Exercise Session 9

$$E \to E' \to E$$

$$\begin{tikzcd} T_eE \arrow[r] \arrow[rr, bend right, "\cdot n"'] & T_eE' \arrow[r] & T_eE \end{tikzcd}$$

(1) Let $K/\mathbb{Q}$ imag. quadr., k any alg. closed field.

(a) Let $\mathcal{O} \subseteq K$ be an order. There is a unique $f \in \mathbb{Z}_{\geq 1}$ s.t.

$$\mathcal{O} \cong \mathbb{Z} + f\mathcal{O}_K$$

- Note $\mathcal{O} \subseteq \mathcal{O}_K$ and $\mathcal{O}_K/\mathcal{O}$ is torsion. Let $f := |\mathcal{O}_K/\mathcal{O}|$.
- Clearly $\mathbb{Z} + f\mathcal{O}_K \subseteq \mathcal{O}$
- $f = |\mathcal{O}_K/\mathcal{O}|$

  $= |\mathcal{O}_K/\mathbb{Z}+f\mathcal{O}_K|$

$\Rightarrow \mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$.

(b) Let $\Lambda \subseteq K$ be a lattice and $\mathcal{O}_\Lambda := \{x \in K \mid x\Lambda \subseteq \Lambda\}$.

- $\mathcal{O}_\Lambda$ is an order in K.

  1. $\mathcal{O}_\Lambda \subseteq \mathcal{O}_K$

  2. W.l.o.g. $\Lambda \subseteq \mathcal{O}_K$. Let $d = |\mathcal{O}_K/\Lambda|$. Then $d\mathcal{O}_K \subseteq \mathcal{O}_\Lambda$.

  $\rightsquigarrow d\mathcal{O}_K \subseteq \mathcal{O}_\Lambda \subseteq \mathcal{O}_K$

$\Rightarrow \mathcal{O}_\Lambda \cong \mathbb{Z}^2$ as $\mathbb{Z}$-module.

Enough to show that $\Lambda/p\Lambda$ has rk 1 over $\mathcal{O}_\Lambda/p$ $\forall p \in \operatorname{Spec} \mathcal{O}_\Lambda$. But fibers do not change under completion.

- Λ is projective of rank 1 over $\mathcal{O}_\Lambda$.

1. Enough to show this after $- \otimes_{\mathbb{Z}} \mathbb{Z}_p$ (for all primes p).

$\rightsquigarrow$ replace K by $K_p := K \otimes_{\mathbb{Z}} \mathbb{Z}_p$, Λ by $\Lambda_p := \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p$, etc.

not necessarily a field!

$$K_p \cong \begin{cases} \mathbb{Q}_p \times \mathbb{Q}_p & \text{if } p \text{ splits in } K \\ \text{unramified ext}/\mathbb{Q}_p & \text{if } p \text{ is inert in } K \\ \text{ramified ext}/\mathbb{Q}_p & \text{if } p \text{ ramifies in } K \end{cases}$$

If $K = \mathbb{Q}(\sqrt{d})$ then $K_p = \mathbb{Q}_p[x]/(x^2 - d)$.

2. Can rescale Λ_p by any $a \in K_p^\times$.

$\rightsquigarrow$ W.l.o.g. $\Lambda_p \subseteq \mathcal{O}_{K_p}$ and $1 \in \Lambda_p$

- If K_p is a field, divide Λ_p by generator with smaller valuation.
- If $K_p \cong \mathbb{Q}_p \times \mathbb{Q}_p$, use explicit computation.

3. By same argument as in (a), $\Lambda_p = \mathbb{Z}_p + f_p \mathcal{O}_{K_p}$ for some $f_p \in \mathbb{Z}$

$\Rightarrow \mathcal{O}_{\Lambda_p} = \Lambda_p$. $\rightsquigarrow \Lambda_p$ is free of rk 1 over $\mathcal{O}_{\Lambda_p}$.

- $$\{\text{lattices } \Lambda \subset K\}/K^\times \cong \coprod_{f \in \mathbb{Z}_{\geq 1}} \operatorname{Pic}(\mathbb{Z} + f\mathcal{O}_K).$$

$$\Lambda \longmapsto (\mathcal{O}_\Lambda = \mathbb{Z} + f\mathcal{O}_K, \Lambda)$$

1. Map is clearly injective.

2. Surjectivity: Pick $f \in \mathbb{Z}$, $P \in \mathrm{Pic}(\mathbb{Z} + f\mathcal{O}_K)$.

Let $\eta \in \mathrm{Spec}(\mathbb{Z}+f\mathcal{O}_K)$ generic point. Then $P_\eta \cong K$.

$\rightsquigarrow P \hookrightarrow P_\eta = K$.

Also P finite over $\mathbb{Z} \Rightarrow P$ free over $\mathbb{Z}$. But $P \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = P_\eta = K$

has $\dim_{\mathbb{Q}} = 2$, hence $\mathrm{rk}\, P = 2$.

(c) Let char $k = 0$.

- All EC's E over k with $\mathrm{End}^0(E) \cong K$ are isogenous.

1. Assume $k = \mathbb{C}$. By lecture 4,

(i) $\mathrm{Hom}(\mathbb{C}/\Lambda, \mathbb{C}/\Lambda') = \{a \in \mathbb{C} \mid a\Lambda \subseteq \Lambda'\}$

(ii) $\mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}$ CM by $K \Leftrightarrow \tau \in K$.

Pick $E_0 := \mathbb{C}/\mathcal{O}_K$. Let $E = \mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}$ with $\tau \in K$. Pick any $a \in \mathbb{Z}$

s.t. $a\tau \in \mathcal{O}_K$. Then a defines $E \to E_0$.

$\rightsquigarrow$ All E with CM by K are isogenous to E_0.

2. Let E, E' EC's over k with $\mathrm{End}^0(E) \cong \mathrm{End}^0(E') \cong K$. Write k

as the union of fin. tr.deg. ext of $\mathbb{Q}$.

$\overset{\text{sheet 8}}{\rightsquigarrow}$ for some fin tr. deg. $k_0/\mathbb{Q}$, $\exists$ EC's E_0, E_0' s.t.

$(E_0)_k = E$, $(E_0')_k = E'$ and $\mathrm{End}^0(E_0) \cong \mathrm{End}^0(E_0') \cong K$.

$\rightsquigarrow$ w.l.o.g k fin. tr. deg$/_{\mathbb{Q}}$ (still $k = \bar{k}$).

Pick $k \hookrightarrow \mathbb{C}$. Then by sheet 8,

$$\begin{array}{ccc} \mathrm{Hom}((E_0)_{\mathbb{C}}, (E_0')_{\mathbb{C}}) & = & \mathrm{Hom}(E_0, E_0') \\ \neq & & \neq \\ 0 & \rightsquigarrow & 0. \end{array}$$

$$\begin{tikzcd}
\bullet \left\{ \text{EC's } E \text{ over } k \text{ with } \mathrm{End}^0(E) \cong K \right\}/_{\cong} \arrow[d, leftrightarrow, "\cong"] \arrow[r, hook, "\sim"] & \coprod\limits_{f \in \mathbb{Z}_{\geq 1}} \mathrm{Pic}(\mathbb{Z} + f\mathcal{O}_K) \arrow[ddl, leftrightarrow, "\cong\ (b)"] \\
\{E \mid \exists E_0 \to E\}/_{\cong} \arrow[d, leftrightarrow, "\cong \text{ lecture}"] & \\
K^\times \backslash \prod_{\ell} \{\Lambda \subseteq V_\ell E_0\} \arrow[r, leftrightarrow, "\sim"] & K^\times \backslash \{\Lambda \subset K\}
\end{tikzcd}$$

E_0 any fixed EC with $\mathrm{End}^0(E_0) \cong K$

not canonical

(d) • Show that p splits in K ($\Leftrightarrow E$ ordinary).

• Then use same arguments as in (c).

② p prime, S noetherian $\mathbb{F}_p$-scheme, E/S EC. Let $F: E \to E^{(p)}$ be the relative Frobenius.

(a) F is finite locally free of degree p.

$$\begin{tikzcd}
E \arrow[r, "F"] \arrow[rr, bend left, "F_X"] \arrow[dr] & E^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d] \\
& S \arrow[r, "F_S"] & S
\end{tikzcd}$$

- F finite. (either use above diagram or note that F is proper + quasi-finite)

- F flat by fiber criterion for flatness (Stacks 039E).

$\Rightarrow \ker F \subseteq E$ is finite loc. free grp scheme $/S$.

- $E^{(p)} = E/\ker F$.

  1. Check $E^{(p)}$ satisfies univ. prop. of $E/\ker F$.

  2. By univ. prop., have $E/\ker F \to E^{(p)}$. Check this is isom.

$$\begin{tikzcd} & E/\ker F \arrow[r] & E^{(p)} \\ E \arrow[ur] \arrow[urr, "F"'] & & \end{tikzcd}$$

(6) By univ. prop. of quotients,

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "F"'] & E \\ E^{(p)} = E/\ker F \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$