Exercise Sheet 1

Discussed on 14.04.2021

Definition. Let k be a field and let (G, m_G) and (H, m_H) be group schemes over k. Then a *group homomorphism* $\varphi\colon G \to H$ is a morphism of k-schemes such that $m_H \circ (\varphi \times \varphi) = \varphi \circ m_G$. Given such a φ, we define $\ker \varphi \hookrightarrow G$ as the fiber product $\ker \varphi = G \times_{H, e_H} \operatorname{Spec} k$, where $e_H\colon \operatorname{Spec} k \hookrightarrow H$ is the neutral element map.

Problem 1. Let k be a field. Show that the group endomorphisms $\mathbb{G}_m \to \mathbb{G}_m$ are precisely the n-th power maps $[n]\colon \mathbb{G}_m \to \mathbb{G}_m$ for $n \in \mathbb{Z}$. More generally, find all group homomorphisms $\mathbb{G}_m^n \to \mathbb{G}_m^l$ for integers $n, l > 0$.

Problem 2. Let k be a field.

(a) Let $\mathbb{G}_a\colon \mathrm{Sch}/k \to \mathrm{Grp}$ be the functor which assigns to every k-scheme S the group $(\mathcal{O}_S(S), +)$ of global sections. Show that $\mathbb{G}_a$ is an affine group scheme by explicitly writing down its coordinate ring and multiplication map.

(b) The underlying scheme of the group scheme GL_n is

$$\mathrm{GL}_n = \operatorname{Spec}(k[T_{i,j}]_{i,j=1}^n[S]/(S \cdot \det(T_{i,j}) - 1)).$$

Write down the multiplication map $m\colon \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n$ as a map on the coordinate rings.

Problem 3. Let k be a field.

(a) Given an group scheme G over k, show that the neutral element map $e_G\colon \operatorname{Spec} k \to G$ is a closed immersion.

(b) Given a group homomorphism $\varphi\colon G \to H$ of group schemes over k, show that $\ker \varphi$ can naturally be made into a group scheme such that for every k-scheme S,

$$(\ker \varphi)(S) = \ker(G(S) \to H(S))$$

as groups.

Problem 4. Let k be a field.

(a) For every integer $n > 0$ we define

$$\mu_n := \ker([n]\colon \mathbb{G}_m \to \mathbb{G}_m),$$

where $[n]$ denotes the n-th power map (see Problem 1). Compute the coordinate ring of μ_n and determine the multiplication map on the coordinate ring.

(b) Show that if k is algebraically closed and n is invertible in k then $\mu_n \cong (\mathbb{Z}/n\mathbb{Z})_k$ is a constant group scheme. What does μ_p look like if k has characteristic p?