Exercise Sheet 13

Discussed on 21.07.2021

Definition. Let K be a field, V a K-vector space and $P\colon V \to K$ a map. We say that P is *polynomial* if for any $n \geq 0$ and any vectors $v_1, \dots, v_n \in V$ the map

$$K^n \to K, \qquad (x_1, \dots, x_n) \mapsto P(x_1 v_1 + \dots + x_n v_n)$$

is given by a polynomial, i.e. by an element of $K[T_1, \dots, T_n]$.

Lemma. *Let K be an infinite field, V a K-vector space and $P\colon V \to K$ a map. Then P is polynomial if and only if for all $v, w \in V$ the map*

$$K \to K, \qquad x \mapsto P(v + xw)$$

is given by a polynomial.

Problem 1. Let X be an abelian variety of dimension g over some field k.

(a) Let $\varphi, \psi \in \operatorname{End}(X)$ and let L be a line bundle on X. Show that there are line bundles L_0, L_1, L_2 on X such that for all $n \in \mathbb{Z}$ we have

$$(\psi + n\varphi)^* L = L_0 \otimes L_1^n \otimes L_2^{n(n-1)/2}.$$

Hint: Apply the Theorem of the Cube with $f = \psi + n\varphi$, $g = h = \varphi$ and induct on n.

(b) Show that $\deg\colon \operatorname{End}(X) \to \mathbb{Z}$ extends to a polynomial function on $\operatorname{End}^0(X) = \operatorname{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (here $\deg \varphi = 0$ if φ is not surjective).

Hint: Fix an ample line bundle L on X and recall that for every $\varphi \in \operatorname{End}(X)$ we have $\deg(\varphi) = (\varphi^* L)^g / (L)^g$, where $(L)^g$ denotes the intersection product of g copies of L. Now apply (a).

(c) For every prime ℓ, define the ℓ-adic Tate module $T_\ell X$ of X. Show that the natural map

$$\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \operatorname{End}(X) \hookrightarrow \operatorname{End}(T_\ell X)$$

is injective. Deduce that $\operatorname{End}(X)$ has rank $\leq 4g^2$.

Hint: Apply the same proof strategy as for elliptic curves. The hard bit is to show the following: If $M \subset \operatorname{End}(X)$ is a finitely generated subgroup then $\mathbb{Q}M \cap \operatorname{End}(X)$ is again finitely generated over $\mathbb{Z}$. To prove this, first reduce to the case that X is simple, so that every $\varphi \in \operatorname{End}(X)$ is an isogeny. Then use (b) to deduce that $\mathbb{Q}M \cap \operatorname{End}(X) \subset \mathbb{Q}M$ is discrete.