Exercise Session 11

Let k alg. closed field, C/k proper smooth connected curve.
Assume that $\mathrm{Pic}^\circ_{C/k}$ is representable by a k-scheme which is locally of finite type.

Thm: $\mathrm{Pic}^\circ_{C/k}$ is an AV of dim $g = g(C)$.

Proof:

(a) Let X be a scheme. Then $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$.

Prop: Let Y be a topological space, $\mathcal{F}$ an abelian sheaf on X. Then

$$H^1(Y, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{F}).$$

$\mathcal{U}: Y = \bigcup_i U_i$ open cover.

$$C^n(\mathcal{U}, \mathcal{F}) := \prod_{i_0, \dots, i_n \in I^{n+1}} \mathcal{F}(\underbrace{U_{i_0 \dots i_n}}_{= U_{i_0} \cap \dots \cap U_{i_n}})$$

$$d^n : C^n(\mathcal{U}, \mathcal{F}) \to C^{n+1}(\mathcal{U}, \mathcal{F})$$

$$(d^n s)_{i_0 \dots i_{n+1}} = \sum_{k=0}^{n+1} (-1)^k s_{i_0 \dots \hat{i}_k \dots i_{n+1}} \big|_{U_{i_0 \dots i_{n+1}}}$$

$\check{H}^n(\mathcal{U}, \mathcal{F}) =$ cohomology of $C^\bullet(\mathcal{U}, \mathcal{F})$.

Sketch: Pick any injective sheaf $\mathcal{G}$ s.t. $\mathcal{F} \hookrightarrow \mathcal{G}$.

$\rightsquigarrow$ SES $\quad 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{R} \to 0$

$$0 \to C^\bullet(\mathcal{U},\mathcal{F}) \to C^\bullet(\mathcal{U},\mathcal{G}) \to \mathcal{D}^\bullet(\mathcal{U}) \to 0$$

$= C^\bullet(\mathcal{U},\mathcal{G})/C^\bullet(\mathcal{U},\mathcal{F})$

G injective

$$\rightsquigarrow \begin{tikzcd} 0 \arrow[r] & H^0(Y,\mathcal{F}) \arrow[r] & H^0(Y,\mathcal{G}) \arrow[r] & H^0(Y,\mathcal{R}) \arrow[r] & H^1(Y,\mathcal{F}) \arrow[r] & H^1(Y,\mathcal{G}) = 0 \\ 0 \arrow[r] & \check{H}^0(Y,\mathcal{F}) \arrow[r] \arrow[u, "\wr"] & \check{H}^0(Y,\mathcal{G}) \arrow[r] \arrow[u, "\wr"] & H^0(\mathcal{D}^\bullet(\mathcal{U})) \arrow[r] \arrow[u, "\alpha"'] & \check{H}^1(Y,\mathcal{F}) \arrow[r] \arrow[u, "\beta"'] & \check{H}^1(Y,\mathcal{G}) = 0 \arrow[u] \end{tikzcd}$$

Exercise

Check: After $\varinjlim_{\mathcal{U}}$, α becomes isom.

$\rightsquigarrow$ β becomes isom. $\square$

Rmk: Allowing more general "hypercoverings" $\mathcal{U}$, we have

$$H^n(\mathcal{C},\mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U},\mathcal{F}) \qquad \forall n \geq 0 \qquad \text{(Stacks Thm 01H0)}$$

on any site $\mathcal{C}$.

Claim: $\check{H}^1(\mathcal{U},\mathcal{O}_X^\times) = \{ \text{l.b. } \mathcal{L} \text{ on } X \mid \mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i} \}$.

Proof:

$$\mathcal{L} \longmapsto \left.\begin{matrix} \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i} \\ \mathcal{L}|_{U_j} \xrightarrow{\sim} \mathcal{O}_{U_j} \end{matrix}\right\} \mathcal{O}_{U_{ij}} \xrightarrow{\sim} \mathcal{O}_{U_{ij}} \mathrel{\hat{=}} \text{element of } \mathcal{O}_X^\times(U_{ij})$$

By def:

$$\check{H}^1(\mathcal{U},\mathcal{O}_X^\times) = \Big\{ (\varphi_{ij})_{ij} \in \prod_{i,j} \mathrm{Aut}(\mathcal{O}_{U_{ij}}) \;\Big|\; \varphi_{ik} = \varphi_{ij} \circ \varphi_{jk} \ \forall i,j,k \Big\} \Big/ \Big\{ (\varphi_{ij})_{ij} \;\Big|\; \begin{matrix} \exists (f_i)_i \in \prod_i \mathrm{Aut}(\mathcal{O}_{U_i}) \\ \text{s.t. } \varphi_{ij} = f_i f_j^{-1} \end{matrix} \Big\}$$

(b) Show that the tangent space of $\mathrm{Pic}^0_{C/k}$ at 0 is $H^1(C,\mathcal{O}_C)$.

By lecture 10, page 11, tangent space is

$$\mathrm{Mor}_0(\mathrm{Spec}\, k[\varepsilon], \mathrm{Pic}^0_{C/k}) = \ker\left(\mathrm{Pic}^0_{C/k}(k[\varepsilon]) \to \mathrm{Pic}^0_{C/k}(k)\right)$$

Consider the SES

$$1 \to 1+\varepsilon\mathcal{O}_C \to \mathcal{O}_C[\varepsilon]^\times \to \mathcal{O}_C \to 1 \qquad \text{on } C$$

$$\begin{tikzcd}
 & & k[\varepsilon]^\times \arrow[r, two heads] \arrow[d, equal] & k^\times \arrow[d, equal] & \\
\rightsquigarrow 1 \arrow[r] & H^0(C, 1+\varepsilon\mathcal{O}_C) \arrow[r] & H^0(C,\mathcal{O}_C[\varepsilon]^\times) \arrow[r] & H^0(C,\mathcal{O}_C^\times) \arrow[r, "0"] & H^1(C, 1+\varepsilon\mathcal{O}_C) \arrow[d] \\
 & & & \mathrm{Pic}_C(k[\varepsilon]) \arrow[r, equal] & H^1(C,\mathcal{O}_C[\varepsilon]^\times) \arrow[d] \\
 & & & \mathrm{Pic}_C(k) \arrow[r, equal] & H^1(C,\mathcal{O}_C^\times)
\end{tikzcd}$$

$$\Rightarrow H^1(C, 1+\varepsilon\mathcal{O}_C) = \ker\left(\mathrm{Pic}_C(k[\varepsilon]) \to \mathrm{Pic}_C(k)\right)$$

$$\begin{matrix} \| & & \| \\ H^1(C,\mathcal{O}_C) & & \ker\left(\mathrm{Pic}^0_C(k[\varepsilon]) \to \mathrm{Pic}^0_C(k)\right) \end{matrix}$$

(c) $\mathrm{Pic}^0_{C/k}$ is smooth over k

Lifting criterion for smoothness: Need to show that for all k-alg. A, ideals $I \subseteq A$ s.t. $I^2=0$, the map

$$\mathrm{Pic}^0_C(A) \to \mathrm{Pic}^0_C(A/I)$$

is surjective.

Consider the SES $0 \to f^*I \to \mathcal{O}_{C_A} \to \mathcal{O}_{C_{A/I}} \to 0$ on $|C_A| = |C_{A/I}|$, where $f: C_A \to \operatorname{Spec} A$ (this is the pullback of $0 \to I \to A \to A/I \to 0$ along f, using that f is flat)

$\rightsquigarrow$ get SES $\quad 1 \longrightarrow 1+f^*I \to \mathcal{O}^\times_{C_A} \to \mathcal{O}^\times_{C_{A/I}} \to 1.$

$$\rightsquigarrow \quad \overset{\operatorname{Pic}_C(A)}{\overset{\shortparallel}{H^1(C_A, \mathcal{O}^\times_{C_A})}} \longrightarrow \overset{\operatorname{Pic}_C(A/I)}{\overset{\shortparallel}{H^1(C_{A/I}, \mathcal{O}^\times_{C_{A/I}})}} \to H^2(C_A, 1+f^*I) = H^2(C_A, f^*I) = 0$$

surj.

$$\begin{tikzcd}
f^*I & & I \\
C_A \arrow[r, "f"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner"] & \operatorname{Spec} A \arrow[d, "g"] \\
C \arrow[r, "h"'] & \operatorname{Spec} k & I
\end{tikzcd}$$

$Rg'_* f^*I = h^*I$

flat base-change (stacks)

$$\rightsquigarrow R\Gamma(C_A, f^*I) = \underbrace{Rh_* Rg'_*}_{=R(hg')_*}(f^*I) = Rh_*(h^*I) = R\Gamma(C, h^*I)$$

C has dim 1, hence this vanishes in degree ≥ 2.

Alternatively, lecture 7 page 8 shows that $Rf_*(f^*I)$ vanishes in degree ≥ 2.

(d) Fix a point $P \in C(k)$. There is a canonical map

$$\varphi: C^g \to \mathrm{Pic}^0_C, \quad (P_1, \dots, P_g) \mapsto \mathcal{O}([P_1] + \dots + [P_g] - g[P])$$

For any $S \in Sch_k$, $P_1, \dots, P_g \in C(S)$, can use above formula because $P_1, \dots, P_g, P_S$ are sections of $C_S \to S$, hence their images are Cartier divisors (lecture 8, page 1,2)

(e) φ is surjective

Enough to check on k-points (because C, $\mathrm{Pic}^0_{C/k}$ are loc. of finite type)

$\rightsquigarrow$ to show: $C(k)^g \longrightarrow \mathrm{Pic}^0_C(k)$ is surjective.

$(P_1, \dots, P_g) \mapsto \mathcal{O}([P_1] + \dots + [P_g] - g[P])$

Take any deg 0 l.b. $\mathcal{L}$ on C. By Riemann-Roch,

$$h^0(\mathcal{L} \otimes \mathcal{O}(g[P])) \geq \underbrace{\deg(\mathcal{L} \otimes \mathcal{O}(g[P]))}_{g} + 1 - g = 1$$

$\rightsquigarrow \exists$ non-zero $f: \mathcal{O}_C \to \mathcal{L} \otimes \mathcal{O}(g[P])$. This is automatically injective as C is integral.

$\rightsquigarrow$ get SES $\quad 0 \to \mathcal{O}_C \to \mathcal{L} \otimes \mathcal{O}_C(g[P]) \to \mathcal{F} \to 0 \qquad (*)$

torsion sheaf, $h^0(\mathcal{F}) = g$

$\rightsquigarrow \mathcal{F}$ supported at points $\underset{C(k)}{P_1, \ldots, P_g \in}$ (with multiplicities)

$$\left(\rightsquigarrow 0 \to \mathcal{O}_C(-[P_1] - \ldots - [P_g]) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F} \to 0 \right)$$

$\Rightarrow \quad \mathcal{L} \cong \mathcal{O}_C([P_1] + \ldots + [P_g] - g[P])$.

$$\left(\begin{array}{l} \text{tensor } (*) \text{ with } \mathcal{O}_C(-g[P]) \\ \rightsquigarrow \quad 0 \to \mathcal{O}_C(-g[P]) \to \mathcal{L} \to \mathcal{F} \to 0. \end{array} \right)$$

Pic^0_C is proper + connected: Follows from C^g proper + connected, and Pic^0_C is loc. fin. type and separated.

shown in lecture

$\dim \mathrm{Pic}^0_C = \dim(\text{tangent space at } 0) = \dim H^1(C, \mathcal{O}_C) = g$

X AV$/k$. $S \in \mathrm{Sch}_k$, $x \in X(S)$.

$$\rightsquigarrow \begin{array}{c} t_x : X_S \to X_S \\ \searrow \; \swarrow \\ S \end{array} \qquad \begin{array}{l} \varphi_{\mathcal{L}}(S) : X(S) \longrightarrow \mathrm{Pic}^0_{X/k}(S) \\ \qquad\qquad x \longmapsto t_x^* \mathcal{L}_S \otimes \mathcal{L}_S^{-1} \end{array}$$