are above. We then have to show that near $x \in X$ (let $s = f(x) \in S$), the bottom row of the sequence,

$$\begin{array}{ccccccc}
\bigoplus_{i=1}^r A \cdot e_i & & & & & & \\
\downarrow & \overset{\mathcal{J}}{\searrow} & & & & & \\
A \otimes_B I = I/I^2 & \longrightarrow & A \otimes_B \Omega^1_{B/R} & \longrightarrow & \Omega^1_{A/R} & \longrightarrow & 0
\end{array},$$

is exact on the left and locally split, i.e. the Jacobian $\mathcal{J}$ has rank r, where $t_1, \ldots, t_r \in I$ are such that they form a basis of $I/I^2 \otimes k(x)$. Notice the bottom row of the sequence,

$$\begin{array}{ccccccc}
\bigoplus_{i=1}^r A_s \cdot \bar{t}_i & & & & & & \\
\downarrow\!\!\downarrow & \overset{\mathcal{J}_s}{\searrow} & & & & & \\
J/J^2 & \longrightarrow & A_s \otimes_{B_s} \Omega^1_{B_s/k(s)} & \longrightarrow & \Omega^1_{A_s/k(s)} & \longrightarrow & 0
\end{array},$$

is exact on the left and locally split, where $A_s = A \otimes_R k(s)$, $B_s = B \otimes_R k(s) = k(s)[X_1, \ldots, X_n]$ and $J = \ker(B_s \to A_s)$. However, from tensoring the exact sequence $I \to B \to A$ defining the ideal I with $k(s)$ over R we see that $J \cong I \otimes_R k(s)$. Now M_s has rank r and $I/I^2 \otimes_R k(s) \cong J/J^2$, which implies M has rank r. This finishes our proof. $\square$

7 Smoothness is Local on Source 15/05/2017

We are reminded of three important results from last week; Proposition 5.14, the Jacobi criterion 6.4 and our criterion for smoothness which was Theorem 6.12. Today we want to cover two things, the first being noetherian approximation, and the second is discussing the difficultly to why proving smoothness is a local on the source property of a map of schemes. In the proof of Proposition 6.12 we discussed why if $f : X \to S$ is smooth and S noetherian, then f is flat. To eliminate this noetherian assumption, we have the following arguments which constitute a general little trick.

Without loss of generality we consider $X = \operatorname{Spec} A \to S = \operatorname{Spec} R$ and A is a finitely presented (and hence noetherian) R-algebra. The idea is to write any ring R as the filtered colimit of finitely generated $\mathbb{Z}$-algebras over R_i, and then any finitely presented structures (modules, algebras, schemes, ...) are already defined over some R_i (as it is defined by a finite piece of data). More precisely this idea is captured by the following two lemmas and subsequent example.

Lemma 7.1. *Let R be a ring, then R is a filtered colimit of finitely generated $\mathbb{Z}$-algebras R_a,*

Proof. We write $R = \mathbb{Z}[X_i, i \in I]/(f_j, j \in J)$ for I, J two arbitrary (potentially infinite) sets. For any finite subset $I' \subseteq I$ we let $J' \subseteq J$ be the set of all j such that $f_i \in \mathbb{Z}[X_i, i \in I']$. We then set,

$$R_{I'} = \mathbb{Z}[X_i, i \in I']/(f_j, j \in J'),$$

which is some finitely generated $\mathbb{Z}$-algebra. These $R_{I'}$ form a filtered system and we have,

$$R = \operatorname{colim}_{I' \subseteq I, \text{finite}} R_{I'}.$$

□

Lemma 7.2. *Given $R_i, i \in I$, some filtered system of rings and R its colimit, and A is a finitely presented R-algebra, then there exists $i \in I$ and a finitely presented R_i-algebra A_i such that $A = A_i \otimes_{R_i} R$.*

Proof. We know A is of the form

$$A = R[X_1, \ldots, X_n]/(f_1, \ldots, f_m),$$

for some collection $f_j \in R[X_1, \ldots, X_n]$, which involve finitely many elements of R. This means there exists some i such that f_j is the image of $f_{j,i} \in R_i[X_1, \ldots, X_n]$, then we set,

$$A_i = R_i[X_1, \ldots, X_n]/(f_{1,i}, \ldots, f_{m,i}),$$

and we're done. □

Our general problem can then be stated as the following.

Given A has some property as an R-algebra then does A_i have this property (up to enlargening i)?

For example, assume we know that smoothness implies flatness over noetherian schemes, then let us see this works in the general case.

Smoothness implies Flatness. Without loss of generality we take $X = \operatorname{Spec} A \to S = \operatorname{Spec} R$, and write

$$R = \operatorname{colim}_{a \in I} R_a,$$

be a filtered colimit of noetherian R_a. Now we use the Jacobi criterion, so locally on X we have

$$A = R[X_1, \ldots, X_n]/(g_1, \ldots, g_m),$$

for some $m \leq n$ and look at the value

$$\Delta = \det\left(\frac{\partial g_i}{\partial X_j}\right)_{j=1,\dots,n, i=1,\dots,m} \in A^\times,$$

where the matrix in question is a minor of the Jacobian matrix. Now there exists an $a \in I$ and $g_{1,a}, \dots, g_{m,a} \in R_a[X_1, \dots, X_n]$ mapping to $g_1, \dots, g_m$ so that

$$A_a = R_a[X_1, \dots, X_n]/(g_{1,a}, \dots, g_{m,a}),$$

then the Jacobi criterion in the opposite direction tells us the value,

$$\Delta_a = \det\left(\frac{\partial g_{i,a}}{\partial X_j}\right)_{i,j=1,\dots,m} \in A_a,$$

such that $\Delta_a \mapsto \Delta$ under the canonical map $A_a \to A$. Since $A = \operatorname{colim}_{a' \geq a} A_{a'}$ by the normal cofinality arguments, we see Δ_a becomes invertible for some large values of a'. After replacing a by a' if necessary, then A_a is smooth over R_a, then the noetherian case tells us A_a are flat over R_a, and then we have $A = A_a \otimes_{R_a} R$ is flat over R. □

In the beginning of the proof of Theorem 6.12 we made a short argument to reduce our result to the noetherian case, but the above method works in more generality. Now we turn our attention to understanding why (formal) smoothness is local on the source (target). The only piece of data we will not prove here is the following theorem of Raynoud and Gruson, alluded to after Proposition 4.6.

Theorem 7.3. *Given a ring A and an A-module M, then M is projective if and only if there is a covering of $\operatorname{Spec} A = \bigcup_i D(f_i)$ such that $M[f_i^{-1}]$ are all projective $A[f_i^{-1}]$-modules.*

Note that M has no finiteness assumptions, which is the hardest and most subtle part of the proof. This is relevant to our current goal as for $f : X = \operatorname{Spec} A \to S = \operatorname{Spec} R$, then f being smooth implies $\Omega^1_{A/R}$ is a projective A-module (Proposition 6.1).

Now let $f : X \to S$ be a morphism of schemes such that there is an open cover $X = \bigcup U_i$ with $U_i = \operatorname{Spec} A_i$ such that all restrictions $f|_{U_i}$ are formally smooth (hence $\Omega^1_{A_i/S}$ is a projective A_i-module). Assume we are given such a diagram,

$$\begin{array}{ccc} T_0 & \xrightarrow{u_0} & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ T & \xrightarrow{u} & S \end{array},$$

where $i : T_0 \hookrightarrow T$ is a closed immersion with T affine and associated ideal subsheaf $\mathcal{I}$ with $\mathcal{I}^2 = 0$. We have $U_i \subseteq X$ open, so by continuity $T_{i,0} \subseteq T_0$ which are the preimage of the U_i's under u_0, are also open. Note that $|T| = |T_0|$ which implies that our lift will happen on open subschemes $T_i \subseteq T$. We know we locally have lifts $u_i : T_i \to X$, but they need not agree on overlaps. This leads us to the definition of a torsor.

Definition 7.4. *We recall the local and global definitions of torsors here.*

1. *Let G be a group. A G-torsor is a set P equipped with an action $G \times P \to P$ such that P is nonempty and this action is simply transitive, so for each $p \in P$ we have $G \to P$ defined by $g \mapsto gp$ is a bijection.*

2. *Given a space T and $\mathcal{G}$ a sheaf of groups over T. Then a $\mathcal{G}$-torsor is a sheaf $\mathcal{P}$ on T equipped with an action $\mathcal{G} \times \mathcal{P} \to \mathcal{P}$ such that all stalks of $\mathcal{P}$ are nonempty and for all open $U \subsetneq T$, and each $p \in \mathcal{P}(U)$, we have an isomorphism $\mathcal{G}(U) \to \mathcal{P}(U)$ defined by $g \mapsto gp$.*

Example 7.5. If T is the usual Möbius band and $\mathcal{G} = \mathbb{Z}/2\mathbb{Z}$ is the constant sheaf and $\mathcal{P}$ is the sheaf of local orientations of T. Notice $\mathcal{P}(T) = \varnothing$ but locally the sections are $\mathbb{Z}/2\mathbb{Z}$ with the transitive $\mathbb{Z}/2\mathbb{Z}$-action.

Proposition 7.6. *Let T be an affine scheme and $\mathcal{P}$ be the sheaf defined by,*

$$V \longmapsto \left\{ \text{set of all liftings of } u_0|_{V_0} : \begin{array}{ccc} V_0 & \xrightarrow{u_0|_{V_0}} & X \\ \downarrow & \nearrow & \downarrow \\ V & \longrightarrow & S \end{array} \right\}.$$

Then $\mathcal{P}$ is a torsor under the sheaf of groups,

$$\mathcal{G} = \mathcal{H}om_{T_0}(u_0^*\Omega^1_{X/S}, \mathcal{I}),$$

where $\mathcal{I}$ is the ideal subsheaf associated to $T_0 \hookrightarrow T$.

Proof. We define an action first. Let $u_V : V \to X$ be a local lift, and let $\phi \in \mathrm{Hom}_V(u_0^*\Omega^1_{X/S}|_{V_0}, \mathcal{I}|_{V_0})$, then we produce a lift $u|_V + \phi$, where we use that we can classify all lifts in terms of one lift using derivations (see Lemma 5.1). This classification also implies that given a section u_V we obtain the desired bijection. All the stalks of $\mathcal{P}$ are also non-empty as $P(T_i) \neq \varnothing$, since f restricted to the U_i's is formally smooth. □

Definition 7.7. *Given a topological space T and $\mathcal{G}$ a sheaf of groups, then $H^1(T, \mathcal{G})$ can be identified as,*

$$H^1(T, \mathcal{G}) = \{\mathcal{G}\text{-torsors } \mathcal{P}\}/\text{isomorphism},$$

the first cohomology of T with coefficients in $\mathcal{G}$ (just a set in general).

Notice there is a distinguished element $* \in H^1(T, \mathcal{G})$ given by the trivial $\mathcal{G}$-torsor, which is just $\mathcal{G}$ itself with left multiplication (hence $H^1(T, \mathcal{G})$ is actually a based set).

Remark 7.8. Moreover, $\mathcal{P}(T) \neq \varnothing$ is equivalent to $\mathcal{P} \cong \mathcal{G}$, so $[\mathcal{P}] = * \in H^1(T, \mathcal{G})$. One direction is obvious, since $\mathcal{G}(T) \neq \varnothing$, and for the other direction, if $p \in \mathcal{P}(T)$, then we can define $\mathcal{G} \to \mathcal{P}$ by $g \mapsto gp$ which by definition is an isomorphism.

Our goal here is to prove $[\mathcal{P}] = * \in H^1(T, \mathcal{H}om_{T_0}(u_0^*\Omega^1_{X/S}, \mathcal{I}))$. It suffices to prove this first cohomology group is in fact just a point itself.

Proof. Given $u_0^*\Omega^1_{X/S}|_{T_{i,0}} = (u_0|_{T_{i,0}})^*\Omega^1_{U_i/S}$, where $\Omega^1_{U_i/S}$ are all projective A_i-modules, so the whole thing is projective over T_i. Theorem 7.3 now implies $u_0^*\Omega^1_{X/S} = \widetilde{M}$ with M some projective A/I-module. We can now reduce this to the following proposition: □

Proposition 7.9. *Given $T = \operatorname{Spec} A$ on an affine scheme (really T_0 and A/I respectively using the notation from above), M a projective A-module, N an A-module, then with $\mathcal{M} = \widetilde{M}$ and $\mathcal{N} = \widetilde{N}$ we have,*

$$H^1(T, \mathcal{H}om_{\mathcal{O}_T}(\mathcal{M}, \mathcal{N})) = \{*\}.$$

Notice that if $\mathcal{M}$ is of finite rank (as it often is) then our hom-sheaf above is quasi-coherent.

Proof. We see M is a direct summand of $\bigoplus A$, so $\mathcal{M}$ is a direct summand of $\bigoplus \mathcal{O}_T$, which implies $\mathcal{H}om_{\mathcal{O}_T}(\mathcal{M}, \mathcal{N})$ is a direct summand of

$$\mathcal{H}om_{\mathcal{O}_T}\left(\bigoplus \mathcal{O}_T, \mathcal{N}\right) = \prod \mathcal{H}om_{\mathcal{O}_T}(\mathcal{O}_T, \mathcal{N}) \cong \prod \mathcal{N}.$$

We then obtain the following sequence of injections,

$$H^1(T, \mathcal{H}om_{\mathcal{O}_T}(\mathcal{M}, \mathcal{N})) \hookrightarrow H^1\left(T, \prod \mathcal{N}\right) \hookrightarrow \prod H^1(T, \mathcal{N}),$$

where we do need a little argument to justify the last injection using the definition of H^1 using torsors. Hence we have reduced this whole problem to the following question about sheaf cohomology:

Proposition 7.10. *Let A be a ring, N be an A-module and $\mathcal{N} = \widetilde{N}$ a quasicoherent sheaf on $T = \operatorname{Spec} A$, then $H^1(T, \mathcal{N}) = \{*\}$.*

□

We will prove the above proposition next lecture (see Propoosition 8.1), and more generality when we approach sheaf cohomology in a systematic way in lecture 12 (see Theorem 12.1).

8 The First Sheaf Cohomology Group 18/05/2017

Last time we spoke of $\mathcal{G}$-torsors over a topological space X for a sheaf of abelian groups $\mathcal{G}$. We defined,

$$H^1(X, \mathcal{G}) = \{\mathcal{G}\text{−torsors}\},$$

which has the distinguished point $\mathcal{G}$. Let us rephrase Proposition 7.10 for today.

Proposition 8.1. *Given X is an affine scheme and $\mathcal{G} = \mathcal{M}$ is a quasi-coherent sheaf (of $\mathcal{O}_X$-modules) on X, then*

$$H^1(X, \mathcal{M}) = \{*\}.$$

To see this we will prove a more general proposition.

Proposition 8.2. *Given a scheme X and a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ on X, then*

$$H^1(X, \mathcal{M}) = \{\textit{extensions of } \mathcal{O}_X\textit{-modules } \widetilde{M}, 0 \to \mathcal{M} \to \widetilde{\mathcal{M}} \to \mathcal{O}_X \to 0\}/\cong,$$

the set of extensions of $\mathcal{M}$ and $\mathcal{O}_X$ modulo isomorphisms. More precisely, and more generally, we actually have an equivalence of categories between that of $\mathcal{M}$-torsors on X, say $\mathrm{Tors}_{\mathcal{M}}$*, and that of extensions of $\mathcal{M}$ and $\mathcal{O}_X$ as $\mathcal{O}_X$-modules, say* $\mathrm{Exten}_{\mathcal{M}}$.

This proposition is in fact true for any ringed space $(X, \mathcal{O}_X)$, but we will not need this generality. The maps in $\mathrm{Exten}_{\mathcal{M}}$ are $f : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$ such that the following diagram commutes,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \widetilde{\mathcal{M}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathrm{id}_{\mathcal{M}}} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \mathrm{id}_{\mathcal{O}_X}} & & \\ 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \widetilde{\mathcal{M}}' & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \end{array},$$

hence by the five-lemma, all such f are isomorphisms. Also notice that all extensions $\widetilde{M}$ are automatically quasi-coherent, as the sequence that defines them is locally split.

Remark 8.3. Given a space X and a sheaf of groups $\mathcal{G}$, where $\mathcal{P}$ and $\mathcal{P}'$ are both $\mathcal{G}$-torsors, then a map $f : \mathcal{P} \to \mathcal{P}'$ in $\mathrm{Tors}_{\mathcal{G}}$, so it must commute with the $\mathcal{G}$-actions, is an isomorphism. We can check this locally, so we have $\mathcal{P}(X) \neq \varnothing \neq \mathcal{P}'(X)$, so we have $\mathcal{P} \cong \mathcal{G} \cong \mathcal{P}'$ by Remark 7.8, so we need to look at maps $f : \mathcal{G} \to \mathcal{G}$ which commute with the $\mathcal{G}$-action (which has always secretly been a left action). Let $g = f(1) \in \mathcal{G}(X)$, then for all $U \subseteq X$, $h \in \mathcal{G}(U)$, from the necessary equivariance of f we obtain the equality,

$$f(h) = f(1) \cdot h.$$

Hence f is simply right multiplication by g, which is an isomorphism with inverse right multiplication by g^{-1}.

Notice that from this proof we also see that the group of automorphisms of $\mathcal{G}$ as a $\mathcal{G}$-torsor is simply $\mathcal{G}(X)$.

Proof of Proposition 8.2. We have a functor $\Phi : \mathrm{Exten}_{\mathcal{M}} \to \mathrm{Tors}_{\mathcal{M}}$ which sends $\widetilde{\mathcal{M}}$ to the $\mathcal{M}$-torsor $\mathcal{P}_{\widetilde{\mathcal{M}}}$ which is defined by,

$$U \longmapsto \{s \in \widetilde{\mathcal{M}}(U) \mid p(s) = 1\},$$

where p is the map $\widetilde{\mathcal{M}} \to \mathcal{O}_X$. So the local sections of $\mathcal{P}_{\widetilde{\mathcal{M}}}$ are local sections of p. This is an $\mathcal{M}$-torsor, which we should of course justify. Let $m \in \mathcal{M}(U)$ then our action sends a section $s \mapsto s + m$ inside

$\mathcal{P}_{\widetilde{\mathcal{M}}}(U)$, which is still inside $\mathcal{P}_{\widetilde{\mathcal{M}}}(U)$ since $m \in \mathcal{M}(U)$ means it is in the kernel of p. If $s \in \mathcal{P}_{\widetilde{\mathcal{M}}}(U)$ then the map,

$$\mathcal{M}(U) \longrightarrow \mathcal{P}_{\widetilde{\mathcal{M}}}(U), \qquad m \longmapsto s + m,$$

is an isomorphism since $\mathcal{P}_{\widetilde{\mathcal{M}}}(U) = s + \mathcal{M}(U)$. To see the stalks are nonempty we let $X = \operatorname{Spec} A$, then our exact sequence defining $\widetilde{\mathcal{M}}$ is equivalent to an exact sequence of A-modules,

$$0 \longrightarrow M \longrightarrow \widetilde{M} \longrightarrow A \longrightarrow 0,$$

which splits as A is a projective A-module. Now we have to check this functor is fully faithful and essentially surjective. For the former, let $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}'$ be two elements of $\text{Exten}_{\mathcal{M}}$, then we want to see,

$$\operatorname{Hom}_{\text{Exten}_{\mathcal{M}}}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}') \cong \operatorname{Hom}_{\text{Tors}_{\mathcal{M}}}(\mathcal{P}_{\widetilde{\mathcal{M}}}, \mathcal{P}_{\widetilde{\mathcal{M}}'}),$$

through this functor Φ. How ever both of these are global sections of the sheaves,

$$\mathcal{H}om_{\text{Exten}_{\mathcal{M}}}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}') \longrightarrow \mathcal{H}om_{\text{Tors}_{\mathcal{M}}}(\mathcal{P}_{\widetilde{\mathcal{M}}}, \mathcal{P}_{\widetilde{\mathcal{M}}'}),$$

respectively. We want to now see this is an isomorphism of sheaves, and we can do this locally, so let $X = \operatorname{Spec} A$. In the affine case though, we have

$$\widetilde{\mathcal{M}} \cong \mathcal{M} \oplus \mathcal{O}_X \cong \widetilde{\mathcal{M}}'.$$

This means,

$$\operatorname{Hom}_{\text{Exten}_{\mathcal{M}}}(\mathcal{M} \oplus \mathcal{O}_X, \mathcal{M} \oplus \mathcal{O}_X) = \left\{ \begin{pmatrix} \mathrm{id}_{\mathcal{M}} & \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) \\ 0 & \mathrm{id}_{\mathcal{O}_X} \end{pmatrix} \right\} \cong \mathcal{M} \cong \operatorname{Hom}_{\text{Tors}_{\mathcal{M}}}(\mathcal{M}, \mathcal{M}),$$

where we notice that $\mathcal{P}_{\widetilde{\mathcal{M}}} = \mathcal{M}$ and $\mathcal{P}_{\widetilde{\mathcal{M}}'} = \mathcal{M}$ in the affine case. This shows Φ is fully faithful, so for essential surjectivity let $\mathcal{P}$ be any $\mathcal{M}$-torsor. Then locally on $X = \cup U_i$ there is a unique (up to unique isomorphism) corresponding extension $\widetilde{\mathcal{M}}_i$ on U_i such that $\mathcal{P}_{\widetilde{\mathcal{M}}} \cong \mathcal{P}|_{U_i}$, as locally $\mathcal{P} \cong \mathcal{G}$. This is unique up to unique isomorphism, and by fully faithfulness all these extensions $\mathcal{M}_i$ glue to a global extension $\widetilde{\mathcal{M}}$ with $\mathcal{P}_{\widetilde{\mathcal{M}}} \cong \mathcal{P}$. □

A fancy way to phrase this last step is to say that the assignment sending the open subset U of X to the local category of extensions $\text{Exten}_{\mathcal{M}|_U}$ and the assignment sending U to $\text{Tors}_{\mathcal{M}|_U}$ are stacks on X. These are generalisations of sheaves, and the backbone of the Stacks Project [8].

Proof of Proposition 8.1. If $X = \operatorname{Spec} A$ is affine, then $H^1(X, \mathcal{M})$ is simply the extensions of $\mathcal{M}$, which are simply A-module extensions,

$$0 \longrightarrow M \longrightarrow \widetilde{M} \longrightarrow A \longrightarrow 0,$$

but $\widetilde{M} = M \oplus A$ as A is a projective A-module. □

Recall that if X is a topological space, and $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of abelian groups, then the global sections functor is only exact on the left, and *not* on the right in general.

Proposition 8.4. *Given the situation above, there is a natural exact sequence,*

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \xrightarrow{\delta} H^1(X, \mathcal{F}').$$

This exact sequence actually continues to the right two more places with our explicit definition of $H^1(X, \mathcal{M})$ (see exercise 6.2), and infinitely on the right with the general tools of sheaf cohomology we'll soon see.

Proof. Let $s \in \Gamma(X, \mathcal{F}'')$ then we can produce an $\mathcal{F}'$-torsor $\mathcal{P}_s$ which is defined as $U \mapsto \{t \in \mathcal{F}(U) | p(t) = s|_u\}$, with $\mathcal{F}'$-action defined as $t \mapsto t + t'$ for $t' \in \mathcal{F}'(U)$. Checking this is an $\mathcal{F}'$-torsor is similiar to checking the functor Φ above, using the fact also that $\mathcal{F} \to \mathcal{F}''$ is an epimorphism. We then set $\delta(s) = [\mathcal{P}_s] \in H^1(X, \mathcal{F}')$, and checking δ continues the exactness of our sequence is simple,

$$\delta(s) = * \qquad \Leftrightarrow \qquad \mathcal{P}_s \cong \mathcal{F}' \qquad \Leftrightarrow \qquad \mathcal{P}_s(X) \neq \varnothing \qquad \Leftrightarrow \qquad s \in p(\Gamma(X, \mathcal{F})).$$

□

Our goal now is to produce a general theory of cohomology groups of quasi-coherent sheaves on schemes. We would like to have cohomology groups ($\mathcal{O}_X(X)$-modules) $H^i(X, \mathcal{M})$ for all $i \geq 0$ such that,

0. $H^0(X, \mathcal{M}) = \Gamma(X, \mathcal{M})$,
1. for each short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ we would like to obtain a long exact sequence of cohomology groups,

$$0 \to H^0(X, \mathcal{M}') \to H^0(X, \mathcal{M}) \to H^0(X, \mathcal{M}'') \to H^1(X, \mathcal{M}') \to H^1(X, \mathcal{M}) \to H^1(X, \mathcal{M}'') \to \cdots .$$

2. The theory is minimal in some functorial sense.

This minimality condition means something like, given another theory $H^i_?$ satisfying 1 and 2, then we want to obtain a canonical injection,

$$H^i(X, \mathcal{M}) \hookrightarrow H^i_?(X, \mathcal{M}),$$

for all schemes X and quasi-coherent sheaves $\mathcal{M}$. We have see that $H^1(X, \mathcal{M})$ is simply all the extensions of $\mathcal{M}$, then we can define $\delta : H^1(X, \mathcal{M}) \to H^1_?(X, \mathcal{M})$ by $\widetilde{\mathcal{M}} \mapsto \delta(1)$, where δ is the map $H^0_?(X, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) \to H^1_?(X, \mathcal{M})$ defining $\widetilde{\mathcal{M}}$. This assignment is injective since if $f(1) = 0 \in H^1_?(X, \mathcal{M})$ then $1 \in \operatorname{im} H^0_?(X, \widetilde{\mathcal{M}}) = \Gamma(X, \widetilde{\mathcal{M}})$ by 1, so the extesion is split and hence 0 inside $H^1(X, \mathcal{M})$. The minimality condition (2) should then say something like $H^1_?(X, \mathcal{M}) = H^1(X, \mathcal{M})$.

This theory is a special case of a much more general theory. If X is a any topological space (or as Grothendieck prefered, a topos) and a sheaf of abelian groups $\mathcal{F}$ on X, then there are cohomology (abelain) groups $H^i(X, \mathcal{F})$ satisfying conditions 0-2, where minimality here means $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ if $\mathcal{F}$ is an injective. In algebraic topology we can look at a CW-complex X and the constant sheaf $\mathbb{Z}$, and then these groups $H^i(X, \mathbb{Z})$ become the singular cohomology groups topologists know and love.

Definition 8.5. *Let X be a space and $\mathcal{F}$ a sheaf of abelian groups on X. Then $\mathcal{F}$ is injective if for all injections $i : \mathcal{F} \hookrightarrow \widetilde{\mathcal{F}}$, where $\widetilde{\mathcal{F}}$ is also a sheaf of abelian groups, then there exists a section $s : \widetilde{\mathcal{F}} \to \mathcal{F}$, so $s \circ i = \mathrm{id}_{\mathcal{F}}$.*

Remark 8.6. Notice that $\mathcal{F}$ is injective if and only if the functor $\mathrm{Hom}_(-, \mathcal{F})$ from sheaves of abelian groups on X to the category of abelian groups is exact. Indeed, if

$$0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0,$$

is exact, then $\mathrm{Hom}(-, \mathcal{F})$ is always left exact, so to show $\mathrm{Hom}(\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{G}', \mathcal{F})$ is an epimorphism, let $f : \mathcal{G} \to \mathcal{F}$ and let $\mathcal{F} \hookrightarrow \widetilde{\mathcal{F}} := (\mathcal{F} \oplus \mathcal{G})/\mathcal{G}'$. If $\mathcal{F}$ is injective then this injection splits and we obtain a map $\mathcal{G} \to \mathcal{F}$ restricting to $f : \mathcal{G}' \to \mathcal{F}$. Conversely, we can look at the exact sequence,

$$0 \longrightarrow \mathcal{F} \longrightarrow \widetilde{\mathcal{F}} \longrightarrow \widetilde{\mathcal{F}}/\mathcal{F} \longrightarrow 0.$$

Proposition 8.7. *Given an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of sheaves of abelian groups on X, and $\mathcal{F}'$ is injective, then*

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \longrightarrow 0,$$

is exact.

Proof. Consider $\mathcal{F}' \to \mathcal{F}$, then we have a splitting $s : \mathcal{F} \to \mathcal{F}'$ so $\mathcal{F} \cong \mathcal{F}' \oplus \mathcal{F}''$ and we're done. $\square$

Next we want to see there are enough injective inside the category of sheaves of abelian groups over X. We'll see why we want this after the statement of this theorem.

Theorem 8.8 (Enough Injectives)**.** *Let X be a space and $\mathcal{F}$ a sheaf of abelian groups, then there exists an injection $\mathcal{F} \hookrightarrow \widetilde{\mathcal{F}}$ such that $\widetilde{\mathcal{F}}$ is injective.*

Remark 8.9. This is super useful, and essentially tells us what we want to do. Given $\mathcal{F}$, we want to compute $H^i(X, \mathcal{F})$, then we embed $\mathcal{F}$ into an injective $\widetilde{\mathcal{F}}$. This gives us a series of exact sequences from the long exact sequence on cohomology from the short exact sequence $0 \to \mathcal{F} \to \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}/\mathcal{F} \to 0$.

$$0 = H^i(X, \widetilde{\mathcal{F}}) \longrightarrow H^i(X, \widetilde{\mathcal{F}}/\mathcal{F}) \longrightarrow H^{i+1}(X, \mathcal{F}) \longrightarrow H^{i+1}(X, \widetilde{\mathcal{F}}) = 0.$$

We know $H^1(X, \mathcal{F})$ from explicit calculations, and we can calculate $H^i(X, \mathcal{F})$ by calculating $H^{i+1}(X, \widetilde{\mathcal{F}}/\mathcal{F})$, and apply the same process to the quotient sheaf $\widetilde{\mathcal{F}}/\mathcal{F}$. Inductively we can calculate all the sheaf cohomology groups of $\mathcal{F}$.

Proof. First we consider the case when $X = *$, then $\mathcal{F}$ is simply any abelian group M. It should be know that there exists an injective abelian group $\widetilde{M}$ and an injection $M \hookrightarrow \widetilde{M}$, and we'll revisit this proof next lecture. In general, for any $x \in X$ we choose injections $\mathcal{F}_x \hookrightarrow \widetilde{M}_x$, where $\widetilde{M}_x$ are all injective abelian groups. Let $i_x : \{x\} \hookrightarrow X$ be the inclusion of the point x, then we set

$$\widetilde{\mathcal{F}} = \prod_{x \in X} (i_x)_* \widetilde{M}_x,$$

and define a map $\mathcal{F} \to \widetilde{F}$ by the collection of maps $\mathcal{F} \to (i_x)_* \widetilde{M}_x$ which are adjoint to the given injections $i_x^* \mathcal{F} = \mathcal{F}_x \to \widetilde{M_x}$. This map is clearly injective since we can check this on stalks, and the following lemma will see that $\widetilde{\mathcal{F}}$ is an injective sheaf. $\square$

Lemma 8.10. *Let $f : Y \to X$ be a map of spaces.*

1. *Arbitrary products of injectives are injective.*
2. *If $\mathcal{F}$ is injective on Y, then $f_*\mathcal{F}$ is injective on X.*

Proof. For part 1, take sheaves $\mathcal{F}_i$ for $i \in I$ and notice,

$$\operatorname{Hom}\left(-, \prod_i \mathcal{F}_i\right) \cong \prod \operatorname{Hom}(-, \mathcal{F}_i),$$

is exact as the product of exact functors is exact. For the second part, we have

$$\operatorname{Hom}(-, f_*\mathcal{F}) \cong \operatorname{Hom}(f^*(-), \mathcal{F}),$$

is as exact as f^* is exact (easily checked on stalks) and $\mathcal{F}$ is injective by assumption. $\square$

Notice that more generally, functors having exact left adjoints preserve injective objects (see Footnote [19]).

9 Homological Algebra I (Derived Functors) 22/05/2017

Before we get into the meat of this lecture, we have some loose ends to wrap up from last time.

Theorem 9.1. *For any abelian group M, there exists an injective map $M \hookrightarrow \widetilde{M}$, where $\widetilde{M}$ is an injective abelian group. Moreover, an abelian group M is injective if and only if M is divisible.*

Proof. First assume M is injective, then the map $\mathbb{Z} \to \mathbb{Z}$ defined by multiplication by some non-zero $n \in \mathbb{Z}$ is injective, hence we have a surjection

$$M \cong \mathrm{Hom}(\mathbb{Z}, M) \longrightarrow \mathrm{Hom}(\mathbb{Z}, M) \cong M,$$

which is just multiplication by n, so M is divisible. Assume now that M is divisible, and take $N \hookrightarrow \widetilde{N}$ be any injection of abelian groups, then we want to show,

$$\mathrm{Hom}(\widetilde{N}, M) \longrightarrow \mathrm{Hom}(N, M),$$

is surjective. Given a map $f : N \to M$, we consider the set of pairs $(N', f : N' \to M)$ which extend f in the sense that $N \subseteq N' \subseteq \widetilde{N}$. This creates a directed system and we apply Zorn's lemma to obtain a maximal such (N', f'), so by replacing N by N' we can assume (N, f) are a maximal pair. Assume that $N \neq \widetilde{N}$, and choose some $x \in \widetilde{N} \backslash N$, then we have the following diagram,

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \widetilde{N} & & \\ \uparrow & & \uparrow & & \\ m\mathbb{Z} & \longrightarrow & N & \xrightarrow{f} & M \end{array},$$

where $\mathbb{Z} \to \widetilde{N}$ maps $1 \mapsto x$, and $m \in \mathbb{Z}$ is the restriction of this to N. The composite map $m\mathbb{Z} \to M$ extends to $\mathbb{Z} \to M$ since if $m = 0$ we can do this trivially, and if $m \neq 0$ we need to divide an element in M by m, which we can do since M is divisible. Hence we obtain an extension to,

$$(N \oplus \mathbb{Z})/m\mathbb{Z} = N + \mathbb{Z}\{x\} \subseteq \widetilde{N},$$

which contradicts the maximality of N, hence $N = \widetilde{N}$. Now for the first statement, let $0 \neq x \in M$, then $x \cdot \mathbb{Z} \subseteq M$ is either $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ for some $n > 0$, and both these groups inject into the divisible (and hence injective) groups $\mathbb{Q}$ and $\mathbb{Q}/n\mathbb{Z}$ respectively. For such an $x \in M$ we set $\widetilde{M_x}$ to be either of these two options, depending on x. The map $x \cdot \mathbb{Z} \hookrightarrow \widetilde{M_x}$ extends to M as $\widetilde{M_x}$ is injective, and we then consider the map,

$$M \xrightarrow{\prod f_x} \prod_{0 \neq x \in M} \widetilde{M_x} = \widetilde{M} \ .$$

This map is injective as all the product factors are, and $\widetilde{M}$ is also injective as the product of injective things are injective (similar to part 2 of Lemma 8.10). ☐

Notice this proof uses Zorn's lemma and hence is equivalent to using the axiom of choice. It is very non-constructive.

Recall now the situation we were in last lecture, where we have a space X and consider the functor of global sections from the category of sheaves of abelian groups on X to the category of abelian groups, and we want to build some cohomology. Today we are going to go to pure abstraction (à la Grothendieck and Cartan-Eilenberg) through homological algebra following the construction and the basic theory of δ-functors to find the answer.

Definition 9.2. *Let $\mathcal{A}$ be a category.*

1. *We say $\mathcal{A}$ is pointed if it has (up to unique isomorphism) an object which is both an initial and a final object (called a zero object in $\mathcal{A}$ and denoted as 0).*

2. *This category is preadditive if it is pointed, and for any $X, Y \in \mathcal{A}$, the coproduct $X \coprod Y$ and the product $X \times Y$ exist, and the canonical map,*

$$\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{pmatrix} : X \coprod Y \longrightarrow X \times Y,$$

   is an isomorphism. We then write $X \oplus Y$ for either the product or the coproduct of X and Y.

3. *If $\mathcal{A}$ is preadditive and for all $X, Y \in \mathcal{A}$ the hom-monoid $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is an abelian group with its natural monoid structure (see the remark below), then we call $\mathcal{A}$ additive.*

4. *Finally, we say $\mathcal{A}$ is abelian if it is additive and for all $f : X \to Y$ in $\mathcal{A}$, both*

$$\ker(f) := \lim \left(\begin{array}{ccc} & & 0 \\ & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \right), \qquad \text{and} \qquad \mathrm{coker}(f) := \mathrm{colim} \left(\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \\ 0 & & \end{array} \right),$$

   exist, and the natural map $\mathrm{coim}(f) := \mathrm{coker}(\ker(f) \to X) \to \ker(Y \to \mathrm{coker}(f)) =: \mathrm{im}(f)$ is an isomorphism.

Remark 9.3. Notice that all of the above definitions are properties of categories, and not extra datum, which is sometimes how this material is presented.

1. If $\mathcal{A}$ is pointed, then for each $X, Y \in \mathcal{A}$ we have a canonical zero maps $X \to 0 \to Y$, hence $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a pointed set. Notice the categories of sets and topological spaces are not pointed, but the categories of based topological spaces, rings, R-modules, groups, etc. are pointed.

2. If $\mathcal{A}$ is preadditive, then for any $X, Y \in \mathcal{A}$, the pointed hom-sets $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ can be given the structure of an abelian monoid through the following composite,

$$f + g : X \xrightarrow{\Delta_X} X \times X \xleftarrow[\cong]{} X \coprod X \xrightarrow{f \coprod g} Y \ .$$

   Given X, Y and $Z \in \mathcal{A}$, the composite map $\mathrm{Hom}_{\mathcal{A}}(X, Y) \times \mathrm{Hom}_{\mathcal{A}}(Y, Z) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$ is a map of abelian monoids.

3. If $\mathcal{A}$ is additive, then it is canonically enriched over abelian groups.

4. The categories of abelian groups, R-modules, finitely generated R-modules if R is noetherian, sheaves of abelian groups on a space X, quasi-coherent sheaves on a scheme X, or coherent sheaves on a noetherian scheme X are all abelian categories.

Also notice the duality involved in all of these definitions. A category $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{\mathrm{op}}$ is abelian. Hence everything we say in this generality from now on will dualise (e.g. left exact to right exact, injective to projective, etc.).

Definition 9.4. *Given two abelian categories $\mathcal{A}$ and $\mathcal{B}$, then an additive functor*[15] *$F : \mathcal{A} \to \mathcal{B}$ is left exact if for all $f : X \to Y$ the canonical map $F(\ker(f)) \to \ker(F(f))$ is an isomorphism.*

[15] An additive functor is the obvious thing: A functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories has to preserve the direct sum $\oplus$ and zero objects. An additive functor then gives us group homomorphisms $\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(FX, FY)$.

To make the above definition more memorable, we can equivalently say an additive functor $F : \mathcal{A} \to \mathcal{B}$ is left exact if given a short exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

in $\mathcal{A}$, then we have the follow exact sequence in $\mathcal{B}$,

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C).$$

Quintessential examples are the left exact functors $\mathrm{Hom}_A(M, -)$ and $\Gamma(X, -)$, and the right exact functor $- \otimes_A M$. There will be many more examples throughout the course.

We have made the above definition explicit, but we leave the following definitions up to the reader to define. In an abelian category $\mathcal{A}$, it makes sense to define cochain complexes, the cohomology of these cochain complexes, and exact sequences.

Definition 9.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories, then a cohomological δ-functor extending F is a sequence $F^i : \mathcal{A} \to \mathcal{B}$ of additive functors such that $F^0 = F$, together with boundary maps (natural in the following short exact sequences) $\delta : F^i(Z) \to F^{i+1}(X)$ for all short exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, such that for all such short exact sequences we obtain the following complex,*

$$0 \to F^0(X) \to F^0(Y) \to F^0(Z) \xrightarrow{\delta} F^1(X) \to F^1(Y) \to F^1(Z) \xrightarrow{\delta} F^2(X) \to \cdots,$$

which is exact. Moreover, such a δ is called universal if it is initial in the category of cohomological δ-functors extending F.

A δ-functor extending a left exact functor F in a way measures how much F fails to be an exact functor. In practice it is hard to check if a given δ-functor is universal, it is somehow just a nice categorical definition to benefit the theory. The following is a definition of Grothendieck from [4], which turns out to be a practical way of checking if a given δ-functor is universal.

Definition 9.6. *An additive functor $G : \mathcal{A} \to \mathcal{B}$ is effaceable if for each $X \in \mathcal{A}$ there exists an injection[16] $X \to \widetilde{X}$ such that $G(\widetilde{X}) = 0$. A cohomological δ-functor is effaceable if F^i are effaceable for all $i \geq 1$ (not including $F^0 = F$).*

Definition 9.7. *Let $\mathcal{A}$ be an abelian category.*

1. *We say $X \in \mathcal{A}$ is injective if $\mathrm{Hom}_{\mathcal{A}}(-, X)$ is exact, which is equivalent to the condition that for all injections $f : X \hookrightarrow \widetilde{X}$, f is split (a direct generalisation of Definition 8.5). Dually we have projective objects.*

2. *We say $\mathcal{A}$ has enough injectives if for each $X \in \mathcal{A}$ there is an injection $X \hookrightarrow \widetilde{X}$ where $\widetilde{X}$ is injective.*

Remark 9.8. Notice that if $G : \mathcal{A} \to \mathcal{B}$ is effaceable and $X \in \mathcal{A}$ is injective, then $G(X) = 0$. To see this, choose an injection $X \hookrightarrow \widetilde{X}$ such that $G(\widetilde{X}) = 0$ from the effaceability of G. The object X is injective, so we obtain a section $\widetilde{X} \to X$ which allows us to see X as a direct summand of $\widetilde{X}$. Since G is additive, $G(X)$ is a direct summand of $G(\widetilde{X})$, which implies $G(X) = 0$.

Notice that if $\mathcal{A}$ has enough injectives, then $G : \mathcal{A} \to \mathcal{B}$ is effaceable if and only if for all injective $X \in \mathcal{A}$ we have $G(X) = 0$. This follows from Remark 9.8. Theorem 8.8 saw that the category of abelian groups over a space X has enough injectives and we will prove in exercise 7.2 that the categories of A-modules and $\mathcal{O}_X$-modules for a ringed space $(X, \mathcal{O}_X)$ both have enough injectives.

The following theorem justifies the definition of an effaceable δ-functor.

[16] An injection here is a monomorphism. A monomorphism u in a category $\mathcal{C}$ is a map in $\mathcal{C}$ such that $u \circ f = u \circ g$ implies $f = g$ for any maps f and g such that these equations make sense. The dual concept is that of an epimorphism.

Theorem 9.9. *If $(F^i)_{i\geq 0}$ is an effaceable cohomological δ-functor, then it is universal.*

Proof. Let G be any cohomological δ-functor extending F, then we want to produce a series of natural transformations $F^i \to G^i$ for all $i \geq 0$, which we will do by induction. Since $F^0 = G^0 = F$ we have the base case. Assume that we can produce these natural transformations for $i = 0, \ldots, i-1$. For any $X \in \mathcal{A}$ we choose an injection $X \hookrightarrow \widetilde{X}$ such that $F^i(\widetilde{X}) = 0$, which we can do since F^i is effaceable. From this we obtain the following diagram with exact rows,

$$\begin{array}{ccccccc} F^{i-1}(\widetilde{X}) & \xrightarrow{a} & F^{i-1}(\widetilde{X}/X) & \xrightarrow{\delta_F} & F^i(X) & \longrightarrow & F^i(\widetilde{X}) = 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} & & \\ G^{i-1}(\widetilde{X}) & \xrightarrow{b} & G^{i-1}(\widetilde{X}/X) & \xrightarrow{\delta_G} & G^i(X) & \longrightarrow & G^i(\widetilde{X}) \end{array} .$$

By the functorality of the cokernel, we obtain a map $F^i(X) = \operatorname{coker}(a) \to \operatorname{coker}(b)$, where the equality comes from the exactness of the top row and the fact that $F^i(\widetilde{X}) = 0$, and this actually maps to $G^i(X)$ by the exactness of δ_G, and hence we obtain a unique map $\phi : F^i(X) \to G^i(X)$. To check this map is independent of our choice of $\widetilde{X}$, we see that two different $\widetilde{X}, \widetilde{X}'$ injective into their direct sum modulo the diagonal image of X, so it suffices to check when we enlarge $\widetilde{X}$, and this is clear. We then need to check for naturality in X, and that these maps commute with δ and stuff, but this is not so hard and just a little tedious. ☐

This theorem directly leads us to the next.

Theorem 9.10. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor of abelian categories, and assume $\mathcal{A}$ has enough injectives. Then an effaceable (and hence universal) cohomological δ-functor extending F exists.*

Definition 9.11. *The ith right derived functor of $F : \mathcal{A} \to \mathcal{B}$ is denoted as $R^iF : \mathcal{A} \to \mathcal{B}$, and is defined as the ith effaceable functor in the theorem above. For $F = \Gamma$ we write $H^i = R^i\Gamma$.*

Proof. Let us see now that we essentially have no choice in our definition of these right derived functors of F. Let $X \in \mathcal{A}$, then we choose an injective $I^0 \in A$ and an injection $X =: X^0 \overset{d^0}{\hookrightarrow} I^0$ so we have an exact sequence

$$0 \longrightarrow X^0 \longrightarrow I^0 \longrightarrow X^1 \longrightarrow 0,$$

where X^1 is the cokernel. From this we obtain the exact sequence,

$$F(I^0) \longrightarrow F(X^1) \longrightarrow F^1(X^0) \longrightarrow 0,$$

if such a F^1 was to exist, using the fact that $F^i(I) = 0$ when I is injective. By continuing this process we obtain exact sequences,

$$0 \longrightarrow X^i \longrightarrow I^i \longrightarrow X^{i+1} \longrightarrow 0.$$

We can summarise this information as the following diagram,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X & \xrightarrow{d^0} & I^0 & & \xrightarrow{\quad d^1 \quad} & & I^1 & \xrightarrow{\quad d^2 \quad} & I^2 \longrightarrow \cdots \\ & & & & & \searrow & & \nearrow & & \searrow \quad \nearrow & \searrow \\ & & & & & & X^1 & & & X^2 & \quad \cdots \end{array} ,$$

where the top row is a complex which we call an injective resolution[17] of X. We then have the chain of isomorphisms,

$$F^i(X^0) \cong F^{i-1}(X^1) \cong \cdots \cong F^1(X^{i-1}) = \operatorname{coker}(F(I^{i-1}) \to F(X^i)).$$

[17] An injective resolution of X is an exact sequence $0 \to X \to I^0 \to I^1 \cdots$ such that $X \to I^0$ is an injective map, and all I^i are injective.

Our only problem with this is this depends on our choice of injective resolution. Let us first remark though that we can rewrite this cokernel as $H^i(F(I^\bullet))$. This is because $X^{i+1} \hookrightarrow I^{i+1}$ is injective by construction so

$$0 \longrightarrow X^i \longrightarrow I^i \xrightarrow{d^{i+1}} I^{i+1},$$

is exact, and by left exactness of F we see that $F(X^i) = \ker(F(d^{i+1}))$, and then it is clear,

$$H^i(F(I^\bullet)) = F(X^i)/\operatorname{im}(F(d^i)) = \operatorname{coker}(F(I^{i-1}) \to F(X^i)).$$

The recipe for construction F^i is then forced to be the following: For each $X \in \mathcal{A}$ we choose an injective resolution,

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots,$$

then apply F to $I^\bullet$ and we have $F^i(X) := H^i(F(I^\bullet))$, but we still need to show this is independent of our choice of $I^\bullet$. This well-definedness comes from the following theorem though. □

Theorem 9.12. *Let $\mathcal{A}$ be an abelian category, and consider the following solid diagram in $\mathcal{A}$,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{d} & I^0 & \xrightarrow{d^0} & I^1 & \xrightarrow{d^1} & \cdots \\ & & \downarrow f & & \downarrow f^0 & & \downarrow f^1 & & \\ 0 & \longrightarrow & Y & \xrightarrow{e} & J^0 & \xrightarrow{e^0} & J^1 & \xrightarrow{e^1} & \cdots \end{array},$$

where $I^\bullet$ and $J^\bullet$ are injective resolutions for X and Y respectively. Then the dotted arrows f^i exist such that the whole diagram commutes, and between any two choices of f^i and f'^i we have a chain homotopy between them.

Proof. To define f^0 we start with the injection $X \hookrightarrow I^0$ and the composition $X \to Y \to J^0$, then the fact J^0 is injective gives us a map $f^0 : I^0 \to J^0$ which that the corresponding diagram commutes. For f^1, we know that $e^0 \circ f^0$ factors through $X^1 = \operatorname{coker}(d)$ since

$$e^0 \circ f^0 \circ d = e^0 \circ e \circ f = 0.$$

We then look at $X^1 \hookrightarrow I^1$ and then this factored map to J^1 and use the injectivity of J^1 to obtain $f^1 : I^1 \to J^1$. We then repeat this construction inductively to obtain $f^\bullet : I^\bullet \to J^\bullet$ of chain complexes. For the chain homotopy, we want maps $h^i : I \to J^{i-1}$ for all $i \geq 1$ such that $f^0 - f'^0 = h^1 \circ d^0$ and $f^i - f'^i = h^{i+1} \circ d^i + e^{i-1} \circ h^i$ for all $i \geq 2$. We will consider chain homotopies with more rigour next lecture. For the first map we consider the following diagram,

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \longrightarrow & I^0 \\ & & \downarrow f & & \downarrow f^0, f'^0 \\ 0 & \longrightarrow & Y & \longrightarrow & J^0 \end{array}.$$

We then have $f^0 - f'^0$ factor over X by exactness, so they are maps from $I^0/X \to J^0$, but $I^0/X = X^1$ which injects into I^1, and hence we obtain a map $h^1 : I^1 \to J^0$ as J^0 is injective. In the next step we consider the diagram,

$$\begin{array}{ccc} I^0 & \xrightarrow{d^0} & I^1 \\ {\scriptstyle f^0, f'^0} \downarrow & \swarrow {\scriptstyle h^1} & \downarrow {\scriptstyle f^1 - f'^1} \\ J^0 & \xrightarrow{e^0} & J^1 \end{array},$$

then $f^1 - f'^1$ restricted to I^0 is simply $e^0 \circ (f^0 - f'^0) = e^0 \circ h^1 \circ d^0$. Hence $(f^1 - f'^1 - e^0 \circ h^1)|_{I^0} = 0$, so $f^1 - f'^1 - e^0 \circ h^1$ factors over the image of d^0, and $d^1 : I^1/I^0 \hookrightarrow I^2$ is an injection, so the map $f^1 - f'^1 - e^0 \circ h^1 : I^1/I^0 \to J^1$ and the injectivity of J^1 give us $h^2 : I^2 \to J^1$. We then continue inductively, making the small remark that $e^i \circ (f^i - f'^i) = e^i \circ (h^{i+1} \circ d^i + e^{i-1} \circ h^i) = e^i \circ h^{i+1} \circ d^i$ as $J^\bullet$ is a complex. □

Note that we didn't strictly need $I^\bullet$ to consist of injective objects. This will prove a useful observation in the future.

10 Homological Algebra II (Homotopy Categories) 29/05/2017

Recall Theorem 9.10 from last time. A key step in the proof was to see the association $X \mapsto I^\bullet$ sending an object $X \in \mathcal{A}$ to an injective resolution defines a functor,

$$\mathcal{A} \longrightarrow K^{\geq 0}(\text{Inj}(\mathcal{A})),$$

where the latter category is the homotopy category of non-negatively graded complexes of injectives from $\mathcal{A}$ (we will define this shortly). We are going to follow this observation in much more detail now.

Definition 10.1. *Let $\mathcal{A}$ be an additive category.*

1. *We define $C(\mathcal{A})$ to be the category of (cochain) complexes, so sequences $\cdots \to X^i \xrightarrow{d^i} X^{i+1} \to \cdots$ of maps and objects in $\mathcal{A}$ such that $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$. Within $C(\mathcal{A})$ we have full subcategories $C^{\geq 0}(\mathcal{A})$ of non-negatively graded complexes, and $C^+(\mathcal{A})$ of complexes with some $j \in \mathbb{Z}$ such that $X^k = 0$ for all $k < j$.*
2. *Given $f, g : X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$ (maps in $C(\mathcal{A})$ are levelwise maps commuting with differentials), then a (cochain) homotopy from f to g is a collection of maps $h^i : X^i \to Y^{i-1}$ such that for all $i \in \mathbb{Z}$ we have*
$$f^i - g^i = d_Y^{i-1} \circ h^i + h^{i+1} \circ d_X^i.$$
If such an h exists, we say f is homotopic to g, and write $f \simeq g$.
3. *We define $K(\mathcal{A})$ to be the category of complexes up to homotopy, which has the same objects of $C(\mathcal{A})$ and the same maps too, but now considered up to homotopy defined above. Notice this is still an additive category, even if there are a few well-definedness checks to be done. We have bounded variants too, such as $K^{\geq 0}(\mathcal{A})$ which are all objects in $K(\mathcal{A})$ literally with $X^i = 0$ for all $i < 0$. This is not a full subcategory in the sense that it is not closed under isomorphisms*[18].

In the topological setting of singular cochain complexes, if two spaces are homotopic, then we can cook up a cochain homotopy as defined above. The following lemma justifies this definition.

Lemma 10.2. *If $f \simeq g$ in $C(\mathcal{A})$, then $f^* = g^* : H^i X \to H^i Y$ for all $i \in \mathbb{Z}$.*

Proof. Consider $f^i - g^i$ as a map from $Z^i(X) \to Z^i(Y)$, the i-cocycles of X to the i-cocycles of Y. We know $f^i - g^i = d_Y^{i-1} \circ h^i + h^{i+1} \circ d_X^i$. On $Z^i(X)$ we have $h^{i+1} \circ d_X^i = 0$, and once we quotient by the i-boundaries $B^i(X)$ and $B^i(Y)$ on each side we have $d_Y^{i-1} \circ h^i$ is also zero, hence $f^i = g^i$ on cohomology. □

This produces the following immediate corollary.

Corollary 10.3. *The map $H^i : C(\mathcal{A}) \to \mathcal{A}$ factors through $K(\mathcal{A})$.*

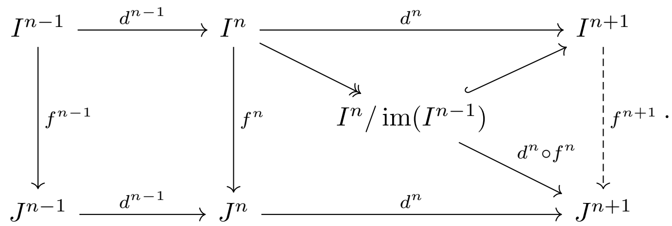
The following theorem then rephrases some of the work we did last time, in particular Theorems 9.10 and 9.12, which we'll explain after the proof.

Theorem 10.4. *Given an abelian category $\mathcal{A}$ with enough injectives, then there is an equivalence of categories from the full subcategory of $K^{\geq 0}(\text{Inj}(\mathcal{A}))$ of all objects with at most one nonzero cohomology group in degree zero, a category we'll call $\mathcal{C}$ for now, and $\mathcal{A}$ itself via the cohomology functor $H^0 : \mathcal{C} \to \mathcal{A}$.*

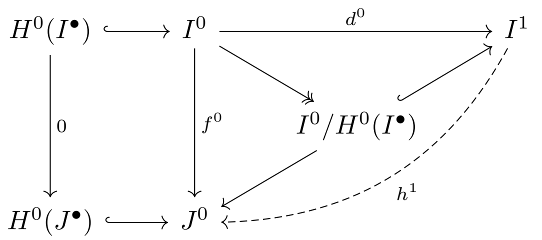
Proof. Essential surjectivity is provided by the existence of injective resolutions. For fullness, if $I^\bullet$ and $J^\bullet$ are in $\mathcal{C}$, then for any given $f : H^0(I^\bullet) \to H^0(J^\bullet)$ we can try to construct $f^\bullet : I^\bullet \to J^\bullet$. For f^0 we use the fact that $H^0(I^\bullet) = Z^0(I^\bullet) \subseteq I^0$, and similarly for J, hence we use the inclusion $H^0(I^\bullet) \subseteq I^0$,

[18] For example the complex $C^\bullet$ with $C^{-1} = C^0 = \mathbb{Z}$ joined by the identity map and $C^i = 0$ for all other levels is homotopic to an element of $C^{\geq 0}(\mathcal{A})$, but does not lie inside this category.

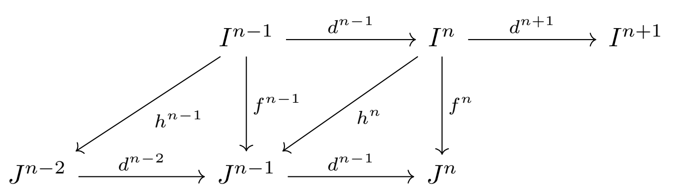
the composite $H^0(I^\bullet) \to H^0(J^\bullet) \hookrightarrow J^0$, and the fact that J^0 is injective to produce a map $f^0 : I^0 \to J^0$. If we have built $f^i : I^i \to J^i$ for all $i \leq n$, then we consider the following diagram,



The factorisation of d^n and f^n through $\mathrm{im}(I^{n-1})$ comes from the fact $I^\bullet$ is a complex and the fact the left hand square commutes, and we obtain f^{n+1} since J^{n+1} is injective. For faithfulness now, we need only show that if $f : I^\bullet \to J^\bullet$ induces the zero map on zeroth cohomology, then f was the zero map in $\mathcal{C}$, since we are working with categories enriched over abelian groups (additive categories here). This is where we need to use the homotopy category structure. If $H^0(f) = 0$, then we have the following commutative diagram,



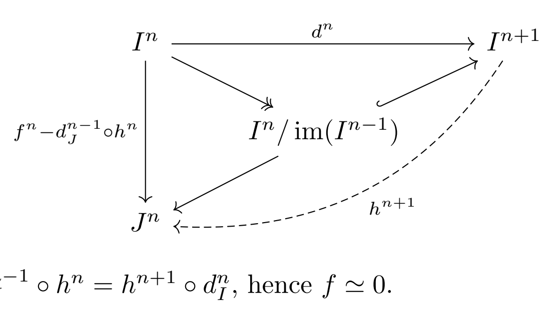
For similar reasons to the previous diagram, we obtain $h^1 : I^1 \to J^0$ such that $f^0 = h^1 \circ d^0$. Set $h^i = 0$ for all $i < 0$. For the inductive step, assume h^i exists for $i \leq n$, then we have the following (non-commutative) diagram,



We notice that,

$$f^n|_{I^{n-1}} = d_J^{n-1} \circ f^{n-1} = d_J^{n-1} \circ (d_J^{n-2} \circ h^{n-2} + h^n \circ d_I^{n-1}) = d_J^{n-1} \circ h^n \circ d_I^{n-1},$$

hence $(f^n - d_J^{n-1} \circ h^n)|_{I^{n-1}} = 0$ so we can factor this as,



From this we have $f^n - d_J^{n-1} \circ h^n = h^{n+1} \circ d_I^n$, hence $f \simeq 0$. □

Notice this is essentially the same argument as used in the proof of Theorem 9.12, which was a critical argument in the proof of Theorem 9.10. From this we now define the ith derived functor of a left-exact functor $F : \mathcal{A} \to \mathcal{B}$ as the following composite,

$$\mathcal{A} \xrightarrow{(H^0)^{-1}} \mathcal{C} \xrightarrow{F} K^{\geq 0}(\mathcal{B}) \xrightarrow{H^i} \mathcal{B} \ ,$$

which matches Definition 9.11 since the inverse of H^0 is taking an injective resolution. We haven't actually seen yet that these derived functors form a δ-functor, so consider the following lemma.

Lemma 10.5. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence inside an abelian category $\mathcal{A}$ with enough injectives. Then we can find injective resolutions for X, Y and Z such that the following diagram commutes and has exact rows and columns,*

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & J^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & K^0 & \longrightarrow & K^1 & \longrightarrow & K^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array} , \tag{10.6}$$

with all I^i, J^i and K^i injective objects of $\mathcal{A}$.

Sketch of a Proof. We only provide a proof sketch here, but the proof can be found in [9] as the dual of Lemma 2.2.8. First we choose an injective resolution for X, and then an injective resolution for Y and extend the map f to an injective map $I^\bullet \hookrightarrow J^\bullet$ using Theorem 9.12 and a little more work. Then we can set $K^\bullet = J^\bullet / I^\bullet$, which certainly comes with an exact sequence,

$$0 \longrightarrow I^\bullet \longrightarrow J^\bullet \longrightarrow K^\bullet \longrightarrow 0,$$

but $I^\bullet$ is injective so this splits and K^i can then be seen as a direct summand of J^i and hence is also injective. $\square$

This theorem gives us a commutative diagram similar to Diagram 10.6 after applying a left-exact functor F with not necessarily exact rows, but still with exact columns, since all the columns except the first were split exact before we applied F and F is additive. Hence we have a short exact sequence of complexes, and when we take the cohomology of a short exact sequence of complexes we obtain a long exact sequence on cohomology. Let us consider our original desired application of derived functors and homological algebra. Let $(X, \mathcal{O}_X)$ be a ringed space, then we'll see in exercise 7.2 that the category of $\mathcal{O}_X$-modules has enough injectives, hence we can consider the following diagram of functors,

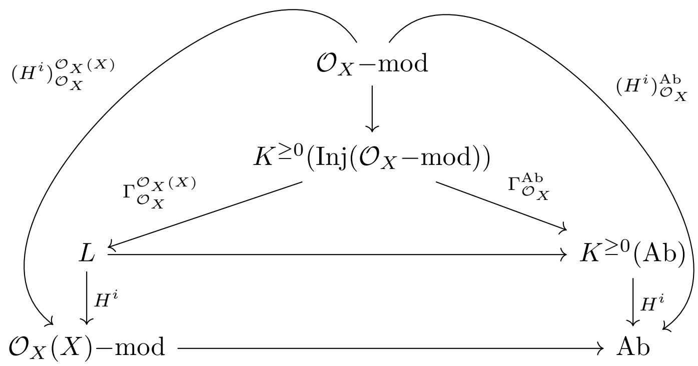
$$\begin{array}{ccc}
\mathcal{O}_X\text{-mod} & \xrightarrow{\Gamma_{\mathcal{O}_X}^{\mathcal{O}_X(X)}} & \mathcal{O}_X(X)\text{-mod} \\
\downarrow{\scriptstyle \text{forg}} & \searrow^{\Gamma_{\mathcal{O}_X}^{\text{Ab}}} & \downarrow{\scriptstyle \text{forg}} \\
\text{Ab}(X) & \xrightarrow[\Gamma_{\text{Ab}(X)}^{\text{Ab}}]{} & \text{Ab}
\end{array} .$$

We can now derive the global sections functor of an $\mathcal{O}_X$-modules $\mathcal{M}$ in three different ways, using the three different global sections functors. The following proposition says they are all the same. We will only use this clumsy notation for the a priori different types of sheaf cohomology for the rest of this lecture.

Proposition 10.7. *For any $\mathcal{O}_X$-module $\mathcal{M}$ over a ringed space $(X, \mathcal{O}_X)$ we have,*

$$(H^i)^{\mathcal{O}_X(X)}_{\mathcal{O}_X}(X, \mathcal{M}) = (H^i)^{\mathrm{Ab}}_{\mathcal{O}_X}(X, \mathcal{M}) = (H^i)^{\mathrm{Ab}}_{\mathrm{Ab}(X)}(X, \mathcal{M}).$$

Proof. First we'll prove the first equality. This is easy though by contemplating the following, obviously commutative, diagram of functors and categories,



,

where $L = K^{\geq 0}(\mathcal{O}_X(X)-\mathrm{mod})$ (for typographical reasons). For the other equality, it is enough to show $(H^i)^{\mathrm{Ab}}_{\mathrm{Ab}(X)}$ is still an effaceable δ-functor of $\mathcal{O}_X$-modules. For this is is enough to prove the following lemma, by the definition of effaceable. □

Lemma 10.8. *If $\mathcal{M}$ is an injective $\mathcal{O}_X$-module, then $H^i_{\mathrm{Ab}(X)}(X, \mathcal{M}) = 0$ for $i > 0$.*

Proof. This proof is quite classical, and is completed by first observing that if $\mathcal{M}$ is injective then $\mathcal{M}$ is flasque, and if $\mathcal{M}$ is flasque then $\mathcal{M}$ is acyclic ($H^i(X, \mathcal{M}) = 0$ for $i > 0$). This will be the content of the rest of the lecture. □

Note that there is no shortcut we can take here. It is not automatically true that an injective $\mathcal{O}_X$-module is an injective sheaf of abelian groups over X. This detour is necessary.

Definition 10.9. *A sheaf $\mathcal{F}$ on a space X is called flasque (welk auf Deutsch) if for all open subsets $U \subseteq V \subseteq X$ the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.*

Lemma 10.10. *If $\mathcal{M}$ is an injective $\mathcal{O}_X$-module, then $\mathcal{M}$ is flasque.*

Proof. We will prove in exercise 7.3 that if $j : U \hookrightarrow X$ is an inclusion of an open subset, then the functor j^* from $\mathcal{O}_X$-modules to $\mathcal{O}_U$-modules has a left adjoint $j_!$ which sends $\mathcal{M}$ to $j_!\mathcal{M}$, which in turn is the sheafification of the presheaf that sends $V \subseteq X$ to 0 if $V \not\subseteq U$ and $\mathcal{M}(V)$ if $V \subseteq U$. Notice that $(j_!\mathcal{M})_x = \mathcal{M}_x$ if $x \in U$ and zero otherwise. For this reason this sheaf is called extension by zero. The adjunction in particular implies,

$$\mathcal{M}(U) \cong \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, j^*\mathcal{M}) \cong \mathrm{Hom}_{\mathcal{O}_X}(j_{U!}\mathcal{O}_U, \mathcal{M}),$$

and likewise $\mathcal{M}(V) \cong \mathrm{Hom}_{\mathcal{O}_X}(j_{V!}\mathcal{O}_V, \mathcal{M})$, but $j_{V!}\mathcal{O}_V \hookrightarrow j_{U!}\mathcal{O}_U$ is injective by inspection. In particular, if $\mathcal{M}$ is injective, then we obtain a surjection

$$\mathcal{M}(U) = \mathrm{Hom}_{\mathcal{O}_X}(j_{U!}\mathcal{O}_U, \mathcal{M}) \twoheadrightarrow \mathrm{Hom}_{\mathcal{O}_X}(j_{V!}\mathcal{O}_V, \mathcal{M}) \cong \mathcal{M}(V).$$

□

Lemma 10.11. *If $\mathcal{F}$ is a flasque sheaf of abelian groups, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. Assume that $0 \to \mathcal{F} \to \overline{\mathcal{F}} \to \mathcal{G} \to 0$ is a short exact sequence of sheaves where $\mathcal{F}$ is flasque, then we claim $\overline{F}(X) \twoheadrightarrow \mathcal{G}(X)$ is surjective. To see this, pick some $t \in \mathcal{G}(X)$ and choose a maximal pair $(U, \overline{s})$ of an open subset $U \subseteq X$ and $\overline{s} \in \overline{F}(U)$ which maps to $t|_U$, which is given to us by Zorn's lemma, using the fact that $\overline{\mathcal{F}} \to \mathcal{G}$ is an epimorphism of stalks. Assume $U \neq X$, and let $x \in X \backslash U$, and let $\overline{s}' \in \overline{\mathcal{F}}(V)$ for some $x \in V$ with $\overline{s}' \mapsto t|_V$, which again exists since $\overline{F} \to \mathcal{G}$ is an epimorphism on stalks. Then we notice $s' = \overline{s}|_{U\cap V} - \overline{s}'|_{U \cap V} \in \ker(\overline{\mathcal{F}} \to \mathcal{G})(U \cap V)$, and this kernel is simply $\mathcal{F}(U \cap V)$ by exactness. Let $s \in \mathcal{F}(V)$ be a lift of s', which exists as $\mathcal{F}$ as flasque, then $\overline{s}' + s \mapsto t|_V$ and

$$(\overline{s}' + s)|_{U\cap V} = \overline{s}'|_{U\cap V} + s' = \overline{s}|_{U\cap V}.$$

This means that $\overline{s}$ and $\overline{s}' + s$ together glue to a section of $\overline{\mathcal{F}}(U \cup V)$ which restricts to $t|_{U\cup V}$, contradicting the maximality of $(U, \overline{s})$. Hence $U = X$. This shows our claim.

We now choose an injection $\mathcal{F} \hookrightarrow \widetilde{\mathcal{F}}$ where $\widetilde{\mathcal{F}}$ is injective and set $\mathcal{G} = \widetilde{\mathcal{F}}/\mathcal{F}$ to be the cokernel, then we have the exact sequence,

$$\widetilde{\mathcal{F}}(X) \longrightarrow \mathcal{G}(X) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \widetilde{F}) = 0,$$

which implies $H^1(X, \mathcal{F}) = 0$ and $H^i(X, \mathcal{G}) \cong H^{i+1}(X, \mathcal{F})$ for all $i \geq 1$. We can conclude the argument here by induction if we know that $\mathcal{G}$ is also flasque, and we claim this is true. Now $\mathcal{F}$ is flasque by assumption, and $\widetilde{\mathcal{F}}$ is flasque by an argument similar to Lemma 10.10. The restrictions $\mathcal{F}|_U$ and $\widetilde{\mathcal{F}}|_U$ are flasque on the space U and we then know $\widetilde{\mathcal{F}}(U) \to \mathcal{G}(U)$ is surjective by the previous claim. For $U \subseteq V \subseteq X$ we then have the following commutative diagram,

$$\begin{array}{ccc} \widetilde{\mathcal{F}}(V) & \longrightarrow & \mathcal{G}(V) \\ \big\downarrow & & \big\downarrow \\ \widetilde{\mathcal{F}}(U) & \longrightarrow & \mathcal{G}(U) \end{array},$$

which recognises $\mathcal{G}(V) \to \mathcal{G}(U)$ as being surjective. □

We could also look at the diagram,

$$\begin{array}{ccc} \mathrm{QCoh}(X) & & \\ \big\downarrow & \searrow^{\Gamma_{\mathrm{QCoh}}} & \\ \mathcal{O}_X\text{-mod} & \xrightarrow{\Gamma_{\mathcal{O}_X}} & \mathrm{Ab} \end{array},$$

but in general these two global sections functors do not have the same derived functors. This is because injective quasi-coherent sheafs are not in general flasque or even acyclic. The moral of the story is to forget about quasi-coherent sheaves when we apply these homological algebra formalism. The argument fails as $j_!$ does not preserve quasi-coherence, hence it does not exist as we used it in the proof of Lemma 10.10. Moreover, it is a non-trivial theorem (due to Gabber) that there even exists enough injective quasi-coherent $\mathcal{O}_X$-modules on a scheme.

11 Homological Algebra III (Derived Categories) 01/06/2017

Last time we focused on the derived functors of the global sections functor(s), because these give us sheaf cohomology. Today we are going to look at another left exact functor we have been using for eight months now and derive this. Doing this will lead us part of the way down two rabbit holes; one of derived categories, and another of spectral sequences.

Let $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a map of ring spaces. We have a left exact functor

$$f_* : \mathcal{O}_Y\text{−mod} \longrightarrow \mathcal{O}_X\text{−mod},$$

the direct image functor, and we know from exercise 7.2 that the domain category has enough injectives.

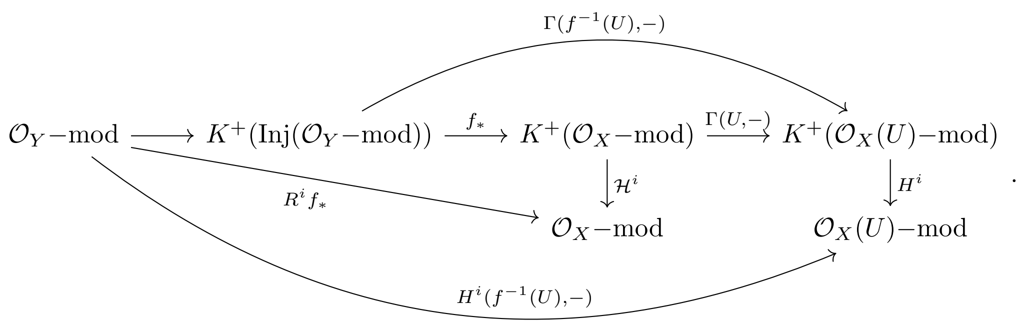
Definition 11.1. *We define the higher direct image functors, $R^i f_*$, to be the ith derived functor of f_* for $i \geq 0$.*

This is a totally acceptable formal definition, and the following lemma makes it a little more concrete.

Lemma 11.2. *For any $\mathcal{O}_Y$-module $\mathcal{N}$, then $R^i f_* \mathcal{N}$ is the sheafification of $U \mapsto H^i(f^{-1}(U), \mathcal{N}|_{f^{-1}(U)})$.*

We will prove later that in the world of algebraic geometry, in other words if f is a qcqs map of schemes, $\mathcal{N}$ is quasi-coherent, and U is affine, then $R^i f_* \mathcal{N}(U) = H^i(f^{-1}(U), \mathcal{N}|_{f^{-1}(U)})$. See Proposition 14.3.

Proof. We first choose an injective resolution $0 \to \mathcal{N} \to \mathcal{J}^0 \to \mathcal{J}^1 \to \cdots$ of $\mathcal{N}$, and note that for all open $V \subseteq Y$, the restriction is still an injective resolution. This is shown in exercise 7.3(ii). We then consider the following commutative diagram,



Above we have written $\mathcal{H}^i$ and H^i for what are formally the same functor, simply taking cohomology in an abelian category, however we like to emphasize that $\mathcal{H}^i$ is a sheaf. This proof is then finished by the following lemma. □

Lemma 11.3. *Let $A^\bullet \in K(\mathcal{O}_X\text{−mod})$. Then the ith cohomology sheaf $\mathcal{H}^i(A^\bullet) \in \mathcal{O}_X$−mod is the sheafification of the presheaf $U \mapsto H^i(A^\bullet(U)) \in \mathcal{O}_X(U)$−mod.*

The proof of this lemma is essentially just unpacking all our definitions, such as how we define the cokernel of sheaves.

Proof. Let $A^\bullet \in K(\mathcal{O}_X\text{−mod})$, so

$$A^\bullet = \cdots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \cdots,$$

each differential d^i and each A^i live inside the category of $\mathcal{O}_X$-modules. We then define the cohomology sheaf $\mathcal{H}^i(A^\bullet)$ to simply be the kernel of d^i modulo the image of d^{i-1}, which is the same as

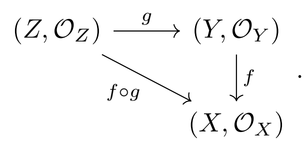
$$\mathrm{coker}(A^{i-1} \xrightarrow{d^{i-1}} \ker(d^i)).$$

We then see that $\mathcal{H}^i(A^\bullet)$ is the sheafification of

$$U \mapsto \operatorname{coker}\left(A^{i-1}(U) \overset{d^{i-1}(U)}{\longrightarrow} \ker(d^i)(U)\right) = H^i(A^\bullet(U)),$$

since the cokernel of sheaves has to be sheafified. $\square$

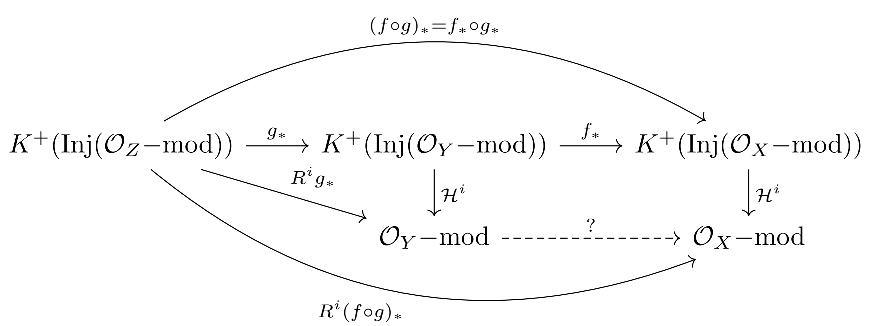
We now make a very natural consideration, which will occupy us for the rest of the lecture. Consider the following commutative diagram of ringed spaces,



Then we have $(f \circ g)_* = f_* \circ g_*$ as functor from $\mathcal{O}_Z$-modules to $\mathcal{O}_X$-modules, so we then ask the following natural question.

Does $(f \circ g)_* = f_* \circ g_*$ imply that $R^i(f \circ g)_* = R^i f_* \circ R^i g_*$?

The answer is no, but there is still a lot we can salvage from this idea. Let us assume for a second that all the structure sheaves of X, Y and Z are the constant $\mathbb{Z}$ sheaf, so that modules over these structure sheaves are simply sheaves of abelian groups. Then f_* has an exact left adjoint $f^* = f^{-1}$, and hence it preserves injectives.[19] In this case we have the following commutative diagram, but the dashed functor does not necessarily exist such that the diagram commutes,



Remark 11.4. If f_* is exact, then $R^i(f \circ g)_* = f_* \circ R^i g_*$, since the ?-functor can simply be f_*, as then f_* commutes with cohomology. In general we need to consider a different type of derived functor.

Definition 11.5. *Let $\mathcal{A}$ be an abelian category. A map $f : X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$ is a quasi-isomorphism if $H^i(f)$ is an isomorphism for all $i \in \mathbb{Z}$. The derived category*[20] *of $\mathcal{A}$ is then defined to be*

$$D(\mathcal{A}) = C(\mathcal{A})[\textit{quasi-isomorphisms}^{-1}],$$

which is $C(\mathcal{A})$ with all quasi-isomorphisms inverted (localising $C(\mathcal{A})$ at the class of quasi-isomorphisms).

Notice there is an obvious functor $C(\mathcal{A}) \to D(\mathcal{A})$ (our localisation functor) which is essentially surjective, and this factors through $K(\mathcal{A})$, which produces a unique factorising functor $K(\mathcal{A}) \to D(\mathcal{A})$.

[19]Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories, which admits an exact left adjoint G, then if I is injective in $\mathcal{A}$, we see FI is injective in $\mathcal{B}$ since,

$$\mathrm{Hom}_{\mathcal{B}}(-, FI) \cong \mathrm{Hom}_{\mathcal{A}}(G(-), I),$$

is exact.

[20]The reader is advised to read chapter 10 of [9] or Tag 05QI of [8].

Theorem 11.6. *If $\mathcal{A}$ has enough injectives, then the functor $K^+(\mathrm{Inj}(\mathcal{A})) \to D(\mathcal{A})$ is fully-faithful, with essential image $D^+(\mathcal{A})$, the full subcategory of $X \in D(\mathcal{A})$ such that $H^i(X) = 0$ for all sufficiently negative i.*

The proof of this theorem is not much harder than the proof of Theorem 10.4, and the equivalence of categories $D^+(\mathcal{A}) \to K^+(\mathrm{Inj}(\mathcal{A}))$ is similar to taking an injective resolution.

Definition 11.7. *Let $F : \mathcal{B} \to \mathcal{A}$ be a left exact functor of abelian categories where $\mathcal{B}$ has enough injectives. Then the (total) derived functor of F,*

$$RF : D^+(\mathcal{B}) \longrightarrow D^+(\mathcal{A}),$$

is defined as the following composite,

$$D^+(\mathcal{B}) \xleftarrow{\cong} K^+(\mathrm{Inj}(\mathcal{B})) \xrightarrow{F} K^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A}),$$

where the first functor is the inverse of the functor from Theorem 11.6, and the last functor is the unique functor factoring the canonical localisation functor $C^+(\mathcal{A}) \to D^+(\mathcal{A})$.

Remark 11.8. We can recover the individual derived functor R^iF as $\mathcal{B} \to D^+(\mathcal{B}) \overset{RF}{\to} D^+(\mathcal{A}) \overset{H^i}{\to} \mathcal{A}$. Also notice that if F happens to preserve injectives, then we can avoid all these derived notions, since the following diagram commutes,

$$\begin{array}{ccc} K^+(\mathrm{Inj}(\mathcal{B})) & \xrightarrow{F} & K^+(\mathrm{Inj}(\mathcal{A})) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ D^+(\mathcal{B}) & \xrightarrow{RF} & D^+(\mathcal{A}) \end{array} .$$

In this way, RF is a generalisation to when F does not preserve injectives.

Now we can again ask ourselves the following question.

Given two left exact functors $\mathcal{C} \overset{G}{\to} \mathcal{B} \overset{F}{\to} \mathcal{A}$, does the following diagram commute?

$$\begin{array}{ccc} D^+(\mathcal{C}) & \xrightarrow{RG} & D^+(\mathcal{B}) \\ & {\scriptstyle R(F\circ G)}\searrow & \downarrow{\scriptstyle RF} \\ & & D^+(\mathcal{A}) \end{array} \tag{11.9}$$

In almost all practical cases this diagram will commute, but it will not in general.

Proposition 11.10. *Assume G preserves injectives, then $R(F \circ G) = RF \circ RG$, i.e. Diagram 11.9 commutes.*

Proof. We simply observe that the following diagram commutes, since G preserves injectives.

$$\begin{array}{ccccc} & & \overset{F\circ G}{\frown} & & \\ K^+(\mathrm{Inj}(\mathcal{C})) & \xrightarrow{G} & K^+(\mathrm{Inj}(\mathcal{B})) & \xrightarrow{F} & K^+(\mathcal{A}) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow \\ D^+(\mathcal{C}) & \xrightarrow{RG} & D^+(\mathcal{B}) & \xrightarrow{RF} & D^+(\mathcal{A}) \\ & & \underset{R(F\circ G)}{\smile} & & \end{array} .$$

□

We can generalise this slightly, as Grothendieck does in his famous Tôhoku paper, [4].

Proposition 11.11. *Assume G maps injectives to F-acyclic objects, i.e. for all injective $Z \in \mathcal{C}$ we have $(R^iF)(G(Z)) = 0$ for all $i > 0$. Then $R(F \circ G) = RF \circ RG$.*

Notice that this condition is necessary. Let $Z \in \mathcal{C}$ be injective, then $R^i(F \circ G)(Z) = 0$ and $R^iG(Z) = 0$ for all $i > 0$ as both are derived functors and Z is injective, then $RG(Z) = G(Z)[0]$. We are using the notation $X[i]$ for $X \in \mathcal{A}$ an object in an abelian category to represent the complex of $C(\mathcal{A})$ with a single X concentrated in degree $-i$. This implies that we have $R^iF(G(Z)) = 0$ for all $i > 0$.

Sketch of the Proof. Let $\mathcal{B}' \subseteq \mathcal{B}$ be the full subcategory of F-acyclic objects, then the following diagram commutes,

$$\begin{array}{ccc} K^+(\mathcal{B}') & \xrightarrow{F} & K^+(\mathcal{A}) \\ \downarrow & & \downarrow \\ D^+(\mathcal{B}) & \xrightarrow{RF} & D^+(\mathcal{A}) \end{array},$$

since derived functors can be computed not just by injective resolutions, but by acyclic resolutions. This is clear for complexes in $\mathcal{B}'$ concentrated in one degree by the definition of acyclicity, and in general we induct using some big exact sequence. This gives us the following commutative diagram, which would finish our proof,

$$\begin{array}{ccccc} & & \overset{F \circ G}{\frown} & & \\ K^+(\mathrm{Inj}(\mathcal{C})) & \xrightarrow{G} & K^+(\mathcal{B}') & \xrightarrow{F} & K^+(\mathcal{A}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \\ D^+(\mathcal{C}) & \xrightarrow{RG} & D^+(\mathcal{B}) & \xrightarrow{RF} & D^+(\mathcal{A}) \\ & & \underset{R(F \circ G)}{\smile} & & \end{array}.$$

□

Let us take all of this abstractness back to the world of ringed spaces and derived direct images.

Corollary 11.12. *Let $g : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ and $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be maps of ringed spaces, then $R(f \circ g)_* = Rf_* \circ Rg_*$ as functors from $D^+(\mathcal{O}_Z\text{−mod})$ to $D^+(\mathcal{O}_X\text{−mod})$.*

Proof. We need to see that if $\mathcal{M}$ is an injective $\mathcal{O}_Z$-module, then $g_*\mathcal{M}$ is f_*-acyclic. From Lemma 10.10 we know $\mathcal{M}$ is flasque and we can quickly check this implies $g_*\mathcal{M}$ is flasque. Obviously then $g_*\mathcal{M}|_V$ is flasque for all open $V \subseteq Y$, hence $g_*\mathcal{M}|_V$ is $\Gamma(V, -)$-acyclic, so for all open $V \subseteq Y$ we have $H^i(V, g_*\mathcal{M}) = 0$ for all $i > 0$. Using Lemma 11.2 we see that $R^if_*(g_*\mathcal{M}) = 0$. □

In some sense this does not answer our initial question about $R^i(f \circ g)_*$, R^if_* and R^ig_*. Let us consider a few cases and try to come to some conclusions about these functors, before stating the general result.

Case 0: If f_* is exact, then we have seen in Remark 11.4 that $R^i(f \circ g)_* = f_* \circ R^ig_*$.

Case 1: Let $\mathcal{M}$ be an $\mathcal{O}_Z$-module which is g_*-acyclic, i.e. $R^ig_*\mathcal{M} = 0$ for all $i > 0$. Then we claim $R^i(f \circ g)_*\mathcal{M} = R^if_*(g_*\mathcal{M})$, and to see this, we notice that $Rg_*\mathcal{M} = g_*\mathcal{M}[0]$, and thus

$$R^i(f \circ g)_*\mathcal{M} = \mathcal{H}^i(R(f \circ g)_*\mathcal{M}) = \mathcal{H}^i(Rf_* \circ Rg_*\mathcal{M}) = \mathcal{H}^i(Rf_*(g_*\mathcal{M}[0])) = R^if_*(g_*\mathcal{M}).$$

Case 2: For our final test case, let $\mathcal{M}$ be an $\mathcal{O}_Z$-module with $R^i g_* \mathcal{M} = 0$ for all $i > 1$. We claim there is a long exact sequence,

$$\cdots \longrightarrow R^i f_*(g_*\mathcal{M}) \longrightarrow R^i(f \circ g)_*\mathcal{M} \longrightarrow R^{i-1} f_*(R^1 g_*\mathcal{M}) \xrightarrow{\delta} R^{i+1} f_*(g_*\mathcal{M}) \longrightarrow \cdots . \tag{11.13}$$

To try to figure this out, let us think about $Rg_*\mathcal{M}$. It is represented by a complex $\mathcal{N}^0 \xrightarrow{d} \mathcal{N}^1$ concentrated in degrees 0 and 1, with $\ker(d) = g_*\mathcal{M}$ and $\operatorname{coker}(d) = R^1 g_*\mathcal{M}$. This gives us the following short exact sequence of complexes,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & g_*\mathcal{M} & \longrightarrow & \mathcal{N}^0 & \longrightarrow & \mathcal{N}^0/g_*\mathcal{M} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{N}^1 & \longrightarrow & \mathcal{N}^1 & \longrightarrow & 0 \end{array} .$$

We notice that the complex $\mathcal{N}^0/g_*\mathcal{M} \to \mathcal{N}^1$ is quasi-isomorphic via the obvious map to $R^1 g_*\mathcal{M}[-1]$. This "remains a short exact sequence" (which means a distinguished triangle in triangulated categorical language) after applying Rf_*, which gives us a long exact sequence on cohomology $\mathcal{H}^i$, so we have

$$\cdots \longrightarrow \mathcal{H}^i(Rf_*(g_*\mathcal{M}[0])) \longrightarrow \mathcal{H}^i(Rf_*(\mathcal{N}^\bullet)) \longrightarrow \mathcal{H}^i(Rf_*(R^1 g_*\mathcal{M}[-1])) \xrightarrow{\delta} \mathcal{H}^{i+1}(Rf_*(g_*\mathcal{M}[0])) \longrightarrow \cdots .$$

Once we make the identifications,

$$\mathcal{H}^i(Rf_*(g_*\mathcal{M}[0])) \cong R^i f_*(g_*\mathcal{M}), \qquad \mathcal{H}^i(Rf_*(R^1 g_*\mathcal{M}[-1])) \cong R^{i-1} f_*(R^1 g_*\mathcal{M}),$$
$$\text{and } \mathcal{H}^i(Rf_*(\mathcal{N}^\bullet)) \cong \mathcal{H}^i(Rf_*(Rg_*\mathcal{M})) \cong R^i(f \circ g)_*\mathcal{M},$$

we see the long exact sequence above directly translates to Sequence 11.13.

In general though, $Rg_*\mathcal{M}$ has a filtration with graded pieces $R^i g_*\mathcal{M}[-i]$, which is a complex whose only nonzero cohomology sheaf is $\mathcal{H}^i$ and is exactly $R^i g_*\mathcal{M}$. Filtrations are like "many short exact sequences" which induce "many long exact sequences" on cohomology, which gives us a spectral sequence[21]. Our spectral sequence has an E_2-page indexed by two non-negative integers p and q, which looks like $E_2^{p,q} = R^p f_*(R^q g_*\mathcal{M})$. The E_2-page of our spectral sequence has maps $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ which are called differentials, since any composition of two of these maps is zero. It turns out these d_2-differentials are just generalisations of δ from Sequence 11.13. Since we have differentials, we can take cohomology, and the E_3-page is defined exactly as that,

$$E_3^{p,q} = \ker(d_2 : E_2^{p,q} \to E_2^{p+2,q-1}) / \operatorname{coker}(d_2 : E_2^{p-2,q+1} \to E_2^{p,q}).$$

Again we have differentials, called $d_3 : E_3^{p,q} \to E_3^{p+3,q-2}$, and this process continues to the E_4-page. In general we have an E_r-page of our spectral sequence defined in the obvious way. This is a first quadrant spectral sequence, so we can see that since these differentials grow larger as r increases, eventually an element $E_r^{p,q}$ where $r > p, q$ cannot be hit or receive nonzero differentials, and hence $E_r^{p,q} \cong E_k^{p,q}$ for all $k \geq r$ in this case. In this situation we define $E_\infty^{p,q} = E_r^{p,q}$, where this position has stabilised. All of this information come to the following theorem, which is true also in the generality of Proposition 11.11.

Theorem 11.14. *The sheaf $R^i(f \circ g)_*\mathcal{M}$ has a decreasing filtration,*

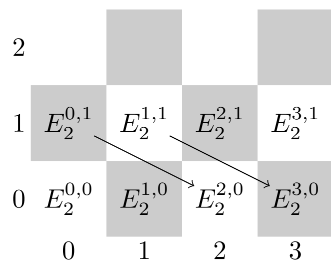
$$F^p R^i(f \circ g)_*\mathcal{M} \subseteq R^i(f \circ g)_*\mathcal{M},$$

with $F^{-1} = R^i(f \circ g)_*\mathcal{M}$ and $F^i = 0$ such that the associated graded $\operatorname{gr}^p R^i(f \circ g)_*\mathcal{M} = E_\infty^{p,i-p}$ in the spectral sequence defined above. In the usual language of spectral sequences we may write,

$$E_2^{p,q} = R^p f_*(R^q g_*\mathcal{M}) \Longrightarrow R^{p+q}(f \circ g)_*\mathcal{M}.$$

[21] See chapter 5 of [9] for more on spectral sequences.

Let us consider this spectral sequence for case 2 above, and see that we come to the same conclusion. We have $R^q g_* \mathcal{M} = 0$ for all $q > 1$, so the spectral sequence looks as follows, where the horizontal axis is the p-axis and the vertical that q-axis,



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We will have $E_2^{p,q} = 0$ for all $q > 1$, so the only possible differentials amoung the whole spectral sequence are the d_2-differentials indicated. Our spectral sequence stabilises on the E_3-page for degree reasons and we then get

$$E_\infty^{p,q} = E_3^{p,q} = \begin{cases} \ker(d_2) & q = 1 \\ \operatorname{coker}(d_2) & q = 0 \\ 0 & \text{otherwise} \end{cases} .$$

If we unpack every, such as the identification of the E_∞-page with the associated graded of $R^i(f \circ g)_* \mathcal{M}$ we obtain the following short exact sequence,

$$0 \longrightarrow \operatorname{coker}(f_*(R^1 g_* \mathcal{M}) \to R^2 f_*(g_* \mathcal{M})) \to R^2(f \circ g)_* \mathcal{M} \longrightarrow \ker(R^1 f_*(R^1 g_* \mathcal{M}) \to R^3 f_*(g_* \mathcal{M})) \longrightarrow 0,$$

and a collection of other, similar short exact sequences, which exend to the long exact sequence of Sequence 11.13 of Case 2.

12 Čech Cohomology 12/06/2017

Let us start today with a theorem.

Theorem 12.1. *Let $X = \operatorname{Spec} A$ be an affine scheme and $\mathcal{M} = \widetilde{M}$ be a quasi-coherent sheaf, then $H^i(X, \mathcal{M}) = M$ for $i = 0$ and zero for all higher i.*

We already know that $H^0(X, \mathcal{M}) = M$ since these are simply the global sections, and $H^1(X, \mathcal{M}) = 0$ since we have alternative definitions of this in terms of $\mathcal{M}$-torsors and certain isomorphism classes of extensions. We want to embed $\mathcal{M}$ into an injective sheaf in order to compute the long exact sequence and prove Theorem 12.1. To prove general results, we need to be able to compute $H^i(X, \mathcal{F})$, and to do this we are going to use Čech cohomology.

Let X be a topological space, and let $\mathcal{U} = \{U_i \subseteq X, i \in I\}$ be a certain collection of open subsets of X, such that they form a cover of X. Assume that I is totally ordered, where usually $I = \{1, 2, \dots, n\}$. Let $\mathcal{F}$ be an abelian sheaf on X.

Definition 12.2. *The Čech complex of $\mathcal{F}$ with respect to $\mathcal{U}$ is then defined as,*

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}) = \left(\prod_{i \in I} \mathcal{F}(U_i) \longrightarrow \prod_{i_1 < i_2} \mathcal{F}(U_{i_1} \cap U_{i_2}) \longrightarrow \cdots \right),$$

where the differentials are simply the alternating sum of restriction maps, so this first differential is a product of $\operatorname{res}^{U_{i_1}}_{U_{i_1} \cap U_{i_2}} - \operatorname{res}^{U_{i_2}}_{U_{i_1} \cap U_{i_2}}$.

Notice that by the sheaf condition, the kernel of the first differential is $\mathcal{F}(X)$, since this is exactly all the local sections that glue to global sections because they agree on restrictions.

Definition 12.3. *We then define the Čech cohomology of $\mathcal{F}$ with respect to $\mathcal{U}$ to be the cohomology of the above Čech complex,*

$$\check{H}^q(\mathcal{U}, \mathcal{F}) = H^q(\check{C}^\bullet(\mathcal{U}, \mathcal{F})).$$

The idea now to to make this definition independent of our chosen cover $\mathcal{U}$, so we would look at the colimit $\operatorname{colim}_{\mathcal{U}} \check{H}^q(\mathcal{U}, \mathcal{F})$ over all collections of covers $\mathcal{U}$, and sometimes, but not all the time, this turns out to be $H^q(X, \mathcal{F})$.

Lemma 12.4. *Let X be a topological space with some cover $\mathcal{U}$, and $\mathcal{F}$ be an abelian sheaf on X.*

1. *If $U_i = X$ for some $i \in I$, then $\check{H}^q(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$ for $q = 0$, and zero otherwise.*
2. *For general $\mathcal{U}$ we have $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.*
3. *If $\mathcal{F}$ is injective then $\check{H}^q(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$ if $q = 0$, and zero otherwise, so $\check{H}^q(\mathcal{U}, \mathcal{F}) = H^q(X, \mathcal{F})$.*

Proof. For part 2 we simply appeal to the sheaf condition as previously remarked. For the first and third parts, we start by letting $f_{i_1, \dots, i_k} : U_{i_1} \cap \cdots \cap U_{i_k} \hookrightarrow X$ be the open immersion, and we claim there is a natural long exact sequence of abelian sheaves on X,

$$\cdots \longrightarrow \bigoplus_{i_1 < i_2} (f_{i_1, \dots, i_k})_! \mathbb{Z} \longrightarrow \bigoplus_{i \in I} (f_i)_! \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0, \tag{12.5}$$

where the first map is the sum of differences of natural maps $(f_{i_1, i_2})_! \mathbb{Z} \to (f_{i_1})_! \mathbb{Z}$ and $(f_{i_1, i_2})_! \mathbb{Z} \to (f_{i_2})_1 \mathbb{Z}$, and $(f_i)_!$ is adjoint to $\mathbb{Z} \to f_i^* \mathbb{Z} = \mathbb{Z}$. To check this claim, we need to see exactness on stalks, but this is easy since $((f_i)_! \mathbb{Z})_x = \mathbb{Z}$ if $x \in U_i$ and 0 otherwise. For each x there is some i with $x \in U_i$, so we can replace X by U_i, and U_j by $U_i \cap U_j$, so without loss of generality we have $U_i = X$ for some

$i \in I$. This reduces us to the following claim; if $U_i = X$ for some $i \in I$, then the identity is homotopic to zero on the complex 12.5, i.e. there is the following diagram,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \bigoplus_{i_1<i_2}(f_{i_1,i_2})_!\mathbb{Z} & \longrightarrow & \bigoplus_i (f_i)_!\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & \swarrow_{h^{-2}} & \downarrow \mathrm{id} & \swarrow_{h^{-1}} & \downarrow \mathrm{id} & \swarrow_{h^{0}} & \downarrow \mathrm{id} & & \\ \cdots & \longrightarrow & \bigoplus_{i_1<i_2}(f_{i_1,i_2})_!\mathbb{Z} & \longrightarrow & \bigoplus_i (f_i)_!\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array},$$

such that $\mathrm{id} = dh^{i-1} + h^i d$. To see this, we take $h^0 : \mathbb{Z} = (f_{i_0})_!\mathbb{Z} \to \bigoplus_i (f_i)_!\mathbb{Z}$ to be the canonical inclusion. For higher homotopies, like h^{-1}, we also define it to be the canonical inclusion,

$$\bigoplus_i (f_i)_!\mathbb{Z} = \bigoplus_i (f_{i,i_0})_!\mathbb{Z} \to \bigoplus_{i_1<i_2} (f_{i_1,i_2})_!\mathbb{Z},$$

with a plus sign if $i < i_0$ and a minus sign if $i_0 < i$. This gives us both of our claims. To show part 1 and part 3 we consider the complex $C^\bullet$ of 12.5, and then the complex of homomorphisms of abelian sheaves on X from $C^\bullet$ into $\mathcal{F}$,

$$\mathrm{Hom}(C^\bullet, \mathcal{F}) = 0 \longrightarrow \mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i_1<i_2} \mathcal{F}(U_{i_1} \cap U_{i_2}) \longrightarrow \cdots, \tag{12.6}$$

using the identification,

$$\mathrm{Hom}\left(\bigoplus (f_i)_!\mathbb{Z}, \mathcal{F}\right) \cong \prod_i \mathrm{Hom}((f_i)_!\mathbb{Z}, \mathcal{F}) \cong \prod_i \mathrm{Hom}(\mathbb{Z}, f_i^*\mathcal{F}) \cong \prod_i \mathcal{F}(U_i).$$

Notice that the complex to the right of $\mathcal{F}(X)$ in 12.6 is precisely the Čech complex of $\mathcal{F}$ with respect to $\mathcal{U}$. For part 1 we notice that the complex 12.5 is zero in the homotopy category of abelian sheaves on X, and we see that the complex 12.6 is then exact. For part 3, we also use that the functor $\mathrm{Hom}(-, \mathcal{F})$ is exact to obtain our desired result. □

What can we do for a general $\mathcal{F}$? We need to take a quick detour now through some more homological algebra which we do not have time to cover to full detail.

We can choose an injective resolution $0 \to \mathcal{F} \to I^0 \xrightarrow{d^0} I^1 \to \cdots$. If $\mathcal{U}$ is an open cover of X where I is totally ordered, then we have the following double complex,

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_i \mathcal{F}(U_i) & \longrightarrow & \prod_i I^0(U_i) & \longrightarrow & \prod_i I^1(U_i) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_{i_1<i_2} \mathcal{F}(U_{i_1} \cap U_{i_2}) & \longrightarrow & \prod_{i_1<i_2} I^0(U_{i_1} \cap U_{i_2}) & \longrightarrow & \prod_{i_1<i_2} I^1(U_{i_1} \cap U_{i_2}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}. \tag{12.7}$$

The first row gives us the sheaf cohomology of $\prod_i H^*(U_i, \mathcal{F})$, since restricting injective resolutions gives us an injective resolution, and similarly the second row gives us the product of the sheaf cohomology $\prod_{i_1<i_2} H^*(U_{i_1} \cap U_{i_2}, \mathcal{F})$. The first column gives us the Čech cohomology of $\mathcal{U}$, so $\check{H}^*(U, \mathcal{F})$, the second and third columns give us the Čech cohomology of I^0 and I^1 respectively, which we can calculate from Lemma 12.4. A double complex is just the obvious thing.

Definition 12.8. *A double complex $X^{\bullet,\bullet}$ (in an abelian category $\mathcal{A}$) is a commutative diagram,*

$$X^{\bullet,\bullet} = \begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & X^{p,q+1} & \xrightarrow{d^h} & X^{p+1,q+1} & \longrightarrow & \cdots \\ & & {\scriptstyle d^v}\uparrow & & {\scriptstyle d^v}\uparrow & & \\ \cdots & \longrightarrow & X^{p,q} & \xrightarrow{d^h} & X^{p+1,q} & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \end{array},$$

such that the composition of two vertical, or two horizontal maps is zero, and each small square is commutative up to a sign, i.e. $d^v \circ d^h = -d^h \circ d^v$.

For us we will have $X^{p,q} = 0$ unless $p, q \geq 0$. We'll call this a first quadrant double complex.

Definition 12.9. *The total complex of a (first quadrant) double complex $X^{\bullet,\bullet}$ is defined as*

$$(\mathrm{Tot}X^{\bullet,\bullet})^i = \bigoplus_{p+q=i} X^{p,q} = \prod_{p+q=i} X^{p,q},$$

where the differentials are alternating sums of the differentials coming from $X^{\bullet,\bullet}$ as a double complex.

Theorem 12.10. *The total complex $\mathrm{Tot}X^{\bullet,\bullet}$ of a double complex $X^{\bullet,\bullet}$ has a filtration, which in turn gives us two spectral sequences,*

$$E_1^{p,q} = H^q(X^{\bullet,p}) \Longrightarrow H^{p+q}(\mathrm{Tot}X^{\bullet,\bullet}), \qquad (E')_1^{p,q} = H^p(X^{q,\bullet}) \Longrightarrow H^{p+q}(\mathrm{Tot}X^{\bullet,\bullet}).$$

These are essentially built by first taking cohomology in the horizontal direction or the vertical direction (see chapter 5.6 of [9] for more on the spectral sequences of a double complex). Notice that both of these spectral sequences converge to the cohomology of the total complex. If we apply these spectral sequences to the double complex 12.7 without the column expressing the Čech complex of $\mathcal{F}$, we need to first take cohomology in the vertical direction, in which case we obtain the global sections of $I^i(X)$ and nothing in higher degrees. Due to lack of higher differentials and extension problems, this implies the cohomology of the total complex is

$$H^q(0 \to I^0(X) \to I^1(X) \to \cdots) = H^q(X, \mathcal{F}).$$

Corollary 12.11. *We have a spectral sequence,*

$$E_1^{p,q} = \prod_{J \subseteq I, |J|=p+1} H^q\left(\bigcap_{j\in J} U_j, \mathcal{F}\right) \Longrightarrow H^{p+q}(X, \mathcal{F}).$$

Proof. Take the spectral sequence where we look at the cohomology in the horizontal direction first, and compare this to the fact discussed above that $H^q(\mathrm{Tot}X^{\bullet,\bullet}) \cong H^q(X, \mathcal{F})$. □

Note that the $q = 0$ line of this page is exactly the Čech complex $\check{C}(\mathcal{U}, \mathcal{F})$. This leads us to the following corollary, which is essential to any explicit sheaf cohomology calculations we do in this course.

Corollary 12.12. *Assume that for all $0 \neq J \subseteq I$ such that J is finite we have $H^q\left(\bigcap_{j\in J} U_j, \mathcal{F}\right) = 0$ for all $q > 0$, then $H^q(X, \mathcal{F}) \cong \check{H}^q(\mathcal{U}, \mathcal{F})$.*

After the proof of Theorem 12.1 we will see that if we have all $\bigcap_i U_i$ are affine and $\mathcal{F}$ is quasi-coherent, then we can use Čech cohomology, e.g. for projective space $\mathbb{P}^n$ with the standard covering.

Proof. The spectral sequence of Corollary 12.11 degenerates immediately after the E_1-page, hence $E_2 = E_\infty$ and we're done. $\square$

Proof of Theorem 12.1. Let $X = \operatorname{Spec} A$ and $\mathcal{M} = \widetilde{M}$ be a quasi-coherent sheaf on X. For induction, assume that for all affine schemes X and all quasi-coherent $\mathcal{M}$ over X we have $H^i(X, \mathcal{M}) = 0$ for all $i = 1, \ldots, q-1$. We want to show that in our case, $H^q(X, \mathcal{M}) = 0$. Choose some $\alpha \in H^q(X, \mathcal{M}) \neq 0$, then we claim there is a cover $\mathcal{U} = \{U_i\}$ of X such that $0 = \alpha|_{U_i}$ for all U_i inside $H^q(U_i, \mathcal{M})$. To see this, take $0 \to \mathcal{M} \to I^\bullet$ to be an injective resolution of $\mathcal{M}$, then α comes from an element $\widetilde{\alpha} \in \ker(d^q : I^q(X) \to I^{q+1}(X))$, which is exact as a complex of sheaves, so $\widetilde{\alpha}$ is locally the image of d^{q-1}, which proves our claim. Without loss of generality, we take $U_i = D(f_i)$ for $i = 1, \ldots, n$. We then look at the spectral sequence of Corollary 12.11, which gives us

$$E_1^{p,q'} = \bigoplus_{J \subseteq \{1,\ldots,n\}, |J| = p+1} H^{q'}\left(D\left(\prod_{j \in J} f_j\right), \mathcal{M}\right) \Longrightarrow H^{p+q'}(X, \mathcal{M}).$$

We easily see the $q' = 0$ row is again the Čech complex $\check{C}(\mathcal{U}, \mathcal{M})$, and between the $q' = 1$ and $q' = q-1$ rows of the E_1-page of this spectral sequence, we have only zeros by induction. The E_2-page then has $\check{H}^*(\mathcal{U}, \mathcal{M})$ on the $q = 0$ row. By inspection, we have a $\prod_i H^q(U_i, \mathcal{M})$ in the E_2^{0q}-position. Upon considering the convergence of this spectral sequence, and the fact that these E_2^{0q} and E_2^{q0} terms will survive to the E_∞-page for degree reasons, we obtain the extension problem,

$$0 \longrightarrow \check{H}^q(X, \mathcal{M}) \longrightarrow H^q(X, \mathcal{M}) \longrightarrow \prod_i H^q(U_i, \mathcal{M}) \longrightarrow 0.$$

As $\alpha \in \ker(H^q(X, \mathcal{M}) \to \prod_i H^q(U_i, \mathcal{M}))$ by the construction of the cover U_i, then α has to come from $\check{H}^q(X, \mathcal{M})$. This theorem then follows quickly from the next lemma. $\square$

Lemma 12.13. *Let $X = \operatorname{Spec} A$, $U_i = D(f_i)$ be a cover of X by open affines, and $\mathcal{M}$ be a quasi-coherent sheaf on X. Then $\check{H}^q(\mathcal{U}, \mathcal{M}) = 0$ for $q > 0$.*

Proof. Since $\mathcal{M} = \widetilde{M}$, then we only need to prove the exactness of,

$$0 \to M \to \bigoplus_i M[f_i^{-1}] \to \bigoplus_{i_1 < i_2} M[(f_{i_1} f_{i_2})^{-1}] \to \cdots,$$

the Čech complex for $\mathcal{M}$. As $A \to \prod_i A[f_i^{-1}]$ is faithfully flat for this finite product, then it is enough to check everything after tensoring with $- \otimes_A A[f_i^{-1}]$ for all i by descent. In other words, we may assume $U_i = X$ for some $i \in I$, but then we always have $\check{H}^q(\mathcal{U}, \mathcal{M}) = 0$ for $q > 0$ by Lemma 12.4 part 1. $\square$

As a final corollary, let us see this all in an algebro-geometric light.

Corollary 12.14. *Let X be a separated scheme, $\mathcal{U}$ be an affine open cover of X, and $\mathcal{M}$ a quasi-coherent sheaf. Then $H^q(X, \mathcal{M}) = \check{H}^q(\mathcal{U}, \mathcal{M})$ for all $q \geq 0$.*

Proof. Since X is separated, then $U, V \subseteq X$ being affine implies that $U \cap V$ is affine. $\square$

13 Finiteness in Cohomology I 19/06/2017

Today we will explore finiteness results in sheaf cohomology, which consists of two main results, and the two big proofs the accompany them. Last semester (for example in the formulation of the Riemann-Roch Theorem) we state that if a scheme X over a field k is projective, and $\mathcal{F}$ is a coherent sheaf over X, then $\dim \Gamma(X, \mathcal{F})$ is finite. Today we will prove a generalisation of this theorem which requires us to use sheaf cohomology.

Theorem 13.1. *Let A be a noetherian ring, and X a projective scheme over A, i.e. there is a closed immersion $i : X \hookrightarrow \mathbb{P}^n_A$ for some $n \geq 1$. Let $\mathcal{F}$ be a coherent sheaf over X, then for all $i \geq 0$ the cohomology group $H^i(X, \mathcal{F})$ is finitely generated as an A-module.*

The proof will proceed by descending induction on i, so as a first step we need vanishing of the sheaf cohomology for large degrees of i. There are two ways we could do this.

We could use Čech cohomology, take $\mathbb{P}^n_A = \bigcup_{i=0}^n U_i$ to be the standard open affine cover, then $X = \bigcup_{i=0}^n (U_i \cap X)$, where $U_i \cap X = V_i$ are all open affines of X. As X is separated we can use the Čech complex to compute cohomolgy, and this looks explicitly like,

$$\check{C}(\{V_i\}, \mathcal{F}) = \quad 0 \to \bigoplus_{i=0}^n \mathcal{F}(V_i) \to \bigoplus_{i<j} \mathcal{F}(V_i \cap V_j) \to \cdots \to \mathcal{F}(V_0 \cap \cdots \cap V_n) \to 0,$$

and it follows that $H^i(X, \mathcal{F}) \cong \check{H}^i(\{V_i\}, \mathcal{F}) = 0$ for all $i > n$. There is another method though, originally due to Grothendieck in the case that X is a noetherian space, and later generalised to all spectral spaces in 1994 by Scheiderer. We will only state and prove the noetherian case.

Theorem 13.2. *If X is a noetherian spectral space, and $\mathcal{F}$ an abelian sheaf on X, then $H^i(X, \mathcal{F}) = 0$ if $i > \dim X$.*

Proof. The proof proceeds with many reduction steps. First, recall exercise 8.4(iii) which states that if $\mathcal{F}_i$ is a direct filtered system of abelian sheaves on a spectral space X, then for any $i \geq 0$ we have,

$$\operatorname{colim}_i H^i(X, \mathcal{F}_i) \cong H^i(X, \operatorname{colim}_i \mathcal{F}_i).$$

We will now run with an induction argument on $\dim X = n$. First we notice that we can assume X is irreducible. If not, then the fact X is noetherian implies that it has finitely many irreducible components, $Z_1, \ldots, Z_m \subseteq X$. By induction on m we may assume the result holds for $Z_1 = Z$, and let $X' = Z_2 \cup \cdots \cup Z_m$. Let $U = X \backslash Z \subseteq X'$, with $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ the canonical immersions. If $\mathcal{F}$ is an abelian sheaf, then we have a short exact sequence,

$$0 \longrightarrow j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow 0.$$

On cohomology we obtain the exact sequence,

$$H^i(X, j_! j * \mathcal{F}) \longrightarrow H^i(X, \mathcal{F}) \longrightarrow H^i(X, i_* i^* \mathcal{F}).$$

In exercise 9.1 we will show that if $f : Y \to X$ as a closed immersion of schemes and $\mathcal{F}$ is an abelian sheaf on Y, then the canonical map

$$H^n(X, f_* \mathcal{F}) \longrightarrow H^n(Y, \mathcal{F}),$$

is an isomorphism for all $n \geq 0$. This means $H^k(X, i_* i^* \mathcal{F}) \cong H^k(Z, i^* \mathcal{F}) = 0$ for $k > \dim Z \leq \dim X$. If we now let $i' : X' \hookrightarrow X$ be the closed immersion, then as $U \subseteq X'$ we have $j_! j^* \mathcal{F} = i'_*(i')^*(j_! j^* \mathcal{F})$, since $i'_*(i')^*$ preserves stalks on U and is 0 elsewhere, the same effect as $j_!$. So we obtain an exact sequence,

$$0 \longrightarrow i'_*(i')^* j_! j^* \mathcal{F} \longrightarrow i'_*(i')^* \mathcal{F} \longrightarrow i'_*(i')^* i_* i^* \mathcal{F} \longrightarrow 0,$$

where we can identify the last term as $i'_*(i')^* i_* i^* \mathcal{F} = i_{(Z\cap X')*} i^*_{(Z\cap X')} \mathcal{F}$. On cohomology we then have the exact sequence,

$$H^{q-1}(X, i_{(Z\cap X')*} i^*_{Z\cap X'} \mathcal{F}) \longrightarrow H^q(X, j_! j^* \mathcal{F}) \longrightarrow H^q(X, i'_*(i')^* \mathcal{F}) \cong H^q(X', (i')^* \mathcal{F}),$$

where we have made the now obvious identification of the first term. The first term is zero if $q - 1 > \dim(Z \cap X')$ which itself is strictly less than the dimension of X, and the last term is zero if $q > \dim X' \leq \dim X$ by induction on m. Hence $H^q(X, j_! j^* \mathcal{F}) = 0$ for $q > \dim X$ and thus $H^q(X, \mathcal{F}) = 0$ for $q > \dim X$. Hence we can assume X is irreducible. If $\dim X = 0$, then X is a point, since X is irreducible, and we are done here too. We may also assume that $\mathcal{F}$ is generated by one single section. To see this, let $s \in \mathcal{F}(U)$ and let $j_U : U \hookrightarrow X$ be the open immersion, then we obtain $\mathbb{Z} \to j_U^* \mathcal{F}$, where $\mathbb{Z}$ is the constant sheaf, which is equivalent to a map $j_{U!}\mathbb{Z} \to \mathcal{F}$. The map

$$\bigoplus_B j_{U!}\mathbb{Z} \longrightarrow \mathcal{F},$$

is surjective, where B is the set of all open subsets $U \subseteq X$ and all $s \in \mathcal{F}(U)$. For any finite subset $S \subseteq B$ of sections, let $\mathcal{F}_S$ be the image of the above map when restricted to only those direct summands where $s \in S$. Then $\mathcal{F}_S$, and $S \subseteq B$ form a filtered direct system, with $\mathcal{F} \cong \operatorname{colim}_S \mathcal{F}_S$, thus by the commutivity of cohomology with filtered colimits, we are reduced to the case where $\mathcal{F} = \mathcal{F}_S$. We can do another induction now on the number of elements of S. If $s \in S$, then we have a short exact sequence,

$$0 \longrightarrow \mathcal{F}_{S\setminus\{s\}} \longrightarrow \mathcal{F}_S \longrightarrow \mathcal{F}' \longrightarrow 0,$$

where $j_{U!}\mathbb{Z} \to \mathcal{F}'$ is a surjection, for some open immersion $j_U : U \hookrightarrow X$ since $\mathcal{F}'$ is generated by one element. On cohomology we then obtain,

$$H^q(X, \mathcal{F}_{S\setminus\{s\}}) \longrightarrow H^q(X, \mathcal{F}_S) \longrightarrow H^q(X, \mathcal{F}').$$

We know when the first term dies by induction, and the last term brings us down to our reduction that $\mathcal{F}$ can be assumed to be generated by one element. We now want to look at the exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow j_!\mathbb{Z} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $j : U \hookrightarrow X$ is some open immersion. For each $x \in X$ we see that $\mathcal{G}_x \hookrightarrow (j_!\mathbb{Z})_x$ is injective, and the later stalks is simply $\mathbb{Z}$ if $x \in U$ and zero elsewhere. Now any $x \in X$ is a specialisation of the unique generic point $\eta \in X$, so we have the following commutative diagram,

$$\begin{array}{ccc} \mathcal{G}_x & \hookrightarrow & (j_!\mathbb{Z})_x \\ \downarrow & & \downarrow = \\ \mathcal{G}_\eta & \hookrightarrow & (j_!\mathbb{Z})_\eta \end{array},$$

where the hooked arrows indicate injections. We now set some $d \in \mathbb{Z}$ such that $\mathcal{G}_\eta = d\mathbb{Z} \subseteq \mathbb{Z}$. If $d = 0$, then $\mathcal{G} = 0$ and $\mathcal{F} = j_!\mathbb{Z}$. If $d > 0$ we then obtain the short exact sequence,

$$0 \longrightarrow \mathcal{G} \longrightarrow j_!(d\mathbb{Z}) \cong j_!\mathbb{Z} \longrightarrow \mathcal{F}' \longrightarrow 0,$$

such that $\mathcal{F}'_\eta$ is exactly the cokernel of $\mathcal{G}_\eta \to d\mathbb{Z}$ which is simply zero. Hence $\mathcal{F}' = i_* i^* \mathcal{F}'$ for some closed $Z \subseteq X$ and $Z \neq X$, since $\mathcal{F}' = 0$ on an open subset of X. We then have $H^q(X, \mathcal{F}) \cong H^q(Z, i^* \mathcal{F}) = 0$ for $q > \dim Z \leq \dim X - 1$. Hence we have an exact sequence,

$$H^{q-1}(X, \mathcal{F}') \longrightarrow H^q(X, \mathcal{G}) \longrightarrow H^q(X, j_!\mathbb{Z}),$$

where the first term is zero for $q > \dim X$. If $H^q(X, j_!\mathbb{Z}) = 0$ for $q > \dim X$, then $H^q(X, \mathcal{G}) = 0$ for all $q > \dim X$. On the other hand we have the exact sequence,

$$H^q(X, j_!\mathbb{Z}) \longrightarrow H^q(X, \mathcal{F}) \longrightarrow H^{q+1}(X, \mathcal{G}).$$

Again, we have $H^q(X,\mathcal{F}) = 0$ for $q > \dim X$ if this holds for $j_!\mathbb{Z}$. It remains to show that if X is irreducible, $j : U \hookrightarrow X$ is open then $H^q(X, j_!\mathbb{Z}) = 0$ for $q > \dim X$. To do this, we look at the short exact sequence,

$$0 \longrightarrow j_!\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow i_*\mathbb{Z} \longrightarrow 0,$$

with $i : Z = X\backslash U \hookrightarrow X$. This gives us the following exact sequence on cohomology,

$$H^{1-q}(Z,\mathbb{Z}) \longrightarrow H^q(X, j_!\mathbb{Z}) \longrightarrow H^q(X,\mathbb{Z}),$$

where the first term is zero for $q - 1 > \dim Z < \dim X$ by induction on the dimension of X, and the final term is zero since $\mathbb{Z}$ is flasque as X is irreducible (we could not play this game for $\mathbb{Z}$ on Z as Z may not be irreducible). This finishes our proof. □

We now go back to Theorem 13.1, so recall the notation of that theorem.

Proof of Theorem 13.1. Since we have $H^i(X,\mathcal{F}) \cong H^q(\mathbb{P}^n_A, i_*\mathcal{F})$ we may assume without loss of generality that $X = \mathbb{P}^n_A$. We then have $H^q(\mathbb{P}^n_A,\mathcal{F}) = 0$ for $q > n$, so we now assume that this holds for $q' > q$, and start a downward induction argument. For some $N >> 0$ we know $\mathcal{F} \otimes \mathcal{O}(N) =: \mathcal{F}(N)$ is generated by global sections, so we have a surjection $\mathcal{O}^{\oplus m}_{\mathbb{P}^n_A} \twoheadrightarrow \mathcal{F}(N)$ which is equivalent to a map $\mathcal{O}_{\mathbb{P}^n_A}(-N)^{\oplus m} \twoheadrightarrow \mathcal{F}$. We then have a short exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A}(-N)^{\oplus m} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{G}$ is coherent, so we have the associated long exact sequence on cohomology,

$$H^q(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-N)^{\oplus m}) \longrightarrow H^q(\mathbb{P}^n_A,\mathcal{F}) \longrightarrow H^{q+1}(X,\mathcal{G}).$$

Now the final term is finitely generated over A by induction, and the following proposition explicitly calculates the first term using the isomorphism,

$$H^q(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-N)^{\oplus m}) \cong H^q(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-N))^{\oplus m}.$$

In particular, we see the first and last terms are finitely generated, hence so is the middle one, and we're done. □

Proposition 13.3. *Let $d > 0$, then we have,*

$$H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-d)) = \begin{cases} 0 & i \neq n \text{ or } d \leq n \\ \left(\frac{1}{x_0 \cdots x_n} A[x_0^{-1}, \ldots, x_n^{-1}]\right)_d & i = n, d > n \end{cases},$$

where each x_i has degree 1. In particular they are all finitely generated as A-modules.

This proposition appears as exercise 9.3, so we only prove the case for $n = 1$ to get a feel for what is going on.

Proof. We use the Čech complex for this calculation, since $\mathbb{P}^n_A$ is proper, so in particular separated. Let $n = 1$, then we have $\mathbb{P}^1_A = U_0 \cup U_1$ where $U_0 = \operatorname{Spec} A[t]$ and $U_1 = \operatorname{Spec} A[t^{-1}]$ (where we could think of $t = x_0/x_1$ and $t^{-1} = x_1/x_0$). The local sections of $\mathcal{O}(-d)$ are then,

$$\Gamma(U_0, \mathcal{O}(-d)) = A[t]t^{-d} \cong A[t], \qquad \Gamma(U_1, \mathcal{O}(-d)) = A[t^{-1}]t^d \cong A[t^{-1}],$$

which have obvious maps to $\Gamma(U_0 \cap U_1, \mathcal{O}(-d)) \cong A[t^{\pm 1}]$. In the Čech complex though, the map from the local sections of $\mathcal{O}(-d)$ at U_1 to those at $U_0 \cap U_1$ is t^{-d}-times the obvious (algebraic map). We then have Čech complex as follows,

$$\check{C}(\{U_0, U_1\}, \mathcal{O}(-d)) = \qquad 0 \longrightarrow A[t] \oplus A[t^{-1}] \longrightarrow A[t^{\pm 1}] \longrightarrow 0,$$

where the map sends (f_0, f_1) to $f_0 - t^{-d} f_1$. This map is certainly injective and the image are finite sums $\sum a_n t^n$ with $a_n = 0$ if $-d < n < 0$, simply by inspection. Hence the global sections are empty (which we already knew), and the first cohomology group is $\bigoplus_{i=-d+1}^{-1} A \cdot t^i$. □

For fun we can calculate the cohomolgy of $\mathcal{O}(d)$ for non-negative d as well.

Proposition 13.4. *Let $d \geq 0$, then we have,*

$$H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d)) = \begin{cases} A[x_0, \ldots, x_n]_d & i = 0 \\ 0 & \text{otherwise} \end{cases} .$$

Proof. For $i = 0$ we know this already, since we calculated it last semester. We now prove that $H^i(\mathbb{P}^n_A, \mathcal{O}(d)) = 0$ for $d \geq 0$ and $i > 0$ by induction on n and d. For $n = 0$ we have $\mathbb{P}^0_A = \operatorname{Spec} A$ and we're done by Theorem 12.1. For $n > 0$ we look at one of the obvious closed immersions $i : \mathbb{P}^{n-1}_A \hookrightarrow \mathbb{P}^n_A$, so we can look at the short exact sequence,

$$0 \longrightarrow \mathcal{O}(d-1) \longrightarrow \mathcal{O}(d) \longrightarrow i_*\mathcal{O}(d) \longrightarrow 0,$$

which comes by tensoring the canonical sequence when $d = 0$ with $O(d)$. On cohomology this gives us the exact sequence,

$$H^i(\mathbb{P}^n_A, \mathcal{O}(d-1)) \longrightarrow H^i(\mathbb{P}^n_A, \mathcal{O}(d)) \longrightarrow H^i(\mathbb{P}^{n-1}_A, \mathcal{O}(d)).$$

The first term is zero by induction on d if $d > 0$, and the last term is zero by induction on n. Hence we only need to calculate $H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = 0$ for $i > 0$, and this is done with an explicit Čech complex calculation. $\square$

14 Finiteness in Cohomology II 22/06/2017

Soon we want to talk about Serre duality, the Riemann-Roch theorem, the theorem of formal functions, Stein factorisations and Zariski's main theorem, but first we need to finish our work on finiteness and base change results. Last time we proved Theorem 13.1. A key calculation we didn't clarify last time is the following lemma.

Lemma 14.1. *Let I be a non-empty totally ordered set, and M be an abelian group. Then,*

$$0 \longrightarrow M \longrightarrow \prod_{i \in I} M \longrightarrow \prod_{i_1 < i_2} M \longrightarrow \cdots,$$

is exact.

Proof. Let X be a point, and let it be covered by a point as well U_i, then $I \neq \varnothing$ is a cover and we see that the sequence of the lemma is the Čech complex of the constant sheaf M on X. We know the cohomology of this space is trivial except in the zero degree, so this Čech complex is exact. □

Today we plan to prove the following theorem.

Theorem 14.2. *Let $f : X \to Y$ be a proper morphism of noetherian schemes, and $\mathcal{F}$ be a coherent sheaf on X. Then $R^i f_* \mathcal{F}$ are all coherent for $i \geq 0$.*

To prepare us for this theorem we will use the following proposition.

Proposition 14.3. *Let $f : X \to Y$ be a qcqs morphism of schemes and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $R^i f_* \mathcal{F}$ is quasi-coherent and for all open affine $V = \operatorname{Spec} B \subseteq Y$ with preimage $U = f^{-1}(V) \subseteq X$ we have,*

$$(R^i f_* \mathcal{F})(V) = H^i(U, \mathcal{F}).$$

Proof. Recall that $R^i f_* \mathcal{F}$ is the sheafification of $V \mapsto H^i(U, \mathcal{F})$ (see Lemma 11.2). We can work locally on Y, so we may assume $Y = \operatorname{Spec} B$ is affine. We now claim that for all $g \in B$ we have,

$$H^i(f^{-1}(D(g)), \mathcal{F}) = H^i(X, \mathcal{F})[g^{-1}].$$

Given the claim, it follows that $D(g) \mapsto H^i(f^{-1}(D(g)), \mathcal{F})$ defines a quasi-coherent sheaf $\widetilde{H^i(X, \mathcal{F})}$ giving us $R^i f_* \mathcal{F} = \widetilde{H^i(X, \mathcal{F})}$ which gives us the result. To prove this claim we first assume that X is separated, which implies that for $J \neq \varnothing$ we see that all intersections of affine opens covering X is again affine, so

$$\bigcap_{i \in J} U_i = U_J,$$

is affine. We then choose a finite open affine cover $X = \bigcup_{i=1}^n U_i$, then $H^i(X, \mathcal{F})$ is computed by the Čech complex $\check{C}(\{U_i\}, \mathcal{F})$, whose terms are $\bigoplus_{J \subseteq I, |J| = k} \mathcal{F}(U_J)$ with $k > 0$. We also see $f^{-1}(D(g)) = \bigcup_{i=1}^n D_{U_i}(g)$ is separated, so for all $J \neq \varnothing$ we have

$$\bigcap_{i \in J} D_{U_i}(g) = D_{U_J}(g),$$

is again affine. Again, we see that $H^i(f^{-1}(D(g)), \mathcal{F})$ is then computed in terms of the Čech complex $\check{C}(\{D_{U_i}(g)\}, \mathcal{F})$, whose terms are similar to the above Čech complex. In fact, we notice a relation between these two Čech complexes,

$$\check{C}(\{D_{U_i}(g)\}, \mathcal{F}) = \check{C}(\{U_i\}, \mathcal{F})[g^{-1}].$$

Since localisation is exact we obtain the desired claim. Now consider a general X, i.e. not assumed to be separated. We can cover X by affines U_i but $U_J = \bigcap_{i \in J} U_i$ for $J \neq \varnothing$ might not be affine, but it will be separated, as an open subset of an affine scheme. We then have two spectral sequences, and maps between them as indicated

$$\begin{array}{ccc} E_1^{pq} = \bigoplus_{J \subseteq I, |J| = p+1} H^q(U_J, \mathcal{F}) & \Longrightarrow & H^{p+q}(X, \mathcal{F}) \\ \downarrow & & \downarrow \\ (E_1^{pq})' = \bigoplus_{J \subseteq I, |J| = p+1} H^q(U \cap f^{-1}(D(g)), \mathcal{F}) & \Longrightarrow & H^{p+1}(f^{-1}(D(g)), \mathcal{F}) \end{array}.$$

From the separated case we know that $H^q(U_J \cap f^{-1}(D(g)), \mathcal{F}) = H^q(U_J, \mathcal{F})[g^{-1}]$, thus we see that $(E_1^{pq})' = E_1^{pq}[g^{-1}]$, and using the fact that localisation is exact we see that $(E_r^{pq})' = E_r^{pq}[g^{-1}]$ for all $r \geq 1$, i.e. for all the pages of our spectral sequence. From this we get the $r = \infty$ case, and from this we use the convergence of our spectral sequence to see that

$$H^j(f^{-1}(D(g)), \mathcal{F}) = H^j(X, \mathcal{F})[g^{-1}].$$

This shows our claim, and thus our proposition too. □

We have been trying to prove Theorem 14.2. By the Proposition 14.3 we know that these higher direct images are quasi-coherent, so without loss of generality we can take $Y = \operatorname{Spec} A$ (as in Theorem 14.2). We then need to show that $(R^i f_* \mathcal{F})(Y) = H^i(X, \mathcal{F})$ is a finitely generated A-module. When f is projective we did this last time, Theorem 13.1. For the general case we could use one of two approaches. The first is to use Chow's lemma to reduce this to the projective case. However this feels like cheating, since we just make many reductions and then compute $H^q(\mathbb{P}^n_A, \mathcal{O}(-d))$. Alternatively there is also the Cartan-Serre argument which is used in complex geometry. This uses a little functional analysis, using a variant of the following statement.

Proposition 14.4. *If V is a Banach space with a compact automorphism, then V is finite dimensional.*

This argument was first used by Kiehl in the 1960's with regard to non-archimedean geometry (e.g. over the p-adic numbers), and by Faltings in the 1990's in algebraic geometry. The trick is to base change from $\mathbb{Z}$ to the ring of Laurant power series $\mathbb{Z}((t))$ and then use the topology we can place on this ring.

Example 14.5. If $X = \mathbb{P}^1_{\mathbb{C}}$ as a complex manifold and $\mathcal{F} = \mathcal{O}_X$ is the structure sheaf, then we can choose two nice open covers $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C}) = \bigcup_i U_i = \bigcup_i V_i$ such that $\overline{V}_i \subseteq U_i$. Then we have the Čech complexes which both compute $H^*(X, \mathcal{F})$, and this containment condition on U_i and V_i means we obtain a map of complexes,

$$\check{C}(\{U_i\}, \mathcal{F}) \longrightarrow \check{C}(\{V_i\}, \mathcal{F}),$$

which is just restriction. All the levels of our chain complex are Banach spaces and the restriction maps are compact operators on each level. We then apply a proposition similar to Proposition 14.4, which states that if $C^\bullet$ and $D^\bullet$ are complexes of Banach spaces, and $f^\bullet : C^\bullet \to D^\bullet$ is a quasi-isomorphism such that all the f^i are compact operators, then the cohomology of both $C^\bullet$ and $D^\bullet$ are isomorphic through f and level-wise finite dimensional.

To finish Theorem 14.2 we will take the path of Chow's Lemma.

Theorem 14.6 (Chow's Lemma)**.** *Let $f : X \to Y = \operatorname{Spec} A$ be a proper map of noetherian schemes, then there exists some $f' : X' \to Y$ that is projective, and a proper birational map $g : X' \to X$ such that $f \circ g = f'$, i.e. the following diagram commutes,*

$$\begin{array}{ccc} X' & & \\ \downarrow{\scriptstyle g} & \searrow^{f'} & \\ X & \xrightarrow{f} & Y \end{array}.$$

To prove this we will recall a few definitions, including an adjective in the above theorem.

Definition 14.7. *A birational map of schemes is a map $f : X \to Y$ of schemes such that for some open dense subset $V \subseteq Y$ we have $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is an isomorphism.*

Recall that given a map $f : X \to Y$ of schemes, then the scheme theoretic image[22] of f is the smallest closed subscheme of X over which f factors. This can be identified as $\underline{\mathrm{Spec}}(\mathcal{O}_Y/\mathcal{I})$ where I is the kernel of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ when f is qcqs.

Definition 14.8. *An open subscheme $U \subseteq X$ is scheme theoretically dense if the scheme theoretic image of $U \hookrightarrow X$ is equal to X.*

This definition behaves as one would hope.

Lemma 14.9. *Given two schemes X, Y over another scheme Z, two maps $f, g : X \to Y$ as schemes over Z, such that $Y \to Z$ is separated, and a scheme theoretically dense $U \subseteq X$, then $f|_U = g|_U$ implies that $f = g$.*

Proof. Consider $Y \hookrightarrow Y \times_Z Y$, which is closed as Y is separated, and the map $(f, g) : X \to Y \times_Z Y$. Pulling back along these maps gives us a scheme X' with a map $X' \to X$ that is closed. Since f and g agree on U, we have a map $U \hookrightarrow X'$, which commutes with the inclusion $U \hookrightarrow X$ and the closed immersion $X' \hookrightarrow X$. The fact that U is scheme theoretically dense in X now implies that $X' = X$, so $f = g$ on all of X. □

This gives us all the ammunition we need to prove Chow's Lemma.

Proof of Theorem 14.6. We can reduce to the case that X is irreducible quite easily. Since X is noetherian we know $X = \bigcup_{i=1}^n X_i$ with each X_i irreducible. Notice the map $\coprod X_i \to X$ is a proper birational map, since it is an isomorphism when restricted to $X \setminus \bigcup_{i \neq j}(X_i \cap X_j)$. Hence we may replace X be this disjoint union, and then we may restrict our attention to one single X_i. Since X is irreducible and noetherian, we take $X = \bigcup_{i=1}^n U_i$ for U_i some open affine cover, with open immersions

$$U_i \hookrightarrow \mathbb{A}_A^{n_i} \hookrightarrow \mathbb{P}_A^{n_i}.$$

Let $P_i \subseteq \mathbb{P}_A^{n_i}$ be the scheme theoretic image of these maps above. Let $U = \cap_{i=1}^n U_i$, then we have a map,

$$h : U \hookrightarrow (P_1 \times_A \cdots \times_A P_n) \times_A X,$$

which is an open embedding since it factors through $U_1 \times_A \cdots \times_A U_n \times_A X$ through open embeddings. It is also an immersion, since it factors through $U_1 \times_A \cdots \times_A U_n \times U$ through first a closed immersion, followed by an open immersion. Let X' be the scheme theoretic image of h, so X' is closed inside $P_1 \times_A \cdots \times_A P_n \times_A X$, and this is proper over X, so $g : X' \to X$ is proper (even projective). Notice that we have the following commutative diagram,

$$\begin{array}{ccc} U & \hookrightarrow & X' \\ & \searrow & \downarrow g \\ & & X \end{array},$$

and U is scheme theoretically dense inside X'. We claim this implies g is birational. To see this, we have a section $s : U \to g^{-1}(U)$ of a restriction of g, and since g is proper we see that s is closed, hence $s(U)$ is closed and $U = s(U) = g^{-1}(U)$. We now claim that $g : X' \to Y = \operatorname{Spec} A$ is projective. More precisely, we want to show,

$$l : X' \hookrightarrow P_1 \times_A \cdots \times_A P_n \times_A X \to P_1 \times_A \cdots \times_A P_n,$$

[22] See footnote 43 on page 101 of [7]

is a closed immersion, since this is followed by the closed immersion $P_1 \times_A \cdots \times_A P_n \hookrightarrow \mathbb{P}^{n_1}_A \times_A \cdots \times_A \mathbb{P}^{n_n}_A$, recognising g as projective. We notice the composition l is obviously proper, as projections and closed immersions are proper, so we only need to see if it is a locally closed immersion. Let $V_i = g^{-1}(U) \subseteq X'$, which as a set cover X'. Notice that the following diagram commutes when restricted to $U \subseteq V_i$,

$$\begin{array}{ccc} V_i & \longrightarrow & U_i \\ \big\downarrow & & \big\downarrow \\ P_1 \times_A \cdots \times_A P_n \times_A X & \longrightarrow & P_i \end{array},$$

but since U is scheme theoretically dense inside these V_i's, we apply Lemma 14.9 and see that it actually commutes on the nose. Without loss of generality now, we restrict our attention to $i = 1$, and consider the following commutative diagram,

$$\begin{array}{ccc} V_1 & \overset{\eta}{\hookrightarrow} & U_1 \times_A P_2 \times_A \cdots \times_A P_n \times_A U_1 \\ & \searrow^{\xi} & \Delta \uparrow \ \downarrow \\ & & U_1 \times_A P_2 \times_A \cdots \times_A P_n \end{array}.$$

The map η is a closed immersion since it was defined by a base change, and ξ is a closed immersion since the diagram map of U_1, Δ is. This means that l is a locally closed immersion, and we're done. □

We can now finally prove Theorem 14.2.

Proof of Theorem 14.2. Assume for simplicity for now that $\dim X < \infty$, then we induct on $\dim X$. We may assume the result is true for all $\mathcal{F}$ with $\dim \operatorname{supp} \mathcal{F} < \dim X$. We then use Chow's lemma to obtain a map $X' \to X$ which is proper and birational (on say $U \subseteq X$) and another map $f' : X' \to Y$ which is projective. This implies that $H^i(X', g^*\mathcal{F})$ are finitely generated A-modules from Theorem 13.1, and we obtain the following commutative diagram,

$$\begin{array}{ccccc} X' & \overset{(g,\mathrm{id})}{\hookrightarrow} & X \times_A X' & \hookrightarrow & X \times_A \mathbb{P}^n_A = \mathbb{P}^n_X \\ & & & \searrow & \big\downarrow \\ & & & & X \end{array}.$$

Hence all the $R^i g_*(g^*\mathcal{F})$ are coherent, and also zero on U if $i > 0$, since g is an isomorphism there. This means these coherent sheaves are concentrated on $X \backslash U$, and we know the result holds here by induction. We now consider the spectral sequence,

$$E_2^{pq} = H^p(X, R^q g_*(g^*\mathcal{F})) \Longrightarrow H^{p+q}(X', g^*\mathcal{F}),$$

using the fact that $R\Gamma(X, -) \circ Rg_* = R\Gamma(X', -)$ on $g^*\mathcal{F}$. We know what our spectral sequence converges to is a finitely generated A-modules, and that for $q \geq 1$ the E_2-page is finitely generated. This means we have a spectral sequence, where only the bottom row is not known to be finitely generated. However, each position is only possibly hit by finitely many differentials from the finitely generated parts of our spectral sequence. Since this spectral sequence converges to something finitely generated, this implies that this bottom row is also finitely generated, i.e. $H^p(X, g_*g^*\mathcal{F})$ is finitely generated. We have a short exact sequence now,

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow g_*g^*\mathcal{F} \longrightarrow \mathcal{G}' \longrightarrow 0,$$

by forming kernels and cokernels. We know that $\mathcal{G}$ and $\mathcal{G}'$ are concentred on $X \backslash U$, so the result holds there, and we just saw by a spectral sequence argument that if holds for $g_*g^*\mathcal{F}$ too. Hence the conclusion holds for $\mathcal{F}$. □

15 Affine Criterion and Base Change 26/06/2017

We would like to prove some fundamental theorems about sheaf cohomology. Specifically Serre's criterion for affine schemes, and the cohomological criterion for ample line bundles.

Theorem 15.1 (Serre's Affine Criterion)**.** *Let X be a qcqs scheme, then the following are equivalent;*

1. *X is affine.*
2. *For all quasi-coherent sheaves $\mathcal{F}$ on X we have $H^i(X,\mathcal{F}) = 0$ for all $i > 0$.*
3. *For all ideal subsheaves $\mathcal{I} \subseteq \mathcal{O}_X$, we have $H^1(X,\mathcal{I}) = 0$.*

Proof. We have seen in Theorem 12.1 that part 1 implies part 2, and the fact that part 2 implies part 3 is logic. To show part 3 implies part 1 we let $A = \Gamma(X,\mathcal{O}_X)$, then we have a map $g : X \to \operatorname{Spec} A$. We claim that there exists $f_1,\dots,f_n \in A$ generating the unit ideal such that the pre-image $g^{-1}(D(f_i)) = D_X(f_i)$ is affine. Notice that this claim implies g in particular is affine, which implies X is affine. To prove this claim, let $x \in X$ be a closed point of X, then we can find an open affine neighbourhood $x \in U \subseteq X$, and we then set $Z = X\backslash U$. This gives us the short exact sequence,

$$0 \longrightarrow \mathcal{I}_{Z\sqcup\{x\}} \longrightarrow \mathcal{I}_Z \longrightarrow i_*k(x) \longrightarrow 0,$$

which is easily checked on stalks, where $i : \{x\} \hookrightarrow X$ is the inclusion of x into X. The corresponding long exact sequence on cohomology then gives us

$$H^0(X,\mathcal{I}_Z) \longrightarrow H^0(X,i_*k(x)) \longrightarrow H^1(X,\mathcal{I}_{Z\sqcup\{x\}}).$$

The first map is surjective since $H^1(X,\mathcal{I}_{Z\sqcup\{x\}}) = 0$ by assumption. Thus we can find an $f \in H^0(X,\mathcal{I}_Z)$ such that $f(x) = 1$ inside $k(x)$. Thus $D_X(f) = D_U(f) \ni x$, since f vanishes on Z, we see $D_U(f)$ is affine, since it is a principal open subset of an affine scheme U. Since X is quasi-compact, we can cover X with finitely many $f_i \in A$ such that $X = \bigcup_{i=1}^n D_X(f_i)$, with each $D_X(f_i)$ affine. Note that it obviously suffices to do this for closed points, which we assumed x is, using the fact that X is qcqs so it has a closed point. Notice also that at this point we can only conclude that $X \hookrightarrow \operatorname{Spec} A$ is an open immersion, so X is quasi-affine. We still need to prove that these f_i's generate A. We have a short exact sequence,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^n \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

where the last map sends $(a_1,\dots,a_n) \mapsto \sum_i a_i f_i$ and $\mathcal{F}$ is defined to be the kernel, simply since these $D_X(f_i)$'s cover X. This gives us the long exact sequence on cohomology,

$$\begin{array}{ccccc} H^0(X,\mathcal{O}_X^n) & \xrightarrow{h} & H^0(X,\mathcal{O}_X) & \longrightarrow & H^1(X,\mathcal{F}) \\ \downarrow{=} & & \downarrow{=} & & \\ A^n & \xrightarrow{(f_1,\dots,f_n)} & A & & \end{array}.$$

Hence it is enough to show here that $H^1(X,\mathcal{F}) = 0$ to obtain surjectivity of h. Let $\mathcal{F}_i = \mathcal{F} \cap \mathcal{O}_X^i$ be a filtration of $\mathcal{F}$, where $\mathcal{O}_X^i \hookrightarrow \mathcal{O}_X^n$ includes elements into the first i coordinates, then $0 = \mathcal{F}_0 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F}$. We now have an injection

$$\mathcal{F}_{i+1}/\mathcal{F}_i \hookrightarrow \mathcal{O}_X^{i+1}/\mathcal{O}_X^i \cong \mathcal{O}_X.$$

Hence $\mathcal{F}_{i+1}/\mathcal{F}_i$ is an ideal sheaf, so it's first cohomology is zero by assumption. By induction on long exact sequences defining these ideal sheaves we obtain $H^1(X,\mathcal{F}) = 0$. □

We have a little bit of machinery built up now, so we can continue proving theorems.

Theorem 15.2. *Let A be a noetherian ring, and $g : X \to \operatorname{Spec} A$ a projective map, and $\mathcal{L}$ an ample line bundle*[23] *on X. Then for all coherent sheaves $\mathcal{F}$ over X, there is some $n_0 \in \mathbb{Z}$ such that for all $n \geq n_0$ and $i > 0$ we have $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$.*

The definition of a projective scheme X over a noetherian ring is only a slight generalisation to that of Definition 25.2 of [7].

Definition 15.3. *If X is a scheme of finite type over a noetherian ring A, then X is quasi-projective if X admits an ample line bundle, which is equivalent to asking for a locally closed immersion $h : X \hookrightarrow \mathbb{P}^n_A$.*

The proof of the equivalence in the above definition is very similar to the same statement over a field presented in Theorem 25.7 from [7].

Proof of Theorem 15.2. Let m be some sufficiently large integer, and $h : X \hookrightarrow \mathbb{P}^N_A$ a closed immersion such that $\mathcal{L}^{\otimes m} = h^*\mathcal{O}(1)$. We may assume that $m = 1$, since we can always reach the $\mathcal{L}^{\otimes m}$ case by induction. Without loss of generality we have $\mathcal{L} = h^*\mathcal{O}(1)$, then we have,

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = H^i(X, \mathcal{F} \otimes h^*\mathcal{O}(n)) \cong H^i(\mathbb{P}^N_A, h_*(\mathcal{F} \otimes h^*\mathcal{O}(n))) = H^i(\mathbb{P}^N_A, h_*\mathcal{F} \otimes \mathcal{O}(n)).$$

Notice that one can construct a canonical map $h_*(\mathcal{F} \otimes h^*\mathcal{O}(n)) \to h_*\mathcal{F} \otimes \mathcal{O}(n)$ using adjunctions, which one can also check is an isomorphism by looking at stalks. Without loss of generality we can take $X = \mathbb{P}^N_A$. We will now procede by descending induction on i. We see that for all $\mathcal{F}$ there is some n_0 such that for all $n \geq n_0$ and $i' \geq i$ we have $H^{i'}(\mathbb{P}^N_A, \mathcal{F}(n)) = 0$, where we write $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}(n)$. For $i = N+1$ we see from the Čech complex that $H^{i'}(\mathbb{P}^N_A, \mathcal{F}(n)) = 0$. Now assume this is true for $i+1$, then there is some j such that $\mathcal{F}(j)$ is globally generated, so that we have a short exact sequence,

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}^r_{\mathbb{P}^N_A} \longrightarrow \mathcal{F}(j) \longrightarrow 0,$$

where $\mathcal{G}$ is simply the kernel. We now choose some $n_0 \geq 0$ such that $H^{i'}(\mathbb{P}^N_A, \mathcal{G}(n)) = 0$ for all $n \geq n_0$ and all $i' \geq i+1$. We then obtain the following short exact sequence by twisting,

$$0 \longrightarrow \mathcal{G}(n) \longrightarrow \mathcal{O}^r_{\mathbb{P}^N_A}(n) \longrightarrow \mathcal{F}(j+n) \longrightarrow 0.$$

This implies, since $i' \geq i+1$ that the first term in the following exact sequence is zero,

$$H^{i'}(\mathbb{P}^N_A, \mathcal{O}^r_{\mathbb{P}^N_A}(n)) \longrightarrow H^{i'}(\mathbb{P}^N_A, \mathcal{F}(j+n)) \longrightarrow H^{i'+1}(\mathbb{P}^N_A, \mathcal{G}(n)),$$

by our calculations of Proposition 13.4. The last term is also zero by assumption, so we see

$$H^{i'}(\mathbb{P}^N_A, \mathcal{F}(j+n)) = 0,$$

for $n \geq n_0$ and $i' \geq i$, and we're now done. □

There is a stronger converse to this theorem, but we want the reader to notice the change in hypotheses.

Proposition 15.4. *Let X be a qcqs scheme and $\mathcal{L}$ a line bundle on X. Assume that for all quasi-coherent sheaves $\mathcal{F}$, there exists an $n > 0$ such that $H^1(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$. Then $\mathcal{L}$ is ample.*

Proof. We can prove that X is covered by open affine subsets of the form $D(s)$ where $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ for some $n > 0$ (again, recall Definition 25.4 from [7]). Let $x \in X$ be a closed point, which exists as X is qcqs, and take some open affine neighbourhood $x \in U \subseteq X$, and set $Z = X \backslash U$. Denoting the inclusion $\{x\} \hookrightarrow X$ by i we obtain the following short exact sequence,

$$0 \longrightarrow \mathcal{I}_{Z \sqcup \{x\}} \longrightarrow \mathcal{I}_Z \longrightarrow i_*k(x) \longrightarrow 0.$$

[23] Recall what an ample line bundle is from Definition 25.4 in [7].

Choosing n correctly, we have the following diagram,

$$\begin{array}{ccccc} H^0(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n}) & \longrightarrow & H^0(X, i_*k(x)) & \longrightarrow & H^1(X, \mathcal{I}_{Z \sqcup \{x\}} \otimes \mathcal{L}^{\otimes n}) \\ \big\downarrow & & \big\downarrow = & & \\ H^0(X, \mathcal{L}^{\otimes n}) & \longrightarrow & k(x) & & \end{array},$$

where the left vertical map is an inclusion. Since the right-most term is zero by assumption, we obtain $s \in H^0(X, \mathcal{L}^{\otimes n})$ such that s vanishes on Z, and $s \neq 0$ on x. This implies that $x \in D_X(s) = D_U(s)$ which is affine by Lemma 26.2 in [7]. □

As hinted at before the statement of this proposition, this is not generally an equivalent definition of an ample line bundle. If X is not projective, then $\mathcal{L}$ can be ample, but $H^n(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \neq 0$ for $n > 0$.

Example 15.5. If X is quasi-affine, then $\mathcal{L} = \mathcal{O}_X$ is ample. If one had $H^1(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = H^1(X, \mathcal{F}) = 0$ for all $n > 0$ then Theorem 15.1 would say that X is affine. We know explicit examples of schemes that are quasi-affine and not affine, for example $X = \mathbb{A}^2_k \backslash (0,0)$. In this case we have $H^1(X, \mathcal{O}_X) \neq 0$, which is closely related to exercise 10.1[24].

We now move onto a slightly different idea, which ties together with some remarks we tried to make at the beginning of the semester. Recall that if $f : X \to Y$ is a qcqs map of schemes and $\mathcal{F}$ is a quasi-coherent sheaf on X, then Proposition 14.3 tells us that $R^i f_* \mathcal{F}$ are all quasi-coherent. Sometimes we like to think of a map $f : X \to Y$ of schemes to be a parametrised family of schemes X_y, each a scheme over $\operatorname{Spec} k(y)$ for all $y \in Y$. If we let $g_y : X_y \to X$, then we can now formulate the following natural question:

How are $R^i f_* \mathcal{F}$ and $H^i(X_y, g_y^* \mathcal{F})$ related?

We can abstract this a little too. Let $f : X \to Y$ be a qcqs map of schemes and $g : Y' \to Y$ be any map, then we have the following pullback diagram,

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow f' & & \big\downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}.$$

Given a quasi-coherent sheaf $\mathcal{F}$ over X, we can now consider $g^* R^i f_* \mathcal{F}$ or $R^i f'_* g'^* \mathcal{F}$ and think about how they are related.

Construction 15.6. We are going to construct a natural map $g^* R^i f_* \mathcal{F} \to R^i f'_* g'^* \mathcal{F}$, which is natural in $\mathcal{F}$, which is going to be called the base change map. First we let $i = 0$ to get an idea of what is going on. We want a map in

$$\operatorname{Hom}(g^* f_* \mathcal{F}, f'_* g'^* \mathcal{F}) \cong \operatorname{Hom}(f_* \mathcal{F}, g_* f'_* g'^* \mathcal{F}) = \operatorname{Hom}(f_* \mathcal{F}, f_* g'_* g'^* \mathcal{F}),$$

using the obvious identifications,

$$g_* f'_* g'^* = (g f')_* g'^* = (f g')_* g'^* = f_* g'_* g'^*.$$

Consider the natural map $\mathcal{F} \to g'_* g'^* \mathcal{F}$ adjoint to the identity map $g'^* \mathcal{F} \to g'^* \mathcal{F}$, then applying f_* to this gives us our natural base change map for $i = 0$. For general i, we need to use the edges maps of spectral sequences. In general, for any maps $f : X \to Y$, $g : Y \to Z$ of topological spaces we have edge maps,

$$R^i g_* f_* \mathcal{F} \longrightarrow R^i (g f)_* \mathcal{F} \longrightarrow g_* R^i f_* \mathcal{F},$$

[24] Ex 10.1: Let k be a field and $j : X \hookrightarrow \mathbb{A}^2_k$ be the natural open immersion. For $i \geq 0$, compute $R^i j_* \mathcal{O}_{\mathbb{A}^2_k \backslash \{(0,0)\}}$.

natural in $\mathcal{F}$. This is part of the spectral sequence,

$$E_2^{pq} = R^p g_* R^q f_* \mathcal{F} \Longrightarrow R^{p+q}(g \circ f)_* \mathcal{F}.$$

By definition of the convergence of this spectral sequence we have an inclusion $E_\infty^{p0} \hookrightarrow R^p(g \circ f)_* \mathcal{F}$, and by the definition of the E_∞-page we see that E_∞^{p0} is a quotient of $R^p g_* f_* \mathcal{F}$ by the image of various differentials. This gives us the natural edge map $R^p g_* f_* \mathcal{F} \to R^p(g \circ f)_* \mathcal{F}$. Dually, we see that E_∞^{0q} is a submodule of $g_* R^q f_* \mathcal{F}$ as the kernel of various differentials, and $R^q(g \circ f)_* \mathcal{F}$ surjects onto E_∞^{0q} by definintion of convergence. Composition then gives the second edge map $R^q(g \circ f)_* \mathcal{F} \to g_* R^q f_* \mathcal{F}$.

Once again, we would like a natural map inside $\operatorname{Hom}(g^* R^i f_* \mathcal{F}, R^i f'_* g'^* \mathcal{F})$, so we start with the map $\mathcal{F} \to g'_* g'^* \mathcal{F}$ adjoint to the identity, and apply $R^i f_*$ to obtain $R^i f_* \mathcal{F} \to R^i f_* g'_* g'^* \mathcal{F}$. Post-composing this with our first edge map, then identifying $R^i(fg')$ and $R^i(gf')$, and then post-composing with the second edge map, we obtain $R^i f_* \mathcal{F} \to g_* R^i f'_* g'^* \mathcal{F}$, which by adjunction gives us a natural map between the desired objects.

These definitions of the base change map are quite abstract. In the outline of the proof of the following theorem we are going to use a different, but equivalent description of this base change map. The proof that both of these descriptions are the same amounts to an enormous diagram chase, as one should expect given the above definition. It is nice to know the definition is natural, and have a concrete description written down.

Theorem 15.7 (Flat Base Change)**.** *Consider a qcqs map of schemes $f : X \to Y$, and a flat map $g : Y' \to Y$. For all quasi-coherent sheaves $\mathcal{F}$ over X, the base change map*

$$g^* R^i f_* \mathcal{F} \longrightarrow R^i f'_* g'^* \mathcal{F},$$

is an isomorphism.

Example 15.8. Let X be a qcqs scheme over a field k, and k' be a field extension of k, where the map g is the flat map of schemes $\operatorname{Spec} k' \to \operatorname{Spec} k$. Then for

$$g' : X' = X \times_{\operatorname{Spec} k} \operatorname{Spec} k' \to X,$$

and any quasi-coherent sheaf $\mathcal{F}$ over X we have,

$$H^i(X, \mathcal{F}) \otimes_k k' \cong H^i(X', g'^* \mathcal{F}).$$

Proof of Theorem 15.7. This is local on Y', so we may assume that $Y' = \operatorname{Spec} A'$ and $Y = \operatorname{Spec} A$ are affine schemes (so A' is a flat A-algebra). We have to prove that $H^i(X', g'^* \mathcal{F}) \cong H^i(X, \mathcal{F}) \otimes_A A'$. The proof is identical to the proof we did last time (in the proof of Proposition 14.3) that

$$H^q(f^{-1}(D(h))) = H^q(X, \mathcal{F})[h^{-1}],$$

for $h \in A$, where we first did the separated case and then used a Čech to sheaf cohomology spectral sequence to finish it off. We will only repeat the separated case here, so assume that X is separated. We can then write $X = \bigcup_{i=1}^n U_i$ for U_i a collection of open affines and for all $\varnothing \neq J \subseteq I = \{1, \ldots, n\}$, $U_J = \bigcap_{j \in J} U_j$ is affine. We can compute $H^q(X, \mathcal{F})$ by $\check{C}^\bullet(\{U_i\}, \mathcal{F})$, which has terms

$$\bigoplus_{J \subseteq I, |J| = k > 0} \mathcal{F}(U_J).$$

We also have $X' = \bigcup_{i=1}^n U'_i$, where $U'_i = U_i \times_{\operatorname{Spec} A} \operatorname{Spec} A'$ are also open affines, hence $H^q(X', g'^* \mathcal{F})$ is computed by $\check{C}^\bullet(\{U'_i\}, g'^* \mathcal{F})$, which has terms,

$$\bigoplus_{J \subseteq I, |J| = k > 0} (g'^* \mathcal{F})(U'_J).$$

We notice now that $(g'^*\mathcal{F})(U'_J) = \mathcal{F}(U_J) \otimes_A A'$, so $\mathcal{F}|_{U_J} = \widetilde{M}$, with M some $\Gamma(U_J, \mathcal{O}_{U_J}) = B_J$-module. Let $B'_J = \Gamma(U'_J, \mathcal{O}_{U'_J}) = B_J \otimes_A A'$. We then see

$$g'^*\mathcal{F}|_{U'_J} = M \widetilde{\otimes_{B_J}} B'_J = M \widetilde{\otimes_A} A'.$$

This implies that,

$$(g'^*\mathcal{F})(U'_J) = \mathcal{F}(U_J) \otimes_A A',$$

and our whole Čech complex has simply been base changed, i.e.

$$\check{C}^\bullet(\{U'_i\}, g'^*\mathcal{F}) = \check{C}^\bullet(\{U_i\}, \mathcal{F}) \otimes_A A'.$$

Since A' is a flat A-algebra, then $- \otimes_A A'$ is exact, and we obtain the result,

$$H^i(X', g'^*\mathcal{F}) \cong H^i(X, \mathcal{F}) \otimes_A A'.$$

□

16 Generalised Base Change and $\otimes^{\mathbb{L}}$ 03/07/2017

Let $f : X \to Y$ and $g : Y' \to Y$ be maps of schemes, with f qcqs, and $\mathcal{M}$ be a quasi-coherent sheaf on X. We will denote $f' : X' \to Y'$ as the base change of f along g. The question we want to ask for today is:

When is the canonical map $g^* R^i f_* \mathcal{M} \to R^i f'_* g'^* \mathcal{M}$ from Construction 15.6 an isomorphism?

Without loss of generality we take $Y' = \operatorname{Spec} A'$ and $Y = \operatorname{Spec} A$ be to affine, which reformulates the above question to asking when,

$$H^i(X, \mathcal{M}) \otimes_A A' \longrightarrow H^i(X', g'^* \mathcal{M})$$

is an isomorphism. The most interesting cases are when $Y' = \operatorname{Spec} k(y)$ for some $y \in Y$. Last time we proved Theorem 15.7, which says that if g is a flat map of schemes then we have this isomorphism. However, this does not usually include the case of $Y' = \operatorname{Spec} k(y)$ unless y is a generic point and Y is reduced at y. One needs some type of flatness hypothesis to hope for a base change result, but this requires a little added generality involving the derived tensor product

$$- \otimes^{\mathbb{L}} -,$$

the left derived functor of $- \otimes -$. This apporach needs to be formulated with derived schemes, which enters into the area of derived algebraic geometry. If one wants to study in the usual world, we would need to assume that

$$- \otimes^{\mathbb{L}} - = - \otimes -,$$

which amounts to flatness assumptions. Let us begin this new approach now.

Definition 16.1. *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{M}$ is flat over Y if one of the following two equivalent conditions are satisfied,*

1. *There exists a cover of X by open affines $U_i = \operatorname{Spec} A_i \subseteq X$ mapping into open affines $V_i = \operatorname{Spec} B_i \subseteq Y$ such that $\mathcal{M}(U_i)$ is a flat B_i-module.*
2. *For all open affines $U = \operatorname{Spec} A \subseteq X$ mapping into an open affine $V = \operatorname{Spec} B \subseteq Y$, $\mathcal{M}(U)$ is a flat B-module.*

Proposition 16.2. *Given the following pullback diagram of schemes,*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array},$$

where f is qcqs, and let $\mathcal{M}$ be flat over Y with $Y = \operatorname{Spec} B$ an affine scheme. Then there exists a bounded complex $N^\bullet$ of flat B-modules (independent of g) such that for all $g : Y' = \operatorname{Spec} B' \to Y$ we see $H^i(X', g'^* \mathcal{M})$ is computed by $N^\bullet \otimes_B B'$. More precisely, $R\Gamma(X', g'^* \mathcal{M}) \cong N^\bullet \otimes_B B'$ in $D(B)$, the derived category of B-modules.

Proof. Assume just for simplicity that f is separated, as to avoid a Čech to sheaf cohomology spectral sequence for now. Let X be covered by open affines $U_i = \operatorname{Spec} A_i \subseteq X$ indexed on the finite set I such that for all non-empty $J \subseteq I$ we have $U_J = \bigcap_{i \in J} U_i$ is affine, then we can compute $R\Gamma(X, \mathcal{M})$ with Čech cohomology. Define $\mathcal{N}^\bullet = \check{C}(\{U_i\}, \mathcal{M})$, so the terms are,

$$\mathcal{N}^k = \bigoplus_{J \subseteq I, |J| = k+1} \mathcal{M}(U_J).$$

From our assumptions we see that each $\mathcal{M}(U_J)$ is a flat B-module. Similarly, for all $g : Y' = \operatorname{Spec} B' \to Y$ we have $X' = \bigcup_{i \in I} U_i'$ with $U_i' = U_i \times_Y Y'$ affine opens, so $R\Gamma(X, g'^*\mathcal{M})$ is computed by the Čech complex $\check{C}(\{U_i'\}, g'^*\mathcal{M})$ with terms,

$$\bigoplus_{J \subseteq I, |J| = k+1} (g'^*\mathcal{M})(U_J').$$

From Theorem 15.7 we see that each $(g'^*\mathcal{M})(U_J') \cong \mathcal{M}(U_J) \otimes_B B'$. Hence we see that $\check{C}(\{U_i'\}, g'^*\mathcal{M}) = N^\bullet \otimes_B B'$. ☐

We would like to take the word flat out of the above proposition.

Lemma 16.3. *Let A be a ring, $M^\bullet, N^\bullet$ be complexes of flat A-modules which are bounded above, i.e. $M^i = N^i = 0$ for all sufficiently large i. Let $f : M^\bullet \to N^\bullet$ be a quasi-isomorphism, then for all A-modules L the map*

$$f \otimes_A L : M^\bullet \otimes_A L \longrightarrow N^\bullet \otimes_A L,$$

is a quasi-isomorphism.

Remark 16.4. There are a few observations to be made about this lemma.

1. This fails without flatness! For example, take $A = \mathbb{Z}$ and the following $f : M^\bullet \to N^\bullet$,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}.$$

   Setting $L = \mathbb{Z}/2$ we obtain the map $f \otimes_A L$,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{\cdot 2 = 0} & \mathbb{Z}/2 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array},$$

   which is clearly not a quasi-isomorphism.

2. If $N^\bullet$ is a bounded above complex of A-modules, then we can choose a quasi-isomorphism $f : M^\bullet \to N^\bullet$ where $M^\bullet$ is a bounded above complex of flat A-modules then define,

$$N^\bullet \otimes_A^{\mathbb{L}} L = M^\bullet \otimes_A L,$$

   which is well-defined in the derived category $D(A)$.[25] This lemma tells us that this definition is independent of our choice of quasi-isomorphism f if $M^\bullet$ is "flat". For example, if we take $N^\bullet$ to be the complex concentrated in degree zero for some A-module N, then

$$\operatorname{Tor}_i^A(N, L) = H_i(N \otimes_A^{\mathbb{L}} L),$$

   for all $i \geq 0$, as the left derived functor of $- \otimes_A^{\mathbb{L}} L$ (notice left derived functors are *homological*).

[25] This is a type of "flat" replacement, which occasionally allows one to define left derived functors in the derived category of many abelian categories.

Proof of Lemma 16.3. Recall that if $f : M^\bullet \to N^\bullet$ is a map of complexes, then we can define the complex $C_f^\bullet$ which has entries and differentials,

$$C_f^i = M^{i+1} \oplus N^i, \qquad d_{C_f} = \begin{pmatrix} -d_M & 0 \\ -f & d_N \end{pmatrix}.$$

This fits into a short exact sequence,

$$0 \longrightarrow N^\bullet \longrightarrow C_f^\bullet \longrightarrow M^\bullet[1] \longrightarrow 0,$$

which gives us a long exact on cohomology[26],

$$\cdots \longrightarrow H^iN \longrightarrow H^iC_f \longrightarrow H^{i+1}M \xrightarrow{H^{i+1}(f)} H^{i+1}N \longrightarrow \cdots.$$

In particular, note that f is a quasi-isomoprhism if and only if C_f is acyclic (i.e. $H^i(C_f) = 0$ for all i). Notice that $C_{f\otimes_A L} = C_f \otimes_A L$, so it is enough to prove the following claim.

Let $M^\bullet$ be a bounded above, acyclic complex of flat A-modules, then for all A-modules L we have $M^\bullet \otimes_A L$ is again acyclic. To see this, without loss of generality we can set $M^i = 0$ for all $i \geq 1$. Let $Z^i = \ker(d^i : M^i \to M^{i+1})$ and $B^i = \operatorname{im}(d^{i-1} : M^{i-1} \to M^i) \cong Z^i$, then $H^i = Z^i/B^i = 0$ by assumption, so we have an exact sequence,

$$0 \longrightarrow Z^i \longrightarrow M^i \longrightarrow B^{i+1} = Z^{i+1} \longrightarrow 0.$$

By descending induction we want to prove that all these Z^i's are flat. This is true for $i \geq 1$ as $Z^i = 0$, and we then want to use the following proposition.

Proposition 16.5. *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of A-modules with M_2 and M_3 flat over A, then M_1 is flat.*

Proof of Propsition 16.5. Using the fact that $\operatorname{Tor}_i^A(N, L) \cong \operatorname{Tor}_i^A(L, N)$[27], we see that $- \otimes_A M_1$ is exact if and only if $\operatorname{Tor}_1^A(L, M_1) = 0$ for all A-modules L, if and only if $\operatorname{Tor}_1^A(M_1, L) = 0$. If we apply the long exact sequence of $\operatorname{Tor}_n^A(-, L)$ to the short exact sequence in the statement of this proposition, we obtain the exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_2^A(M_3, L) \longrightarrow \operatorname{Tor}_1^A(M_1, L) \longrightarrow \operatorname{Tor}_1^A(M_2, L) \longrightarrow \cdots.$$

The outer two terms are zero by Theorem 16.6 and the fact that M_2 and M_3 are flat. To show all higher Tor vanish we need to see that the sequence,

$$0 \longrightarrow Z^i \otimes_A L \longrightarrow M^i \otimes_A L \longrightarrow Z^{i+1} \otimes_A L \longrightarrow 0,$$

is still exact, as then $Z^i \otimes_A L = \ker(d^i \otimes_A L) = \operatorname{im}(d^{i-1} \otimes ._A L)$. This follows since the obstruction to the first map being injective is the existence of $\operatorname{Tor}_1^A(Z^{i+1}, L)$, but this is zero since Z^{i+1} is flat by induction. □

The proof of this lemma is then finished by the following theorem. □

Theorem 16.6. *Let A be a ring, and M and N some A-modules. Recall that $\operatorname{Tor}_i^A(-, N)$ is the ith left derived functor of $M \mapsto M \otimes_A N$. There is a natural isomorphism*

$$\operatorname{Tor}_i^A(M, N) \cong \operatorname{Tor}_i^A(N, M).$$

[26]Recall exercise 8.3.

[27]This might seem almost tautological since $N \otimes_A L \cong L \otimes_A N$, but what we are really saying here is that we can calculate the Tor functor by taking projective resolutions in either variable. See Theorem 16.6

Proof. Let $P^\bullet \to M$ and $Q^\bullet \to N$ be projective resolutions of M and N respectively, and consider the double complex $C^{p,q} = M^p \otimes_A N^q$. We have two spectral sequences which compute the cohomology of the total complex of $C^{\bullet,\bullet}$. One spectral sequence looks at $C^{\bullet,\bullet}$ and calculates cohomology in the horizontal direction first, which, using the fact that $- \otimes N^i$ is exact (as N^i is projective which implies it is flat) we obtain an E_0-page with only entries $M \otimes_A N^p$ along the $q = 0$ column. Hence this spectral sequence collapses immediately and we obtain $\mathrm{Tor}_i^A(N, M) \cong H^{-i}(\mathrm{Tot}C^{\bullet,\bullet})$. Similarly from the other spectral sequence we obtain $\mathrm{Tor}_i^A(M, N) \cong H^{-i}(\mathrm{Tot}C^{\bullet,\bullet})$, which gives us our result. □

There should be a nice way of proving Lemma 16.3 without using Theorem 16.6, but the proof of Lemma 16.3 is quite nice and Theorem 16.6 is interesting in its own right.

Remark 16.7. Lemma 16.6 is a special case of the assertion that derived functors (such as $- \otimes_A^{\mathbb{L}} L$) can be computed by acyclic resolutions. We now reformulate Theorem 15.7 to obtain a fully derived statement. Consider the set-up from the beginning of the lecture, where $Y' = \mathrm{Spec}\, B'$ and $Y = \mathrm{Spec}\, B$ and $\mathcal{M}$ is a quasi-coherent sheaf on X, which is flat over Y, then obtain the statement

$$R\Gamma(X, \mathcal{M}) \otimes_B^{\mathbb{L}} B' \cong R\Gamma(X', g'^*\mathcal{M}).$$

Theorem 16.8. *Let $f : X \to Y = \mathrm{Spec}\, B$ be a proper map, with B noetherian, and let $\mathcal{M}$ be a coherent sheaf on X which is flat over Y. Then $R\Gamma(X, \mathcal{M})$ is computed by a complex $N^\bullet$ of finite projective B-modules. Moreover, $\mathcal{N}^\bullet$ can be used in the previous base change proposition (Proposition 16.2) and one can assume $N^i = 0$ for $i > \dim f = \sup_{y \in Y} \dim f^{-1}(y)$.*

We can reformulate this theorem in slightly nicer language.

Definition 16.9. *Let A be a ring. An object $C \in D(A)$ is called a perfect complex if it can be represented by a complex of finite projective A-modules.*

Hence the theorem above simply says $R\Gamma(X, \mathcal{M})$ is a perfect complex of B-modules, and this theorem is equivalent to the following lemma.

Lemma 16.10. *Let B be a noetherian ring and $N^\bullet$ a bounded (in both directions!) complex of flat B-modules such that all $H^i(N^\bullet)$ are finitely generated for all $i \in \mathbb{Z}$. Then $N^\bullet \in D(B)$ is a perfect complex.*

We will prove this next lecture, and in the meantime we make an observation.

Remark 16.11. We can calculate $H^i(X, \mathcal{M}) = 0$ for $i > \dim f$. If not, let $i > \dim f$ be the maximal i with $H^i(X, \mathcal{M}) \neq 0$. Then let $y \in Y$ be in the support of some non-zero section of $H^i(X, \mathcal{M})$, then we have $H^i(X, \mathcal{M}) \otimes_B k(y) = H^i(X_y, \mathcal{M}_y)$. As $- \otimes_B k(y)$ is right exact this implies that $H^i(X_y, \mathcal{M}_y) \neq 0$, which is a contradiction.

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A corollary of Remark 16.7 is the following.

Corollary 17.1. *There is a spectral sequence*

$$E_2^{p,q} = \operatorname{Tor}_p^B(H^q(X, \mathcal{M}), B') \Longrightarrow H^{p+1}(X', g'^*\mathcal{M}),$$

which is non-zero only if $p \leq 0$ and $q \geq 0$.

By looking at the upper right corner for maximal i such that $H^i(X, \mathcal{M}) \neq 0$ we have

$$H^i(X', g'^*\mathcal{M}) = H^i(X, \mathcal{M}) \otimes_B B',$$

and $H^j(X', g'^*\mathcal{M}) = 0$ for all $j > i$. Also notice that if all $H^q(X, \mathcal{M})$ are flat over B, then $\operatorname{Tor}_p^B(-,-) = 0$ for $p \neq 0$, so we obtain $H^q(X', g'^*\mathcal{M}) = H^q(X, \mathcal{M}) \otimes_B B'$. This gives us the following corollary.

Corollary 17.2. *If $f : X \to Y$ is proper, and $\mathcal{M}$ a coherent sheaf on X which is flat over Y, then $R^i f_* \mathcal{M} = 0$ for all $i > \dim f$.*

This is true in more generality, but we only need it as stated.

Proof. Without loss of generality we can take $Y = \operatorname{Spec} B$ to be affine. Consider a maximal i such that $H^i(X, \mathcal{M}) \neq 0$, and notice this cohomology is a finitely generated B-module. Take $y \in Y$ in the support of $H^i(X, \mathcal{M})$ so that $H^i(X, \mathcal{M}) \otimes_B k(y) \neq 0$. However, by the above we have

$$H^i(X, \mathcal{M}) \otimes_B k(y) = H^i(X_y, g'^*\mathcal{M}),$$

where $h' : X_y = X \times_Y \operatorname{Spec} k(y) \to X$ and $H^i(X_y, g'^*\mathcal{M}) = 0$ for $i > \dim X_y$. Thus $i \leq \dim X_y \leq \dim f$. □

This is essentially the proof outlined at the end of the last lecture. Now we would like to prove the most important finiteness result we have seen, or will see in this course.

Theorem 17.3. *Let $f : X \to Y = \operatorname{Spec} B$ be a proper map, with B noetherian, $\mathcal{M}$ and coherent sheaf over X which is flat over Y, and set $d = \dim f$. Then $R\Gamma(X, \mathcal{M})$ is quasi-isomorphic to a bounded complex $N^\bullet$ of finite projective B-modules, with $N^i = 0$ for $i < 0$ or $i > n$.*

Proof. We know that $R\Gamma(X, \mathcal{M})$ is quasi-isomorphic to a complex of the form,

$$\cdots \longrightarrow 0 \longrightarrow N'^0 \longrightarrow \cdots \longrightarrow N'^m \longrightarrow 0 \longrightarrow \cdots,$$

where all N'^i are flat B-modules which are acyclic in degrees greater than d. All we need to know now (since $H^i(X, \mathcal{M})$ is finitely generated) is the following.

For all $0 \leq i \leq d$, there is a commutative diagram, whose rows are complexes and N^j are finite free B-modules,

$$\begin{array}{ccccccccccccccc} & & & & 0 & \longrightarrow & N^i & \longrightarrow & \cdots & \longrightarrow & N^d & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N'^0 & \longrightarrow \cdots \longrightarrow & N'^{i-1} & \longrightarrow & N'^i & \longrightarrow & \cdots & \longrightarrow & N'^d & \longrightarrow & N'^{d+1} & \longrightarrow & \cdots \end{array}, \quad (17.4)$$

such that the cone of the vertical map of complex is acyclic in degrees larger than i. We will show this by descending induction on i. For $i = d$ we consider $\ker(N'^d \to N'^{d+1}) \twoheadrightarrow H^d(X, \mathcal{M})$ and let

$s_1, \ldots, s_k \in H^d(X, \mathcal{M})$ be generators. We lift these to elements in this kernel and we obtain a map $f : N^d = B^k \to N'^d$ which does the job. To check this we see the differentials of the cone are

$$C_f^{d-1} = N^{d-1} \oplus N^d \longrightarrow C_f^d = N'^d \longrightarrow C_f^{d+1} = N'^{d+1},$$

and this is exact. Now assume we have a situation as in Diagram 17.4, then we obtain the following exact sequence,

$$H^i(N'^\bullet) \longrightarrow H^i(C_f) \longrightarrow H^{i+1}(N^\bullet),$$

which implies the cohomology $H^i(C_f)$ is finitely generated. We can then find a N^i which surjects onto $H^i(C_f)$, which gives us the following diagram,

$$\begin{array}{ccccc} N^i & \longrightarrow & N^{i+1} & \longrightarrow & N^{i+2} \\ \downarrow & & \downarrow & & \downarrow \\ N'^i & \longrightarrow & N'^{i+1} & \longrightarrow & N'^{i+2} \end{array},$$

as desired. This shows our claim, so we have Diagram 17.4 for $i = 0$. We now consider the cone C_f which is a bounded complex of flat B-modules which are acyclic outside of degree 0, so we have the following diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_f^0 & \longrightarrow & \cdots & \longrightarrow & C_f^m & \longrightarrow & 0 \\ & \searrow & \nearrow & & & \searrow & \nearrow & & \\ & Z^1 & & & & Z^m & & & \end{array},$$

where Z^j are the kernels (and also images) of the differentials of C_f. Observe the map $Z^m \to C_f^m$ is flat. As we did last time, we would use descending induction to see that all Z^i are flat B-modules. Now let $N^0 = \ker(C_f^0 \to Z^1) = H^0(C_f)$. This is alse a flat B-module, as a kernel of a surjective map of flat modules, which is also finitely generated over a noetherian ring B, at which stage we recall that flat and finitely presented is equivalent to finite projective. Notice that $N^0 = H^0(C) = \ker(N'^0 \oplus N^q \to N'^1 \oplus N^2)$, so the diagram,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & N^0 & \longrightarrow & N^1 & \longrightarrow & \cdots & \longrightarrow & N^d & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & N'^0 & \longrightarrow & N'^1 & \longrightarrow & \cdots & \longrightarrow & N'^d & \longrightarrow & 0 \end{array},$$

gives us the map of complexes which does the job. □

We then obtain the following corollaries of this theorem.

Corollary 17.5. *In the situation of Theorem 17.3 we see the function*

$$y \in Y \mapsto \chi(X_y, \mathcal{M}|_{X_y}) = \sum_{i \geq 0} (-1)^i \dim H^i(X_y, \mathcal{M}|_{X_y}),$$

is locally constant.

Corollary 17.6. *In the situation of Theorem 17.3 we see the function,*

$$y \in Y \mapsto \dim H^i(X_y, \mathcal{M}|_{X_y}),$$

for all $i \geq 0$ is upper semi-continuous (may jump up under specialisation).

We come back to the latter corollary above one of these in lecture 22, see Corollaries 22.3 and 22.5, and a version of Corollary 17.5 in exercise 11.4. Now we move onto the big finale of this course; the Riemann-Roch Theorem and Serre Duality.

For the rest of this lecture we let k be algebraically closed, and X be a smooth projective curve[28]. Recall that $\mathrm{Div}(X)$ is the free abelian group on the closed points of X, and $K(X)$ is the function field of X. We then have the following theorem proved last semester as Theorem 28.7 in [7].

Theorem 17.7. *The following sequence is exact,*

$$0 \longrightarrow k^\times \longrightarrow K^\times \longrightarrow \mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

We will skip a proof of the following proposition. A similar statement can be found in [6], Proposition II.6.4.

Proposition 17.8. *The degree map* $\mathrm{Div}(X) \to \mathbb{Z}$ *which sends* $\sum n_x[x] \mapsto \sum n_x$ *factors over* $\mathrm{Pic}(X)$.

The Riemann-Roch theorem can then be stated as follows.

Theorem 17.9 (Riemann-Roch)**.** *For any line bundle* $\mathcal{L}$ *on* X*, we have*

$$\dim_k H^0(X, \mathcal{L}) - \dim_k H^1(X, \mathcal{L}) = \chi(X, \mathcal{L}) = \deg \mathcal{L} + 1 - g,$$

where $g = \dim_k H^1(X, \mathcal{O}_X)$.

Proposition 17.8 and the Riemann-Roch Theorem are equivalent to the following theorem.

Theorem 17.10. *For any divisor* D *we have,*

$$\chi(X, \mathcal{O}(D)) = \deg D + 1 - g.$$

Indeed, Theorem 17.10 implies that $\deg(D)$ depends only on the line bundle $\mathcal{O}(D)$, thus Proposition 17.8 amounts to choosing D such that $\mathcal{L} \cong \mathcal{O}(D)$.

Proof of Theorem 17.10. We first do the base case, where $D = 0$, from which we obtain $\chi(X, \mathcal{O}_X) = \dim_k H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X) = 1 - g$ since $\deg D = 0$. For some induction we assume $D' = D + [x]$. We then have the short exact sequence,

$$0 \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{O}(D') \longrightarrow i_* k(x) \longrightarrow 0,$$

where $i : \{x\} \hookrightarrow X$ is the inclusion of a point. This gives us a long exact sequence on cohomology,

$$0 \longrightarrow H^0(\mathcal{O}(D)) \longrightarrow H^0(\mathcal{O}(D')) \longrightarrow H^0(i_* k) \longrightarrow H^1(\mathcal{O}(D)) \longrightarrow H^1(\mathcal{O}(D')) \longrightarrow H^1(i_* k) = 0,$$

where all the cohomologies are taken over X. This implies $\chi(\mathcal{O}(D')) = \chi(\mathcal{O}(D)) + 1$, which gives us what we want. Similarly we induct in the other direction by "subtracting" $[x]$, but this is a similar computation. $\square$

Notice that with the power of sheaf cohomology we have just proved the Riemann-Roch theorem in a few lines. Suppose now that k is not algebraically closed, X is a smooth projective curve which is geometrically connected (i.e. $X \otimes_k \overline{k}$ is connected), then we still have,

$$0 \longrightarrow k^\times \longrightarrow K(X)^\times \longrightarrow \mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

However, we need to define the degree function slightly differently,

$$\deg\left(\sum n_x[x]\right) = \sum n_x \deg(k(x)/k).$$

[28] Recall a curve over k is an integral scheme over $\mathrm{Spec}\, k$ of dimension 1 (see Definition 20.9 in [7].

This is clearly a generalisation of the previous definition of the degree map when k is algebraically closed. Again we see the degree function factors over $\mathrm{Pic}(X)$ and we have $\chi(X,\mathcal{L}) = \deg \mathcal{L} + 1 - g$ and the proof is the same, just noting that $\chi(C, i_*k(x)) = \dim_k k(x) = \deg(k(x)/k)$. Notice that $\chi(X,\mathcal{L}) = \chi(X_{\overline{k}}, \mathcal{L}_{\overline{k}})$, as $H^i(X_{\overline{k}}, \mathcal{L}_{\overline{k}}) = H^i(X,\mathcal{L}) \otimes_k \overline{k}$, which implies these degrees, $\deg \mathcal{L}$ and $\deg \mathcal{L}_{\overline{k}}$ are also equal. We could also check this directly and deduce the Riemann-Roch theorem over k from the Riemann-Roch theorem over $\overline{k}$. To get the Riemann-Roch theorem in terms of the language of last semester, we actually need some more machinery.

Theorem 17.11 (Serre Duality I)**.** *Let k be a field and X a proper smooth scheme over k, then for any vector bundle ξ over X there is a canonical isomorphism*

$$H^i(X, \omega_{X/k} \otimes \xi^\vee) = \mathrm{Hom}_k(H^{d-i}(X,\xi), k),$$

where $0 \leq i \leq d = \dim X$ and $\omega_{X/k} = \Lambda^d \Omega^1_{X/k}$ is the highest exterior power of $\Omega^1_{X/k}$, which we call the canonical line bundle.

Specialising X to a curve, $i = 0$ and $\xi = \mathcal{L}$ we obtain

$$\dim_k H^1(X,\mathcal{L}) = \dim_k H^0(X, \Omega^1_{X/k} \otimes \mathcal{L}^\vee).$$

There is a even a formulation for all coherent sheaves $\mathcal{M}$ in the case that $\mathcal{M}^\vee$ is well-behaved. We need to use the Ext-functor in this case.

Definition 17.12. *Let A be a ring and M, N two A-modules. Then we define $\mathrm{Ext}^i(M,N)$ to be the ith right derived functor of $N \mapsto \mathrm{Hom}(M,N)$. Similarly, if $(X,\mathcal{O}_X)$ is a ringed space, then we define $\mathrm{Ext}^i_{(X,\mathcal{O}_X)}(\mathcal{M},\mathcal{N})$ to be the ith right derived functor of $\mathcal{N} \mapsto \mathrm{Hom}(\mathcal{M},\mathcal{N})$.*

Proposition 17.13. *Let X be a scheme and $\mathcal{N}$ a quasi-coherent sheaf. Then*

$$\mathrm{Ext}^i_X(\mathcal{O}_X, \mathcal{N}) \cong H^i(X,\mathcal{N}).$$

More generally, if ξ is an vector bundle (a locally finite free $\mathcal{O}_X$-module), then

$$\mathrm{Ext}^i_X(\xi,\mathcal{N}) \cong H^i(X, \mathcal{N} \otimes \xi^\vee).$$

Proof. For the first claim we notice that both sides of the equation are the derived functors of

$$\mathcal{N} \mapsto \mathrm{Hom}_X(\mathcal{O}_X,\mathcal{N}) \cong N(X),$$

where the last isomorphism is obtained by the Yoneda lemma. For the second part we notice that $- \otimes_{\mathcal{O}_X} \xi^\vee$ is exact and preserves injectives (as it has an exact left adjoint, $- \otimes_{\mathcal{O}_X} \xi$) and hence both sides are derived functors of

$$\mathcal{N} \mapsto \mathrm{Hom}_X(\xi,\mathcal{N}) \cong \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{N} \otimes \xi^\vee) \cong (\mathcal{N} \otimes \xi^\vee)(X).$$

□

Serre duality can now be reformulated as the following theorem.

Theorem 17.14 (Serre Duality II)**.** *Let k be a field, X a proper smooth scheme over $\mathrm{Spec}\, k$ with $H^0(X,\mathcal{O}_X) \cong k$, $d = \dim X$, and $\omega_{X/k} = \Lambda^d \Omega^1_{X/k}$, the canonical line bundle (dualising sheaf).*

1. *There is a (canonical) isomorphism for any coherent sheaf ξ,*

$$H^d(X, \omega_{X/k}) \longrightarrow k.$$

2. *The natural pairing,*

$$\mathrm{Ext}^i_X(\xi, \omega_{X/k}) \times H^{d-i}(X,\xi) \longrightarrow H^d(X,\omega_{X/k}) \cong k,$$

is perfect, i.e. it induced a canonical isomorphism $\mathrm{Ext}^i(-,-) \cong H^{d-i}(-,-)^\vee$.

The full proof of this will occupy the next few lectures.

18 Ext Functor and Serre Duality 10/07/2017

Recall Theorem 17.14, which in a way is like an algebraic Poincaré duality. Today we will focus on studying these Ext groups in Theorem 17.14, and from this we will obtain our natural pairing. Let $\mathcal{A}$ be an abelian category with enough injectives (such as the category of $\mathcal{O}_X$-modules over a scheme X).

Definition 18.1. *The functor* $\operatorname{Ext}^i_{\mathcal{A}}(X,-) : \mathcal{A} \to \mathrm{Ab}$ *is the ith right derived functor of* $\operatorname{Hom}_{\mathcal{A}}(X,-)$ *for some* $X \in \mathcal{A}$*.*

Recall the following diagram of subcategories and equivalences,

$$\begin{array}{ccc} D^+(\mathcal{A}) & \xrightarrow{=} & C^+(\mathcal{A})/\{\mathrm{quasi-iso.}\} \\ \downarrow & & \downarrow \\ D(\mathcal{A}) & \xrightarrow{=} & C(\mathcal{A})/\{\mathrm{quasi-iso.}\} \end{array},$$

and recall we saw in Theorem 11.6 that $D^+(\mathcal{A}) \cong K^+(\mathrm{Inj}(\mathcal{A}))$. In fact, if $X^\bullet \in C(\mathcal{A})$ is any complex, $I^\bullet \in C^+(\mathrm{Inj}(\mathcal{A}))$, then we have

$$\operatorname{Hom}_{D(\mathcal{A})}(X^\bullet, I^\bullet) = \operatorname{Hom}_{C(\mathcal{A})}(X^\bullet, I^\bullet)/\mathrm{homotopy}.$$

This is related to the observation that in all lemmas about complexes of injectives (from lectures 10 and 11) it was only really necessary to assume that target was injective.

Moral: Mapping into injectives, and mapping out of projectives is well-behaved.

If X is any object in $\mathcal{A}$ and $i \in \mathbb{Z}$, then we have a complex $X[i]$ which has a single X concentrated in degree $-i$. Given a complex C, we define $C[i]$ to be C shifted to the left by $-i$ where the differentials have a sign of $(-1)^i$. This gives us a functor $D(\mathcal{A}) \to D(\mathcal{A})$, and we want to analyse the maps to and from the image of this functor in $D(\mathcal{A})$.

Proposition 18.2. *Let* $X, Y \in \mathcal{A}$*, then we have*

$$\operatorname{Hom}_{D(\mathcal{A})}(X, Y[i]) = \begin{cases} \operatorname{Ext}^i_{\mathcal{A}}(X,Y) & i \geq 0 \\ 0 & i < 0 \end{cases}.$$

Proof. Let $Y \hookrightarrow I^\bullet$ be an injective resolution of Y so that $Y \cong I^\bullet$ inside $D(\mathcal{A})$. Then we have,

$$\operatorname{Hom}_{D(\mathcal{A})}(X, Y[i]) \cong \operatorname{Hom}_{D(\mathcal{A})}(X, I^\bullet[i]) \cong \operatorname{Hom}_{C(\mathcal{A})}(X, I^\bullet)/\mathrm{homotopy}.$$

What is a map between these complexes though? It is simply a map $X \to I^i$ inside $\mathcal{A}$, such that $X \to I^i \to I^{i+1}$ is zero, up to homotopy. This is then exactly,

$$\ker(\operatorname{Hom}_{\mathcal{A}}(X, I^i) \to \operatorname{Hom}_{\mathcal{A}}(X, I^{i+1}))/\operatorname{im}(\operatorname{Hom}_{\mathcal{A}}(X, I^{i-1}) \to \operatorname{Hom}_{\mathcal{A}}(X, I^i)),$$

which by definition is $\operatorname{Ext}^i_{\mathcal{A}}(X,Y)$. □

Notice that this proposition allows us to define $\operatorname{Ext}^i_{\mathcal{A}}$ in an abelian category $\mathcal{A}$ even if $\mathcal{A}$ does not have enough injectives. Notice also that this shift functor $X \mapsto X[1]$ is a self-equivalence of categories on $C(\mathcal{A})$ and $D(\mathcal{A})$, so for all $X, Y \in \mathcal{A}$ and $i, j \in \mathbb{Z}$ we obtain,

$$\operatorname{Hom}_{D(\mathcal{A})}(X[i], Y[j]) \cong \operatorname{Hom}_{D(\mathcal{A})}(X, Y[j-i]) = \begin{cases} \operatorname{Ext}^{j-i}_{\mathcal{A}}(X,Y) & j \geq i \\ 0 & j < i \end{cases}.$$

Corollary 18.3. *Given* $X, Y, Z \in \mathcal{A}$ *and* $i, j \in \mathbb{Z}$*, there is a natural bilinear pairing,*

$$\operatorname{Ext}^i_{\mathcal{A}}(X,Y) \times \operatorname{Ext}^j_{\mathcal{A}}(Y,Z) \longrightarrow \operatorname{Ext}^{i+j}_{\mathcal{A}}(X,Z).$$

Proof. The left hand side is simply $\mathrm{Hom}_{D(\mathcal{A})}(X, Y[i]) \times \mathrm{Hom}_{D(\mathcal{A})}(Y[i], Z[i+j])$, and we can apply composition and obtain a map into $\mathrm{Hom}_{D(\mathcal{A})}(X, Z[i+j]) \cong \mathrm{Ext}^{i+j}_{\mathcal{A}}(X, Z)$. □

Just like in the case where $\mathcal{A}$ is the category of R-modules, there is an explicit description of $\mathrm{Ext}^1_{\mathcal{A}}(X, Y)$, from which the functor gets its name.[29]

Proposition 18.4. *For $X, Y \in \mathcal{A}$ we have the group $\mathrm{Ext}^1_{\mathcal{A}}(X, Y)$ can be identified with the isomorphism classes of extensions of the form,*

$$0 \to Y \to Z \to X \to 0,$$

where $Z \in \mathcal{A}$, and two extensions are isomorphic if we have an isomorphism $Z \to Z'$ which commutes with all the maps between Y and X.

Proof. Let $\mathrm{YExt}^1_{\mathcal{A}}(X, Y)$ denote the isomorphism classes of extensions, then we have a natural map,

$$\alpha : \mathrm{YExt}^1_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(X, Y),$$

which assigns to an extension $0 \to Y \to Z \to X \to 0$ the element $\delta(\mathrm{id}_X)$ from the long exact sequence of $\mathrm{Ext}^i_{\mathcal{A}}$ associated to the short exact sequence,

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, X) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{A}}(X, Y) \longrightarrow \cdots.$$

This is clearly well-defined under isomorphism classes of extensions. First notice α is injective, since if $\delta(\mathrm{id}_X) = 0$ then $\mathrm{id}_X : X \to X$ lifts to a map $X \to Z$, i.e. associated exact sequence splits, hence was zero inside $\mathrm{YExt}^1_{\mathcal{A}}(X, Y)$. Next notice that α is surjective. To see this, we first embed $Y \hookrightarrow I$ where I is something injective, and by taking the cokernel we obtain the short exact sequence,

$$0 \longrightarrow Y \longrightarrow I \longrightarrow Q \longrightarrow 0.$$

From this we obtain the following long exact sequence on cohomology,

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, Q) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(X, I).$$

This last group is zero since I is injective so we have a map $X \to Q$. By pulling back we obtain $Z = I \times_Q X$, which fits into the follow commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & I & \longrightarrow & Q & \longrightarrow & 0 \end{array}.$$

This gives us an element in $\mathrm{YExt}^1_{\mathcal{A}}(X, Y)$ with the desired image. □

We now make a quite formal argument about the functor Ext.

Proposition 18.5. *Let $X \in \mathcal{A}$ and given the following short exact sequence in $\mathcal{A}$,*

$$0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0,$$

then we have the following long exact sequences,

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, Y') \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, Y'') \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(X, Y') \longrightarrow \cdots,$$

and

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Y'', X) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Y, X) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Y', X) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(Y'', X) \longrightarrow \cdots.$$

[29] There is a similar description for higher $\mathrm{Ext}^i_{\mathcal{A}}(X, Y)$ in terms of exact sequences starting with Y, ending with X and with i-many pieces in between, but the actual description becomes a little cumbersome and is relatively unuseful. This is sometimes called the Yoneda extension, hence the YExt notiation appearing here.

Proof. The first long exact sequence comes from the fact that $\mathrm{Ext}^i_{\mathcal{A}}(X,-)$ is defined as a right derived functor, so it is in fact a δ-functor. For the second sequence, let $X \hookrightarrow I^\bullet$ be an injective resolution then we have the following diagram with exact rows,

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(Y'', I^0) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(Y'', I^1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(Y, I^0) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(Y, I^1) & \longrightarrow & \cdots . \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(Y', I^0) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(Y', I^1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

This diagram actually also has exact columns since taking a $\mathrm{Hom}_{\mathcal{A}}$ into an injective object is exact. From this diagram we obtain a short exact sequence of complexes, and then the desired long exact sequence on cohomology. □

Proposition 18.6. *Let $\mathcal{A}$ have enough projectives, then $\mathrm{Ext}^i_{\mathcal{A}}(-,X)$ is the ith right derived functor of* $\mathrm{Hom}_{\mathcal{A}}(-,X) : \mathcal{A}^{op} \to \mathrm{Ab}$.

Proof. The second long exact sequence of Proposition 18.5 says that this is a δ-functor, so it suffices to see it is also effaceable, for which it suffices to see whenever $Y \in \mathcal{A}$ is projective we have $\mathrm{Ext}^i_{\mathcal{A}}(Y,X) = 0$ for $i > 0$. For $i = 1$ we have Proposition 18.4 which say that $\mathrm{Ext}^1_{\mathcal{A}}(Y,X) = 0$ since all the short exact sequences

$$0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0,$$

are split as Y is projective. For $i > 1$ we choose a short exact sequence $0 \to X \to I \to Q \to 0$ with I injective, and then look at the associated long exact sequence,

$$\cdots \longrightarrow \mathrm{Ext}^{i-1}_{\mathcal{A}}(Y,Q) \longrightarrow \mathrm{Ext}^i_{\mathcal{A}}(Y,X) \longrightarrow \mathrm{Ext}^i(Y,I) \longrightarrow \cdots .$$

The first group is zero by induction, and the last group since I is injective. □

This proposition is not super useful to use, since the category of $\mathcal{O}_X$-modules does not always have enough projectives, but R-mod always does, so there is some salvation. Let us specialise now. Consider the pairing of Corollary 18.3, in the case where $\mathcal{A}$ is the category of $\mathcal{O}_X$-modules, $i = d - j$, $X = \mathcal{O}_X$, $Y = \xi$ is some coherent sheaf, and $Z = \omega_X$ is our dualising sheaf,

$$H^{d-i}(X,\xi) \times \mathrm{Ext}^i(\xi,\omega_X) \cong \mathrm{Ext}^{d-i}(\mathcal{O}_X,\xi) \times \mathrm{Ext}^i(\xi,\omega_X) \longrightarrow \mathrm{Ext}^d(\mathcal{O}_X,\omega_X) = H^d(X,\omega_X).$$

Before we tackle this, let us consider the isomorphism $H^d(X,\omega_X) \cong k$ in the case when $k = \mathbb{C}$ and $d = 1$, since this example contains a lot of geometry that is lost in the general case.

Let X be a projective smooth connected curve over $\mathbb{C}$, then we call $X(\mathbb{C})$ a (compact) Riemann surface. The goal of this little excursion is going to be to construct a natural map

$$H^1(X, \Omega^1_{X/\mathbb{C}}) \longrightarrow \mathbb{C}.$$

Let $\Delta^* = \{z \in \mathbb{C} | 0 < |z| < 1\}$ be the punctured open unit disc in $\mathbb{C}$, then we say $f : \Delta^* \to \mathbb{C}$ is meromorphic at 0 if there exists some $n \geq 1$ such that $f(x) \cdot z^n$ extends to a holomorphic function on $\delta = \{z \in \mathbb{C} || z| < 1\}$. This is equivalent to the existence of $n \geq 1$ such that $|f(z)| < |z^{-n}|$ as $|z| \to 0$ by Riemann's theory of bounded holomorphic functions extending over a puncture. Let $\omega = f(z)dz$ be a meromorphic differential form on Δ^*, i.e. $f(z)$ is a meromorphic function on Δ^*.

Definition 18.7. *The residue of ω at 0 is* $\mathrm{res}_{z=0}\omega = a_{-1} \in \mathbb{C}$ *where* $f(z) = \sum_{n=-N}^{\infty} a_n z^n$*, with* $a_n \in \mathbb{C}$ *and some growth condition.*

A priori, it is not clear this is invariant under coordinate transformations such as $z \mapsto z + z^2$, which is an auotmorphism in a small ball around 0.

Theorem 18.8 (Cauchy's Residue Theorem)**.** *Let* $\gamma : [0,1] \to \mathbb{C}$ *be a smooth small circle around 0, e.g.* $t \mapsto e^{2\pi i t}$*, then*

$$\int_\gamma \omega = 2\pi i \ \mathrm{res}_{z=0}\omega.$$

Proof. Simply calculating we obtain,

$$\int_\gamma \omega = \int_0^1 f(e^{2\pi i t}) d(e^{2\pi i t}) = \sum_{n=-N}^{\infty} a_n \int_0^1 e^{2\pi i t(n+1)} 2\pi i dt = 2\pi i a_{-1},$$

where the second equality comes from some convergence result in analysis, and $\int_0^1 e^{2\pi i t m} dt$ is zero if $m \neq 0$ and one if $m = 0$. We then do some complex analysis to show this integral is invariant under small perturbation. □

Corollary 18.9. *The map* $\mathbb{C}((z))dz \to \mathbb{C}$ *which sends a Laurent series to* a_{-1} *is invariant under automorphisms of* $\mathbb{C}((z)) = \Omega^1_{\Delta^*}$ *that are given by* $z \mapsto b_1 z + b_2 z^2 + \cdots$ *for* $b_1 \in \mathbb{C}^\times$ *and* $b_i \in \mathbb{C}$ *for* $i \geq 2$.

A priori this only holds if all these series converge, but all such series are dense here. The same statement holds true for any field, but at least the classical proofs reduce this to the case over $\mathbb{C}$ (see exercise 12.4). Back to our curves X over $\mathbb{C}$. Let $x, y \in X$ be two distinct points, then $U = X\backslash\{x\} = \mathrm{Spec}\, A$ and $V = X\backslash\{y\} = \mathrm{Spec}\, B$ are (affine[30]) opens covering X. We then see that $H^1(X, \Omega^1_{X/\mathbb{C}})$ is computed by taking the cokernel of,

$$\Omega^1_{A/\mathbb{C}} \oplus \Omega^1_{B/\mathbb{C}} \longrightarrow \Omega^1_{D/\mathbb{C}},$$

by Meyer-Vietoris, where $U \cap V = \mathrm{Spec}\, D$. One can think of $\Omega^1_{D/\mathbb{C}}$ as differential forms on $X(\mathbb{C})\backslash\{x, y\}$ which are meromorphic at x and y. We have two maps $\Omega^1_{D/\mathbb{C}} \to \mathbb{C}$, taking the residues at x or y respectively. We claim these maps only differ by a sign.

Lemma 18.10. *Given the set-up above, we have*

$$\mathrm{res}_x + \mathrm{res}_y = 0 : \Omega^1_{D/\mathbb{C}} \longrightarrow \mathbb{C}.$$

Proof. Take some $z \in X(\mathbb{C})$ and integrate some $\omega \in \Omega^1_{D/\mathbb{C}}$ around some small disc with boundary γ, which contains z but not x or y. Then we have $\int_\gamma \omega = 0$ since γ doesn't contain either poles of ω, but by changing the orientation of γ we obtain

$$0 = -\int_\gamma \omega = 2\pi i(\mathrm{res}_x\omega + \mathrm{res}_y\omega).$$

□

A corollary of this lemma is the following.

Corollary 18.11. *The map* $\mathrm{res}_x : \Omega^1_{D/\mathbb{C}} \to \mathbb{C}$ *vanishes on the image of* $\Omega_{A/\mathbb{C}} \oplus \Omega_{B/\mathbb{C}}$*, so it factors over a map,*

$$H^1(X, \Omega^1_{X/\mathbb{C}}) \longrightarrow \mathbb{C}.$$

[30]Removing a point from any proper smooth curve over a field is an affine curve. To see this we notice that $\mathcal{O}(nx)$ is ample for some n sufficiently large, hence we obtain a closed embedding into $\mathbb{P}^n_k$ for some n. With this embedding there is a hyperplane $H \subseteq \mathbb{P}^n_k$ with the property that $H \cap X = \{x\}$ set theoretically, and so $X\backslash\{x\} \subseteq \mathbb{P}^n_k\backslash\mathbb{P}^n_k \cong \mathbb{A}^n_k$.

Proof. This is clear for $\Omega^1_{B/\mathbb{C}}$ as these are holomorphic at x and $\mathrm{res}_x = -\mathrm{res}_y$, and a similar argument works for $\Omega^1_{A/\mathbb{C}}$. ☐

This approach to constructing this map relating duality is worked out in full generality in Hartshorne's "Residues and Duality". We will now give a short proof outline of Theorem 17.14.

Sketch of a Proof of Theorem 17.14. Let X be projective (a hypothesis we really need for this arguement). We then procede in a few steps.

1. Let $X = \mathbb{P}^n_k$, then $\omega_X = \mathcal{O}(-n-1)$ and we explicitly know $H^n(\mathbb{P}^n_k, \mathcal{O}(-n-1)) \cong k$, and we then use a reduction argument to reduce this to the case when $\xi = \mathcal{O}(d)$ for some $d > 0$. In this case it is again an explicit computation. This is duality for $\mathbb{P}^n_k \to \operatorname{Spec} k$.

2. In general, we now choose a closed embedding $i : X \hookrightarrow \mathbb{P}^n_k$. We then know that

$$\mathrm{Ext}^j_{\mathbb{P}^n_k}(i_*\xi, \omega_{\mathbb{P}^n_k}) \times H^{n-j}(\mathbb{P}^n_k, i_*\xi) \cong \mathrm{Ext}^j_{\mathbb{P}^n_k}(i_*\xi, \omega_{\mathbb{P}^n_k}) \times H^{n-j}(X, \xi) \longrightarrow k,$$

   is a perfect pairing, and what remains is an identification $\mathrm{Ext}^{j+n-d}_{\mathbb{P}^n_k}(i_*\xi, \omega_{\mathbb{P}^n_k}) \cong \mathrm{Ext}^j_X(\xi, \omega_X)$. This is duality for $i : X \hookrightarrow \mathbb{P}^n_k$.

3. The previous two parts then combine to give us duality for $X \to \operatorname{Spec} k$.

☐

This is a standard embedding trick, used to prove Poincaré duality and Grothendieck-Hirzebruch-Riemann-Roch.

19 $\mathcal{E}$xt Functor and the Proof of Serre Duality 13/07/2017

Let us reformulate Serre Duality on more time which drops the condition that $H^0(X, \mathcal{O}_X) \cong k$. Today we will see a proof of this theorem in full.

Theorem 19.1 (Serre Duality III). *Let k be a field and X a smooth projective scheme over k of dimension d. There is a trace map,*

$$\mathrm{tr} : H^d(X, \omega_X) \longrightarrow k,$$

such that for all coherent sheaves ξ on X, $0 \le i \le d$,

$$\mathrm{Ext}^i(\xi, \omega_X) \times H^{d-i}(X, \xi) \longrightarrow H^d(X, \omega_X) \longrightarrow k,$$

is a perfect pairing.

Proof of Theorem 19.1 where $X = \mathbb{P}^n_k$. Recall the short exact sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}(1)^{n+1} \longrightarrow \left(\Omega^1_{\mathbb{P}^n_k}\right)^\vee = T_{\mathbb{P}^n_k} \longrightarrow 0,$$

where we call the dual of the sheaf of Kähler differentials $T_{\mathbb{P}^n_k}$ the tangent bundle of $\mathbb{P}^n_k$. Passing to determinants (so taking the highest exterior powers possible), we obtain,

$$\mathcal{O}(n+1) = \Lambda^{n+1}(\mathcal{O}(1)^{n+1}) = \Lambda^1 \mathcal{O}_{\mathbb{P}^n_k} \otimes \Lambda^n T_{\mathbb{P}^n_k} = \omega^\vee_{\mathbb{P}^n_k},$$

using the fact that if we have an exact sequence of finite locally free modules, then the det is multiplicative in the sense indicated above, and det commutes with duals. The above explicitly implies that $\omega_{\mathbb{P}^n_k} = \mathcal{O}(-n-1)$. Recall that we have calculated the cohomology of $\mathcal{O}(-m)$ for $m > 0$ in Proposition 13.3. In particular we have,

$$H^n(\mathbb{P}^n_k, \mathcal{O}(-n-1)) = k \cdot x_0^{-1} \cdots x_n^{-1} \xrightarrow{\cong} k,$$

and this is our trace map. Notice that the identification of $\omega_{\mathbb{P}^n_k} \cong \mathcal{O}(-n-1)$ and this trace map are not canonical, but the composite $H^n(\mathbb{P}^n_k, \omega_{\mathbb{P}^n_k}) \to k$ is canonical. We can see this by taking an explicit generator of $H^n(\mathbb{P}^n_k, \omega_{\mathbb{P}^n_k})$ such as,

$$\frac{x_0^n}{x_1 \cdots x_n} d\left(\frac{x_1}{x_0}\right) \wedge \cdots \wedge d\left(\frac{x_n}{x_0}\right),$$

in the Čech complex. One can then check this is independent of coordinate transformations. We now have maps,

$$\mathrm{Ext}^i(\xi, \omega_{\mathbb{P}^n_k}) \longrightarrow \left(H^{d-i}(X, \xi)\right)^\vee,$$

coming from our pairing of Corollary 18.3 and this trace map. We claim both sides of the above can be regarded as δ-functors from the opposite category of coherent sheaves on $\mathbb{P}^n_k$ to abelian groups, with zeroth functor $\mathrm{Ext}^0(\xi, \omega_{\mathbb{P}^n_k}) = \mathrm{Hom}(\xi, \omega_{\mathbb{P}^n_k})$ respectively $H^n(\mathbb{P}^n_k, \xi)^\vee$. It is clear the left-hand-side is a δ-functor, and for the right-hand-side we have $H^j(\mathbb{P}^n_k, \xi) = 0$ for all $j > n$ and $H^i(\mathbb{P}^n_k, \xi)$ are all finite dimensional k-vector spaces, so $V \mapsto V^\vee$ is exact. This information together tells us that given a short exact sequence,

$$0 \longrightarrow \xi' \longrightarrow \xi \longrightarrow \xi'' \longrightarrow 0,$$

of coherent sheaves on $\mathbb{P}^n_k$ we obtain the long exact sequence,

$$0 \longrightarrow H^n(\mathbb{P}^n_k, \xi'')^\vee \longrightarrow H^n(\mathbb{P}^n_k, \xi)^\vee \longrightarrow H^n(\mathbb{P}^n_k, \xi')^\vee \longrightarrow H^{n-1}(\mathbb{P}^n_k, \xi')^\vee \longrightarrow \cdots,$$

hence the right-hand-side is also a δ-functor. To see these functors are equal, if suffices to check they agree for $i = 0$ and they are effaceable, hence universal. First for effaceability. For any ξ, we can find a surjective $\mathcal{O}(-N)^r \twoheadrightarrow \xi$ for some large N, and then we have

$$\mathrm{Ext}^i(\mathcal{O}(-N), \omega_{\mathbb{P}^n_k}^r) \cong \mathrm{Ext}^i(\mathcal{O}_{\mathbb{P}^n_k}, \omega_{\mathbb{P}^n_k}(N)^r) \cong H^i(\mathbb{P}^n_k, \omega_{\mathbb{P}^n_k}(N)^r).$$

These are zero for $i > 0$ and N sufficiently large though by the calculation of Proposition 13.4 or the vanishing statement of Theorem 15.2. So the left-hand-side is effaceable. For the right-hand-side we want $H^i(\mathbb{P}^n_k, \mathcal{O}(-N)^r) = 0$ for $i < n$, but this follows by direct computation (essentially from similar calculations to Proposition 13.3). To show these universal δ-functors agree for $i = 0$ we choose an exact sequence,

$$O(-N_1)^{r_1} \longrightarrow \mathcal{O}(-N_0)^{r_0} \longrightarrow \xi \longrightarrow 0.$$

We look at the result of this sequence after applying the functor $\mathrm{Hom}(-\omega_{\mathbb{P}^n_k})$,

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}(\xi, \omega_{\mathbb{P}^n_k}) & \longrightarrow & \mathrm{Hom}(\mathcal{O}(-N_0)^{r_0}, \omega_{\mathbb{P}^n_k}) & \longrightarrow & \mathrm{Hom}(\mathcal{O}(-N_1)^{r_1}, \omega_{\mathbb{P}^n_k}) \\ & \downarrow \phi & & \downarrow \psi & & \downarrow \eta \\ 0 \longrightarrow & H^n(\mathbb{P}^n_k, \xi)^\vee & \longrightarrow & H^n(\mathbb{P}^n_k, \mathcal{O}(-N_0)^{r_0})^\vee & \longrightarrow & H^n(\mathbb{P}^n_k, \mathcal{O}(-N_1)^{r_1})^\vee \end{array}.$$

Now ϕ is an isomorphism once ψ and η are isomorphisms, so we have reduced this question to something about these twisted sheaves. We want to check that

$$H^0(\mathbb{P}^n_k, \mathcal{O}(N-n-1)) \times H^n(\mathbb{P}^n_k, \mathcal{O}(-N)) = \mathrm{Hom}(\mathcal{O}(-N), \mathcal{O}(-n-1)) \times H^n(\mathbb{P}^n_k, \mathcal{O}(-N)) \longrightarrow k$$

is a perfect pairing. A basis for the first factor on the left is $\prod_{i=0}^n x_i^{n_i}$ where $n_i \geq 0$ and $\sum n_i = N-n-1$, and a basis for the other factor is $\prod_{i=0}^n x_i^{n_i}$ with $n_i < 0$ and $\sum n_i = N$. This pairing is then given by multiplying two polynomials together and looking at the coefficient of $(x_0 \cdots x_n)^{-1}$ from the definition of tr. However $n_i \mapsto -1 - n_i$ gives a map from the left factor to a dual basis, hence this is a perfect pairing. □

For the general step, we need to work with localised Ext's, so sheaf variants of Ext.

Definition 19.2. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{M}$ an $\mathcal{O}_X$-module, then we define*

$$\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{M}, -) : \mathcal{O}_X\text{−Mod} \longrightarrow \mathcal{O}_X\text{−Mod},$$

to be the ith right derived functor of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, -)$.

Proposition 19.3. *There is a spectral sequence (a Grothendieck spectral sequence)*

$$E_2^{pq} = H^p(X, \mathcal{E}\mathrm{xt}^q_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})) \Longrightarrow \mathrm{Ext}^{p+q}_X(\mathcal{M}, \mathcal{N}).$$

Proof. The Grothendieck spectral sequence for $F = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, -)$ and $G = \Gamma(X, -)$, and note that

$$\Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, -)) = \mathrm{Hom}(\mathcal{M}, -),$$

by definition. Of course, we also have to check that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, -)$ maps injective sheaves to acyclic sheaves, which follows from the next lemma. □

Lemma 19.4. *Let $\mathcal{M}$ be an $\mathcal{O}_X$-module and $\mathcal{I}$ an injective $\mathcal{O}_X$-module, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{I})$ is flasque.*

Proof. We need to check for $U \subseteq V$, that the following map is surjective,

$$\mathrm{Hom}_(j_{V!}\mathcal{M}|_V, \mathcal{I}) \cong \mathrm{Hom}(\mathcal{M}|_V, \mathcal{I}|_V) \xrightarrow{\phi} \mathrm{Hom}(\mathcal{M}|_U, \mathcal{I}|_U) \cong \mathrm{Hom}(j_{U!}\mathcal{M}|_U, \mathcal{I}),$$

where $j_U : U \hookrightarrow X$ and $j_V : V \hookrightarrow X$ are the inclusions. We have a natural map $j_{u!}\mathcal{M}|_U \to j_{V!}\mathcal{M}|_V$ which is injective by inspection on stalks, such that ϕ is simply equal to precomposition by this map, followed by a restriction. Since I is injective, we conclude that ϕ above is surjective. □

For schemes we want to relate $\mathcal{E}\mathrm{xt}^i$ to Ext^i, and we can do this on an affine level.

Proposition 19.5. *For X a noetherian scheme, and $\mathcal{M},\mathcal{N}$ quasi-coherent schemes, such that $\mathcal{M}$ is coherent, then $\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$ are quasi-coherent, and coherent if $\mathcal{N}$ is also coherent. We also have that for all open affines $U = \operatorname{Spec} A \subseteq X$, our sheaf ext $\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})|_U$ restricted to U is isomorphism to the sheaf associated to the A-module $\mathrm{Ext}^i_A(\mathcal{M}(U),\mathcal{N}(U))$.*

Compare this with Example 17.4 in [7].

Proof. It follows from the definitions of $\mathcal{E}\mathrm{xt}^i$ that it commutes with localisation, meaning for all $U \subseteq X$ open, then $\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})|_U = \mathcal{E}\mathrm{xt}^i_{\mathcal{O}_U}(\mathcal{M}|_U,\mathcal{N}|_U)$, since restrictions of injective resolutions are injective, and restricting is also exact. Without loss of generality then, we may take $X = \operatorname{Spec} A$ with $\mathcal{M} = \widetilde{M}$ and $\mathcal{N} = \widetilde{N}$. Fix N, then we have two δ-functors from the opposite abelian category[31] of coherent sheaves on X, which is equivalent to the opposite category of finitely generated A-modules, to the category of $\mathcal{O}_X$-modules,

$$M \mapsto \widetilde{\mathrm{Ext}^i_A(M,N)}, \qquad M \mapsto \mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\widetilde{M},\widetilde{N}).$$

They agree for $i = 0$ by the Example 17.4 in [7], so we again just need to check that both these functors are effaceable. Given some $M \in A$-mod, which is finitely generated then we have a surjection $A^n \twoheadrightarrow M$, so it suffices that both functors vanish for $M = A^n$, or by additivity of both functors, if they vanish for $M = A$. However A is a projective A-module, so by Proposition 18.6 we have $\mathrm{Ext}^i_A(A,N) = 0$ for all $i > 0$. For the other functor, we have

$$\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\widetilde{A},\widetilde{N}) = \mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{N}),$$

is the ith derived functor of $\mathcal{N} \mapsto \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{N}) \cong \mathcal{N}$, which is exact, hence zero for $i > 0$. If $\mathcal{N}$ is coherent, then N is finitely generated and we can check for coherence of $\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$. For this, recall $X = \operatorname{Spec} A$ with A noetherian and M and N are finitely generated A-modules, then $\mathrm{Ext}^i_A(M,N)$ is finitely generated for all $i \geq 0$. To obtain this explitly we compute Ext^i_A here using projective resolutions of M which we can do in A-mod using Proposition 18.6. We can choose projective resolutions of N and M such that each entry P is finite free, from which we notice $\mathrm{Hom}_A(P,N)$ is also finitely generated. The functors Ext are then computed by taking cohomology of this complex of finitely generated A-modules, hence finitely generated. □

Back to the proof of Serre duality. For the moment we let $i : X \hookrightarrow \mathbb{P}^n_k = \mathbb{P}$ be any closed subscheme (like our X in Theorem 19.1), and ξ be a coherent sheaf on X.

Proposition 19.6. *There is a spectral sequence,*

$$E_2^{pq} = \mathrm{Ext}^p_X(\xi, \mathcal{E}\mathrm{xt}_{\mathcal{O}_\mathbb{P}}(i_*\mathcal{O}_X,\omega_\mathbb{P})) \Longrightarrow \mathrm{Ext}^{p+q}_\mathbb{P}(i_*\xi,\omega_\mathbb{P}),$$

where we are purposely ommiting some i^*.

Something like $\mathcal{E}\mathrm{xt}^p_X(\xi, ?)$ should arise in the Serre duality for our X, so after we prove this we will have to explicitly identify this ? with something more desirable. This will happen after we finish the proof of Serre duality, but first things are first.

Proof. The same proposition holds with $\omega_\mathbb{P}$ replaced by any $\mathcal{O}_\mathbb{P}$-module $\mathcal{M}$,

$$E_2^{pq} = \mathrm{Ext}^p_X(\xi, \mathcal{E}\mathrm{xt}_{\mathcal{O}_\mathbb{P}}(i_*\mathcal{O}_X,\mathcal{M})) \Longrightarrow \mathrm{Ext}^{p+q}_\mathbb{P}(i_*\xi,\mathcal{M}).$$

This is simply the Grothendieck spectral sequence again, from the equality at the $p = q = 0$ level

$$\mathrm{Hom}_X(\xi, \mathcal{H}om_\mathbb{P}(i_*\mathcal{O}_X,\mathcal{M})) = \mathrm{Hom}_\mathbb{P}(i_*\xi,\mathcal{M}). \tag{19.7}$$

[31] Note that this is only an abelian category because X is noetherian here.

This equality holds true as $\mathcal{H}om_{\mathbb{P}}(i_*\mathcal{O}_X, \mathcal{M}) \subseteq \mathcal{M} = \mathcal{H}om_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}, \mathcal{M})$ as $\mathcal{O}_{\mathbb{P}} \twoheadrightarrow i_*\mathcal{O}_X$. Hence we have one containment for this equality, and the other containment comes from the fact that any $i_*\xi \to \mathcal{M}$ factors over the subsheaf $i_*\mathcal{O}_X$ as $i_*\xi$ is an $\mathcal{O}_X$-module. We still have to check the injectives to acyclics condition for the Grothendieck spectral sequence, but we can do better hand show that if $\mathcal{I}$ is an injective $\mathcal{O}_{\mathbb{P}}$-module, then $\mathcal{H}om_{\mathbb{P}}(i_*\mathcal{O}_X, \mathcal{I})$ is an injective $\mathcal{O}_X$-module. Using Equation 19.7 we obtain,

$$\mathrm{Hom}_X(\xi, \mathcal{H}om_{\mathbb{P}}(i_*\mathcal{O}_X, \mathcal{I})) = \mathrm{Hom}_{\mathbb{P}}(i_*\xi, \mathcal{I}),$$

and the latter is exact in ξ since i is a closed immersion and $\mathcal{I}$ is injective. □

Let us identify an $\mathcal{E}\mathrm{xt}$ term which will give us Serre duality.

Proposition 19.8. *Assume that X is a smooth projective scheme over a field k, with $r = n - d$ where n is the dimension of the projective space X embeds into and d is the dimension of X. Then*

$$\mathcal{E}\mathrm{xt}^q_{\mathbb{P}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}}) \cong \begin{cases} \omega_X & q = r \\ 0 & \text{else} \end{cases}.$$

Proof of Theorem 19.1. In this case the spectral sequence of Proposition 19.6 degenerates into a simple isomorphism

$$\mathrm{Ext}^p_X(\xi, \omega_X) \cong \mathrm{Ext}^{p+r}_{\mathbb{P}}(i_*\xi, \omega_{\mathbb{P}}).$$

In particular, for $\xi = \mathcal{O}_X$ and $p = d$ we obtain,

$$\mathrm{tr} : H^d(X, \omega_X) \cong \mathrm{Ext}^d_X(\mathcal{O}_X, \omega_X) \cong \mathrm{Ext}^{d+r=n}_{\mathbb{P}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}}) \cong H^0(\mathbb{P}, i_*\mathcal{O}_X)^\vee \cong H^0(X, \mathcal{O}_X)^\vee \longrightarrow k.$$

The third isomorphism of the above composition is Serre duality on $\mathbb{P}$ ($= \mathbb{P}^n_k$), which we have already seen, and the last map is dual to the canonical map $k \to H^0(X, \mathcal{O}_X)$. This is our trace map. For general ξ we see that for $0 \le i \le d$ we have,

$$\mathrm{Ext}^i_X(\xi, \omega_X) \cong \mathrm{Ext}^{i+r}_{\mathbb{P}}(i_*\xi, \omega_{\mathbb{P}}) \cong H^{n-i-r}(\mathbb{P}, i_*\xi)^\vee \cong H^{d-i}(X, \xi)^\vee.$$

To check this comes from the trace map we defined above is a simply a big diagram chase. It should seem reasonable though, since this trace pairing was defined in essentially the same way as the perfect pairing above. □

Remark 19.9. More generally, if X is not smooth, but we have $\mathcal{E}\mathrm{xt}^p_{\mathbb{P}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}}) = 0$ for $q \neq r$ (i.e. mimicing Proposition 19.8), then if we define $\omega_X^0 = \mathcal{E}\mathrm{xt}^r_{\mathbb{P}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}})$, the dualising sheaf on X, we still have a Serre duality. We have a trace map,

$$\mathrm{tr} : H^d(X, \omega_X^0) \longrightarrow k,$$

and for all coherent ξ a perfect pairing,

$$\mathcal{E}\mathrm{xt}^i_X(\xi, \omega_X^0) \times H^{d-i}(X, \xi) \longrightarrow H^d(X, \omega_X^0) \xrightarrow{\mathrm{tr}} k.$$

The proof essentially follows from the argument above. This happens if and only if X is Cohen-Macauley, and ω_X^0 is a line bundle if and only if X is Gorenstein.

Proof of Proposition 19.8. We can check that $\mathcal{E}\mathrm{xt}^q_{\mathbb{P}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}})$ is zero for $q \neq r$ and a line bundle for $q = r$ locally, so without loss of generality we swap out $\omega_{\mathbb{P}}$ with $\mathcal{O}_{\mathbb{P}}$ and work in little open affine neighbourhoods $X_0 \subseteq X$ and $P_0 \subseteq \mathbb{P}$. These fit into the following commutative pullback diagram,

$$\begin{array}{ccc} \mathrm{Spec}\, A = X_0 & \hookrightarrow & P_0 = \mathrm{Spec}\, B \\ \downarrow & & \downarrow f \\ \mathrm{Spec}\, A_0 = \mathbb{A}^d_k & \longrightarrow & \mathbb{A}^n_k = \mathrm{Spec}\, B_0 \end{array}, \tag{19.10}$$

where we choose f such that it is étale and the bottom map is the inclusion of the first d-coordinates. We want to compute $\mathrm{Ext}^q_B(A,B)$, where $A = B \otimes_{B_0} A_0$, where B is flat over B_0 since f is étale. This flatness means we have $\mathrm{Ext}^q_B(A,B) = \mathrm{Ext}^q_{B_0}(A_0,B_0) \otimes_{B_0} B$. This is because we can just choose a projective resolution $P_0^\bullet$ of A_0 over B_0, and then by flatness of B over B_0 the complex $P^\bullet = P_0^\bullet \otimes_{B_0} B$ gives us a projective resolution of A. We then notice $\mathrm{Hom}_B(P^\bullet, B) = \mathrm{Hom}_{B_0}(P_0^\bullet, B_0) \otimes_{B_0} B$, and then the fact B is flat over B_0 implies $- \otimes_{B_0} B$ commutes with cohomology. It then remains to compute,

$$\mathrm{Ext}^q_{k[X_1,\dots,X_n]}(k[X_1,\dots,X_d], k[X_1,\dots,X_n]) \cong \mathrm{Ext}^q_{k[X_{d+1},\dots,X_n]}(k, k[X_{d+1},\dots,X_n]) \otimes_k k[X_1,\dots,X_d],$$

where the isomorphism above comes from a similar flat base change argument. This is just an explicit calculation, which we will see now. Next lesson we will see $\mathcal{E}\mathrm{xt}^q_{\mathbb{P}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}}) \cong \omega_X$ explicity. □

Lemma 19.11. *Let k be a field, then $\mathrm{Ext}^q_{k[X_1,\dots,X_r]}(k, k[X_1,\dots,X_r]) = 0$ for $q \neq r$ and k for $q = r$.*

Proof. For this explicit computation, it is best to use the Koszul complex, we looks as follows,

$$\cdots \longrightarrow \bigoplus_{i_1<i_2} k[X_1,\dots,X_r] \longrightarrow \bigoplus_{i=1}^{r} k[X_1,\dots,X_r] \longrightarrow k[X_1,\dots,X_r] \longrightarrow k \longrightarrow 0.$$

This is always a finite free resolution. The maps are simply alternating sums where we omit one basis element at a time. For example, if $r = 1$ we have,

$$0 \longrightarrow k[x] \xrightarrow{\cdot x} k[x] \longrightarrow k \longrightarrow 0,$$

where the map is multiplication by x. For $r = 2$ we have,

$$0 \longrightarrow k[X_1,X_2] \xrightarrow{(X_2,-X_1)} k[X_1,X_2] \oplus k[X_1,X_2] \xrightarrow{(X_1,X_2)} k[X_1,X_2] \longrightarrow k \longrightarrow 0.$$

When we take $\mathrm{Hom}_k(-, k[X_1,\dots,X_r])$ we again obtain a Koszul-like complex, where the cohomology is concentrated in the top degree, r. We will discuss this in more detail next lecture, and chapter 4.5 of [9] also has more information. □

20 Formal Functions 17/07/2017

Before we get onto the main topic of this lecture, the theory of formal functions, we need to finish Proposition 19.8 from last time, which comes with a little more theory too.

Remark 20.1. Let $i : Z \hookrightarrow X$ be a closed immersion of schemes, then the functor $i_* : \mathcal{O}_Z-\text{mod} \to \mathcal{O}_X-\text{mod}$ does not just have a left adjoint i^*, but also a right adjoint $i^!$. We define the latter as,

$$i^!\mathcal{M} = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{M}),$$

which can be explicitly written as,

$$i^!\mathcal{M}(U) = \{m \in \mathcal{M}(U) \mid \forall V \subseteq U, \forall f \in \ker(\mathcal{O}_X(V) \to i_*\mathcal{O}_Z(V)), f \cdot m|_V = 0\}.$$

A motto for this could be,

$$i^!\mathcal{M} \subseteq \mathcal{M} \textit{ is the subsheaf of sections killed by } \mathcal{I} = \ker(\mathcal{O}_X \to i_*\mathcal{O}_Z).$$

We then notice that $\mathcal{E}\text{xt}^q_{\mathcal{O}_X}(i_*\mathcal{O}_Z, -)$ is the qth right derived functor of $i^!$, which is now clear from our definitions. As $i^!$ is a right adjoint, it is left exact and is therefore entitled for right derived functors.

We can then reformulate Proposition 19.8 as follows.

Proposition 20.2. *Let X be a smooth projective scheme over a field k, and r be the codimension of X inside $\mathbb{P}^n_k$. Then*

$$R^q i^! \omega_{\mathbb{P}^n_k} \cong \begin{cases} \omega_X & q = r \\ 0 & \text{else} \end{cases}.$$

Proof. We already saw the beginning of this proof in the proof of Proposition 19.8, but recall that we are working locally with $\operatorname{Spec} B \subseteq \mathbb{P}^n_k$ and the following diagram,

$$\begin{array}{ccc} V(f_1, \dots, f_r) = \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} B \\ & \searrow & \downarrow \\ & & \operatorname{Spec} k \end{array}.$$

The vertical map is smooth of dimension n and the diagonal map is smooth of dimension $d = \dim X$, and $f_1, \dots, f_r$ form a regular sequence[32] in B. We come to this local situation by find a pullback diagram such as Diagram 19.10, as we have previously seen. We then want to form the Koszul complex $K(f_1, \dots, f_r; B)$ by induction. This is a different approach to last time, where we hide the potentially confusing maps in the Koszul complex in some previously defined machinery; the cone of a map of complexes. First we let $K(f_1; B) = C_{f_1} = \text{cone}(B \xrightarrow{\cdot f_1} B)$, considering B as complexes concentrated in degree zero. We then define,

$$K(f_1, \dots, f_r; B) = \text{cone}(f_r : K(f_1, \dots, f_{r-1}; B) \to K(f_1, \dots, f_{r-1}; B)).$$

For example, for $r = 2$ we have,

$$K(f_1, f_2; B) = \text{cone}(f_2 : K(f_1; B) \to K(f_1; B)) = \text{Tot}\begin{pmatrix} B & \xrightarrow{f_1} & B \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_1} \\ B & \xrightarrow{f_1} & B \end{pmatrix}.$$

[32] A sequence of elements $a_1, \dots, a_r$ form a regular sequence in a ring A if a_1 is not a zero divisor of A, and inductively a_i is not a zero divisor of $A/(a_1, \dots, a_{i-1})$.

By induction we can easily see all the terms of these complexes are finite free B-modules. We can also show by induction that,

$$K(f_1,\dots,f_i;B) \simeq B/(f_1,\dots,f_i)[0],$$

where $\simeq$ means quasi-isomorphic. For $i=1$ we have,

$$0 \longrightarrow B \xrightarrow{f_1} B \longrightarrow 0,$$

and the cokernel of this is B/f_1 since f_1 is not a zero divisor. For $i>1$ we have,

$$K(f_1,\dots,f_i;B) = \mathrm{cone}(f_i : K(f_1,\dots,f_{i-1};B) \to K(f_1,\dots,f_{i-1};B))$$
$$\simeq \mathrm{cone}(f_i : B/(f_1,\dots,f_{i-1})[0] \to B/(f_1,\dots,f_{i-1})[0]),$$

which is quasi-isomorphic to $B/(f_1,\dots,f_i)[0]$ since f_i is part of this regular sequence. In particular, $K(f_1,\dots,f_r;B)$ is a finite free resolution of $A = B/(f_1,\dots,f_r)$. We want to compute

$$R^q i^! \omega_{\mathbb{P}^n_k} = \mathcal{E}\mathrm{xt}^q_{\mathcal{O}_{\mathbb{P}^n_k}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^n_k}),$$

which is locally given by $\mathrm{Ext}^q_B(A,\omega_B)$, which we can now compute using $\mathrm{Hom}_B(K(f_1,\dots,f_r;B),\omega_B)$, with an application of Proposition 18.6. Another piece of induction starts with,

$$\mathrm{Hom}_B(K(f_1;B),\omega_B) \cong \omega_B/f_1[-1],$$

since the Hom_B gives us a shift of -1. Induction continues by identifying $\mathrm{Hom}_B(K(f_1,\dots,f_i;B),\omega_B)$ as,

$$= \mathrm{cone}(f_i : \mathrm{Hom}_B(K(f_1,\dots,f_{i-1};B),\omega_B) \to \mathrm{Hom}_B(K(f_1,\dots,f_{i-1};B),\omega_B))[-1]$$
$$\simeq \mathrm{cone}(f_i : \omega_B/(f_1,\dots,f_{i-1})[-i+1] \to \omega_B/(f_1,\dots,f_{i-1})[-i+1]) \simeq \omega_B/(f_1,\dots,f_i)[-i].$$

From this we see that $\mathrm{Ext}^q_B(A,\omega_B) = \omega_B/(f_1,\dots,f_r) = \omega_B \otimes_B A$ for $q=r$, and zero for $q \neq r$. We need to analyse this dependence on $f_1,\dots,f_r$, i.e. we need to show that

$$\mathrm{Ext}^r_B(A,\omega_B) \cong \omega_B \otimes_B A,$$

is a canonical isomorphism. Choosing $f_1,\dots,f_r$ gives us a trivialisation of $I/I^2 = \bigoplus_{i=1}^r A\cdot f_i$, where $I = \ker(B \to A)$. The outcome is then that

$$\mathrm{Ext}^r_B(A,\omega_B) \cong (\omega_B \otimes_B A) \otimes \Lambda^r(I/I^2)^\vee,$$

is canonical. Essentially the choices in both isomorphisms cancel each other out. For $r=1$ we have,

$$0 \longrightarrow f\cdot B = I \longrightarrow B \longrightarrow A \longrightarrow 0, \tag{20.3}$$

which is a finite free resolution of A, and applying $\mathrm{Hom}_B(-,\omega_B)$ gives us,

$$0 \longrightarrow \omega_B \longrightarrow \omega_B \otimes I^\vee \longrightarrow \mathrm{Ext}^1_B(A,\omega_B) \longrightarrow 0.$$

This is the same sequence as Sequence 20.3 tensored with $\omega_B \otimes I^\vee$ over B, which implies that,

$$\mathrm{Ext}^1_B(A,\omega_B) \cong (\omega_B \otimes_B A) \otimes (I/I^2)^\vee.$$

Recall the short exact sequence of finite projective A-modules,

$$0 \longrightarrow I/I^2 \longrightarrow \Omega^1_{B/A} \otimes_B A \longrightarrow \Omega^1_{A/k} \longrightarrow 0,$$

from Proposition 5.14 for example. Taking determinants of this we obtain,

$$\omega_B \otimes_B A \cong \omega_A \otimes \Lambda^r(I/I^2),$$

which implies that,

$$\omega_A \cong (\omega_B \otimes_B A) \otimes \Lambda^r(I/I^2)^\vee,$$

using the fact that exterior powers commute with duals. This gives us our proposition locally, and since we have seen this is a canonical choice, then it glues uniquely. □

This is the end of our chapter on Riemann-Roch and Serre Duality. So for the rest of today we are going to talk about formal functions, which will give us Zariski's main theorem and Stein factorisations eventually. For now, let $f : X \to Y$ be a proper map of noetherian schemes, with ξ a coherent sheaf on X. Given some $y \in Y$, then in general the canonical map,

$$(R^i f_* \xi) \otimes_{\mathcal{O}_Y} k(y) \longrightarrow H^i(X_y, \xi|_{X_y})$$

may not be an isomorphism. The situation improves when we restrict to infinitesimally small neighbourhoods of y. Let $\mathfrak{m}_y \subseteq \mathcal{O}_{Y,y}$ be the maximal ideal, and $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n)$, then we have another canonical map,

$$(R^i f_* \mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n \longrightarrow H^i(X_n, \xi|_{X_n}).$$

Theorem 20.4 (Theorem on Formal Functions)**.** *The canonical map on inverse limits induced from the map above is an isomorphism,*

$$(R^i f_* \xi)_y^\wedge = \lim_n (R^i f_* \xi) \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n \longrightarrow \lim_n H^i(X_n, \xi|_{X_n}).$$

This can be thought of as a base change result along $\{\operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n\}_{n\geq 0} \to Y$, so maybe a pro-scheme base change, or some result about pro-flatness over Y. We will talk about this "pro-" language shortly. Let us try to reduce this statement to something a little easier to handle. We may assume $Y = \operatorname{Spec} A$ is affine, and then we have,

$$H^i(X, \xi)_y^\wedge := \lim_n H^i(X, \xi) \otimes_A A_y/\mathfrak{m}_y^n \cong \lim_n H^i(X_n, \xi|_{X_n}).$$

By flat base change we may replace A by A_y, so without loss of generality A is a local ring and $y \in Y$ is the unique closed point. Now

$$H^i(X, \xi)_y^\wedge = \lim_n H^i(X, \xi) \otimes_A A/\mathfrak{m}_A^n,$$

is simply the $\mathfrak{m}_A$-adic completion of A. If A is noetherian, and $I \subseteq A$ an ideal, then $\widehat{A}$ is flat over A (recall example 2.5), and $M \otimes_A \widehat{A} \to \widehat{M}$ is an isomorphism for M finitely generated, thus

$$H^i(X, \xi)_y^\wedge = H^i(X, \xi) \otimes_A \widehat{A}.$$

Flat base change again allows us to assume that A is a complete local ring, so our theorem has been reduced to the following statement.

Theorem 20.5. *Let A be a complete local noetherian ring, $f : X \to \operatorname{Spec} A$ be a proper map, $X_n = X \times_{\operatorname{Spec} A} \operatorname{Spec} A/\mathfrak{m}^n$, and ξ a coherent sheaf on X. Then the canonical map,*

$$H^i(X, \xi) \longrightarrow \lim_n H^i(X_n, \xi|_{X_n}),$$

is an isomorphism.

Actually something stronger is true.

Definition 20.6. *Let $\{M_n\}_{n\geq 0}$ and $\{N_n\}_{n\geq 0}$ be sequences of abelian groups with sequential maps between them*[33]*, called pro-abelian groups, then a map*

$$\{f_n\}_{n\geq 0} : \{M_n\}_{n\geq 0} \longrightarrow \{N_n\}_{n\geq 0},$$

of pro-abelian groups, which is just a sequences of maps $f_n : M_n \to N_n$ of abelian groups commuting with the maps within $\{M_n\}_{n\geq 0}$ and $\{N_n\}_{n\geq 0}$, is called a pro-isomorphism if $\{\ker_n\}_{n\geq 0}$ and $\{\operatorname{coker}_n\}_{n\geq 0}$ are all pro-zero. A pro-abelian group $\{K_n\}_{n\geq 0}$ is pro-zero if for all $n \geq 0$ there is a $m \geq n$ such that $K_m \to K_n$ is zero.

[33] A neater definition might be; let $\{M_n\}_{n\geq 0}$ be a functor $\mathrm{Po}\mathbb{N}^{op} \to \mathrm{Ab}$, from the opposite poset $\mathrm{Po}\mathbb{N}^{op}$ of natural numbers to the category of abelian groups.

The same definition goes for sheaves of $\mathcal{O}_X$-modules or R-modules, but we hesitate before we work with this "pro-"setting in a general abelian category. If $\{K_n\}_{n\geq 0}$ is pro-zero, then it is a consequence that $\lim_n K_n = 0$ and $\lim^1_n K_n = 0$.[34] Notice that $\lim_n K_n = 0$ for diagramatic reasons.

Remark 20.7. An improvement on Theorem 20.5 is the statement; the canonical pro-map,

$$\{H^i(X,\xi)/\mathfrak{m}^n\}_{n\geq 0} \longrightarrow \{H^i(X_n, \xi|_{X_n})\}_{n\geq 0},$$

is a pro-isomorphism.

Now we can try to prove Theorem 20.5, for which we need the following two lemmas.

Lemma 20.8. *For A a noetherian ring, $I \subseteq A$ an ideal, then the functor from finitely generated A-modules to pro-A-modules, sending M to $\{M/I^nM\}_{n\geq 0}$ is pro-exact, i.e., if we have an exact sequence of finitely generated A-modules,*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

then the cohomology groups of the pro-complex,

$$0 \longrightarrow \{M'/I^nM'\}_{n\geq 0} \longrightarrow \{M/I^nM\}_{n\geq 0} \longrightarrow \{M''/I^nM''\}_{n\geq 0} \longrightarrow 0,$$

are pro-zero.

Notice that now the phrase "$A \to \{A/I^n\}_{n\geq 0}$ is pro-flat" makes sense.

Proof. We always have exactness on the right, so we just need to show exactness on the left. In other words, letting

$$K_n = \ker(M'/I^nM' \to M/I^nM) = (M' \cap I^nM)/I^nM',$$

we want to show $\{K_n\}_{n\geq 0}$ is pro-zero. This is a consequence of the following lemma.

Lemma 20.9 (Artin-Rees Lemma)**.** *In this situation, there is some integer $c > 0$ such that for all $n \geq c$,*

$$M' \cap I^nM = I^{n-c}(M' \cap I^cM).$$

Assuming this is true for now, then the map,

$$K_{n+c} = (M' \cap I^{n+c}M)/I^{n+c}M' = I^n(M' \cap I^cM)/I^{n+c}M' \longrightarrow K_n = (M' \cap I^nM)/I^nM',$$

is zero, as $I^n(M' \cap I^cM) \subseteq I^nM'$, and this is clearly killed inside K_n. Hence $\{K_n\}_{n\geq 0}$ is pro-zero. □

Proof of Lemma 20.9. Consider $B = A \oplus I \oplus I^2 \oplus \cdots$. If $f_1, \dots, f_r$ generate I, then B is a quotient of $A[X_1, \dots, X_n]$ by $X_i \mapsto f_i$, so B is noetherian. Let $N = M \oplus IM \oplus I^2M \oplus \cdots$, then N is a B-module which is finitely generated as an B-module, as M is a finitely generated A-module. Consider now,

$$N' = \bigoplus_{n\geq 0}(I^nM \cap M') \subseteq N,$$

which is a B-submodule of N. Now N' is a finitely generated B-module, so let $t_1, \dots, t_s$ be generators. Without loss of generality, we may take each t_i to be in $I^{n_i}M \cap M'$, so each t_i is homogeneous of degree n_i, simply by decomposing them into their homogeneous components. Let $c = \max n_i$, then for $n \geq c$ any $x \in I^nM \cap M' \subseteq N'$ is of the form,

$$x = \sum_{j=1}^{s} h_j t_j,$$

for some $h_j \in I^{n-n_j} \subseteq I^{n-c}$. This implies that $M' \cap MI^n \subseteq I^{n-c}(M' \cap I^cM)$. The converse containment is clear. □

[34]Notice that lim as a functor is only left exact, so it deserves a right derived functor, whose first level we simply call $\lim^1$.

We can now prove our important theorem of the day, Theorem 20.5.

Proof of Theorem 20.5. Assume that X is projective (this can be proved for $X \to \operatorname{Spec} A$ only a proper map, but our proof need X to be projective.). Without loss of generality then, we can take $X = \mathbb{P}^N_A$ by replacing ξ by $i_*\xi$ for $i : X \hookrightarrow \mathbb{P}^N_A$ a closed immersion. First we verify the result for $\xi = \mathcal{O}(-d)$ for $d > 0$. This can be done in one of three ways. First, we could use explicit calculations of the cohomology of $\mathcal{O}(-d)$ over $\mathbb{P}^N_A$ and $\mathbb{P}^N_{A/\mathfrak{m}^n}$ using Proposition 13.3. Second, we could use a generalised base change result from lecture 16 in the highest degree to see that

$$H^i(\mathbb{P}^n_A, \mathcal{O}(-d)) \otimes_A A/\mathfrak{m}^n \longrightarrow H^i(\mathbb{P}^N_{A/\mathfrak{m}^n}, \mathcal{O}(-d))$$

is an isomorphism. Third, we could use that the cohomology of $\mathcal{O}(-d)$ is free, which gives us our naïve base change results. Regardless, we have the desired result for $\xi = \mathcal{O}(-d)$ and we now want to come back to general ξ, which we will approach using descending induction on the dimension of our cohomology. Let $i > N$, then everything is zero and we're done, so assume the result holds for $i' > i$. We obtain the following short exact sequence using the usual tricks,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}(-d)^r \longrightarrow \xi \longrightarrow 0.$$

Let $\xi_n = \xi \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ and the same for $\mathcal{F}_n$ and $\mathcal{O}(-d)^r$, then we have a complex,

$$0 \longrightarrow \{\mathcal{F}_n\}_{n \geq 0} \longrightarrow \{\mathcal{O}(-d)^r_n\}_{n \geq 0} \longrightarrow \{\xi_n\}_{n \geq 0} \longrightarrow 0,$$

which is pro-exact. More explicitly, we have,

$$0 \longrightarrow K_n \longrightarrow \mathcal{F}_n \longrightarrow \mathcal{O}(-d)^r_n \longrightarrow \xi_n \longrightarrow 0,$$

is exact, where $K_n \to \mathcal{F}_n$ factors through some $\mathcal{F}_n \twoheadrightarrow \mathcal{G}_n \hookrightarrow \mathcal{O}(-d)^r_n$, simply by taking the cokernel of $K_n \to \mathcal{F}_n$. By Lemma 20.8 we see that for all n, there is some m such that $K_m \to K_n$ is zero. We then take our exact sequence above and look at the effect on cohomology groups,

$$\begin{array}{ccccccc}
H^i(X,\mathcal{F}) & \longrightarrow & H^i(X,\mathcal{O}(-d)^r) & \longrightarrow & H^i(X,\xi) & \longrightarrow & H^{i+1}(X,\mathcal{F}) \\
\downarrow \psi & & & & & & \downarrow \cong \\
\lim H^i(X_n,\mathcal{F}_n) & & \downarrow \cong & & \downarrow \phi & & \lim H^{i+1}(X_n,\mathcal{F}_n) \\
\downarrow \alpha & & & & & & \downarrow \beta \\
\lim_n H^i(X_n,\mathcal{G}_n) & \longrightarrow & \lim H^i(X_n,\mathcal{O}(-d)^r_n) & \longrightarrow & \lim H^i(X_n,\xi_n) & \longrightarrow & \lim H^{i+1}(X_n,\mathcal{G}_n)
\end{array},$$

where we let $X = \mathbb{P}^N_A$ for typographical reasons. The lower line is exact as lim is exact when the Mittag-Leffer condition is satisfied (see Definition 3.5.5 in [9]). The maps above that are isomorphisms come from either induction or the $\xi = \mathcal{O}(-d)$ case, but we need to work a little harder before we can apply the five-lemma. The maps α and β are isomorphisms, as the only possible obstruction terms are $\lim H^j(X_n, K_n) = 0$ as $\{H^j(X_n, K_n)\}_{n \geq 0}$ is pro-zero (we need $\{K_n\}_{n \geq 0}$ to be pro-zero here, as it would not suffice to see $\lim K_n = 0$!). This then formaly implies ϕ is surjective by a diagram chase. We apply this argument to $\mathcal{F}$ as well to see that ψ is also an epimorphism, and this puts us in the correct situation to apply the five-lemma. The five-lemma comes from extending this sequence one more term to the right which is an isomorphism by our $\xi = \mathcal{O}(-d)$ calculations. □

We didn't prove the pro-isomorphism discussed in Remark 20.7, but this is possible.

21 Zariski's Main Theorem and Consequences 20/07/2017

We will now try to apply Theorem 20.5 to prove Zariski's main theorem.

Theorem 21.1 (Zariski's Main Theorem)**.** *Let $f : X \to Y$ be a birational projective morphism of integral noetherian schemes, where Y is normal. Then, for all $y \in Y$, X_y is connected.*

Remark 21.2. A stronger statement is that the fibres X_y are actually geometrically connected, which means that $X_y \times_Y \operatorname{Spec} k$ is connected where k is any algebraically closed field extension of $k(y)$. See Remark 21.3.

Proof. This is a local claim, so without loss of generality we can take $Y = \operatorname{Spec} A$ for some noetherian ring A. We first claim that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$, or equivalently that $A \to H^0(X, \mathcal{O}_X)$ is an isomorphism. We now that $H^0(X, \mathcal{O}_X) = \widetilde{A}$ is a finite A-algebra, i.e. finitely generated an A-module, by general finiteness in cohomology, see Theorem 13.1. We know $\widetilde{A}$ is also integral, and $K \to \widetilde{A} \otimes_A K$ where $K = \operatorname{Frac}(A)$ is an isomorphism since f is birational. We see this last point by noticing that f being birational implies the base change map $X \times_{\operatorname{Spec} A} \operatorname{Spec} K \to \operatorname{Spec} K$ is an isomorphism, since $\operatorname{Spec} K$ only has one point, and this map corresponds to the isomorphism advertised above. Using the fact that $\widetilde{A}$ is integral now, we soon that $A \subseteq \widetilde{A} \subseteq \widetilde{A} \otimes_A K = K$, but A is normal, and $A \subseteq \widetilde{A}$ is finite, so then $\widetilde{A} = A$. Hence $A \cong H^0(X, \mathcal{O}_X)$.

Now notice, that more generally if $f : X \to Y$ is simply a projective map of noetherian schemes and $f_*\mathcal{O}_X \to \mathcal{O}_Y$ is an isomorphism, then for all points $y \in Y$ we see X_y is connected. To show this, assume it is false, that $X_y = Z_0 \sqcup Z_1$ is a chosen disjoint union of X_y into two subsets, both of which are open and closed. Then once we set

$$X_n = X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n,$$

we see this is disconnected as $|X_n| = |X_y|$, so $X_n = Z_{0,n} \sqcup Z_{1,n}$. So for all n we get idempotents $e_{0,n}, e_{1,n} \in H^0(X_n, \mathcal{O}_{X_n})$, such that

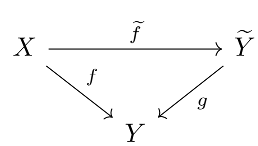
$$e_{0,n} = \begin{cases} 1 & \text{on } Z_{0,n} \\ 0 & \text{on } Z_{1,n} \end{cases}, \qquad e_{1,n} = \begin{cases} 0 & \text{on } Z_{0,n} \\ 1 & \text{on } Z_{1,n} \end{cases}.$$

These are compatible for varying n, so we obtain $e_0, e_1 \in \lim H^0(X_n, \mathcal{O}_{X_n})$ which by Theorem 20.5 is simply $(f_*\mathcal{O}_X)_y^\wedge \cong \mathcal{O}_{Y,y}^\wedge$, where the second isomorphism come from the $f_*\mathcal{O}_X \cong \mathcal{O}_Y$ assumption. So we have $e_0, e_1 \in \mathcal{O}_{Y,y}^\wedge$. This is a local ring, but $\operatorname{Spec} \mathcal{O}_{Y,y}^\wedge$ is connected, however e_0 and e_1 define a disconnection, a contradiction. $\square$

Remark 21.3. To obtain the statement with geometrically connected fibres, we use the assumption that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$ is preserved under flat change change $Y' \to Y$, and if $\overline{k}$ is the algebraic closure of $k(y)$, one can find a flat map $Y' \to Y$, mapping a point $y' \mapsto y$ such that $k(y') = \overline{k}$. Then we simply write up the above result for Y' and we're done. To see a flat map $Y' \to Y$ exists, we can without loss of generality take $Y = \operatorname{Spec} O_{Y,y}$, to be local, then $\overline{k}$ is the increasing union of (x) for $x \in \overline{k}, k(x) \cong k[x]/p$ for some $p \in k[x]$ a monic polynomial. One can lift p to $\widetilde{p} \in \mathcal{O}_{Y,y}[x]$ another monic polynomial, then $\mathcal{O}_{Y,y}[x]/\widetilde{p}$ is finite free over $\mathcal{O}_{Y,y}$, so we may assume $k = k(x)$. We then "take the union" to obtain the desired map.

Another application of Theorem 20.5 is the following theorem, called Stein factorisations.

Theorem 21.4 (Stein Factorisations)**.** *Let $f : X \to Y$ be a projective map of noetherian schemes. Then there is a unique (up to unique isomorphism) factorisation,*



,

such that g is finite and $\widetilde{f}_*\mathcal{O}_X \cong \mathcal{O}_{\widetilde{Y}}$.

Proof. For uniquenness, it suffices to work locally, so withoutl oss of generality we take $Y = \operatorname{Spec} A$, so the fact g is finite implies $\widetilde{Y} = \operatorname{Spec} \widetilde{A}$ affine so $f_*\mathcal{O}_X \cong \mathcal{O}_{\widetilde{Y}}$, i.e. $\widetilde{A} \cong H^0(X, \mathcal{O}_X)$. Existence can now be shown locally, and as this existence is unique up to unique isomorphism, we can glue to a global statement for free. Let $\operatorname{Spec} A = Y$ be affine, and set $\widetilde{Y} = \operatorname{Spec} H^0(X, \mathcal{O}_X)$. This is finite over X by Theorem 13.1 again. A map $X \to \widetilde{Y}$ is equivalent to a map $H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X)$, so we take the identity map here and obtain the following diagram,

$$\begin{tikzcd} X \arrow[rr, "\widetilde{f}"] \arrow[dr, "f"'] & & \widetilde{Y} \arrow[dl, "g"] \\ & Y & \end{tikzcd} .$$

Also we have $\widetilde{f}_*\mathcal{O}_X \cong \mathcal{O}_{\widetilde{Y}}$ as this is equivalent to $H^0(X, \mathcal{O}_X) \cong \widetilde{A}$, our definition of $\widetilde{Y}$. □

The upshot here is that possible variations of connected components of geometric fibres of a projective morphism are completely controlled by a finite morphism. One interpretation of Zarkiski's main theorem is then that if the base is normal, then there are no non-trivial finite birational extensions. For example, if we have a family of curves over a base scheme that is also a proper smooth curve, then singularities may occur from time to time, but the fibres will never 'become' disconnected.

Corollary 21.5. *For $f : X \to Y$ a projective morphism of noetherian integral schemes, where Y is normal, and $X \times_Y \operatorname{Spec} \overline{k(y)}$ is connected for each $y \in Y$ (i.e. X_η is geometrically connected for $\eta \in X$ the unique generic point), then for all $y \in Y$, X_y is geometrically connected.*

Proof. Let $Y = \operatorname{Spec} A$, then we can take a Stein factorisation of f to obtain the diagram,

$$\begin{tikzcd} X \arrow[rr, "\widetilde{f}"] \arrow[dr, "f"'] & & \widetilde{Y} \arrow[dl, "g"] \\ & Y & \end{tikzcd} .$$

We have $\widetilde{A} = H^0(X, \mathcal{O}_X)$, so it is enough to show now that $\widetilde{Y} \cong Y$. We notice that $\widetilde{A}$ is finite over A, and is also integral, and also $\widetilde{A} \otimes_A \overline{k}$ for $k = \overline{k(y)}$ is connected over $\overline{k}$. Similar to the proof of Theorem 21.1, by flat base change we also have $\widetilde{A} \otimes_A \overline{k} = H^0(X \times_Y \operatorname{Spec} \overline{k}, \mathcal{O}_{X \times_Y \operatorname{Spec} \overline{k}})$. If $X \times_Y \operatorname{Spec} \overline{k}$ was reduced, then we get $\widetilde{A} \otimes_A \overline{k} \cong \overline{k}$ which by faithfully flat descent implies that $\widetilde{A} \otimes_A k = k$. We then have $A \subseteq \widetilde{A} \subseteq k$ where $A \subseteq \widetilde{A}$ is finite and A is normal, so $A = \widetilde{A}$. In general though, we only have

$$(\widetilde{A} \otimes_A \overline{k})_{\mathrm{red}} \cong \overline{k}.$$

Then setting $\widetilde{k} = \operatorname{Frac}(\widetilde{A})$ we then only have that the field extension $k \subseteq \widetilde{k}$ is purely inseparable, so we need to use the following lemma.

Lemma 21.6. *If A is normal and noetherian, and $K = \operatorname{Frac}(A)$ and K'/K is a purely separable field extension, and $A \subseteq \widetilde{A} \subseteq K$ is finite, then $\operatorname{Spec} \widetilde{A} \to \operatorname{Spec} A$ is a universal homeomorphism.*

Assuming this lemma, we then see for each $y \in Y$ we have $\overline{k} = \overline{k(y)}$, and then we base change our whole Stein factorisation to obtain,

$$\begin{tikzcd} Y_{\overline{k}} \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dr] \arrow[dd] & \\ & \widetilde{Y}_{\overline{k}} \arrow[dl] & & \widetilde{Y} \arrow[dl] \\ \operatorname{Spec} \overline{k} \arrow[rr] & & Y & \end{tikzcd} .$$

The lemma above then implies that $|\widetilde{Y}_{\overline{k}}|$ is a point, and Stein factorisation implies $|X_{\overline{k}}|$ is connected. $\square$

Remark 21.7. If k in the proof above is a perfect field, then X being reduced as a scheme over $\operatorname{Spec} k$ implies that $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is also reduced. To see this, we see for any finite field separable field extension k' of k, then $\operatorname{Spec} k'$ is finite étale over $\operatorname{Spec} k$ as k is perfect. This implies that $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is finite étale over $\operatorname{Spec} k$. We then use the fact that if X is reduced, and $Y \to X$ is a finite étale map, then Y is reduced. Now if $X = \operatorname{Spec} A$ if affine now, we have $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k} = \operatorname{Spec} A \otimes_k \overline{k}$, and $A \otimes_k \overline{k}$ is simply the colimit of $A \otimes_k k$ over all finite separable extensions k' over k. The elements of this colimit are reduced, hence the colimit is reduced.

For general fields though, notice this fails. For example if $k = \mathbb{F}_p(T)$ and $X = \operatorname{Spec} \mathbb{F}_p(T^{1/p})$, then we have

$$X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k} = \operatorname{Spec} \overline{k}[X]/(X^p - T) = \operatorname{Spec} \overline{k}[X]/(X - T^{1/p})^p,$$

is not reduced. This is why we had to include Lemma 21.6 in the proof above.

Remark 21.8. Everything we have done so far in the lecture also works with proper replacing the adjective projective. We also notice at this stage that we have not seen a definition of a projective morphism in general, i.e. when the target scheme is non-affine, but it is somewhat unclear what the best globalisation is. Peter notices that is is not true that if $X \to \operatorname{Spec} A = Y$ is proper and locally on Y projective, then f is projective. Strange things can happen with families of curves of genus 1. For example, the following proposition.

Proposition 21.9. *Let $f : X \to \operatorname{Spec} A = Y$ be a smooth proper morphism of dimension 1 of noetherian schemes, where Y is connected and $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Then $g_y = g(X_y)$ is independent of $y \in Y$, and if $g \neq 1$ then f is projective. If $g = 1$ and we have a section $Y \to X$, then f is projective also.*

Note that this proposition doesn't hold in general if $g = 1$, and we will see the obstacle clearly exhibited during the course of the proof.

Proof. We have $1 - g_y = \chi(X_y, \mathcal{O}_{X_y})$, by $\mathcal{O}_X$ is flat over $\mathcal{O}_X$, so $R\Gamma(X, \mathcal{O}_X)$ is a perfect complex and it was shown in exercises[35] that this implies $\chi(X_y, \mathcal{O}_{X_y})$ is locally constant. If Y is connected, then $\chi(X_y, \mathcal{O}_{X_y})$ is simply constant, so g_y is also constant. Now we set $g = g_y$ for some $y \in Y$, and we want to find an ample line bundle over these fibres to show X is quasi-projective (see Theorem 25.7 in [7]), and hence projective since it is proper. Since f is smooth of dimension 1, then $\Omega^1_{X/Y}$ is a line bundle. We then claim that if $g = 0$ then $(\Omega^1_{X/Y})^\vee$ is ample, and if $g \geq 2$ then $\Omega^1_{X/Y}$ is ample. This is clear when $Y = \operatorname{Spec} k$ for k a field, since then we have the Riemann-Roch Theorem to tell us that $\deg \Omega^1_{X/k} = 2g - 2$, and the statement that a line bundle $\mathcal{L}$ is ample if and only if $\deg \mathcal{L} > 0$, which is a corollary of the Riemann-Roch Theorem again (seen in Peter's seminar "Jacobians of Curves" for example). For a general Y though, we have the following proposition.

Proposition 21.10. *Let $f : X \to Y = \operatorname{Spec} A$ be a proper smooth map of noetherian schemes, with Y connected, $f_*\mathcal{O}_X \cong \mathcal{O}_Y$ and $\mathcal{L}$ a line bundle on X. Then,*

1. *The map $y \mapsto \deg \mathcal{L}|_{X_y}$ is constant, so let $d = \deg \mathcal{L}|_{X_y}$ for any $y \in Y$.*
2. *If $d > 0$, then $\mathcal{L}$ is ample.*

Proof of Proposition 21.10. For the first part, notice the Riemann-Roch Theorem tells us that

$$\chi(X_y, \mathcal{L}|_{X_y}) = \deg \mathcal{L}|_{X_y} + 1 - g_y.$$

[35] Exercise 11.4(i) reads: Let A be a ring and $C \in D(A)$ be a perfect complex. Prove the function,

$$\mathfrak{p} \in \operatorname{Spec} A \mapsto \sum_{i \in \mathbb{Z}} (-1)^i \dim_{k(\mathfrak{p})} H^i(C \otimes^{\mathbb{L}}_A k(\mathfrak{p}))$$

is locally constant on $\operatorname{Spec} A$.

Both the genus and Euler characteristic functions have been seen to be locally constant on Y, so this degree function is constant as well, as Y is connected. For part 2, we claim that if $\deg \mathcal{L} > 2g-2$, then $R^1 f_* \mathcal{L} = 0$ and $f_* \mathcal{L}$ is locally free. To see this, notice that $R^i f_* \mathcal{L} = 0$ for $i > 1$, and by base change in the highest degree we see that,

$$R^1 f_* \mathcal{L} \otimes k(y) \longrightarrow H^1(X_y, \mathcal{L}|_{X_y}),$$

is an isomorphism. However, the latter is zero by Serre duality, indeed this cohomology group is dual to $H^0(X_y, \mathcal{L}|_{X_y}^\vee \otimes \Omega^1_{X_y/k(y)})$ which is zero for degree reasons by hypothesis. Hence all fibres vanish and we have $R^1 f_* \mathcal{L} = 0$. This then implies $R\Gamma(X, \mathcal{L})$ is computed by a complex of finite projective A-modules of length 1, so $f_* \mathcal{L}$ is locally free. This proves our claim. We then notice that we can compute,

$$\mathrm{rk}(f_* \mathcal{L}) = \dim H^0(X_y, \mathcal{L}|_{X_y}) = d + 1 - g.$$

In particular, $H^0(X, \mathcal{L}) = H^0(Y, f_* \mathcal{L})$ is a finitely generated A-module of rank $d+1-g$. Setting $M = H^0(X, \mathcal{L})$ and using highest degree base change to obtain $M \otimes_A k(y) \to H^0(X_y, \mathcal{L}|_{X_y})$ is an isomorphism, gives us our ampleness from Theorem 22.2 which we'll prove next time. □

If $g = 1$, but we have a section $s : Y \to X$, then notice s is a closed immersion since f is proper, and we notice $(\mathcal{I}_Y)^\vee$ is an ample sheaf, since this line bundle has degree 1. □

Remark 21.11. In the situation of Proposition 21.9 above, we notice that $f_* \mathcal{O}_X \cong \mathcal{O}_Y$ if and only if f has geometrically connected fibres. One direction was hidden in the proof of Theorem 21.1, so conversely, we have for all $y \in Y$,

$$H^0(X \times_Y \operatorname{Spec} \overline{k(y)}, \mathcal{O}_{X \times_Y \operatorname{Spec} \overline{k(y)}}) = \overline{k(y)}.$$

The scheme $X \times_Y \operatorname{Spec} \overline{k}$ is connected and smooth, hence reduced. We then have

$$H^0(X \times_Y \operatorname{Spec} k(y), \mathcal{O}_{X \times_Y \operatorname{Spec} k(y)}) = k(y).$$

We then have the factorisation diagram,

$$k(y) \longrightarrow f_* \mathcal{O}_Y \otimes k(y) \xrightarrow{\alpha} H^0(X \times_Y \operatorname{Spec} k(y), \mathcal{O}_{X \times_Y \operatorname{Spec} k(y)}) = k(y),$$

where the composition is an isomorphism. The map α is then surjective so then some base change in cohomology that we will talk about next lecture implies that $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism in a neighbourhood of y, and then an isomorphism everywhere.

22 Relatively Ample Line Bundles 24/07/2017

We need to prove some statements to wrap up some lose ends from last time. This leads us to the following generalised definition of an ample line bundle.

Definition 22.1. *Let $f : X \to S$ be a map of schemes and $\mathcal{L}$ a line bundle on X. Then $\mathcal{L}$ is relatively ample if the following equivalent conditions hold.*

1. *For all open affines $V = \operatorname{Spec} A \subseteq S$ the restricted line bundle $\mathcal{L}|_{f^{-1}(V)}$ is ample.*
2. *There exists a cover of S by open affines $V = \operatorname{Spec} A \subseteq S$ such that the restricted line bundle $\mathcal{L}|_{f^{-1}(V)}$ is ample.*

The equivalence of these definitions is standard practice by now, but it is not immediate.

Proof. The fact that condition 1 implies condition 2 is obvious. For the other direction we notice the statement is local, so we take $S = \operatorname{Spec} A$ is affine, so $f : X \to S$ is quasi-compact and separated (as it is so locally), which implies X is quasi-compact and separated. Let $Z \subseteq X$ be a closed subset, and chose a point $x \in U = X \backslash Z$. We want to find some $s \in \mathcal{L}^{\otimes n}(X)$ for some $n \geq 1$ such that $s = 0$ on Z but is nonzero at x. Denote by $y \in S$ the image of x under f, then for some i we have $y \in V_i$, and without loss of generality we can take $V_i = \operatorname{Spec} A[f_i^{-1}]$ since such open affines form a basis of the topology on $\operatorname{Spec} A$. We then see there exists $s' \in \mathcal{L}^{\otimes n}(f^{-1}(V_i))$ such that $s' = 0$ on $Z \cap f^{-1}(V_i)$ and s' is also nonzero at x, from the fact $\mathcal{L}$ is ample when restricted to $f^{-1}(V_i)$. As f is qcqs, we see

$$\mathcal{L}^{\otimes n}(f^{-1}(V_i)) = \mathcal{L}^{\otimes n}(X)[f_i^{-1}],$$

from the proof of Proposition 14.3, and so $s' = f/f_i^n$ for some $s \in \mathcal{L}^{\otimes n}(X)$. We then see $sf_i \in \mathcal{L}^{\otimes n}(X)$ vanishes

$$(Z \cap f^{-1}(V_i)) \cup f^{-1}(V(f_i)) \supseteq Z,$$

and sf_i is nonzero at x. □

We then have the following proposition which finishes our proof from last time officially.

Theorem 22.2. *Let $f : X \to Y$ be a proper map of schemes, $\mathcal{L}$ a line bundle on X, $y \in Y$ a point with fibre $X_y = X \times_Y \operatorname{Spec} k(y)$ and $\mathcal{L}_y$ the pullback of $\mathcal{L}$ to X_y. If $\mathcal{L}_y$ is ample, then there exists an open neighbourhood $x \in V \subseteq Y$ such that $\mathcal{L}|_{f^{-1}(V)}$ is relatively ample. In particular, if $\mathcal{L}_y$ is ample for all $y \in Y$, then $\mathcal{L}$ is relatively ample.*

Proof. This proof begins with several reductions. Without loss of generality we can let $Y = \operatorname{Spec} A$, and we can use the usual noetherian approximation argument, so we can assume A is noetherian. Next we would like to restrict to the case when A is local, and $y \in Y$ is simply the closed point of $\operatorname{Spec} A$. To see this, let $Y' = \operatorname{Spec} A_{\mathfrak{p}}$ where $\mathfrak{p}$ corresponds to $y \in Y$. Then if we know the result in this case, we get that the base change of $\mathcal{L}$ to $X \times_Y Y'$ is ample and we're off.
So we know there is some $n >> 0$ and sections $s_0, \ldots, s_r \in \Gamma(X \times_Y Y', \mathcal{L}^{\otimes n})$ defining a closed immersion,

$$i : X \times_Y Y'` \longrightarrow \mathbb{P}^r_{Y'}.$$

We already have $\Gamma(X \times_Y Y', \mathcal{L}^{\otimes n}) = \Gamma(X, \mathcal{L}^{\otimes n}) \otimes_A A_{\mathfrak{p}}$ by flat base change (Theorem 15.7). This means we can find a principal open subset $D(f) = V \subseteq Y$ of y, and sections $t_0, \ldots, t_r \in \Gamma(X \times_Y V, \mathcal{L}^{\otimes n})$ mapping to s_i. We then have,

$$D(t_0) \cup \cdots \cup D(t_r) \subseteq X,$$

is open, and contains X_y. Thus the complement is closed and the image in Y is still closed, since our map f is proper, and does not contain y. After shrinking to V' if necessary, we may assume that

$X = D(t_0) \cup \cdots \cup D(t_r)$. We now replace Y by V, and X by $X \times_Y V$, then the sections $t_0, \ldots, t_r$ give us a map,

$$j : X \longrightarrow \mathbb{P}^r_Y,$$

with $i = j \times_Y Y'$. It remains to show that after replacing Y by an open neighbourhood of y, j is a closed immersion. To see this, we need to show $\mathcal{O}_{\mathbb{P}^r_Y} \to j_*\mathcal{O}_X$ is surjective, and $|j|$ is a closed immersion topologically. For the former, we know j is proper, so $j_*\mathcal{O}_X$ is coherent, so we know the kernel $\mathcal{F}$ of the map in question

$$\mathcal{O}_{\mathbb{P}^r_Y} \longrightarrow j_*\mathcal{O}_X$$

is coherent. This means the support of $\mathcal{F}$ is closed, but we know $\mathrm{supp}\mathcal{F} \cap X_y = \varnothing$, so $y \notin f(\mathrm{supp}\mathcal{F})$ which is closed inside Y. We then base change to $Y \backslash f(\mathrm{supp}\mathcal{F})$, and we see $\mathcal{F} = 0$ hence our required map is surjective. We know $|j|$ is closed, so it remains only to see that j is injective. Assume we have $x_1 \neq x_2 \in X$ which are mapped to the same $z \in \mathbb{P}^r_Y$, then we consider the following diagram,

$$\begin{array}{ccc} X & \overset{\Delta_j}{\hookrightarrow} & X \times_{\mathbb{P}^r_Y} X \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^r_Y & \end{array} .$$

Since j is separated, we know Δ_j is a closed immersion, and we also know that the scheme theoretic product $|X \times_{\mathbb{P}^r_Y} X|$ surjects onto the topological product $|X| \times_{|\mathbb{P}^r_Y|} |X|$. There then exists some $x' \in |X \times_{\mathbb{P}^r_Y} X$ which maps to (x_1, x_2) and x' is not in the image of Δ_j. Hence, it is enough to see that Δ_j is an isomorphism after base change to an open neighbourhood of y. For this we need the map,

$$\mathcal{O}_{X \times_{\mathbb{P}^r_Y} X} \longrightarrow \Delta_{j*}\mathcal{O}_X,$$

is an isomorphis. This is already surjective, and the kernel is a coherent sheaf not supported on the fibre X_y, so by the same argument to show a certain cokernel is zero, we see this kernel is also zero. This long diversion has now shown us that we can assume A is a noetherian local ring. We now have the following statement.

Let $f : X \to \operatorname{Spec} A$ be a proper map, where A is noetherian and local, $y \in Y$ is the unique closed point, and $\mathcal{L}$ a line bundle on X. If $\mathcal{L}_y$ is ample, then $\mathcal{L}$ is ample. Equivalently, for all coherent sheaves $\mathcal{F}$ on X there is some $n_0 \geq 1$ such that for all $n \geq n_0$, $i \geq 1$,

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0,$$

from Proposition 15.4. This cohomology is finitely generated as an A-module from a slight generalisation of Theorem 13.1, so it is enough to show $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})^\wedge_y = 0$ as this is simply our cohomology under the faithfully flat base change $A \to \widehat{A}$. Hence, without loss of generality, we may also assume that A is complete. By the affine version of the theorem on formal functions, Theorem 20.5, we then have,

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = \lim_r H^i(X_r, \mathcal{F} \otimes \mathcal{L}^{\otimes n}|_{X_r}),$$

where $X_r = X \times_{\operatorname{Spec} A} \operatorname{Spec} A/\mathfrak{m}^r$, where $\mathfrak{m}$ corresponds to the closed point y of $Y = \operatorname{Spec} A$. It is enough to show that if $\mathcal{L}_y$ is ample then there exists some $n_0 \geq 1$ for that for all $n \geq n_0, i \geq 1, r \geq 1$,

$$H^i(X_r, \mathcal{F} \otimes \mathcal{L}^{\otimes n}|_{X_r}) = 0.$$

We need a trick here though, because if we go in blindly, our n_0 will depend on r, and that won't help us evaluate the inverse limit above. We need to go to associated gradeds, and base change to $\mathrm{gr}A$. Let

$$\mathrm{gr}A = \bigoplus_{r \geq 0} \mathrm{gr}^r A, \qquad \text{where } \mathrm{gr}^r A = \mathfrak{m}^r/\mathfrak{m}^{r+1},$$

which we call the associated graded of A. Now gr$A$ is a k algebra, where $k = A/\mathfrak{m}$, which is generated by $\mathrm{gr}^1 A = \mathfrak{m}/\mathfrak{m}^2$, which is a finite dimensional k-vector space, hence gr$A$ is a finitely generated k algebra. We then have the sheaf,

$$\mathrm{gr}\mathcal{F} = \bigoplus_{r\geq 0} \mathfrak{m}^r\mathcal{F}/\mathfrak{m}^{r+1}\mathcal{F},$$

which is a sheaf of gr$A$-modules on $X_1 = X \times_Y \operatorname{Spec} k = X_y$. This is finitely generated over $\mathcal{O}_{X_1} \otimes_k \mathrm{gr}A$, since as an module over this sheaf it is generated by $\mathcal{F}/\mathfrak{m}$. Hence gr$\mathcal{F}$ corresponds to a coherent sheaf $\mathcal{F}'$ on $X_1 \times_{\operatorname{Spec} k} \operatorname{Spec} \mathrm{gr}A$. Notice the maps

$$f' : X_1 \times_{\operatorname{Spec} k} \operatorname{Spec} \mathrm{gr}A \to \operatorname{Spec} \mathrm{gr}A, \qquad g : X_1 \times_{\operatorname{Spec} k} \operatorname{Spec} \mathrm{gr}A \longrightarrow X,$$

are proper, and so $g^*\mathcal{L}$ is ample, as it is a pullback of $\mathcal{L}_y$. The cohomological criterion then implies there is some $n_0 \geq 1$ such that for all $n \geq n_0$ and $i \geq 1$,

$$H^i(X_1 \times_{\operatorname{Spec} k} \operatorname{Spec} \mathrm{gr}A, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = H^i(X_1, \mathrm{gr}\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = \bigoplus_{r\geq 0} H^i(X_1, \mathfrak{m}^r\mathcal{F}/\mathfrak{m}^{r+1}\mathcal{F}) = 0,$$

where we have not notated any restrictions of the sheaves above. In other words, for all $r \geq 0$,

$$H^i(X_1, \mathfrak{m}^r\mathcal{F}/\mathfrak{m}^{r+1}\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = H^i(X, \mathfrak{m}^r\mathcal{F}/\mathfrak{m}^{r+1}\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0.$$

However, we have the following equality,

$$H^i(X_r, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = H^i(X_r, \mathcal{F}/\mathfrak{m}^r\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = H^i(X, \mathcal{F}/\mathfrak{m}^r\mathcal{F} \otimes \mathcal{L}^{\otimes n}),$$

so looking at the long exact sequence,

$$0 \longrightarrow \mathfrak{m}^r\mathcal{F}/\mathfrak{m}^{r+1}\mathcal{F} \otimes \mathcal{L}^{\otimes n} \longrightarrow \mathcal{F}/\mathfrak{m}^{r+1}\mathcal{F} \otimes \mathcal{L}^{\otimes n} \longrightarrow \mathcal{F}/\mathfrak{m}^r\mathcal{F} \otimes \mathcal{L}^{\otimes n} \longrightarrow 0,$$

we see by induction on r that $H^i(X, \mathcal{F}/\mathfrak{m}^r\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0, i \geq 1$ and $r \geq 1$. This finishes this proof. □

Recall the situation of our lectures on base change. We have a map $f : X \to Y = \operatorname{Spec} A$ which is proper, and where the schemes are noetherian, and $\mathcal{F}$ is a coherent sheaf on X which is flat over Y. We then know the cohomology $R\Gamma(X, \mathcal{F})$ is computed by a perfect complex of A-modules,

$$\cdots \longrightarrow 0 \longrightarrow M^0 \longrightarrow \cdots \longrightarrow M^d \longrightarrow 0 \longrightarrow \cdots,$$

with $d = \dim f$ and each M^i is a finitely generated projective A-module. For any $A \to A'$, then $R\Gamma(X \times_{\operatorname{Spec} A} \operatorname{Spec} A', \mathcal{F})$ is computed by $M^\bullet \otimes_A A'$. This was the content in and around Theorem 17.3.

We are going to now state and prove Corollary 17.6 from lecture 17.

Corollary 22.3. *The function $Y \to \mathbb{Z}_{\geq 0}$ defined by,*

$$y \longmapsto \dim_{k(y)} H^i(X_y, \mathcal{F}_y),$$

for $\mathcal{F}_y = \mathcal{F}|_{X_y}$ and all $i \geq 0$, is upper semicontinuous, i.e. for all $r \in \mathbb{Z}$, the set,

$$\{y \in Y \mid \dim_{k(y)} H^i(X_y, \mathcal{F}_y) \leq r\},$$

is open.

The idea is that the dimension is sort of locally constant, but it can jump up under specialisation of points.

Proof. We will begin this proof with a reminder of some commutative algebra, and then we will restrict the general case to this one. Recall that if N is a finitely generated A-module, then $y \mapsto \dim_{k(y)}(N \otimes_A k(y))$ is upper semicontinuous, i.e. if $x'_1, \ldots, x'_r$ freely generate $N \otimes_A k(y)$, then in a neighbourhood of y, $x'_1, \ldots, x'_r$ all lift to $x_1, \ldots, x_r \in N$. We then take N' to be the cokernel $A^r \to A$ defined using the elements $x_1, \ldots, x_r$, which is also a finitely generated A-module, with $N' \otimes_A k(y) = 0$. By Nakayama's lemma, we see that $N' \otimes_A A_y = 0$, so $N' = 0$ in a neighbourhood of y. Hence $A^r \to N$ is a surjection locally around y, so for y' in this neighbourhood, $k(y')^r \to N \otimes_A k(y')$ is surjective, hence,

$$\dim_{k(y')}(N \otimes_A k(y')) \leq r.$$

For the general case, we choose a perfect complex (from Theorem 17.3) as above, and let W^i be the cokernel of the differentials $d^{i-1} : M^{i-1} \to M^i$ of this complex. We then have $W^i = M^i/B^i \supseteq Z^i/B^i = H^i = H^i(X, \mathcal{F})$. From, and a quick observation, we have the following exact sequence,

$$0 \longrightarrow H^i \longrightarrow W^i \xrightarrow{d^i} M^{i+1} \longrightarrow W^{i+1} \longrightarrow 0.$$

For all $A \to A'$, we see that

$$(W^i)' = \operatorname{coker}(d^{i-1} \otimes_A A' : M^{i-1} \otimes_A A' \longrightarrow M^i \otimes_A A') = W^i \otimes_A A',$$

and our exact sequence above becomes,

$$0 \longrightarrow H^i(M^\bullet \otimes_A A') \longrightarrow W^i \otimes_A A' \longrightarrow M^{i+1} \otimes_A A' \longrightarrow W^{i+1} \otimes_A A' \longrightarrow 0.$$

In particular, take $A' = k(y)$ for some $y \in Y = \operatorname{Spec} A$, we obtain the equation,

$$\dim_{k(y)}(H^i(M^\bullet \otimes_A k(y))) = \dim_{k(y)}((W^i \otimes_A k(y)) + \dim_{k(y)}(W^{i+1} \otimes_A k(y)) - \dim_{k(y)}(M^{i+1} \otimes_A k(y)). \tag{22.4}$$

Since the first two terms on the right-hand-side of the above equation are upper semicontinuous by our commutative algebra observation above, and the last term is simply locally constant, we see the left-hand-side is upper semi-continuous. □

Corollary 22.5. *Consider the same hypotheses as discussed before Corollary 22.3, with the added condition that A is integral. If the function,*

$$y \longmapsto \dim_{k(y)} H^i(X_y, \mathcal{F}_y),$$

is locally constant, then $H^i(X, \mathcal{F})$ is a finite projective A-module, and $H^i(X, \mathcal{F}) \otimes_A k(y) \to H^i(X_y, \mathcal{F}_y)$ is an isomorphism for all $y \in Y$.

Proof. Again, we will start with a fact from commutative algebra, and then apply this to the general case with the help of Theorem 17.3. If A is a noetherian integral domain, and N is a finitely generated A-module such that the function which sends $y \in \operatorname{Spec} A$ to $\dim_{k(y)}(N \otimes_A k(y))$ is constant, then N is a finitely projective A-module. Equivalently, is a locally free A-module of finite rank. To see this, pick some $y \in Y$, and $x_1, \ldots, x_r \in N$ such that $x'_1, \ldots, x'_r \in N \otimes_A k(y)$ form a basis, then we see the map $A^r \to N$ is surjective after further localisation. This gives us the short exact sequence,

$$0 \longrightarrow N' \longrightarrow A^r \longrightarrow N \longrightarrow 0.$$

If $N' \neq 0$, then we have the following commutative diagrams,

$$\begin{array}{ccc} N' & \hookrightarrow & A^r \\ \downarrow & & \downarrow \\ N' \otimes_A K & \longrightarrow & K^r \end{array},$$

where $K = \operatorname{Frac} A$ and the arrows with hooks are injections, since A is an integral domain. Hence $N' \otimes_A N \neq 0$, and K is flat over A we obtain the short exact sequence,

$$0 \longrightarrow N' \otimes_A K \longrightarrow K^r \longrightarrow N \otimes_A K \longrightarrow 0.$$

By counting the dimensions of the above modules as K-vector spaces, we see that $N' \otimes_A K = 0$, a contradiction, so $N' = 0$ and $N \cong A^r$. In general, we use Equation 22.4 and our hypotheses about locally constant functions to conclude that the functions,

$$y \longmapsto \dim_{k(y)}(W^i \otimes_A k(y)), \quad \dim_{k(y)}(W^{i+1} \otimes_A k(y)),$$

are constant. Our commutative algebra proposition above then states that W^i and W^{i+1} are both finite projective A-modules, so we have the following exact sequence,

$$0 \longrightarrow H^i \longrightarrow W^i \longrightarrow M^{i+1} \longrightarrow W^{i+1} \longrightarrow 0.$$

Since W^{i+1}, M^{i+1} and W^i are all finite projective A-modules, we can conclude that H^i is a finitely projective A-module. We then notice that we can compute $H^i(M^\bullet \otimes_A k(y))$ by tensoring the exact sequence above with $k(y)$ over A, which will stay exact, hence

$$H^i(M^\bullet) \otimes_A k(y) = H^i(M^\bullet \otimes_A k(y)).$$

□

23 Left Derived Functors and $\mathcal{T}$or 27/07/2017

Today was mostly a problems session, with Peter answering various questions, one of which is the following.

Since $\mathcal{O}_X$-mod does not have enough projectives, how do we define left derived functors?

The answer is we cannot with the general machinary of homological algebra, but there are some exceptions.

Lemma 23.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. For any $\mathcal{O}_X$-module $\mathcal{M}$, there is a flat $\mathcal{O}_X$-module $\widetilde{\mathcal{M}}$ and a surjective map $\widetilde{\mathcal{M}} \to \mathcal{M}$.*

Proof. We suggest the $\mathcal{O}_X$-module and map,

$$\widetilde{\mathcal{M}} = \bigoplus_{j_U : U \to X, s \in \mathcal{M}(U)} j_{U!}\mathcal{O}_U \longrightarrow \mathcal{M}.$$

This is clearly surjective by construction, and direct sums of flat $\mathcal{O}_X$-modules are flat $\mathcal{O}_X$-modules, so we only need to see $j_{U!}\mathcal{O}_U$ is flat. For any $\mathcal{N}$, then we have an isomorphism,

$$j_{U!}\mathcal{N}|_U \xrightarrow{\cong} \mathcal{N} \otimes_{\mathcal{O}_X} j_{U!}\mathcal{O}_U,$$

which can be checked on stalks. The domain of the isomorphism above is clearly exact in $\mathcal{N}$, as $j_{U!}$ is exact. Thus $j_{U!}\mathcal{O}_U$ is flat. □

We then define $\mathcal{T}\mathrm{or}_i^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ for $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{N}$ by taking flat resolutions in either of the variables. This works as if $\mathcal{M}$ is flat, then $\mathcal{T}\mathrm{or}_i^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) = 0$ for $i > 0$. This is not obvious, and the justification comes from the next lemma.

Lemma 23.2. *Consider the following exact sequence of $\mathcal{O}_X$-modules,*

$$\cdots \longrightarrow \mathcal{M}^{-2} \longrightarrow \mathcal{M}^{-1} \longrightarrow \mathcal{M}^0 \longrightarrow 0.$$

Then for all $\mathcal{O}_X$-modules $\mathcal{N}$, the following sequence is exact,

$$\cdots \longrightarrow \mathcal{M}^{-2} \otimes_{\mathcal{O}_X} \mathcal{N} \longrightarrow \mathcal{M}^{-1} \otimes_{\mathcal{O}_X} \mathcal{N} \longrightarrow \mathcal{M}^0 \otimes_{\mathcal{O}_X} \mathcal{N} \longrightarrow 0.$$

The proposed proof is essentially a sheafified version of Lemma 16.3, where we don't care that $\mathcal{T}\mathrm{or}_i^{\mathcal{O}_X}$ is not necessarily well-defined, i.e. not independent of the choice of flat resolution. Notice that the name $\mathcal{T}\mathrm{or}_i^{\mathcal{O}_X}$ is appropriate, since if $X = \operatorname{Spec} A$, then M is a flat A-modules if and only if $\widetilde{M}$ is a flat $\mathcal{O}_X$-module, i.e.

$$\widetilde{\mathrm{Tor}_i^A(M, N)} \cong \mathcal{T}\mathrm{or}_i^{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}).$$

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