# Algebraic Geometry I - Wintersemester 2016/17

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# Introduction

This course was taught in Bonn, Germany over the Wintersemester 2016/17, by Prof. Dr. Peter Scholze.

Our plan was to learn the basics of algebraic geometry, so about sheaves, schemes, $\mathcal{O}_X$-modules, affine/separated/proper morphisms, and eventually to show that proper normal curves over $k$ can be naturally associated to a type of field extension of $k$, and separated curves are quasi-projective.

The author of these notes would like to thank Dr. Johannes Anschütz and Alice Campigotto for reading through and editing these notes. The author also takes full responsibility for any and all inaccuracies, mistakes and typos in this write up.

Peter had a lot more to say in lectures than what could be captured here.

# 1 Affine Algebraic Varieties 18/10/2016

Algebraic geometry is the study about solution sets to systems of polynomial equations. The algebra and the geometry play a sort of dual role to each other. To explore this, we'll first revisit the (now outdated) mathematical objects that are varieties.

For this lecture we fix an algebraically closed field $k$.

**Definition 1.1.** *A subset $V \subseteq k^n$ is an affine algebraic set if it can be written as the set of common zeros of a set of polynomials. In other words, if there is a set $M \subseteq k[X_1, \ldots, X_n]$ of polynomials in $n$-variables such that*

$$V = V(M) := \{(x_1, \ldots, x_n) \in k^n \ \mid \forall f \in M : f(x_1, \ldots, x_n) = 0\}.$$

There are two simple (given Hilbert's basis theorem) consequences of this definition.

**Proposition 1.2.** *1. For any subset $M \subseteq k[X_1, \ldots, X_n]$, let $\mathfrak{a} = \mathfrak{a}(M) = Mk[X_1, \ldots, X_n]$ be the ideal generated by $M$, then*

$$V(M) = V(\mathfrak{a}).$$

*2. For any $M \subseteq k[X_1, \ldots, X_n]$, there exists a finite subset $\{f_1, \ldots, f_n\} \subseteq M$ such that $V(f_1, \ldots, f_n) = V(M)$.*

*Proof.* 1. The containment $V(M) \supseteq V(\mathfrak{a})$ is obvious. The conditions of the set $V(\mathfrak{a})$ are stronger than the conditions of the set $V(M)$, because $M \subseteq \mathfrak{a}$. For the converse, write $f \in \mathfrak{a}$ as $\sum_{i=1}^m f_i g_i$ with $f_i \in M$ and $g_i \in k[X_1, \ldots, X_n]$, then for all $x = (x_1, \ldots, x_n) \in V(M)$ we have $f(x) = \sum f_i(x) g_i(x) = 0$, i.e. $f = 0$.

2. One translation of Hilbert's basis theorem says that $k[X_1, \ldots, X_n]$ is noetherian, since $k$ is noetherian (it is a field). Let $M$ be some arbitrary subset of $k[X_1, \ldots, X_n]$, and let $\mathfrak{a} = \mathfrak{a}(M)$ be the ideal it generates. Recall that a ring $R$ is noetherian if every ideal is finitely generated, or equivalently, if every non-empty set of ideals has a maximal element. Let $f_1, \ldots, f_m \in M$ be a set of generating elements of $\mathfrak{a}$, then

$$V(M) = V(\mathfrak{a}) = V(f_1, \ldots, f_n)$$

follows from part 1 above.

□

Let us now consider a handful of examples of affine algebraic sets.

**Example 1.3.** Any finite subset of $k^n$.[1]) Conversely, any affine algebraic set of $k$ is either finite or all of $k$. In fact, $k[x]$ is a principal ideal domain and every $f \in k[x]$ factors as $f = \prod_{i=1}^n (x - \alpha_i)$, so then $V(f) = \{\alpha_1, \ldots, \alpha_n\}$.

**Example 1.4.** For $n = 2$ and $k = \mathbb{C}$, with $x, y$ coordinates, we have a range of classical examples. Keep in mind that we are really just looking at the purely real solutions. The reader is asked to graph these solution sets by hand or using wolfram alpha etc.

1. The equation $x + y = 0$ gives us a straight line through the origin of gradient $-1$.
2. The unit circle in $\mathbb{R}^2$ can be represented by $x^2 + y^2 = 1$.
3. For the equation $x^2 + y^2 = -1$, we of course have a non-empty solution set in $\mathbb{C}^2$, but there are no real solutions.

[1]See exercise sheet 1 problem 1(i): Let $Z \subseteq \mathbb{A}^n(k))$ be a finite set. Prove that $Z$ is Zariski closed in $\mathbb{A}^n(k)$.

4. When we consider curves of degree three (the degree of a curve now will be the degree of the equation that defines it), we have non-singular elliptic curves such as $y^2 = x^3 - x$.

5. The elliptic curve $y^2 = x^3 - x^2$ is quite different, and has a singularity called a node at the origin.

6. Another elliptic curve with a singularity is $y^2 = x^3$, which has a cusp at the origin (somehow even worse than a node).

7. The equation $x^2 = y^2$ can be factorised as $(x+y)(x-y) = 0$, so the solution set of this polynomial are two lines through the origin of gradient $-1$ and 1. The intersection of the two lines is still considered to be a singularity of this curve.

Let's now jump into something that we should have been expecting all the lecture so far: the Zariski topology on $k^n$.

**Proposition 1.5.** *There is a unique topology on $k^n$ for which the closed subsets are exactly the affine algebraic sets.*

*Proof.* Of course we can just claim we have a topology on $k^n$ by letting the closed sets be exactly the affine algebraic sets, but now we have to check these sets contain $\varnothing$ and all of $k^n$, that affine algebraic sets are closed under arbitrary intersection and finite unions.

Firstly we note that $V(\varnothing) = k^n$ and $V(1) = \varnothing$, so we're done with that part.

We also have the equality

$$\bigcap_{i \in \mathcal{I}} V(M_i) = V\left(\bigcup_{i \in \mathcal{I}} M_i\right)$$

once we unjumble the set-theoretic definitions of objects above.

Finally we have to show that affine algebraic sets are closed under finite unions, but by induction we need only worry about the union of two affine algebraic sets. It is clear that

$$V(M_1) \cup V(M_2) \subseteq V(M_1 \cdot M_2), \qquad \text{where } M_1 \cdot M_2 = \{f \cdot g \mid f \in M_1,\ g \in M_2\}.$$

To check the converse, take $x \in V(M_1 \cdot M_2) \backslash V(M_1)$. Take $f \in M_1$ such that $f(x) \neq 0$, then for all $g \in M_2$ we have $fg \in M_1 \cdot M_2$, so

$$0 = (fg)(x) = f(x)g(x).$$

Since $f(x) \neq 0$ and we're in a field (which is necessarily an integral domain), we have $g(x) = 0$ for each $g \in M_2$, so $x \in V(M_2)$. □

**Definition 1.6.** *When we write $\mathbb{A}^n(k)$, we mean the space $k^n$ with the Zariski topology. We call $\mathbb{A}^n(k)$ an $n$-dimensional affine space.*

Given a closed subset $V$ of $\mathbb{A}^n(k)$ (so an affine algebraic subset), we have $V = V(\mathfrak{a})$ for some ideal $\mathfrak{a} \in k[X_1, \ldots, X_n]$. There is a fundamental question we now want to ask ourselves about varieties.

*How tight is the relationship between $V$ and $\mathfrak{a}$?*

The answer to this question is: quite tight. To be precise about this, let us remember a definition from commutative algebra.

**Definition 1.7.** *The radical of an ideal $\mathfrak{a}$ contained in a (commutative unital) ring $R$ is*

$$rad(\mathfrak{a}) = \sqrt{\mathfrak{a}} := \{x \in R \mid \exists m > 0 : x^m \in \mathfrak{a}\}.$$

If we consider $\mathbb{A}^n(k)$ we have $V(\mathfrak{a}) = V(rad(\mathfrak{a}))$. The containment $V(\mathfrak{a}) \supseteq V(rad(\mathfrak{a}))$ is obvious, as $\mathfrak{a} \subseteq rad(\mathfrak{a})$. For the converse note that $f^n(x) = 0$ implies $f(x) = 0$ for each $x \in \mathbb{A}^n(k)$. The following version of Hilbert's Nullstellensatz (zero-position-theorem) will be proved by the end of the lecture.

**Theorem 1.8** (Hilbert's Nullstellensatz)**.** *The map*

$$\Phi : \{\textit{ideals } \mathfrak{a} \subseteq k[X_1, \ldots, X_n] \textit{ with } \mathfrak{a} = rad(\mathfrak{a})\} \longrightarrow \{\textit{closed sets } V \subseteq \mathbb{A}^n(k)\}$$

*defined by* $\mathfrak{a} \mapsto V(\mathfrak{a})$ *is a bijection, with inverse* $\Phi^{-1}$ *defined by*

$$V \longmapsto \{f \in k[X_1, \ldots, X_n] \;\mid\; \forall x \in V : f(x) = 0\}.$$

Before we prove this theorem, let's see an easy corollary, which shows us that we can always find non-empty affine algebraic sets for all proper ideals $\mathfrak{a} \subseteq k[X_1, \ldots, X_n]$.

**Corollary 1.9.** Let $\mathfrak{a} \subsetneq k[X_1, \ldots, X_n]$ be a proper ideal, then there exists $x \in k^n$ such that $f(x) = 0$ for all $f \in \mathfrak{a}$.

*Proof.* Since $\mathfrak{a}$ is a proper ideal, $1 \notin \mathfrak{a}$, so clearly $1 \notin rad(\mathfrak{a})$, which also implies that $rad(\mathfrak{a})$ is a proper ideal of $k[X_1, \ldots, X_n]$. By Hilbert's Nullstellensatz, this implies that $V(\mathfrak{a}) = V(rad(\mathfrak{a})) \neq V(k[X_1, \ldots, X_n]) = \varnothing$, hence we have $x \in k^n$ with $x \in V(\mathfrak{a})$. $\square$

A consequence of this corollary tells us how a set of polynomials has to behave.

Given a finite collection of polynomials $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$, then exactly one of the following two things happens:

1. There exists $x \in k^n$ such that $f_1(x) = \ldots = f_m(x) = 0$.
2. There exists $g_1, \ldots, g_m \in k[X_1, \ldots, X_n]$ such that

$$f_1 g_1 + \cdots + f_m g_m = 1.$$

Part 2 is a clear obstruction to this collection of polynomials having a common zero.

Soon we will see a proof Hilbert's Nullstellensatz. In the proof we will first make a claim, and finish the proof assuming this claim to be true. We will then prove this claim, which will require a small Lemma that we will prove now.

**Lemma 1.10.** *Let $R$ be a non-zero, finitely generated $k$-algebra, then there is always a map $R \to k$ of $k$-algebras.*

*Proof.* We know from commutative algebra that $R$ has a non-zero maximal ideal $\mathfrak{m}$, so by replacing $R$ with $R/\mathfrak{m}$ (which is also a non-zero finitely generated $k$-algebra) we can assume that $R$ is a field. By Noether normalisation[2] we have a finite injective map $k[X_1, \ldots, X_m] \hookrightarrow R$ for some $m \geq 0$. If $m > 0$, then the image of the element $X_1$ becomes invertible in $R$ (it's a field don't forget), so $X_1^{-1} \in R$. Since $R$ is a finitely generated $k[X_1, \ldots, X_m]$-module, $R$ is integral over $k[X_1, \ldots, X_m]$ and amount the $X_1, \ldots, X_m$ we have some $a_1, \ldots, a_n \in k[X_1, \ldots, X_m]$ such that

$$\left(X_1^{-1}\right)^n + a_1 \left(X_1^{-1}\right)^{n-1} + \cdots + a_n = 0.$$

If we multiply the above equation by $X_1^n$ we obtain an algebraic relation amoung $X_1$ and 1, which contradicts the fact these elements are algebraically independent.

[2]Here is a reminder of Noether normalisation: Given a field $k$ and a finitely generated $k$-algebra $A$, then there exists a non-negative integer $m$ and a set of algebraically independent elements $X_1, \ldots, X_m \in A$ such that $A$ is a finitely generated module over the polynomial ring $k[X_1, \ldots, X_m]$. This can be rephrased as done in our first lecture by saying there exists a finite (so that $R$ is a finitely generated $k[X_1, \ldots, X_m]$-module) injective map $k[X_1, \ldots, X_m] \hookrightarrow R$.

This means $m = 0$ and, by the finiteness of the map $k \to R$, that $R$ is a finite field extension of $k$. Recall though from the very beginning of the lecture that $k$ is algebraically closed, so there does not exist a non-trivial finite field extension of $k$. Hence $k \cong R$ and we have our map $R \to k$ of $k$-algebras. $\square$

Note that if $k$ is not algebraically closed, then we fall flat right at the end of that proof, and we can only conclude that $R$ is some finite field extension of $k$. Now we are ready to prove the Nullstellensatz.

*Proof of Hilbert's Nullstellensatz.* We claim the following formula holds for all ideals $\mathfrak{a} \in k[X_1, \ldots, X_n]$, and where $V = V(\mathfrak{a})$,

$$rad(\mathfrak{a}) = \{f \in k[X_1, \ldots, X_n] \ \mid \ \forall x \in V(\mathfrak{a}) : f(x) = 0\} = \Phi^{-1}(V(\mathfrak{a})). \tag{1.11}$$

If we know this, then it is clear that $\Phi^{-1} \circ \Phi$ is the identity,

$$\Phi^{-1} \circ \Phi(V(\mathfrak{a})) = \Phi^{-1}(V(\mathfrak{a})) = rad(\mathfrak{a}) = \mathfrak{a},$$

since $\mathfrak{a} = rad(\mathfrak{a})$ from our hypotheses, and Claim 1.11 says that $\Phi^{-1}(\mathfrak{a}) = rad(\mathfrak{a})$. The map $\Phi$ is also clearly surjective by definition of the closed sets in $\mathbb{A}^n(k)$ and the fact that $V(\mathfrak{a}) = V(rad(\mathfrak{a}))$ for *all* ideals $\mathfrak{a}$ (not just with $\mathfrak{a} = rad(\mathfrak{a})$). Hence, $\Phi$ is injective and surjective, i.e., bijective.

Now we have the task of showing Claim 1.11 holds. To show,

$$rad(\mathfrak{a}) \subseteq \{f \in k[X_1, \ldots, X_n] \ \mid \ \forall x \in V(\mathfrak{a}) : f(x) = 0\},$$

let $f \in rad(\mathfrak{a})$ and take $x \in V := V(\mathfrak{a})$. Since $f \in rad(\mathfrak{a})$ we have $f^m \in \mathfrak{a}$ for some $m > 0$, hence $0 = f^m(x) = (f(x))^m$ in a field $k$, and we're done.

Conversely, let $f \notin rad(\mathfrak{a})$, then by contrapositive we need to find $x \in V$ with $f(x) \neq 0$. Let $R$ be the following quotient ring,

$$R = k[X_1, \ldots, X_n, Y]/(f(X_1, \ldots, X_n)Y - 1, \mathfrak{a}).$$

In other words, we've adjoined an inverse to $f(X_1, \ldots, X_n)$ and killed $\mathfrak{a}$. Along these lines we can rewrite $R$ as

$$R = (k[X_1, \ldots, X_n]/\mathfrak{a})\left[\overline{f}^{-1}\right],$$

where $\overline{f}$ is the image of $f$ in the quotient ring $k[X_1, \ldots, X_n]/\mathfrak{a}$. If $R = 0$ we would have $1 = 0$ , i.e. in this localised ring there would exist an $m > 0$ such that

$$\overline{f}^m = \overline{f}^m \cdot 1 = \overline{f}^m \cdot 0 = 0.$$

This would imply that $f^m \in \mathfrak{a}$ inside $k[X_1, \ldots, X_n]$, i.e. $f \in rad(\mathfrak{a})$, a contradiction. Hence $R$ is a non-zero ring. Also notice that if we had a map of $k$-algebras $R \to k$, then we would obtain $x_1, \ldots, x_n, y \in k$ such that $f(x_1, \ldots, x_n)y = 1$ which implies that $f(x_1, \ldots, x_n) \neq 0$, but $x = (x_1, \ldots, x_n) \in V(\mathfrak{a})$. These facts are just formal consequences from the construction of $R$. However, we do have a map $R \to k$ of $k$-algebras, since $R$ is a non-zero finitely generated $k$-algebra and we can apply Lemma 1.10. This concludes that $\Phi$ and $\Phi^{-1}$ are mutual inverses. $\square$

We can construct an even nicer correspondence however, and one that we will see again in scheme theoretic language.

**Definition 1.12.** *Let $V \subseteq \mathbb{A}^n(k)$ be closed. The algebra of functions on $V$ is defined as*

$$\mathcal{O}(V) := k[X_1, \ldots, X_n]/\{f \ \mid \ \forall x \in V : f(x) = 0\}.$$

Recall that a ring $R$ is called reduced if given $x \in R$ with $x^m = 0$ for some $m > 0$, then $x = 0$. Notice that $\mathcal{O}(V)$ is always reduced, because if $f^m = 0$ in $\mathcal{O}(V)$, then $f^m(x) = 0$ for all $x \in V$, which implies that $f(x) = 0$ for all $x \in V$ (since $k$ is a field), so $f = 0$ in $\mathcal{O}(V)$. Note also that $\mathcal{O}(V)$ is clearly finitely generated as a $k$-algebra.

**Definition 1.13.** *A map between affine algebraic sets $V \subseteq \mathbb{A}^n(k)$ and $W \subseteq \mathbb{A}^m(k)$ is a map $f : V \to W$ of sets of solutions, such that there exists a collection of polynomials $f_1, \dots, f_m \in k[X_1, \dots, X_n]$ with*

$$f(x) = f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$$

*for all $x = (x_1, \dots, x_n)$ in $V$.*

This definition of a morphism clearly has the properties we demand a morphism in a category to have (i.e. composition, associativity and unitial properties), so for each algebraically closed field $k$ we obtain a category AffVar(k) (the authors notation for now) of affine algebraic subsets in $\mathbb{A}^n(k)$ for all $n \geq 0$.

**Remark 1.14.** The $f_i$'s in the definition above are not determined by $f$, since they need only have certain properties on $V$, and the closed subsets $V \subseteq \mathbb{A}^n(k)$ should be thought of as quite small in $\mathbb{A}^n(k)$. The images of the $f_i$'s are however well defined in $\mathcal{O}(V)$, since the quotient in the definition of $\mathcal{O}(V)$ identifies polynomials with the same value on $V$.

Also note that the map $f : V \to W$ determines a map $\widetilde{f} : k[Y_1, \dots, Y_m] \to \mathcal{O}(V)$ which sends $Y_i \mapsto \overline{f}_i$, the image of $f_i$ in the quotient algebra $\mathcal{O}(V)$. Notice that this map $\widetilde{f}$ factors through the quotient $\mathcal{O}(W)$. In other words, we have the following commutative diagram,

$$\begin{tikzcd} k[Y_1, \dots, Y_m] \arrow[r, "\widetilde{f}"] \arrow[d, two heads] & \mathcal{O}(V) \\ \mathcal{O}(W) \arrow[ur, "f^*"'] & \end{tikzcd}.$$

The pullback map $f^*$ is given by $g \mapsto g \circ f$, so simply pre-composition.

A corollary of Hilbert's Nullstellensatz is the following equivalence of categories.

**Corollary 1.15.** There is an equivalence of categories between $AffVar(k)$ and the category of reduced finitely generated $k$-algebras and $k$-algebra homomorphisms, defined by the (contravariant) functor $F$ which sends $V \mapsto \mathcal{O}(V)$ and $f : V \to W$ to $f^* : \mathcal{O}(W) \to \mathcal{O}(V)$.

*Proof.* We will prove this equivalence of categories by showing that $F$ is fully[3] faithful[4], and essentially surjective[5].

**Fully Faithful** Let $V \subseteq \mathbb{A}^n(k)$ and $W \subseteq \mathbb{A}^n(k)$, and consider,

$$F_{V,W} : Hom(V, W) \to Hom(\mathcal{O}(V), \mathcal{O}(W)).$$

To check this map is injective, we take $f, f' : V \to W$, such that $f^* = (f')^*$. In this case we have for all $g \in \mathcal{O}(W), x \in V$, $g(f(x)) = f^*g(x) = (f')^*g(x) = g(f'(x))$. Notice now that $g = \overline{Y}_i$ implies $f(x) = (\overline{Y}_1 f(x), \dots, \overline{Y}_m f(x)) = f'(x)$. To show $F_{V,W}$ is surjective, take a $k$-algebra homomorphism $G : \mathcal{O}(W) \to \mathcal{O}(V)$, which we can specify by simply saying where that $Y_i \to \overline{f}_i$ for all $Y_i \in \mathcal{O}(W)$ (which we remember is a quotient of $k[Y_1, \dots, Y_m]$). We lift these $\overline{f}_i$ to $f_i \in k[X_1, \dots, X_n]$, and obtain a map,

$$(f_1, \dots, f_m) : k^n \to k^m,$$

which we restrict to a map $f : V \to W$. It's then a simple check that $f^* = G$. Take some $Y_i \in \mathcal{O}(W)$ and any $x = (x_1, \dots, x_n) \in V$, then

$$f^*Y_i(x) = Y_i \circ f(x) = Y_i(f_1(x), \dots, f_m(x)) = \overline{f}_i(x) = G(Y_i)(x).$$

[3] A functor $F : \mathcal{C} \to \mathcal{D}$ is full if the induced maps on morphism classes are always a surjection.

[4] Sort of dual, we call a functor $F : \mathcal{C} \to \mathcal{D}$ faithful if the induced map on morphism classes is always an injection.

[5] A functor $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective if for each $Y \in \mathcal{D}$ there is an $X \in \mathcal{C}$ with $F(X) \cong Y$ in $\mathcal{D}$.

**Essentially Surjective** Let $R$ be some reduced finitely generated $k$-algebra, so then $R$ is simply

$$R \cong k[X_1, \ldots, X_n]/\mathfrak{a}$$

where the ideal $\mathfrak{a}$ has the property that $\mathfrak{a} = rad(\mathfrak{a})$, or else $R$ would not be reduced. Then set $V = V(\mathfrak{a}) \subseteq k^n$, and we have $\mathcal{O}(V) \cong R$ as a consequence of Hilbert's Nullstellensatz. $\square$

## 2 Affine Schemes 20/10/2016

Recall from the last lecture that for a closed subset $V \subseteq \mathbb{A}^n(k)$ we have the algebra of functions on $V$ denoted as $\mathcal{O}(V)$. Let $\mathrm{Map}(V,k)$ be the set of all maps of sets $V \to k$, which we can turn into a ring with point-wise addition and multiplication in $k$. Then $\mathcal{O}(V)$ has the following equivalent description,

$$\mathcal{O}(V) = im\left(k[X_1,\dots,X_n] \longrightarrow \mathrm{Map}(V,k)\right),$$

where we include polynomials on $\mathbb{A}^n(k)$ into the ring of all set-theoretic functions $V \to k$. We can also re-write $V(\mathfrak{a})$ in a similar way, as

$$V(\mathfrak{a}) = \mathrm{Hom}_{k-\mathrm{alg.}}(A,k),$$

where $A = k[X_1,\dots,X_n]/\mathfrak{a}$. We would like to extend Corollary 1.15 to involve general rings, which is compatible with including the full subcategory of affine algebraic sets into our new, more general category. In general rings do not have an underlying field $k$.

We could now start talking about general varieties, but since we want to work with schemes, it seems more efficient to define an affine scheme and then we'll get to general schemes.

**Convention: All rings will now be commutative and unital, unless otherwise specified. All maps will be ring homomorphisms.**

**Definition 2.1.** *Let $A$ be a ring, then $\operatorname{Spec} A$ is defined as the collection of all ring homomorphisms $A \to K$, where $K$ is some field, and where we identify two maps $A \to K$ and $A \to K'$ if there exists the following commutative diagram.*

$$\begin{tikzcd} & K' \\ A \arrow[ur] \arrow[r] & K \arrow[u] \end{tikzcd}.$$

This is a rather categorical definition, and in fact we could rephrase it as

$$\operatorname{Spec} A = \mathrm{colim}_{K \in \mathrm{Fld}} \mathrm{Hom}_{\mathrm{Rng}}(A,K),$$

where Fld is the category of fields, and Rng is the category of rings. There is a problem here though, that $\operatorname{Spec} A$ might not necessarily be a set, but we shall rectify that now, with an alternative definition.

**Proposition 2.2.** *The map $\operatorname{Spec} A \to \{$prime ideals in $A\}$ defined by $(f : A \to K) \mapsto \ker(f)$ is a well-defined bijection.*

*Proof.* If $f : A \to K$ is a map, where $K$ is a field, then $\ker(f)$ is a proper ideal in $A$, and if we have $xy$ in $\ker(f)$, then $0 = f(xy) = f(x)f(y)$ which imples $x$ or $y$ is in $\ker(f)$ since $K$ is a field (and an integral domain). Hence $\ker(f)$ is prime. If we have a diagram

$$\begin{tikzcd} & K' \\ A \arrow[ur, "f'"] \arrow[r, "f"'] & K \arrow[u] \end{tikzcd}$$

then the map $K \to K'$ is injective, so $\ker(f) = \ker(f')$, which concludes our map is well-defined.

To check bijectivity, we shall construct an inverse map, which in the process will construct a distinguished representative from each equivalence class. If $\mathfrak{p} \subseteq A$ is a prime ideal, then $A/\mathfrak{p}$ is an integral domain, so we have an inclusion

$$A/\mathfrak{p} \hookrightarrow \mathrm{Frac}(A/\mathfrak{p}) =: k(\mathfrak{p}),$$

from $A/\mathfrak{p}$ into a field[6]. Hence we have the composition

$$f_{\mathfrak{p}} : A \to A/\mathfrak{p} \hookrightarrow k(\mathfrak{p}),$$

which we define as $f_{\mathfrak{p}}$, which is a map to a field with $\ker(f_{\mathfrak{p}}) = \mathfrak{p}$. This gives us surjectivity of the map in question. In general, for any map $f : A \to K$, then we can factor this map through the quotient of $\ker(f) = \mathfrak{p}$, and the map $A/\mathfrak{p} \hookrightarrow K$ can be factored again through $\mathrm{Frac}(A/\mathfrak{p})$. In this way we have the following commutative diagram.

$$\begin{tikzcd}
 & K & \\
A \arrow[ur, "f"] \arrow[r] \arrow[dr, "f_p"'] & A/\mathfrak{p} \arrow[u] \arrow[d] & \\
 & \mathrm{Frac}(A/\mathfrak{p}) \arrow[uu, bend right=60] &
\end{tikzcd}$$

This diagram concludes that $f_p$ is equivalent to $f$. Hence, if we have two maps $f : A \to K$ and $f' : A \to K'$ with $\ker(f) = \ker(f')$, then $f\ f'$. □

The canonical representative $f_{\mathfrak{p}} : A \to k(\mathfrak{p})$ is our natural choice of map into a field given a prime ideal in a ring $A$ (i.e. a point in $\operatorname{Spec} A$).

**Definition 2.3.** 1. *We now define the set* $\operatorname{Spec} A$ *as the set of all prime ideals in* $A$*, called the spectrum of the ring* $A$*.*

2. *For* $x \in \operatorname{Spec} A$*, let* $\mathfrak{p}_x$ *be the prime ideal which is* $x$*, then*

$$k(x) := k(\mathfrak{p}_x) = \mathrm{Frac}(A/\mathfrak{p}_x),$$

*is called the residue field at* $x$*.*

3. *For* $x \in \operatorname{Spec} A$*, we have a natural map* $A \to k(x)$ *which maps* $g \mapsto g(x)$*. In slightly different terms, we want to consider any element* $g \in A$ *as a function on* $\operatorname{Spec} A$*, which value at* $x \in \operatorname{Spec} A$ *is* $g(x) \in k(x)$*. In some more words, for each* $g \in A$*, we can map* $x \in \operatorname{Spec} A$ *to* $g(x) \in k(x)$ *which is essentially just viewing* $g$ *modulo* $\mathfrak{p}_x$*, inside the field of fractions of* $A/\mathfrak{p}$*.*

**Example 2.4.** Let's consider the simplest ring we know, $\mathbb{Z}$, and look at the spectrum of $\mathbb{Z}$. The elements are all the non-zero prime ideals $(2), (3), (5), (7), \ldots$ and the ideal $(0)$ which we think of as a generic point, arbitrarily close to all others (we'll see why it's special once we have some topology). The residue fields at these points are simply $\mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/5, \mathbb{Z}/7, \ldots$ and $\mathbb{Q}$. Remember that elements of $\mathbb{Z}$ act as functions on $\operatorname{Spec} \mathbb{Z}$, so if $n \in \mathbb{Z}$, we have $n(p) = n$ modulo $p$.

There is a relation between the spectrum of a ring and the affine algebraic subsets of $k^n$, so long as $k$ is an algebraically closed field. First we need a quick definition.

**Definition 2.5.** *Given a ring* $A$*, then* $\operatorname{Spec}_{max} A$ *is defined as the subset of* $\operatorname{Spec} A$ *consisting of all the maximal ideals of* $A$*.*

**Proposition 2.6.** *Let* $k$ *be an algebraically closed field, and let* $A$ *be a finitely generated* $k$*-algebra, so* $A = k[X_1, \ldots, X_n]/\mathfrak{a}$*. Then*

$$k^n \supseteq V(\mathfrak{a}) = Hom_{k-alg.}(A, k) \longrightarrow \operatorname{Spec}_{max} A \subseteq \operatorname{Spec} A$$

*the map which sends* $f : A \to k$ *to* $\ker(f)$ *is a bijection.*

[6]Recall the fraction field (or field of fractions) $\mathrm{Frac}(A)$ of an integral domain $A$ is the collection of elements $a/b$ with $a, b \in A$ and $b \neq 0$ modulo the relation $a/b \sim a'/b'$ if and only if $ab' = a'b$. Addition and multiplication is defined the same as how one adds and multiplies rational numbers.

*Proof.* Clearly given $f : A \to k$ we have $\ker(f)$ is a maximal ideal, and for any maximum ideal $\mathfrak{m} \subseteq A$, we've seen that $k \cong A/\mathfrak{m}$ (from the proof of Lemma 1.10). Hence for any maximal ideal $\mathfrak{m} \subseteq A$, we have a map of $k$-algebras $A \to A/\mathfrak{m} \cong k$. It is easy to check that these maps define inverse bijections. $\square$

**Remark 2.7.** It might seem natural to study $\operatorname{Spec}_{\max} A$ instead of $\operatorname{Spec} A$, but $\operatorname{Spec}_{\max} A$ has a big problem: it is not functorial. Consider the inclusion $\mathbb{Z} \to \mathbb{Q}$, then we'll see that the induced map on $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$ simply identifies the generic point $(0) \in \operatorname{Spec} \mathbb{Z}$, since $\operatorname{Spec} \mathbb{Q} = *$ is just a point. We have $\operatorname{Spec}_{\max} \mathbb{Q} = \operatorname{Spec} \mathbb{Q}$, but there is no map into $\operatorname{Spec}_{\max} \mathbb{Z}$ since $\operatorname{Spec}_{\max} \mathbb{Z}$ has no generic point, i.e. no $(0)$.

Let's now at least state why $\operatorname{Spec}(-)$ is functorial.

**Proposition 2.8.** *Given $f : A \to B$, then* $\operatorname{Spec} f$ *defined by $q \mapsto f^{-1}(\mathfrak{q})$ is a well-defined map.*

*Proof.* If $x, y \in A$ and $xy \in f^{-1}(\mathfrak{q})$, then $f(x)f(y) = f(xy) \in \mathfrak{q}$, so $f(x)$ or $f(y)$ are in $\mathfrak{q}$, hence $x$ or $y$ are in $f^{-1}(\mathfrak{q})$. Therefore $f^{-1}(\mathfrak{q})$ is a prime ideal in $A$. $\square$

**Remark 2.9.** For any ring $A$ we have a unique map $\mathbb{Z} \to A$, so we have a unique map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$. Somehow all of algebraic geometry lives over $\operatorname{Spec} \mathbb{Z}$, where we can start to perform number theory and arithmetic algebraic geometry.

We would like to define a topology on $\operatorname{Spec} A$ such that if $A$ is a finitely generated $k$-algebra, then $V(A)$ (which has the Zariski topology of an affine variety) has the subspace topology of $\operatorname{Spec} A$ (from Proposition 2.6).

**Definition 2.10.** *Let $A$ be a ring, for any subset $M \subseteq A$, define*

$$V(M) := \{\mathfrak{p} \subset A \ \mid \ M \subseteq \mathfrak{p}\} = \{x \in \operatorname{Spec} A \ \mid \ \forall g \in M, g(x) = 0\},$$

*called the vanishing locus of $M$.*

The equality above comes from the canonical representative we constructed in Proposition 2.2. It is clear that if $M \subseteq \mathfrak{p}$, then $g(x) = 0$ for all $x \in M$, since the value of $g(x)$ takes place in a field where we previously quotiented out $\mathfrak{p}$. Conversely, the fact that we are working in a field implies that if $g(x) = 0$ then $g = 0$ in $A/\mathfrak{p}$ so $g \in \mathfrak{p}$.

**Example 2.11.** Take $\operatorname{Spec} \mathbb{Z}$ and notice that $V(12) = \{(2), (3)\}$, $V(64) = \{(2)\}$, and $V(210) = \{(2), (3), (5), (7)\}$.

**Proposition 2.12** (Definition/Proposition)**.** *There is a unique topology on* $\operatorname{Spec} A$ *called the Zariski topology, where the closed sets are $V(M)$.*

*Proof.* This proof is very similar to that of Proposition 1.5. $\square$

**Remark 2.13.** It is quite clear that $\operatorname{Spec} f$ is a continuous map, given $f : A \to B$ of rings. This is because the preimage of a closed set $V(M)$ in $\operatorname{Spec} A$ is simply $V(f(M))$ in $\operatorname{Spec}(B)$.

**Remark 2.14.** The topology of $\operatorname{Spec} A$ is different to what we are used to. For exampe there does not exist a ring $A$ such that $\operatorname{Spec} A$ contains $[0, 1]$ with the subspace topology. In Example 2.4 we noted that $(0)$ was a generic point. What we mean by that is that all non-empty open subsets contain $(0)$. To see this, notice that given $M$ and $n \in M$ with $n \neq 0$, then $V(M) \not\ni (0)$, so any non-empty open subset does contain $(0)$.

Similarly, if we consider $A = k[X]$, with $k$ algebraically closed. In this case we have countably many points (so long as $k$ is countable), and a generic point holding them all together, so in this case $\operatorname{Spec} \mathbb{Z} \cong \operatorname{Spec} k[X]$ topologically.

**Proposition 2.15.** *Given a ring $A$, then* $\operatorname{Spec} A$ *is quasi-compact.*

Note that we don't simply say compact, because we don't want to trick people into thinking Spec $A$ is also Hausdorff, which is usually goes hand-in-hand with compactness.

*Proof.* This finiteness statement really just boils down to a finiteness statement in algebra, about polynomials.

Let $\operatorname{Spec} A = \bigcup_{i \in \mathcal{I}} U_i$ be a cover of Spec $A$ by open sets, and let $Z_i = \operatorname{Spec} A \backslash U_i = V(M_i)$, then we have

$$\varnothing = \bigcap_{i \in \mathcal{I}} V(M_i) = V\left(\bigcup_{i \in \mathcal{I}} M_i\right).$$

Let $\mathfrak{a} \subseteq A$ be the ideal now generated by the union of all these $M_i$'s, so $V(\mathfrak{a}) = V(\bigcup_i M_i)$. First let's assume that $\mathfrak{a} \neq A$, then there exists a maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$, but then $\mathfrak{m}$ is also a prime ideal and $\mathfrak{m} \in V(\mathfrak{a}) = \varnothing$, a contradiction. Hence $\mathfrak{a} = A$, so $1 \in \mathfrak{a}$. This means there exists $f_1, \ldots, f_m \in \bigcup_i M_i$ and $g_1, \ldots, g_m \in A$ such that

$$1 = f_1 g_1 + \cdots + f_m g_m. \tag{2.16}$$

We can now choose a finite set $\mathcal{J} \subset \mathcal{I}$ such that $f_1, \ldots, f_m \in \bigcup_{j \in \mathcal{J}} M_j$. Equation 2.16 then tells us that the union of $M_j$'s over $\mathcal{J}$ now generates the unit ideal, so

$$V\left(\bigcup_{i \in \mathcal{J}} M_i\right) = V(A) = \varnothing.$$

Hence $\operatorname{Spec} A = \bigcup_{j \in \mathcal{J}} U_j$. $\square$

## 3 Topological Properties of $\operatorname{Spec} A$ 25/10/2016

We begin the lesson today with a proposition.

**Proposition 3.1.** *Let $M$ be a subset of a ring $A$, and let $\mathfrak{a}$ be the ideal generated by $M$.*

1. $V(M) = V(\mathfrak{a}) = V(rad(\mathfrak{a}))$
2. *The map* $\operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A$ *induced by the quotient map is a homeomorphism onto* $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$.
3. *Closed subsets of* $\operatorname{Spec} A$ *are in one-to-one correspondence with radical ideals of $A$, by mapping*

$$V \longmapsto I(V) := \bigcap_{\mathfrak{p} \in V} \mathfrak{p},$$

*with inverse* $\mathfrak{a} \to V(\mathfrak{a})$.

*Proof.* The proof of part 1 is similar to the first lecture. For part 2, we know that the ideals of $A/\mathfrak{a}$ correspond to the ideals of $A$ containing $\mathfrak{a}$, and prime ideals in $A/\mathfrak{a}$ correspond to prime ideals in $A$ containing $A$, which is the definition of $V(\mathfrak{a})$. Hence we have a continuous bijection, so for this map to be a homeomorphism, it suffices to show that it is a closed map. Let $Z \subseteq \operatorname{Spec}(A/\mathfrak{a})$ be closed, so $Z = V(\mathfrak{b}')$ for some ideal $\mathfrak{b}' \subseteq A/\mathfrak{a}$, so there is an ideal $b \subseteq A$ containing $\mathfrak{a}$, and we now have $Z = V(\mathfrak{b})$ inside $\operatorname{Spec} A$.

The proof for part 3 is similar to the proof of Hilbert's Nullstellensatz (Theorem 1.8). We have that $rad(\mathfrak{a}) \subseteq I(V(\mathfrak{a}))$, and now let's prove the other direction. Let $f \in I(V(\mathfrak{a}))$ with $f \notin rad(\mathfrak{a})$, and let $R = (A/\mathfrak{a})\left[\overline{f}^{-1}\right]$, where $\overline{f}$ is the image of $f$ inside $A/\mathfrak{a}$. It is clear that $R \neq 0$, since this would imply that $\overline{f}^m = 0$ using $1 = 0$, and then we'd have $f^m \in a$ which means that $f \in rad(\mathfrak{a})$, a contradiction. If $R \neq 0$, then $\operatorname{Spec} R \neq \varnothing$, because there exists a maximal ideal $\mathfrak{m} \subseteq R$ as an element of $\operatorname{Spec} R$. Now look at $x \in \operatorname{Spec} R$ and $y \in \operatorname{Spec} A$, then for all $g \in \mathfrak{a}$ we have $g(y) = 0$, since we have the following commutative diagram.

$$\begin{array}{ccc} A & \longrightarrow & R \\ \big\downarrow & & \big\downarrow \\ k(y) & \xrightarrow{=} & k(x) \end{array}$$

This diagram implies that $g(y)$ can be factored through $R$ where $\mathfrak{a}$ has been quotiented out, so $g(y) = 0$. However, this contradicts the fact that $f(y) = \overline{f}(x) \neq 0$ as $\overline{f}$ is invertible in $R$. □

This proof gives us some ideas concerning when the spectrum of a ring is empty, and when the function $f(x)$ is zero for all $x \in \operatorname{Spec} A$ and a $f \in A$.

**Lemma 3.2.** *Let $A$ be a ring and let $f \in A$.*

1. $\operatorname{Spec} A = \varnothing$ *if and only if* $A = 0$.
2. *$f$ is invertible in $A$ if and only if for all* $x \in \operatorname{Spec} A$ *we have* $f(x) \neq 0$.
3. *$f$ is nilpotent in $A$ if and only if for all* $x \in \operatorname{Spec} A$ *we have* $f(x) = 0$.

*Proof.* Part 1 follows from the existence of maximal ideals. In part 2, let $fg = 1$, then for all $f(x)g(x) = 1$ for all $x \in \operatorname{Spec} A$ hence $f(x) \neq 0$ for all $x \in \operatorname{Spec} A$. Remember that these facts are happening in the residue field $k(x)$ which is in particular an integral domain. Conversely, if $f$ is not invertible, then $A/(f)$ is not the zero ring, so then part 1 says that $\operatorname{Spec} A/(f)$ is non-empty. We also have a non-trivial ring map $\operatorname{Spec}(A/(f)) \to \operatorname{Spec} A$ induced by the quotient, and for all $x \in \operatorname{Spec}(A/(f))$

we know that $f(x) = 0$, so $f(x)$ is not always non-zero for each $x \in \operatorname{Spec} A$.

For part 3, if $f$ is nilpotent, then $f^m = 0$ for some natural number $m$. This tells us that $f(x)^m = f^m(x) = 0$ for all $x \in \operatorname{Spec} A$, but we are working over a field $k(x)$ so $f(x) = 0$ for all $x \in \operatorname{Spec} A$. Now assume that $f(x) \neq 0$ for all $x \in \operatorname{Spec} A$, and that $f$ is not nilpotent, such that $A[f^{-1}] \neq 0$. We now proceed similarly to the proof of part 3 of Proposition 3.1, which states that $\varnothing \neq \operatorname{Spec} A[f^{-1}] \to \operatorname{Spec} A$. Since $f$ is invertible in $A[f^{-1}]$ then $f(x) \neq 0$ for some $x \in \operatorname{Spec} A$. $\square$

Since we have been working with some localisations, we would like a proposition corresponding to Proposition 3.1 Part 2, this time relating a specific localisation of a ring with some subset of $\operatorname{Spec} A$.

**Proposition 3.3.** *Given a ring $A$ and $f \in A$.*

1. *The map $\operatorname{Spec} A[f^{-1}] \to \operatorname{Spec} A$ induced by the localisation is an open embedding, with image $D(f) = \operatorname{Spec} A \backslash V(f)$.*
2. *The $A$-algebra $A[f^{-1}]$ is initial amoung all $A$-algebras such that the induced action map $\operatorname{Spec} B \to \operatorname{Spec} A$ factors over $D(f)$.*

**Remark 3.4.** We should remark about part 2. This implies that $A[f^{-1}]$ depends only on $D(f)$, not actually on $f$. So if we have $f, g \in A$ with $D(f) = D(g)$, then $A[f^{-1}] = A[g^{-1}]$.

*Proof.* Recall from commutative algebra that the prime ideals of $A[f^{-1}]$ are in one-to-one correspondence with the prime ideals of $A$ such that $f \notin \mathfrak{p}$, by sending a prime ideal $\mathfrak{q} \in A[f^{-1}]$ to $\mathfrak{q} \cap A$ (not a literal intersection), and a prime ideal $\mathfrak{p} \subset A$ to $\mathfrak{p}[f^{-1}]$. By definition, this second set is the complement of $V(f)$, which we define as $D(f)$. Again, we have a continuous bijection, so let's show it is a closed map too. Let $Z \subset \operatorname{Spec} A[f^{-1}]$ be closed, then $Z = V(M)$ for some $M \subseteq A[f^{-1}]$. Let's write this generating set as

$$M = \left\{ \frac{g_i}{f^{n_i}} \;\middle|\; g_i \in A, n_i \in \mathbb{N} \right\}.$$

Let $N = \{g_i\}$ be the subset of $A$ featuring only the numerators of the elements in $M$, then we consider the closed set $V(N) \subseteq \operatorname{Spec} A$.

$$V(N) \cap \operatorname{Spec} A[f^{-1}] = V(\{g_i\}) = V\left(\left\{\frac{g_i}{f^{n_i}}\right\}\right) = V(M) = Z$$

Hence our bijection map is also a homeomorphism.

Now let $B$ be some $A$-algebra, with structure map $\phi : A \to B$, such that $\operatorname{Spec} \phi$ factors through $D(f)$, then $\phi(f)$ is non-zero everywhere on $\operatorname{Spec} B$. In other words $\phi(f)(y) \neq 0$ for all $y \in \operatorname{Spec} B$. From Lemma 3.2 we see that $\phi(f)$ must be invertible in $B$, so we have a unique map $l$ in the following diagram.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A[f^{-1}] \arrow[d] \arrow[dl, dashed, "l"'] \\ B \arrow[r, "\cong"] & B[\phi(f)^{-1}] \end{tikzcd}$$

$\square$

These open subsets of $D(f)$ are an important feature of the topology on $\operatorname{Spec} A$.

**Definition 3.5.** *A principal open subset of $\operatorname{Spec} A$ is an open subset $U \subseteq \operatorname{Spec} A$ of the form $U = D(f)$ for some $f \in A$. If $U = D(f)$ in $\operatorname{Spec} A$, then we define the coordinate ring $\mathcal{O}(U) = A[f^{-1}]$, which is well defined up to unique isomorphism by Remark 3.4.*

In particular we have $\mathcal{O}(\operatorname{Spec} A) = A$ with $f = 1$.

Currently we want to define affine schemes such that we have an equivalence of categories between rings and affine schemes, but a topological space $\operatorname{Spec} A$ by itself doesn't remember enough about the ring $A$. For example, if $\phi$ is any map between fields, it will induce a homeomorphism. Also, consider $A \to A_{red} = A/rad(0)$, then the induced quotient map on Spec is again a homeomorphism. Next lesson we'll fix that problem, and the solution will be these coordinate rings $\mathcal{O}(U)$. For now, we have more topology to do.

**Proposition 3.6.** *Let $A$ be a ring.*

1. *The principal open subsets of* $\operatorname{Spec} A$ *form a basis $\mathcal{B}$ for the topology.*
2. *If $U, V \in \mathcal{B}$, then $U \cap V \in \mathcal{B}$.*
3. *All the principle open subsets of* $\operatorname{Spec} A$ *are quasicompact.*

*Proof.* For part 1, an arbitrary open subset of $\operatorname{Spec} A$ is written as the complement of a closed set $V(M) = \bigcap_{f \in M} V(f)$, so set theory tells us that $U = \bigcup_{f \in M} D(f)$. Part 2 comes straight from the previous calculation that $V(f) \cup V(g) = V(fg)$ which one would do in the proof of Definition/Proposition 2.12, $D(f) \cap D(g) = D(fg)$. Part 3 is also clear, because we identified the subspaces $D(f)$ of $\operatorname{Spec} A$ is $\operatorname{Spec} A[f^{-1}]$. $\square$

We now come to some purely topological definitions, which will help us try to understand just how strange the spaces $\operatorname{Spec} A$ are.

**Definition 3.7.** *Let $X$ be a topological space.*

1. *$X$ is irreducible if all non-empty proper open subsets of $X$ have non-trivial intersection. This is equivalent to asking that no finite union of non-empty proper closed subsets is all of $X$.*[7]
2. *A point $x \in X$ is a generic point if $x$ is contained in each non-empty open set of $X$.*
3. *$X$ is $T_0$ separable if for all $x, y \in X$ with $x \neq y$, there is an open set $U$ of $X$ which contains either $x$ or $y$, but not both.*
4. *$X$ is Hausdorff if for all distinct points $x, y \in X$ there are open sets $U, V \subset X$ such that $x \in U, y \in V$ and $U \cap V = \varnothing$.*

**Remark 3.8.** There are some obvious consequences of these definitions, which are really quick logic exercises. If $X$ is a Hausdorff space, and $X$ is not a point or empty, then $X$ is not irreducible. If $X$ has a generic point, then $X$ is irreducible. If $X$ is $T_0$, then $X$ has at most one generic point. A point $x \in X$ is generic if and only if $\overline{\{x\}} = X$, in other words, if and only if the closure of $\{x\}$ is equal to $X$.

Last time we saw that $\operatorname{Spec} \mathbb{Z}$ and $\operatorname{Spec} k[X]$ have a generic point, so they are irreducible. We have a more general statement to make about this.

**Lemma 3.9.** *Let $A$ be an integral domain, then* $(0)$ *is an element of* $\operatorname{Spec} A$ *and is a generic point.*

*Proof.* Let $U$ be a non-empty open subset of $\operatorname{Spec} A$, then $U = V(M)^c$ in $\operatorname{Spec} A$ with $M \neq 0$, since $U \neq \varnothing$. Hence $M \not\subseteq (0)$, so $(0) \notin V(M)$ and $(0) \in U$. $\square$

In what follows, if $X$ is a topological space, and $S \subseteq X$ some subset, then we say that $S$ is irreducible (or $x \in S$ is a generic point in $S$) if this condition holds for $S$ with the subspace topology.

**Proposition 3.10.** *Let $A$ be any ring.*

[7] With this equivalent definnition, it is then clear why schemes such as $\operatorname{Spec}(\mathbb{C}[x, y]/(xy))$ are not irreducible, as it can be seen as the union of the $x$ and $y$ axis.

1. *The irreducible closed subsets of* $\operatorname{Spec} A$ *are in one-to-one correspondence with the prime ideas of* $A$*, by sending a prime ideal* $\mathfrak{p} \subset A$ *to* $V(\mathfrak{p})$*.*

2. *For* $\mathfrak{p} \subseteq A$ *a prime ideal, then* $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$*, the closure of* $\{\mathfrak{p}\}$*. In other words,* $\mathfrak{p} \in V(\mathfrak{p})$ *is the unique generic point of* $V(\mathfrak{p})$*.*

*Proof.* Part 1 is similar to part 3 in Proposition 3.1. Let $\mathfrak{p} \subseteq A$ be some prime ideal, then $V(\mathfrak{p}) \cong \operatorname{Spec} A/\mathfrak{p}$ is irreducible since $A/\mathfrak{p}$ is an integral domain. Conversely, if $V \subseteq \operatorname{Spec} A$ is an irreducible closed subset, then we let $\mathfrak{a} = I(V)$. We want to show that $\mathfrak{a}$ is prime, so we let $f, g \in A$ with $f, g \notin \mathfrak{a}$, but with $fg \in \mathfrak{a}$. This implies that $D(f) \cap V(\mathfrak{a}) \neq \varnothing$ and $D(g) \cap V(\mathfrak{a}) \neq \varnothing$, but we have $D(f) \cap D(g) \cap V(\mathfrak{a}) = D(fg) \cap V(\mathfrak{a}) = \varnothing$, which is a contradiction, since it shows $V(\mathfrak{a})$ is be reducible. The fact these two maps are inverses to each other is Proposition 3.1.

For part 2 If $V \subseteq \operatorname{Spec} A$ is closed, and $\mathfrak{p} \in V$, then $\mathfrak{a} := I(V) \subseteq \mathfrak{p} \subseteq A$. This means that we have $V = V(\mathfrak{a}) \supseteq V(\mathfrak{p})$, which implies that $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$. It is simple to check that $\operatorname{Spec} A$ is $T_0$ (using the fact that for each point $x \in \operatorname{Spec} A$ then $x \in V(\mathfrak{p}_x)$) so any subspace is also $T_0$, which gives us the uniqueness of this generic point. □

If we combine all of these topological properties of affine schemes together, we come to a theorem which states when we can identify a space $X$ as $\operatorname{Spec} A$ for some ring $A$.

**Theorem 3.11** (Hochster)**.** *Let* $X$ *be a topological space. Then the following are equivalent:*

1. $X \cong \operatorname{Spec} A$ *for some ring* $A$*.*

2. $X$ *is quasicompact, with a basis* $\mathcal{B}$ *consisting of quasicompact open sets stable under finite intersection, and* $X$ *is sober, so every irreducible closed subset of* $X$ *has a unique generic point.*

3. $X$ *can be written as the inverse limit of finite* $T_0$*-spaces.*

The reference for this theorem is "Prime Ideal Structures in Commutative Rings" (1969), which is Hochster's PhD Thesis. The proof of this theorem involves building some strange ring $A$ such that $\operatorname{Spec} A$ is homeomorphic to $X$ for a specific space $X$, and we will not see the proof here. In particular, this theorem tells us about some strong finiteness properties of $\operatorname{Spec} A$.

**Definition 3.12.** *A topological space* $X$ *is called spectral if it satisfies any of the three equivalent conditions listed in Theorem 3.11.*

**Example 3.13.** Let $A = k[X, Y]/(XY)$, then $\operatorname{Spec} A$ constists of all the points of lying on the $X$ and $Y$ axes inside $k^2$, which represent the maximal ideals, and the whole axes themselves. Both the $X$ and $Y$ axes are irreducible closed subsets of $\operatorname{Spec} A$, so they contain a unique generic point $\eta_X$ and $\eta_Y$, which we can think of as the whole axis. In this case we have

$$\operatorname{Spec} A = \operatorname{Spec}_{\max} A \cup \{\eta_X, \eta_Y\}.$$

Of course we know that $\operatorname{Spec}_{\max} A$ is simply all the $k$-algebra homomorphisms from $A \to k$ which are just all the points $(x, y) \in k^2$ such that $xy = 0$, which of couse is $(x, 0)$ and $(0, y)$ for all $x, y \in k$.

**Example 3.14.** Now let $A = k[X, Y]$ with $k$ algebraically closed, then $\operatorname{Spec} A$ consists of all the points of $k^2$ as recognised by maximal ideals, all the irreducible curves $C \subseteq \operatorname{Spec} A$, which are the irreducible closed subsets of $\operatorname{Spec} A$ with unique generic point $\eta_C$ of height 1, and the generic point (0) for the whole space $\operatorname{Spec} A$.

# 4 $\operatorname{Spec} A$ has a Natural Sheaf 27/10/2016

Today we are going to study sheaves and presheaves in some generality, and then notice that we are already working with a sheaf on $\operatorname{Spec} A$, and this together with the space $\operatorname{Spec} A$ will define an affine scheme.

**Definition 4.1.** *Let $X$ be some topological space.*

1. *The category* Ouv *is the category whose objects are the open subsets of $X$ and the morphisms are inclusions. This comes from the French ouvert', which means open.*
2. *A presheaf of $\mathcal{C}$ (usually $\mathcal{C}$ is Sets, Rings, Groups, Algebras, Modules, ...) on $X$ is a functor*

   $$\mathcal{F} : \mathrm{Ouv}^{op} \longrightarrow \mathcal{C}.$$

   *We call the elements in $\mathcal{F}(U)$ the sections of $\mathcal{F}$ over $U$, and the maps $\mathcal{F}(i) : \mathcal{F}(U) \to \mathcal{F}(V)$ restriction map $res_V^U$, given $V \subseteq U$ in* Ouv.

There are many basic examples of presheaves.

**Example 4.2.** Let $\mathcal{F} = C^0$ which maps an open set $U \mapsto C^0(U, \mathbb{R})$ to the set of continuous maps from $U \to \mathbb{R}$. We can give $C^0(U, \mathbb{R})$ the structure of a ring, and then $C^0$ is a presheaf from $\mathrm{Ouv}(X)$ to the category of rings, or even $\mathbb{R}$-algebras. The restriction maps just take a continuous map $f : U \to \mathbb{R}$ and restrict this map to the open subspace $V \subseteq U$, $f|_V : V \to \mathbb{R}$. If $X$ were a smooth manifold, then we could consider the presheaf which takes $U$ to $C^\infty(U, \mathbb{R})$, the ring of smooth maps $U \to \mathbb{R}$, again with the same restriction maps.

**Definition 4.3.** *Let $\mathcal{F}$ be a presheaf on $X$. Then $\mathcal{F}$ is a sheaf on $X$ if for all $U \in$ Ouv and all open covers $U = \bigcup_\alpha U_\alpha$, then $\mathcal{F}(U)$ is the limit of the following diagram (equaliser of the following two maps),*

$$\prod_{\alpha \in \mathcal{I}} \mathcal{F}(U_\alpha) \underset{res_{U_\beta}^{U_\alpha}}{\overset{res_{U_\beta}^{U_\alpha}}{\rightrightarrows}} \prod_{\alpha, \beta \in \mathcal{I}} \mathcal{F}(U_\alpha \cap U_\beta) \ .$$

Recall that in Set, the equaliser of two maps $f, g : X \to Y$ is all the $x \in X$ such that $f(x) = g(x)$. We can rephrase this sheaf condition as follows. A presheaf $\mathcal{F}$ is a sheaf if given any open subset $U \subseteq X$ and an open cover $U = \bigcup_\alpha U_\alpha$, then

1. if $s, t \in \mathcal{F}(U)$ are such that $s|_{U\alpha} = t|_{U_\alpha}$ for all $\alpha \in \mathcal{I}$, then $s = t$ in $\mathcal{F}(U)$, and
2. if $s_\alpha \in \mathcal{F}(U_\alpha)$ is a collection of sections which agree when restricted to their intersections $U_\alpha \cap U_\beta$, then there exists $s \in \mathcal{F}(U)$ which restricts to each $s_\alpha$ on each open subset $U_\alpha \subseteq U$.

If a presheaf only satisfies part 1 above, then we call this a separated presheaf.

**Example 4.4.** Let $\mathcal{F} = C^0$ again, then we can prove that $C^0$ is indeed a sheaf. Let $s, t : U \to \mathbb{R}$, then if $s$ and $t$ agree on each open subset in a cover of $U$, then $s$ and $t$ necessarily agree on all of $U$. Take $x \in U$, then there exists $U_\alpha \ni x$ and we have $s(x) = s|_{U_\alpha}(x) = t|_{U_\alpha}(x) = t(x)$, so $s = t$ in $C^0(U, \mathbb{R})$. Hence $C^0$ is at least a separated presheaf. Now let $U_\alpha$ be a open cover of $U$, and let $s_\alpha$ be a collection of continuous map $s_\alpha : U_\alpha \to \mathbb{R}$ which agree on the intersections, then we can define a continuous map $s : U \to \mathbb{R}$ by taking $x \in U$, some $U_\alpha$ which contains $x$, and setting $s(x) = s_\alpha(x)$. This is independent of $U_\alpha$ chosen, because all our $s_\alpha$'s agree on intersections, and it's easy to prove it is continuous using this property and the continuity of all the $s_\alpha$'s.

We see a quick proposition now, which makes us really think about some terminology.

**Proposition 4.5.** *If $\mathcal{F}$ is a sheaf, then $\mathcal{F}(\varnothing) = *$ is a single point (the one point set, trivial group, trivial ring, ...).*

*Proof.* Keep in mind, this proof is sort of dumb. Take $U = \varnothing$, then we can cover $U$ by all the open sets $U_i$, indexed over the set $\mathcal{I} = \varnothing$. Now the sheaf condition says that

$$\mathcal{F}(\varnothing) = \mathrm{eq}\left( \prod_{\varnothing} \rightrightarrows \prod_{\varnothing} \right).$$

It might seem silly, but the product over nothing is a single point, so we are simply taking the equaliser of the unique map from a point to a point with itself, which is itself a point. Hence $\mathcal{F}(\varnothing) = *$. $\square$

The power of geometry is that we like to solve global problems, by first solving them locally, and then using a sheaf or something similar to lift these to global solutions. This is what Grothendieck introduced to algebraic geometry (he was originally an analyst/geometer) when he developed schemes.

Our problem right now, is that we have $\mathcal{O}_{\mathrm{Spec} A}$, but it is only defined on principal open subsets $D(f)$ for some $f \in A$. We want this is be a sheaf, but at present this is not even a presheaf!

**Definition 4.6.** *Let $X$ be a topological space, and let $\mathcal{B}$ be a basis for $X$ (stable under finite intersection), where we consider $\mathcal{B}$ as a full subcategory of* $\mathrm{Ouv}(X)$.

1. *A presheaf $\mathcal{F}$ on $\mathcal{B}$ is a functor from $\mathcal{B}^{op} \to \mathcal{C}$.*
2. *A presheaf on $\mathcal{B}$ is a sheaf if it satisfies the sheaf condition(s) for all $U \in \mathcal{B}$ and any open cover of $U$ by elements in this same basis.*

**Lemma 4.7.** *Given a space $X$ and a basis for this space $\mathcal{B}$ stable under intersections, then the functor from sheaves on $X$ to sheaves on $\mathcal{B}$ sending a sheaf on $X$ simply to the restricted sheaf on $\mathcal{B}$ is an equivalence of categories.*

The inverse of this functor takes a sheaf on $\mathcal{B}$, say $\mathcal{G}$ to the sheaf on $X$ defined at $U \in \mathrm{Ouv}(X)$ as the inverse limit of $\mathcal{G}(V)$ for all $V$ contained in $U$ with $V \in \mathcal{B}$,

$$\mathcal{G}(U) = \lim_{V \subseteq U, V \in \mathcal{B}} \ \mathcal{G}(V).$$

*Proof.* Exercise sheet 3 problem 2. $\square$

**Theorem 4.8.** *Let $A$ be a ring, and let the basis $\mathcal{B}$ of* $\mathrm{Spec}\, A$ *be the one made of principle open subsets, then $\mathcal{O}_{\mathrm{Spec} A}$ is a sheaf on $\mathcal{B}$.*

Using Lemma 4.7, we have a sheaf of rings on $\mathrm{Spec}\, A$, which on principal open subsets $D(f)$ is simply

$$\mathcal{O}_{\mathrm{Spec} A}(D(f)) = \mathcal{O}(D(f)) = A[f^{-1}].$$

*Proof.* Let $U = D(f) = \bigcup_{i \in \mathcal{I}} U_i$, with $U_i = D(f_i)$ all inside $\mathrm{Spec}\, A$. We have to show that $A[f^{-1}]$ is the equaliser of

$$\prod_{i \in \mathcal{I}} A[f_i^{-1}] \rightrightarrows \prod_{i,j \in \mathcal{I}} A\left[(f_i f_j)^{-1}\right],$$

where one map localises $A[f_i^{-1}]$ at $f_j$ and the other localises $A[f_j^{-1}]$ at $f_i$. This problem is now pure commutative algebra. First we can make some reductions however.

1. We can assume that we can replace $A[f^{-1}]$ by $A$, and $U$ by $X$, since $U$ is the spectrum of a ring.
2. Since $X$ is quasicompact, there exists $\mathcal{J} \subset \mathcal{I}$ such that $\mathcal{J}$ is finite, so we can assume that $\mathcal{I}$ is finite. This reduction actually requires a little more work to write out formally.

So let's assume that $f = 1$ and $\mathcal{I}$ is finite. To show that $A \to \prod_i A[f_i^{-1}]$ is injective we assume we have $a \in A$ which maps to zero in $A[f_i^{-1}]$ for all $i \in \mathcal{I}$, then there exists $n \geq 1$ such that $f_i^n a = 0$ by the definition of equality in these localised rings. Our $n$ can be chosen to be independent of $i$ since $\mathcal{I}$ is finite. Then we have

$$V(\{f_i^n\}_{i\in\mathcal{I}}) = \bigcap_{i\in\mathcal{I}} V(f_i) = \varnothing,$$

since $X = \bigcup_i D(f_i)$. Hence $\{f_i^n\}_{i\in\mathcal{I}}$ must generate the unit ideal in $A$, so we have $g_i \in A$ such that

$$1 = \sum_{i\in\mathcal{I}} g_i f_i^n.$$

By multiplying both sides by $a$, we obtain $a = \sum g_i f_i^n a = 0$, so we have injectivity. Now we take $s_i \in A[f_i^{-1}]$ such that $s_i = s_j \in A[(f_i f_j)^{-1}]$. Let's write $s_i = \frac{a_i}{f_i^n}$, where again this $n$ can be chosen independent of $i$ since $\mathcal{I}$ is finite. The fact that $s_i = s_j \in A[(f_i f_j)^{-1}]$ gives us the equation

$$(f_i f_j)^m (f_j^n a_i - f_i^n a_j) = 0$$

in $A$. By making some rearrangements to this, by specifically replacing $f_i^m a_i$ by $a_i$ and $n+m$ by $n$ we obtain

$$f_j^n a_i = f_i^n a_j \tag{4.9}$$

for all $i, j \in \mathcal{I}$. From the same reasoning as above we know that the collection of $f_i^n$ 's generates the unit ideal, so we have (a 'partition of unity') $g_i \in A$ such that

$$1 = \sum_{i\in\mathcal{I}} g_i f_i^n.$$

Set $s \in A$ to be

$$s = \sum_{i\in\mathcal{I}} g_i a_i,$$

then for any $j \in \mathcal{I}$ we have

$$f_j^n s = \sum_{i\in\mathcal{I}} g_i f_j^n a_i = \sum_{i\in\mathcal{I}} g_i f_i^n a_j = \left(\sum_{i\in\mathcal{I}} g_i f_i^n\right) a_j = a_j,$$

where the second equality comes from Equation 4.9. This tells us that $s \mapsto \frac{a_j}{f_j^n} = s_j$ in $A[f_j^{-1}]$. $\square$

There was a discussion in a lecture now about whether or not $\mathcal{O}_{\operatorname{Spec} A}$ defined as we have now is different to defining $\mathcal{F}(D(S)) = A[S^{-1}]$ for some subset $S \subseteq A$. Clearly these definitions match up on principal open subsets $D(f)$ for an $f \in A$, but in general these are not the same.

**Example 4.10.** Let

$$A = k[X,Y,Z] \big/ XZ, YZ, Z^2.$$

Then we have the quotient map $A \to A_{red} = k[X,Y]$ where $A_{red}$ is the reduced ring of $A$. We necessarily have $\operatorname{Spec} A \cong \operatorname{Spec} A_{red}$ (as a general fact). If we take $U = \operatorname{Spec} A - \{p\}$ where $p = (0,0) \in \operatorname{Spec} A \cong \operatorname{Spec} A_{red}$, then we have

$$\mathcal{O}_{\operatorname{Spec} A}(U) = \mathcal{O}_{\operatorname{Spec} A_{red}}(U),$$

which we can see from decomposing $U$ as $U = D(X) \cup D(Y)$. Note that $D(X)$ and $D(Y)$ necessarily tells us that $Z = 0$ here too from the $XZ = 0$ and $YZ = 0$ in $A$. The only difference here is that $\operatorname{Spec} A$ has a slightly 'thicker' origin (whatever that means for now). We also have

$$\mathcal{F}_{\operatorname{Spec} A}(U) = A \neq A_{red} = \mathcal{F}_{\operatorname{Spec} A_{red}}(U).$$

This failure is reflected by the fact that $\mathcal{F}_{\operatorname{Spec} A} \neq \mathcal{O}_{\operatorname{Spec} A}$, but we do have $\mathcal{F}_{\operatorname{Spec} A_{red}} = \mathcal{O}_{\operatorname{Spec} A_{red}}$. Of course we've skipped through lots of calculations in this example, but these things will come later.

**Example 4.11.** Alternatively, we can take a ring $A$ and $f, g \in A$, and consider $D(f) \cup D(g)$, which is just $D(S)$ if $S = \{f, g\}$. Assume that $D(f) \cap V(g) \neq \varnothing$ then $\mathcal{O}_X(D(S)) \neq A[S^{-1}]$, because $g \in \mathcal{O}_X(D(S))$ is not invertible, since $V(g) \cap D(S) \neq \varnothing$, but $g$ is invertible in $A[S^{-1}]$.

# 5 General Sheaf Theory 03/11/2016

**Definition 5.1.** *Let $X$ be a topological space and let $\mathcal{C}$ be a category (of sets, abelian groups, rings, etc.) and let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on $X$. A morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ is a family of morphisms (in $\mathcal{C}$) $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for each $U \in Ouv(X)$ such that given an inclusion of open sets $U \subseteq V$ in $X$ we have the following commutative diagram.*

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \\ \Big\downarrow{\scriptstyle res^V_U} & & \Big\downarrow{\scriptstyle res^V_U} \\ \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \end{array}$$

*If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then a morphism of sheaves is simply a morphism of the underlying presheaves.*

Since we defined presheaves as functors $Ouv(X) \to \mathcal{C}$, then we could alternatively define a morphism of presheaves as a natural transformation of functors. The category of sheaves on $X$ embeds fully-faithfully into the category of presheaves on $X$. Next we define a tool to study sheaves and presheaves locally.

**Definition 5.2.** *Let $X$ be a topological space with a presheaf $\mathcal{F}$, and let $x \in X$, then the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at $x$ is defined to be*

$$\mathcal{F}_x := \operatorname{colim}_{U \ni x, U \in \mathrm{Ouv}(X)} \ \mathcal{F}(U).$$

**Remark 5.3.** The above colimit is taken over a filtered indexing set, so alternatively we have,

$$\mathcal{F}_x \cong \{(U, s) \ \mid \ x \in U \in \mathrm{Ouv}(X), s \in \mathcal{F}(U)\}/\sim,$$

where $(U, s) \sim (U', s')$ if and only if there exists $V \in \mathrm{Ouv}(X)$ contained in $U \cap U'$ with $x \in V$ such that $s|_V = s'|_V$. Write $\overline{(U, s)}$ for the equivalence class of $(U, s)$ in the stalk. This filteredness also allows us to say the the functor defined by taking the stalk of a sheaf is exact, using a similar idea to that in problem 4 on exercise sheet 2[8].

If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then for each $x \in X$ we have a well-defined induced map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ defined by $\overline{(U, s)} \mapsto \overline{(U, \phi_U(s))}$. The well-definedness of this map comes from the naturality properties of morphisms of presheaves.

**Definition 5.4.** *Given $s \in \mathcal{F}(U)$ and $x \in U \subseteq X$, then we define the germ of $s$ at $x$ to be $s_x = \overline{(U, s)} \in \mathcal{F}_x$.*

If $\mathcal{B}$ is a basis for our topology on $X$, then we can rewrite the stalk as

$$\mathcal{F}_x \cong \operatorname{colim}_{U \ni x, U \in \mathcal{B}} \ \mathcal{F}(U),$$

which comes from the fact that the set of all $U \in \mathcal{B}$ containing $x$ is cofinal in the set of all open sets containing $x$. This will come in handy when we want to look at the structure sheave of the spectrum of a ring $A$, since this structure sheaf takes very nice values on the basis of principal open subsets.

**Proposition 5.5.** *Let $X$ be a space with presheaves $\mathcal{F}$ and $\mathcal{G}$, and let $\phi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of presheaves.*

1. *If $\mathcal{F}$ is a sheaf and $U \subseteq X$ is open, then*

$$\begin{aligned} \rho_U : \mathcal{F}(U) &\longrightarrow \prod_{x \in U} \mathcal{F}_x \\ s &\longmapsto (s_x)_{x \in U} \end{aligned}$$

*is injective.*

[8] Show that filtered colimits of abelian groups are exact.

*2. If $\mathcal{F}$ is a sheaf, then $\phi_U$ is injective for all open $U \subseteq X$ if and only if $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$, i.e. sections of sheaves are determined by their germs.*

*3. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $\phi_U$ is bijective for all open $U \subseteq X$ if and only if $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is bijective for all $x \in X$.*

*4. If $\mathcal{G}$ is a sheaf, then $\phi = \psi$ if and only if $\phi_x = \psi_x$ for all $x \in X$.*

*Proof.* 1. We need to show $\rho_U$ is injective, so let $s, t \in \mathcal{F}(U)$ be sections, with $s_x = t_x$ for all $x \in U$, then for each $x \in U$ we have $V_x$ containing $x$ open in $U$ such that $s|_{V_x} = t|_{V_x}$, from the equivalent definition of the stalk in Remark 5.3. Clearly $U$ is covered by all these $V_x$'s, and the fact that $\mathcal{F}$ is a sheaf tells us the map

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}(V_x)$$

is injective. Under this map $s, t \mapsto (s|_{V_x})_{x \in U} = (t|_{V_x})_{x \in U}$, so we have $s = t$.

2. First let's suppose that $x \in X$, then $\phi_x$ is injective, since taking stalks is exact, which comes from the fact that filtered colimits are exact from exercise sheet 2 problem 4.[9][10] Conversely, let $U$ be an open subset of $X$, then we have the commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "\phi_U"] & \prod_{x \in U} \mathcal{F}_x \arrow[d, "\prod \phi_x"] \\ \mathcal{G}(U) \arrow[r] & \prod_{x \in U} \mathcal{G}_x \end{tikzcd} .$$

The top morphism is injective by part 1 of this proposition, and the product of injective maps in injective, so this implies that $\phi_U$ is injective.

3. If $\phi_U$ are all bijections then clearly $\phi_x$, which is just the colimit of a whole suite of bijections, is also a bijection. Suppose instead that $\phi_x$ are all bijective for all $x \in X$. We can conclude that $\phi_U$ are all injective for all open subsets $U \subseteq X$, by part 2 of this proposition, so we are left with surjectivity. Let $t \in \mathcal{G}(U)$ and $x \in U$. By assumption there is $V_x \ni x$ open inside of $U$ and a section $s^x \in \mathcal{F}(V_x)$ such that $\phi_x(s^x_x) = t_x$. This equality of germs extends to a small neighbourhood, so there exists $W_x$ contained in $V_x$ also containing $x$ such that $\phi(s^x|_{W_x}) = t|_{W_x}$ inside of $\mathcal{G}(W_x)$. Now we replace $V_x$ with $W_x$. We have a whole bunch of $s|_{V_x}$ now, and we'd like to glue them together, so we need to check they agree on intersections. Take $x, x' \in U$, then

$$\phi_{V_x \cap V_{x'}}\left(s^x|_{V_x \cap V_{x'}}\right) = t|_{V_x \cap V_{x'}} = \phi_{V_x \cap V_{x'}}\left(s^{x'}|_{V_x \cap V_{x'}}\right).$$

Since these $\phi_{V_x \cap V_{x'}}$ are injective maps, then we have $s^x|_{V_x \cap V_{x'}} = s^{x'}|_{V_x \cap V_{x'}}$. Applying the sheaf property of $\mathcal{F}$ allows us to glue these together to obtain $s \in \mathcal{F}(U)$. This means that $s|_{V_x} = s^x$ so by the naturality of $\phi$ we have $\phi(s)|_{V_x} = t|_{V_x}$. The fact that $\mathcal{G}$ is a sheaf then tells us that $\phi(s) = t$, and hence $\phi_U$ is surjective, and in fact bijective.

[9] Let $I$ be a flitered partially ordered set. Show that for each inductive system of short exact sequences indexed over $I$, the colimits also form a short exact sequence.

[10] One might hope that we can repeat this argument with the word surjective, and the answer so far is yes. However, the converse of this statement is false if we replace the injective with surjective. The following is problem 4(ii) on exercise sheet 3: For a holomorphisc function $f : U \to \mathbb{C}$ let $f'$ be its derivative. Show that $f \mapsto f'$ defines a surjective morphism $D : \mathcal{O}_X \to \mathcal{O}_X$ (where $\mathcal{O}_X$ sends an open subset $U \subseteq X$ where $X$ is an open subset of $\mathbb{C}$ to $\{f : U \to \mathbb{C} \text{ holomorphic}\}$) of sheaves. Give an example of an open subset $X \subseteq \mathbb{C}$ such that $D$ is not surjective on global sections.

4. If $\phi = \psi$ then obviously $\phi_x = \psi_x$. Conversely, take $\phi_x = \psi_x$ for all $x \in X$, then for an open set $U \subseteq X$ we have the following commutative diagram.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, shift right, "\psi_U"'] \arrow[d, shift left, "\phi_U"] & \prod_{x \in U} \mathcal{F}_x \arrow[d, "\prod \phi_x = \prod \psi_x"] \\ \mathcal{G}(U) \arrow[r] & \prod_{x \in U} \mathcal{G}_x \end{tikzcd}.$$

The bottom map is injective from part 1 of this proposition, which implies that $\phi_U = \psi_U$ for all open subsets $U \subseteq X$.

□

We leave another warning for the reader here that surjectivity of the stalk function is a big deal. We can describe the failure in an equivalence of surjectivity in terms of homological algebra, and this is where we get sheaf cohomology from. We'll spend most of next semester talking about sheaf cohomology.

**Definition 5.6.** *Let $X$ be a topological space, and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves, and let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.*

1. *The map $\phi$ is called injective (resp. bijective) if for all $x \in X$ $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. bijective). This is equivalent to $\phi_U$ being injective (resp. bijective) for all open subsets $U \subseteq X$.*

2. *The map $\phi$ is called surjective if for all $x \in X$ we have $\phi_x$ is surjective. This is* not *equivalent to $\phi_U$ being surjective for all $U \subseteq X$ (see problem 4(ii) on exercise sheet 3, and Footnote 10).*

There is a way to construct a sheaf from a presheaf, through a process called 'sheafification'. Now we have the correct language to define this construction.

**Proposition 5.7.** *Let $X$ be a topological space and $\mathcal{F}$ be a presheaf on $X$, then there exists a sheaf $\widetilde{\mathcal{F}}$ with a morphism of presheaves $\iota_{\widetilde{\mathcal{F}}} : \mathcal{F} \to \widetilde{\mathcal{F}}$ such that for every morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ where $\mathcal{G}$ is a sheaf, there exists a unique morphism $\overline{\phi} : \widetilde{\mathcal{F}} \to \mathcal{G}$ such that $\phi = \overline{\phi} \circ \iota_{\widetilde{\mathcal{F}}}$. The following also holds.*

1. *The map $\iota_{\widetilde{\mathcal{F}}}$ induces bijections on stalks.*

2. *The pair $(\widetilde{\mathcal{F}}, \iota_{\widetilde{\mathcal{F}}})$ is unique up to unique isomorphism.*

3. *The pair $(\widetilde{\mathcal{F}}, \iota_{\widetilde{\mathcal{F}}})$ is natural in the presheaf variable $\mathcal{F}$ and morphisms of presheaves.*

4. *The assignment $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$ and $\phi \mapsto \overline{\phi}$ is a functor, left adjoint to the inclusion functor from the category of sheaves into the category of presheaves.*

*Proof.* For some open set $U \subseteq X$ we define

$$\widetilde{\mathcal{F}}(U) = \left\{ (s_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x \,\middle|\, \forall x \in U, \exists V \in \mathrm{Ouv}(U) \text{ such that } x \in V, t \in \mathcal{F}(V) : \forall y \in V, s_y = t_y \right\}.$$

Heuristically, this definition is simply a collection of germs, with the condition that they don't vary too much in a small neighbourhood. Define $\iota_{\widetilde{\mathcal{F}}}$ to send the section $s \in \mathcal{F}(U)$ to its germ $(s_x)_{x \in U} \in \widetilde{\mathcal{F}}$. This map is what we were calling $\rho_U$ in Proposition 5.5. It is easy to check that $\widetilde{\mathcal{F}}$ is a sheaf. The first condition is obvious, so take an open set $U \subseteq X$ and an open cover $U_i$ of $U$ with $s_i = (s^i_x)_{x \in U_i}$ that agree on overlaps, then we can define $s = (s^i_x)_{x \in U}$ where $i$ is chosen such that $x \in U_i$. It is also easy to check that $\iota_{\widetilde{\mathcal{F}}}$ is a morphism of presheaves and induces bijections on stalks.

We now show that the pair $(\widetilde{\mathcal{F}}, \iota_{\widetilde{\mathcal{F}}})$ have the desired universal property. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves, where $\mathcal{G}$ is a sheaf, then we obtain the following commutative diagram.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\iota_{\widetilde{\mathcal{F}}}"] \arrow[d, "\phi"'] & \widetilde{\mathcal{F}} \arrow[d, "\prod \phi_x"] \arrow[dl, dotted, "\overline{\phi}"'] \\
\mathcal{G} \arrow[r, "\iota_{\widetilde{\mathcal{G}}}"'] & \widetilde{\mathcal{G}}
\end{tikzcd}$$

If $\mathcal{G}$ is a sheaf, then $\iota_{\widetilde{\mathcal{G}}}$ is a bijection. This follows from the observation that $\iota_{\widetilde{\mathcal{F}}}$ induces a bijection on stalks for each presheaf $\mathcal{F}$, and then we use the fact that $\mathcal{G}$ is a sheaf and part 3 of Proposition 5.5. We construct $\overline{\phi}$ to literally be $\left(\iota_{\widetilde{\mathcal{G}}}\right)^{-1} \circ \prod \phi_x$. This morphism $\overline{\phi}$ is unique because we showed that morphisms agreeing on stalks of sheaves are equal. Part 2, 3 and 4 are purely formal category theory. Since we have defined this sheafification functor with some universal property, then we obtain these extra properties. $\square$

Notice that part 4 tells us that sheafification commutes with all colimits as a left adjoint functor. Now let's quickly mention the spectrum of a ring, or else we may as well be in a topology course right now.

**Proposition 5.8.** *Let $A$ be a ring, $X = \operatorname{Spec} A$ and $x \in X$, then*

$$\mathcal{O}_{X,x} \cong A_{p_x}.$$

*Proof.* Consider the basis for $X$ of principal open subsets, so $\mathcal{B} = \{D(f) \subseteq X | f \in A\}$. Then we can rewrite the definition of the stalk of the structure sheaf at a point $x \in X$ as

$$\mathcal{O}_X|_x = \operatorname{colim}_{x \in U, U = D(f)} \mathcal{O}_X(U) = \operatorname{colim}_{x \in U \in \mathcal{B}} A[f^{-1}] =: B.$$

The above colimit is taken over the structure maps $\frac{a}{f^n} \mapsto \frac{a}{f^n} = \frac{ag^n}{(fg)^n}$. Maybe it already seems obvious that $B \cong A_{\mathfrak{p}_x}$, but we can show this explicitly too.

Let $x \in D(f)$, so equivalently $f \notin \mathfrak{p}_x$, so in particular we have a map $A[f^{-1}] \to A_{\mathfrak{p}_x}$ defined by $\frac{a}{f^n} \mapsto \frac{a}{f^n}$, where of course we are using the same notation for two different equivalence classes of elements in two different rings. These maps are compatible with the structure maps in the direct limit defining $B$ above, so we have a map $\alpha : B \to A_{\mathfrak{p}_x}$. We also have a map $\beta : A \mapsto B$ defined by sending $a \mapsto \frac{a}{1}$. If $f \notin \mathfrak{p}_x$, then $x \in D(f)$ and $\beta(f) = \frac{f}{1}$ in $A[f^{-1}]$, in particular $\beta(f) \in B^\times$, so we obtain a map $\overline{\beta} : A_{\mathfrak{p}_x} \to B$ defined by $\frac{a}{f^n} \mapsto \frac{a}{f^n}$. Clearly $\overline{\beta}$ and $\alpha$ are mutual inverses to each other, so we have

$$B \cong A_{\mathfrak{p}_x}.$$

$\square$

# 6 Locally Ringed Spaces and Schemes 08/11/2016

Let's remind ourselves about local rings. (There will be lots of definitions to start us off today).

**Definition 6.1.** *A ring $A$ is a local ring if it has a unique maximal ideal $\mathfrak{m}_A$.*

In this case, note that all the elements of $A\backslash\mathfrak{m}_A$ are invertible. Conversely, if $A$ is a ring and $I \subsetneq A$ is an ideal such that all elements of $A\backslash I$ are invertible, then $A$ is a local ring with $\mathfrak{m}_A = I$.

**Definition 6.2.** *A spectral space $X$ is called local if it has a unique closed point.*

The following lemma is an obvious consequence of the above two definitions, and motivation for the naming conventions (or an extreme coincidence).

**Lemma 6.3.** *A ring $A$ is a local ring if and only if $\operatorname{Spec} A$ is a local spectral space.*

*Proof.* The closed points of an affine scheme $\operatorname{Spec} A$ are exactly the maximal ideals. □

**Definition 6.4.** *A morphism $\phi : A \to B$ of local rings is a local morphism if $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$ (so it preserves the local structure). A map $f : X \to Y$ of local spectral spaces is called local if it maps the closed point of $X$ to the closed point of $Y$.*

We don't need a proof for the next lemma.

**Lemma 6.5.** *A morphism of local rings $\phi : A \to B$ is local if and only if ${}^a\phi : \operatorname{Spec} B \to \operatorname{Spec} A$ is local.*

**Definition 6.6.** *Given a ring $A$ and a prime ideal $\mathfrak{p} \subseteq A$, then the localisation of $A$ at $\mathfrak{p}$ is*

$$A_{\mathfrak{p}} = A[(A\backslash\mathfrak{p})^{-1}].$$

*Given a spectral space $X$ and $x \in X$, then the localisation of $X$ at $x$ is*

$$X_x = \bigcap_{U \ni x} U.$$

The following predictable proposition is actually worthy of a few lines of dialogue.

**Proposition 6.7.** *Given a spectral space $X$, a point $x \in X$, a ring $A$ and prime ideal $\mathfrak{p} \subseteq A$.*

1. *The ring $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.*
2. *The spectral space $X_x$ is a local spectral space.*
3. *If $X = \operatorname{Spec} A$, then we have a map $A \to A_{\mathfrak{p}}$, and $\operatorname{Spec} A_{\mathfrak{p}} \cong X_x$ where $x$ corresponds to $\mathfrak{p}$ in $X = \operatorname{Spec} A$.*

*Proof.* For part 3, we rewrite $A_{\mathfrak{p}}$ as the following filtered colimit,

$$\operatorname{colim}_{f \notin \mathfrak{p}} A[f^{-1}].$$

This implies that $\operatorname{Spec} A_{\mathfrak{p}}$ can be re-written as the following cofiltered limit.

$$\operatorname{Spec} A_{\mathfrak{p}} = \lim_{f \notin \mathfrak{p}} \operatorname{Spec} A[f^{-1}] = \lim_{f \notin \mathfrak{p}} D(f) = \bigcap_{f \notin \mathfrak{p}} D(f) = X_x$$

For part 2, let $x \neq y \in X_x$ be a closed point, then $X_x\backslash\{y\}$ is open, which implies the existence of a $U \subseteq X$ such that $x \in U$, and $y \notin U$. This implies that $y \notin \bigcap_{U \ni x} U = X_x$, a contradiction. In partilcular, $X_x$ has at most one closed point, but $x \in X_x$ is easily seen to be closed as $\overline{\{x\}} \cap X_x = \{x\}$. Part 1 now follows from part 2 and 3. □

Now comes one of the most important definitions in the course, because it's used in the definition of a scheme, which is actually the most important definition in the course.

**Definition 6.8.** *1. A ringed space is a pair $(X, \mathcal{O}_X)$ of a topological space $X$ and a sheaf of rings $\mathcal{O}_X$.*

*2. A locally ring space is a ring space $(X, \mathcal{O}_X)$, such that for all $x \in X$, the stalk $\mathcal{O}_{X,x}$ at $x$ is a local ring.*

*3. If $A$ is a ring, then* $\operatorname{Spec} A$ *is the locally ringed space* $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

Our dream is to have a fully faithful functor from the opposite category of rings to locally ringed spaces, and for this we need to define what a morphism of ringed spaces are.

**Definition 6.9.** *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces, then a map of ringed spaces is a continuous map of underlying topological spaces $f : X \to Y$ plus a map $f^{\#}_{V \to U} : \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ for all open $U \subseteq X$, $V \subseteq V$ and $f(U) \subseteq V$, such that for all open set containments $U' \subseteq U \subseteq X$ and $V' \subseteq V \subseteq Y$, and $f(U') \subseteq V'$ and $f(U) \subseteq V$, we have the following commutative diagram.*

$$\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r, "f^{\#}_{V \to U}"] \arrow[d, "res^V_{V'}"] & \mathcal{O}_X(U) \arrow[d, "res^U_{U'}"] \\
\mathcal{O}_Y(V') \arrow[r, "f^{\#}_{V' \to U'}"] & \mathcal{O}_X(U')
\end{tikzcd}$$

The intuition here is the we take functions $g$ on $V$ to a function $g \circ f$ on $U$, where $f : X \to Y$ is our continuous map. This map is simply precomposition if we start talking about manifolds and sheafs of smooth functions.

**Definition 6.10.** *Let $f : X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a preshead on $X$ and $\mathcal{G}$ be a presheaf on $Y$.*

*1. The pushforward $f_*\mathcal{F}$ is the presheaf on $Y$ defined by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ for $V \in \operatorname{Ouv}(Y)$.*

*2. The pullback $f^+\mathcal{G}$ is the presheaf on $X$ defined by*

$$(f^+\mathcal{G})(U) = \operatorname{colim}_{f(U) \subseteq V, V \in \operatorname{Ouv}(Y)} \ \mathcal{G}(V)$$

*for an open set $U \subseteq X$.*

Notice that the colimit in part 2 of the definition above is filtered, since given $V, V'$ which both contain $f(U)$, then $V \cap V'$ still contains $f(U)$ and obvious contained in both $V$ and $V'$.

**Proposition 6.11.** *Given the situation of the previous definition, we have the following natural correspondence*

$$PreSh(X)(f^+\mathcal{G}, \mathcal{F}) \cong PreSh(Y)(\mathcal{G}, f_*\mathcal{F}).$$

*We say that $f^+$ is left adjoint to $f_*$, it's right adjoint.*

*Proof.* To prove this, we need to build an isomorphism $\Phi : PreSh(X)(f^+\mathcal{G}, \mathcal{F}) \to PreSh(Y)(\mathcal{G}, f_*\mathcal{F})$ for every $\mathcal{F} \in PreSh(X)$ and every $\mathcal{G} \in PreSh(Y)$, which is natural in the variables $\mathcal{F}$ and $\mathcal{G}$. This is done in Görtz and Wedhorn, [2, Remark 2.26, p.53-4]. $\square$

Recall that the fact we have an adjoint pair of functors tells us a lot of information, like left adjoint preserve all colimits and right adjoints preserve all limits, and stuff like that.

**Proposition 6.12.** *Let $x \in X$, then we have a natural identification $(f^+\mathcal{G})_x = \mathcal{G}_{f(x)}$.*

With natural identifications, we often denote equality, even though it is strictly naturally isomorphic up to a unique natural isomoprhism.

*Proof.* This falls out almost straight from the definitions.

$$(f^+\mathcal{G})_x = \operatorname{colim}_{U\ni x}(f^+\mathcal{G})(U) = \operatorname{colim}_{U\ni x}\operatorname{colim}_{V\supseteq f(U)}\mathcal{G}(V) \cong \operatorname{colim}_{V\ni f(x)}\mathcal{G}(V) = \mathcal{G}_{f(x)}$$

□

We do need a little argument for the isomorphism above, but it is not hard to justify.

**Proposition 6.13.** *Let $f : X \to Y$ be a continuous function of topological spaces, with a sheaf $\mathcal{F}$ of $X$ and a sheaf $\mathcal{G}$ on $Y$.*

1. *$f_*\mathcal{F}$ is a sheaf on $Y$.*
2. *$f^{-1}\mathcal{G}$ is the sheafification of $f^+\mathcal{G}$.*
3. *We have the following adjunction*

$$Sh(X)(f^{-1}\mathcal{G}, \mathcal{F}) \cong Sh(Y)(\mathcal{G}, f_*\mathcal{F}).$$

4. *For all $x \in X$ we have $(f^{-1}\mathcal{G})_x = G_{f(x)}$.*

*Proof.* Let $V = \bigcup_{i\in\mathcal{I}} V_i$ be an open covering of $V$ in $Y$, then we have

$$U = f^{-1}(V) = \bigcup_{i\in\mathcal{I}} U_i,$$

where $U_i = f^{-1}(V_i)$. This means that we have

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V)) = \mathcal{F}(U) \qquad\qquad (f_*\mathcal{F})(V_i) = \mathcal{F}(f^{-1}(V_i)) = \mathcal{F}(U_i),$$

and the sheaf conditions for $f_*\mathcal{F}$ follow straight from the sheaf conditions for $\mathcal{F}$. This show part 1, and for part 2 is actually secretly a definitions, not part of our proposition. Part 3 follows the universal property of sheafification, and our adjunctions.

$$Sh(X)(f^{-1}\mathcal{G}, \mathcal{F}) \cong PreSh(X)(f^+\mathcal{G}, \mathcal{F}) \cong PreSh(Y)(\mathcal{G}, f_*\mathcal{F}) \cong Sh(Y)(\mathcal{G}, f_*\mathcal{F})$$

Finally, part 4 come from the fact that the sheafification process preserves stalks, and then we basically use Proposition 6.12. □

**Corollary 6.14.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces, then

$$Hom((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) \cong \{f : X \to Y \text{ and } f^\# : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X\} \cong \{f : X \to Y \text{ and } f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X\}$$

*Proof.* This follows straight from the formalism of the previous proposition. □

**Definition 6.15.** *Let $\phi : B \to A$ be a map of rings, and let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, then ${}^a\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces given by ${}^a\phi : X \to Y$ on the level of topological spaces, and $({}^a\phi)^b : \mathcal{O}_Y \to ({}^a\phi)_*\mathcal{O}_X$ defined on a basis of principal opens by the natural map for all $s \in B$*

$$\begin{tikzcd}
\mathcal{O}_Y(D(s)) \arrow[r] \arrow[d, "="] & \mathcal{O}_X(({}^a\phi)^{-1}(D(s))) \arrow[d, "="] \\
B[s^{-1}] \arrow[r, "{\phi[s^{-1}]}"] & A[\phi(s)^{-1}]
\end{tikzcd}$$

*where we can identify $\mathcal{O}_X(({}^a\phi)^{-1}(D(s))) = \mathcal{O}_X(D(\phi(s)))$.*

**Remark 6.16** (Warning!)**.** The functor from the category of rings to the category of ringed spaces is not fully faithful yet, because there are maps between affine schemes (now consider only as ringed spaces) that are not yet induced by maps of rings.

**Example 6.17.** Take $\mathfrak{p}$ some prime, then there is a morphism of ringed spaces $f_{\mathfrak{p}} : (\operatorname{Spec} \mathbb{Q}, \mathcal{O}_{\operatorname{Spec} \mathbb{Q}}) \to (\operatorname{Spec} \mathbb{Z}, \mathcal{O}_{\operatorname{Spec} \mathbb{Z}})$ which is defined by sending the point $\operatorname{Spec} \mathbb{Q} = *$ to $p \in \operatorname{Spec} \mathbb{Z}$. Indeed, take $f^{\#} : f_{\mathfrak{p}}^{-1}\mathcal{O}_{\operatorname{Spec} \mathbb{Z}} \to \mathcal{O}_{\operatorname{Spec} \mathbb{Q}}$ to be the natural map, then this does not come from a ring map $\mathbb{Z} \to \mathbb{Q}$.

We need to restrict the set of morphisms. We need some condition relating the geometry of $X$ with the algebra of $\mathcal{O}_X$.

**Definition 6.18.** *A morphism of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a collection of maps $f : X \to Y$ and $f^{\#} : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of ringed spaces, such that for all $x \in X$, the map of local rings*

$$f_x^{\#} : \mathcal{O}_{Y,f(x)} = (f^{-1}\mathcal{O}_Y)_x \to \mathcal{O}_{X,x}$$

*is a local map.*

After almost 12 hours of lectures and 3 assignments, we come to the main definition we are going to need in this course (and for the rest of our algebraic-geometric lifes).

**Definition 6.19.** *An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that $(X, \mathcal{O}_X) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring $A$. A morphism of affine schemes is exactly a morphism of locally ringed spaces. A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists a covering $X = \bigcup_{i \in \mathcal{I}} U_i$ of $X$ by open subsets such that each $(U_i, \mathcal{O}_{X|_{U_i}})$ is isomorphic to an affine scheme. A morphism of schemes is simply a morphism of locally ringed spaces.*

**Remark 6.20.** If $\phi : B \to A$ is a map of rings, then ${}^a\phi : (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$ is a morphism of locally ringed spaces. For all $x \in \operatorname{Spec} A$, we have $\mathfrak{p} \subseteq A$, and $\phi^{-1}(\mathfrak{p}) = \mathfrak{q} \subseteq B$, and we have $({}^a\phi)_x^{\#} : \mathcal{O}_{Y,f(x)} = B_{\mathfrak{q}} \to A_{\mathfrak{p}} = \mathcal{O}_{X,x}$ is a local map. On spectral spaces we have $x \mapsto y = \phi^{-1}(\mathfrak{p}) \in \operatorname{Spec} B$, which on localised spectral spaces is a map $\operatorname{Spec} A_{\mathfrak{p}} = X_x \to Y_y = \operatorname{Spec} B_{\mathfrak{q}}$.

We now have everything we've been dreaming of since we first built Corollary 1.15.

**Theorem 6.21.** *The contravariant functor which sends $A \mapsto (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is an equivalence of categories between the category of rings and the category of affine schemes.*

# 7 Affine Schemes with Strucutre Sheaf are Rings 10/11/2016

Today we'll provide the details behind the equivalence of the categories of rings and affine schemes, i.e. we prove Theorem 6.21.

**Theorem 7.1.** *Consider* Spec *as a contravariant functor from the category of rings and ring homomorphisms to the category of locally ringed spaces and morphisms of locally ringed spaces. Then* Spec *is fully faithful onto its image, which we'll call the category of affine schemes.*

Once we have this theorem, we'll know that the category of affine schemes in equivalent to the category of rings. The inverse of Spec, now considered as a functor from the category of rings to the category of affine schemes, is the global sections functor $\Gamma$. We will use both the notation $\Gamma(X, \mathcal{F})$ and $\mathcal{F}(X)$ for the global sections of a sheaf $\mathcal{F}$ over the topological space $X$.

**Example 7.2.** Let $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ be two affine schemes, and let $f \in A, g \in B$ such that we have a map of rings $\alpha : A[f^{-1}] \to B[g^{-1}]$ which induces an isomorphism on spectra, so ${}^a\alpha : D(g) \to D(f)$ is an isomorphism of spectra. Then we can form the scheme $X = U \cup_\alpha V$ glued along this induced map ${}^a\alpha$ (very similar, in fact a special case of problem 4 on exercise sheet 4[11]). For an explicit example, take $A = \mathbb{C}[T]$ and $B = \mathbb{C}[S]$, and $\alpha : A[T^{-1}] \to B[S^{-1}]$ the map sending $T \to S^{-1}$. Then $U = V = \operatorname{Spec} A = \mathbb{A}^1_{\mathbb{C}}$ is the affine line over $\mathbb{C}$ (similar constructions exist for other fields $k$), and the set of closed points of $U$ and $V$ are simply $\mathbb{C}$. By gluing $U$ and $V$ along ${}^a\alpha$ we obtain a scheme $X$, we call $X$ the projective line $\mathbb{P}^1_{\mathbb{C}}$ over $\mathbb{C}$, and the set of closed points of $X$ are homeomorphic to the sphere $S^2$ once we supply some type of analytic topology. This is a scheme that acts like the Riemann sphere.

We are not actually going to prove Theorem 7.1 today, instead we are going to prove the following.

**Theorem 7.3.** *Let $(X, \mathcal{O}_X)$ be a locally ringed space and let $(Y, \mathcal{O}_Y)$ be an affine scheme, say $Y = \operatorname{Spec} B$, then we have a natural bijection between*

$$Hom\left((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\right) \longrightarrow Hom(B, \Gamma(X, \mathcal{O}_X)).$$

*Proof of Theorem 7.3$\Longrightarrow$Theorem 7.1.* When $(X, \mathcal{O}_X) = \operatorname{Spec} A$ for some ring $A$, then the isomorphism of Theorem 7.3 shows us the functor Spec is fully faithful. $\square$

*Proof of Theorem 7.3.* Both of the following parts of this theorem rely on us building maps between spaces by only knowing what these maps do on global sections. This requires the hypothesis for our ringed spaces to be local quite heavily, as one will see in the proof.

**(Injectivity)** Given $(f, f^\#)$ and some other $(g, g^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ which induce the same map $\phi : B = \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ on global sections. We first claim that $f = g$. To see this, let $x \in X$, then we have the following commutative diagram,

$$\begin{tikzcd} \mathcal{O}_X(X) \arrow[r] & \mathcal{O}_{X,x} \\ \mathcal{O}_Y(Y) = B \arrow[u, "\phi"] \arrow[r] & B_{\mathfrak{q}} = \mathcal{O}_{Y,f(x)} \arrow[u, "f^\#_x"'] \end{tikzcd} \quad , \qquad (7.4)$$

where $\mathfrak{q}$ is the prime ideal of $B$ corresponding to $f(x) \in Y = \operatorname{Spec} B$, $\mathfrak{m}_x$ denote the maximal ideal of $\mathcal{O}_{X,x}$, and $f^\#_x$ is a local map[12]. The preimage of $\mathfrak{m}_x$ in $B$ is $\mathfrak{q}$, which corresponds to $f(x)$, by following

[11]In this exercise we show that the affine line over a ring $A$ with double origin, and the projective line over $A$ are both not affine schemes. We constructed these schemes by gluing $\operatorname{Spec} A[T]$ with $\operatorname{Spec} A[T]$ along the isomorphism $\operatorname{Spec} A[T^{\pm 1}] \to \operatorname{Spec} A[T^{\pm 1}]$ sending $T$ to $T$ and $T'$ respectively.

[12]The proceeding argument needs this map to be local, hence the importance of using our hypothesis that our ringed spaces are in fact locally ringed spaces.

the bottom right corner of the diagram, whilst on the other hand the top left corner of the diagram tells us this only depends on $\phi$. Repeating this argument but with $g$, we see that $f(x) = g(x)$, since both these arguments only depend on $\phi$, hence $f = g$. Now we want to show that $f^\# = g^\#$, or equivalently that $f^\flat = g^\flat$. It suffices to check this on $\mathcal{O}_Y(U)$ for $U = D(s)$ for all $s \in B$ since these principal opens form a basis of our topology on $\operatorname{Spec} B = Y$. We have the following commutative diagram.

$$\begin{tikzcd}
B = \mathcal{O}_Y(Y) \arrow[r, "\phi"] \arrow[d, "\mathrm{res}^Y_U"] & \mathcal{O}_X(X) \arrow[r, "="] \arrow[d, "\mathrm{res}^Y_{f^{-1}(U)}"] & (f_*\mathcal{O}_X)(Y) \arrow[d, "\mathrm{res}^Y_U"] \\
B[s^{-1}] = \mathcal{O}_Y(U) \arrow[r, "{f^\flat(U), g^\flat(U)}"] & \mathcal{O}_X(f^{-1}(U)) \arrow[r, "="] & (f_*\mathcal{O}_X)(U)
\end{tikzcd}$$

We notice that $\mathcal{O}_X(f^{-1}(U))$ obtains a $B$-algebra structure from the map of rings $B \to \mathcal{O}_X(f^{-1}(U))$. In general, there is at most one map of $B$-algebras $B[s^{-1}] \to C$, and it exists if and only if $s$ is invertible in $C$. This implies that $f^\flat(U) = g^\flat(U)$ for all $U$ in the basis of our topology, hence $f^\flat = g^\flat \Leftrightarrow f^\# = g^\#$. This shows that our map of Theorem 7.3 is injective.

The fun part is surjectivity, which comes in three separates sections. Let $\phi : B \to \mathcal{O}_X(X)$ be a map of rings, then we need to produce a map $(f, f^\#) : (X, \mathcal{O}_X) \to (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$.

**(Definition of $f : X \to Y$)** From the argument surrounding Diagram 7.4 we know what we have to do with $x \in X$. We have

$$B \xrightarrow{\phi} \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{X,x} \supseteq \mathfrak{m}_x.$$

We then let $\mathfrak{q} \in B$ be the preimage of $\mathfrak{m}_x$ and define $f(x) = y$, where $y \in \operatorname{Spec} B = Y$ corresponds to $\mathfrak{q} \subseteq B$.

**(Continuity)** Given a general locally ringed space $X$, for $t \in \mathcal{O}_X(X)$, we define

$$D(t) = \{x \in X \mid t_x \notin \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}\} = \{x \in X \mid t_x \neq 0 \in k(x)\},$$

which are analogous to our principal opens. When $X$ is an affine scheme these are simply the principal open subsets. We claim that $D(t) \subseteq X$ is always open, and that the section $t|_{D(t)}$ is invertible inside $\mathcal{O}_X(D(t))$. Both these properties can be checked locally around $x \in D(t)$, since an inverse is unique if it exists, so local inverses automatically glue. As $t_x \notin \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$, we notice that $t_x \in \mathcal{O}_{X,x}$ is invertible[13], so there exists some open subset $U$ of $X$ containing $x$ such that we have $v \in \mathcal{O}_X(U)$ with $t_x v_x = 1 \in \mathcal{O}_{X,x}$, and after taking $U$ small enough, we have $t|_U v|_U = 1$ in $\mathcal{O}_X(U)$. Thus $t|_U$ is invertible, which implies that $U \subseteq D(t)$, so $D(t)$ contains an open neighbourhood of $U$, and $t$ is invertible on $U$.

To prove the continuity of our assignment, we need to check that for each $s \in B$, we have $f^{-1}(D(s))$ is open in $X$. However we find that,

$$f^{-1}(D(s)) = \{x \in X \mid \phi(s)_x \notin \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}\} = D(\phi(s)).$$

Hence this assignment is continuous.

**(Definition of $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$)** To suffices to define compatible maps $f^\flat(D(s)) : \mathcal{O}_Y(D(s)) \to (f_*\mathcal{O}_X)(D(s))$ such that the following diagram commutes,

$$\begin{tikzcd}
B[s^{-1}] = \mathcal{O}_X(D(s)) \arrow[r, "f^\flat|_{(D(s))}"] & (f_*\mathcal{O}_X)(D(s)) = \mathcal{O}_X(D(\phi(s)) \\
B \arrow[u] \arrow[r, "\phi"] & \mathcal{O}_X(X) = (f_*\mathcal{O}_X)(Y) \arrow[u]
\end{tikzcd}.$$

[13]We really need locally ringed spaces here.

Again, there is at most one $f^\flat(D(s))$ with the above commutative diagram, which exists if and only if $\phi(s)|_{D(\phi(s))} \in \mathcal{O}_X(D(\phi(s)))$ is invertible, but we proved this is true above. This gives us a map of ringed spaces $f^\flat$, but we need to check the adjoint map $f^\#$ induces a local map on stalks.

**(The Adjoint Map $f^\# : (f^{-1}\mathcal{O}_Y)_x \to \mathcal{O}_{X,x}$ is Local)** Recall that $f(x) \in Y$ corresponds to $\mathfrak{q} \in B$ given by the preimage of $\mathfrak{m}_x$ under the composition $B \to \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$. We now recall that $\mathcal{O}_{Y,f(x)} = B_\mathfrak{q}$, so we have the following diagram which determines $f_x^\#$ uniquely.

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d] & \mathcal{O}_X(X) \arrow[d] \\ B_\mathfrak{q} = \mathcal{O}_{Y,f(x)} \arrow[r, "f_x^\#"] & \mathcal{O}_{X,x} \end{tikzcd} . \tag{7.5}$$

We want to have $(f_x^\#)^{-1}(\mathfrak{m}_x = \mathfrak{m}_{f(x)} = \mathfrak{q}B_\mathfrak{q} \subseteq B_\mathfrak{q}$, and to see this we note that for affine spectra, the canonical map $\operatorname{Spec} B_\mathfrak{q} \to \operatorname{Spec} B$ is injective. This means we can check the equality in $\operatorname{Spec} B$ after pulling this back to $B_\mathfrak{q}$. This is then trivial from Diagram 7.5. Hence our map is local, and we're done! $\square$

Now that we know why we want to study affine schemes (to utilise geometry, topology and homological algebra for commutative algebras sake, and vice-versa), and general schemes, we can look at new ways to build schemes.

**Definition 7.6.** *A morphism of ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an open immersion if $f : X \to Y$ is an open embedding and $f^\# : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism, or equivalently, there exists an open subset $U \subseteq Y$ such that $(X, \mathcal{O}_X) \cong (U, \mathcal{O}_Y|_U)$, and this isomorphism factors $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ through the inclusion $(U, \mathcal{O}_Y|_U) \to (Y, \mathcal{O}_Y)$.*

**Proposition 7.7.** *Let $(Y, \mathcal{O}_Y)$ be a scheme and $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be an open immersion, then $(X, \mathcal{O}_X)$ is a scheme also.*

**Remark 7.8** (Caution)**.** This is not as obvious as it looks! In fact if $(X, \mathcal{O}_X)$ is a scheme such that it admits an embedding into an affine scheme, then $(X, \mathcal{O}_X)$ is called a quasi-affine scheme (see problem 2 on exercise sheet 14[14]), but is not necessarily affine. Consider the following example.

**Example 7.9.** Let $(Y, \mathcal{O}_Y) = \operatorname{Spec} k[T_1, T_2] = \mathbb{A}_k^2$ be the affine plan over $k$, for an algebraically closed field $k$, then take $U = Y - \{(0,0)\} = D(T_1) \cup D(T_2)$, then $(U, \mathcal{O}_Y|_U)$ is a scheme that is not affine. If it was affine then $U \cong \operatorname{Spec} A$ and $\mathcal{O}_Y(U) = A$, but we can calculate these global sections of $U$ using the covering $U_1 = D(T_1)$ and $U_2 = D(T_2)$ with intersection $U_1 \cap U_2 = D(T_1T_2)$. Hence $\mathcal{O}_Y(U)$ is the equaliser of the pair of maps $\mathcal{O}_Y(U_1) \times \mathcal{O}_Y(U_2) \to \mathcal{O}_Y(U_1 \cap U_2)$, which is really a pair of maps

$$k[T_1^{\pm 1}, T_2] \times k[T_1, T_2^{\pm 1}] \longrightarrow k[T_1^{\pm 1}, T_2^{\pm 1}].$$

This means that the pair $f = \sum_{i,j} a_{i,j} T_1^i T_2^j \in k[T_1^{\pm 1}, T_2]$ and $g = \sum_{i,j} b_{i,j} T_1^i T_2^j \in k[T_1^{\pm 1}, T_2]$ agree if and only if they are strictly polynomials, not Laurent polynomials. But this would imply that $\mathcal{O}_Y(U) = A = k[T_1, T_2]$. This would imply that $(U, \mathcal{O}_Y|_U) \to \operatorname{Spec} A = (Y, \mathcal{O}_Y)$ is an isomorphism, but it is not, so $U$ is not affine.

*Proof of Proposition 7.7.* Let $(Y, \mathcal{O}_Y)$ be a scheme, and $U$ be an open subset of $Y$, then we'll show $(U, \mathcal{O}_Y|_U)$ is a scheme. In the proposition, $(U, \mathcal{O}_Y|_U)$ is the image of $(X, \mathcal{O}_X)$. Now let $\{V_i\}$ be a collection of open sets such that $Y = \bigcup_i V_i$ and $V_i \cong \operatorname{Spec} B_i$, then for all $x \in U$ we can choose $i$ such

[14] These exercises classify affine schemes are those with an open immersion $X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$, or equivalently where $\mathcal{O}_X$ is an ample line bundle.

that $x \in V_i$, so $V_i \cap U \subseteq V_i$ is an open neighbourhood of $x$. This implies that these exists $f \in B_i$ such that $x \in D_{V_i}(f) \subseteq V_i \cap U$, so call $U_x = D_{V_i}(f) \subseteq V_i$. Then we have

$$\Gamma(U_x, \mathcal{O}_Y|_{U_x}) = \Gamma(U_x, (O_Y|_{V_i})|_{U_x}) = B_i[f^{-1}],$$

which is an affine scheme, hence as $x$ varies, all the $U_x$ cover $U$. $\square$

# 8 Scheme Valued Points and Fibre Products 15/11/2016

Given a scheme $X$, then we want to actually figure out what this object $X$ is. We would like some intutition. For every field $k$, we can look at $X(k)$ which is defined as the set of scheme homomorphisms from $\operatorname{Spec} k \to X$, also known as the $k$-valued points of $X$. For example, if $X = \operatorname{Spec} \mathbb{Z}[X_1, \dots, X_n]/(f_1, \dots, f_m)$, then

$$X(k) = \{x = (x_1, \dots, x_n) \in k^n | f_1(x) = \cdots = f_m(x) = 0\}.$$

If $k = \mathbb{R}$ or $\mathbb{C}$, then we can literally draw pictures, but for arbitrary fields, we have to be content with this conceptual picture (aren't we mathematicians anyway?). This $k$-valued points construction does lose quite a bit of information from the scheme $X$, for example if $X = \operatorname{Spec} \mathbb{Z}[T]/p$ for some prime ideal $\mathfrak{p} \subseteq \mathbb{Z}[T]$, then $X(k) = \varnothing$ unless $\mathfrak{p}$ is zero in $k$. Knowing $X(k)$ for all fields $k$ is not nearly enough information to retain.

**Definition 8.1.** *A scheme $X$ is called reduced if for all $U \subseteq X$ open the ring $\mathcal{O}_X(U)$ is a reduced ring (so $f^m = 0$ if and only if $f = 0$).*

**Proposition 8.2.** *1. An affine scheme $X = \operatorname{Spec} A$ is reduced if and only if $A$ is reduced.*

*2. A scheme $X$ is reduced if and only if for all open affines $U = \operatorname{Spec} A \subseteq X$, the ring $A$ is reduced if and only if $X$ admits a cover by reduced affine schemes.*

*Proof.* 1. If $A$ is reduced then $A[f^{-1}]$ is reduced for all $f \in A$, and if $U \subseteq X$ is an open subset, then there exists $f_i \in A$ such that $U = \bigcup_i D(f_i)$. We have an injection

$$\mathcal{O}_X(U) \hookrightarrow \prod_i \mathcal{O}_X(D(f_i)) = \prod_i A[f_i^{-1}],$$

since $\mathcal{O}_X$ is a sheaf, and since the product of reduced rings is reduced, we see that $\mathcal{O}_X(U)$ is reduced. Conversely, if $X = \operatorname{Spec} A$ is reduced then in particular $\mathcal{O}_X(X) = A$ is reduced.

2. First assume that $X$ is reduced, then $\mathcal{O}_X(U) = A$ is reduced if $U = \operatorname{Spec} A$. It is clear that all $\mathcal{O}_X(U_\alpha)$ are reduced for an affine cover $\{U_\alpha\}$ of $X$ if $\mathcal{O}_X(U)$ is reduced for all affine opens $U$. Finally, if $U \subseteq U_\alpha$ for some affine cover $\{U_\alpha\}$ of $X$ where $\mathcal{O}_X(U_\alpha)$ is reduced, then clearly $\mathcal{O}_X(U)$ is reduced, but for any general $U \subseteq X$ the injection,

$$\mathcal{O}_X(U) \hookrightarrow \prod_\alpha \mathcal{O}_X(U \cap U_\alpha),$$

again shows us that $\mathcal{O}_X(U)$ is reduced.

□

Just as there is a canonical way to obtain a reduced ring from any commutative ring $A$ (just take $A_{red} = A/N$ where $N$ is the ideal of nilpotents of $A$), there is a canonical way to reduce a scheme.

**Proposition 8.3.** *1. Given a scheme $X = (|X|, \mathcal{O}_X)$, then the scheme $X_{red} = (|X|, \mathcal{O}_{X_{red}})$ is a reduced scheme where $\mathcal{O}_{X_{red}}$ is defined as the sheafification of the presheaf $\mathcal{O}^0_{X_{red}}$, which is defined on some open subset $U \subseteq X$ as $\mathcal{O}^0_{X_{red}}(U) = (\mathcal{O}_X(U))_{red}$.*

*2. If $X = \operatorname{Spec} A$ is an affine scheme, then*

$$X_{red} = \operatorname{Spec} A_{red}.$$

*3. For any reduced scheme $Y$, we have a bijection*

$$Hom(Y, X_{red}) \longrightarrow Hom(Y, X).$$

*This means that $X_{red}$ has the same (although dual) universal property with respect to $X$, that $A_{red}$ does with respect to $A$.*

*Proof.* 2. We know that $|\operatorname{Spec} A|$ and $|\operatorname{Spec} A_{red}|$ are homeomorphic as topological spaces, so for any $f \in A$, we have $(A[f^{-1}])_{red} = A_{red}[\overline{f}^{-1}]$. This means that

$$\mathcal{O}^0_{(\operatorname{Spec} A)_{red}}(D(f)) = \mathcal{O}_{\operatorname{Spec}(A)_{red}}(D(f)),$$

which implies $(\mathcal{O}_{\operatorname{Spec} A})_{red} = \mathcal{O}_{(\operatorname{Spec}(A)_{red})}$.

1. If $U = \operatorname{Spec} A \subseteq X$ is an open affine then,

$$(|U|, \mathcal{O}_{X_{red}}|U) = (|U|, \mathcal{O}_{U_{red}}) = \operatorname{Spec} A_{red},$$

where the last equality come from part 2. This implies that $\operatorname{Spec} A_{red}$ is an open affine of $X_{red}$, and choosing an open affine cover of $X$ will give us a reduced open affine cover of $X_{red}$.

3. Let $Y$ be a reduced scheme, then for all map $f : Y \to X$, the map of sheaves $f^\flat : \mathcal{O}_X \to f_*\mathcal{O}_Y$ factors uniquely through $\mathcal{O}_{X_{red}}$, so for all open subsets $U$ of $X$ we have the following diagram,

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[rr, "f^\flat(U)"] \arrow[dr] & & (f_*\mathcal{O}_Y)(U) = \mathcal{O}_Y(f^{-1}(U)) \\
& \mathcal{O}_X(U)_{red} = \mathcal{O}^0_{X_{red}}(U) \arrow[ur, "\widetilde{f}(U)"'] &
\end{tikzcd}.$$

These factorisations $\widetilde{f}(U)$'s glue to a unique map $Y \to X_{red}$, a map of schemes, using the universal property of sheafification. $\square$

The above proposition tells us that we cannot tell the difference between a scheme and its associated reduced scheme using $k$-valued points. We can rescue this intuition though, by allowing $T$-valued points for arbitrary schemes $T$.

**Example 8.4** (Construction)**.** Recall that given a category $\mathcal{C}$ and an object $S \in \mathcal{C}$, then the category $\mathcal{C}/S$ is defined with objects the pairs $A \to S$, and morphisms $f : A \to B$ with the following commutative diagram.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr] & & B \arrow[dl] \\
& S &
\end{tikzcd}$$

Fix a scheme $S$, and define for any $T \in \mathrm{Sch}$ (the category of schemes) the $T$-valued point of $X/S$ as

$$X_S(T) = Hom_{\mathrm{Sch}/S}(T, X).$$

We are being a little vague above, we really mean $X_S(T)$ to be all morphisms of schemes $f : T \to X$ such that the following diagram commutes.

$$\begin{tikzcd}
T \arrow[rr, "f"] \arrow[dr] & & X \arrow[dl] \\
& S &
\end{tikzcd}$$

Since $\operatorname{Spec} \mathbb{Z}$ is final in the category of schemes, if we take $S = \operatorname{Spec} \mathbb{Z}$ then we will write $X(T)$, since the commutative diagram above becomes irrelevant. If $Y = \operatorname{Spec} B$ or $S = \operatorname{Spec} C$ we might also write

$$X_S(Y) = X_C(Y) = X_C(B) = X_S(B).$$

This explains our earlier notation for $k$-valued points as $X(k)$. Given a base scheme $S$, and any $X \in \mathrm{Sch}/S$, we have a functor $X_S(-) : (\mathrm{Sch}/S)^{op} \to Sets$ defined by $T \mapsto X_S(T)$. We can use the Yoneda lemma to say something concrete about this.

**Lemma 8.5.** *[Yoneda Lemma] Let $\mathcal{C}$ be a (locally small)*[15] *then the functor $\mathcal{C} \to Fun(\mathcal{C}^{op}, \mathrm{Sets})$ which*

[15] This means that all the hom sets in $\mathcal{C}$ are literally sets, not proper classes.

*sends* $X \mapsto Hom_{\mathcal{C}}(-, X)$ *is fully faithful.*

*Proof.* This is part (i) of problem 2 in exercise sheet 5. □

Applying the Yoneda lemma to $\mathrm{Sch}/S$ we see that $X/S$ is determined by its functor of $T$-points. We actually have a strengthening of this which works in our specific case.

**Proposition 8.6.** *Let* $\mathrm{Sch}^{aff}/S$ *be the full subcategory of* $\mathrm{Sch}/S$ *of affine schemes over* $S$*, then the functor* $\mathrm{Sch}/S \to Fun\left(\left(\mathrm{Sch}^{aff}/S\right)^{op}, \mathrm{Sets}\right)$ *is also fully faithful.*

*Proof.* This is part (ii) of problem 2 in exercise sheet 5. □

In particular, if we take $S = \operatorname{Spec}\mathbb{Z}$ and use our equivalence of the category of affine schemes with the category of rings, we find that a scheme is equivalent to giving a functor from the category of rings to the category of sets, $X \mapsto (R \mapsto X(R))$. This can be a very useful, alternative way to look at schemes, using only the Yoneda lemma and our equivalence of categories from Theorem 7.1.

There are some huge advantages to working with schemes in the relative sense rather than the absolute sense. One reason is that if we consider $X \to S \in \mathrm{Sch}/S$ and if $S = \operatorname{Spec} A$, then all of the sections $\Gamma(U, \mathcal{O}_X)$ have the structure of $A$-algebras. This is just one of the many advantages we'll come across over time.

Recall now that in a category $\mathcal{C}$ with finite limits, we can take the pullback, or fibre product of two maps $X \to S$ and $Y \to S$ to obtain an objects $X \times_S Y$ which is universal in some sense.

**Definition 8.7.** *Given two schemes* $X$ *and* $Y$ *over a base scheme* $S$*, then* $X \times_S Y$ *is the scheme with the universal property that if we have* $T \to X$ *and* $T \to Y$ *for another scheme* $T$ *and the following diagram commutes,*

$$\begin{array}{ccc} T & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array},$$

*then there exists a unique map* $T \to X \times_S Y$ *which factorises the diagram above.*

Recall that we have canonical projection maps $p_X : X \times_S Y \to X$ and $p_Y : X \times_S Y \to Y$. The universal property above states that the $T$-valued points of $X \times_S Y$ are canonically in bijection with the fibre product of the $T$-valued points of $X$ and $Y$ over $S$ in the category of sets. In symbols this reads,

$$(X \times_S Y)(T) \xrightarrow{\cong} X(T) \times_{S(T)} Y(T),$$

where the canonical isomorphism has the prefered direction indicated.

**Theorem 8.8.** *The category of schemes has all fibre products. If* $X = \operatorname{Spec} A$*,* $Y = \operatorname{Spec} B$ *and* $S = \operatorname{Spec} R$*, then* $X \times_S Y \cong \operatorname{Spec}(A \otimes_R B)$*.*

*Proof.* The affine case is a formal consequence of the universal properties of both the fibre product of schemes, the tensor product of rings, the Yoneda Lemma (Lemma 8.5), and Theorem 7.1. The more general existence of the fibre product of schemes can be found in Hartshorne, [3][p. 87, Theorem II.3.2]. The basic idea is to look at affine covers of $X, Y$ and $S$, and then cover $X \times_S Y$ with $X_i \times_{S_j} Y_k$, which we show are affine since we have already proved this case. These pieces all then glue together from general gluing lemmas. □

We see in exercise sheet 5 problem 3 that the fibre product of schemes is not the fibre product of spaces with extra structure, as we see the canonical map $|X \times_S Y| \to |X| \times_{|S|} |Y|$ is surjective, and whose fibres can even be infinite or disconnected. This means the forgetful functor from the category of schemes to topological spaces, or even sets does not admit a left adjoint. In other words, there is no functorial way, left adjoint to the forgetful functor, that will produce a scheme from a general set or topological space.

# 9 Examples, History, and Motivation 17/11/2016

Historically it was very hard to find a language for algebraic geometry which combines the study of objects over $\mathbb{F}_p$ and objects over $\operatorname{Spec}\mathbb{Q}$.

**Theorem 9.1** (Weil Conjectures)**.** *For a scheme $X$ of finite type over $\mathbb{Z}$, (locally* $\operatorname{Spec} A$ *where $A$ is a finitely generated $\mathbb{Z}$-algebra), then the (analytic) topology on $X(\mathbb{C})$ (the complex points of $X$) has a strong influence on the number of $\mathbb{F}_p$-valued points, $X(\mathbb{F}_p)$.*

This was proved mostly by Grothendieck, working with Emil Artin and Jean-Louis Verdier, in a paper published in 1965.

**Example 9.2.** Fix some $a, b \in \mathbb{Z}$, such that $\Delta = 4a^3 + 27b^3 \neq 0$. Then consider the equation $y^2 = x^3 + ax + b$, whose solutions are encoded in the ring

$$A = \mathbb{Z}\left[\tfrac{1}{\Delta}\right][x,y] \Big/ (y^2 - x^3 - ax - b)\cdot$$

Let $X = \operatorname{Spec} A$, then $X(\mathbb{C})$ is the set of all ring homomorphisms $A \to \mathbb{C}$, i.e.

$$X(\mathbb{C}) = \{(x,y) \in \mathbb{C}^2 \mid y^2 = x^3 + ax + b\}.$$

In fact $X(\mathbb{C})$ is homeomorphic to $S^1 \times S^1 \backslash *$ where $*$ is a single point. This looks like a complex torus, whose real points are our classic cubic curves in $\mathbb{R}^2$. We want now to embed $X \hookrightarrow \overline{X}/\mathbb{Z}[1/\Delta]$ by adding a point at $\infty$. The space $\overline{X}$ will be glued from two affine open subsets $X = X_1$ and $X_2$. In this case we take $U_1 = D(y) = \operatorname{Spec} A[y^{-1}] \subseteq X_1$ and $U_2 = D(v) = \operatorname{Spec} B[v^{-1}] \subseteq \operatorname{Spec} B = X_2$, where $B$ is simply $A$ but with variables $u$ and $v$ replacing $x$ and $y$. This gives us

$$A[y^{-1}] = \mathbb{Z}\left[\tfrac{1}{\Delta}\right][x, y^{\pm 1}] \Big/ (y^2 - x^3 - ax - b), \qquad B[v^{-1}] = \mathbb{Z}\left[\tfrac{1}{\Delta}\right][u, v^{\pm 1}] \Bigg/ \left(\left(\tfrac{1}{v}\right)^2 - \left(\tfrac{u}{v}\right)^3 - a\tfrac{u}{v} - b\right)\cdot$$

Now we define an isomorphism $\phi : A[y^{-1}] \to B[v^{-1}]$, by $(x, y) \mapsto \left(\frac{u}{v}, \frac{1}{v}\right)$. When we glue $X_1$ and $X_2$ along $U_1$ and $U_2$, we obtain our scheme $\overline{X}$ over $\operatorname{Spec}\left(\mathbb{Z}\left[\frac{1}{\Delta}\right]\right)$. Note that the complement of $X$ in $\overline{X}$ is equal to the complement of $U_2$ in $X_2$, which is simply $V(v) \subseteq X_2$, which is just $\operatorname{Spec} B/v$. Topologically, the spaces $\operatorname{Spec}(B/v)$ and $\operatorname{Spec}\mathbb{Z}[1/\Delta]$ are homeomorphic. For all fields $k$ we have

$$\overline{X}(k) = X(k) \cup \{\infty\}.$$

We notice that $\infty$ has coordinates $(u, v) = (0, 0)$, so in our $(x, y)$-coordinates we would have $x_y = 0$ and $1/y = 0$, so $y$ acts like $'\infty$. It's a fact that $\overline{X}(\mathbb{C}) \cong S^1 \times S^1$, which one learns in a standard course on Riemann surfaces.

For a scheme $X$, the object $X(k)$ for a field $k$ is in general just a set, but if $k$ is a topological field, then we can place a topology on $X(k)$ "induced from the topology on $k$".

**Example 9.3.** Let $X(\mathbb{C}) = \{(x,y) \in \mathbb{C}^2 \mid y^2 - x^3 - ax - b = 0\}$, we can give $X(\mathbb{C})$ the subspace topology of $\mathbb{C}^2$ in a sensible way.

**Example 9.4.** Let's rename $E := \overline{X}$, and call it the elliptic curve over $\operatorname{Spec}\left(\mathbb{Z}\left[\frac{1}{\Delta}\right]\right)$. What are the fibres of $\mathbb{F}_p$ over $\mathbb{F}_p^2$? Our mental image is that $|\mathbb{F}_p|$ should look like a big generic point with (possibly infintely many) closed points. In this case we have,

$$\begin{aligned} E(\mathbb{F}_p) &= \{(x,y) \in \mathbb{F}_p \mid y^2 - x^3 - ax - b = 0\} \cup \{\infty\} \\ &= \{(x,y) \in x, y \in \{0, 1, \ldots, p-1\} \mid y^2 - x^3 - ax - b \equiv 0 \bmod p\} \cup \{\infty\}. \end{aligned}$$

What is the size of $E(\mathbb{F}_p)$? The heuristics imply there are $p^2$ possibilities for $(x, y)$ that $y^2 = x^3 + ax + b$, but module $p$ we have $p^2/p = p$. Including $\infty$, we have $\#E(\mathbb{F}_p) = p + 1$.

**Theorem 9.5** (Hasse $\sim$ 1930)**.** *For all $p \not| \Delta$, we have $\#E(\mathbb{F}_p) - (p+1)| \leq 2\sqrt{p}$.*

**Theorem 9.6** (Tayler and Collab., 2006)**.** *Assume that $E$ does not have complex multiplication*[16]*, then*

$$\frac{\#E(\mathbb{F}_p) - (p+1)}{\sqrt{p}}$$

*is equidistributed with respect to the measure $\frac{1}{\pi}\sqrt{4-x^2}$ on $[-2,2]$ for varying $p$.*

We can try to explicitly compute some $\#E(\mathbb{F}_p)$, but there are many open conjectures above such $E$'s. For example the Birch-Swinnerton-Dyer conjecture. Even Fermat's last theorem was proved using some of these ideas. In this last case, the solution was given by studying elliptic curves $y^2 = x(x-a^n)(x+b^n)$, where $(a,b,c)$ is a solution of the Fermat equation $x^n + y^n = z^n$. Let's see another example.

**Example 9.7.** Let $A = \mathbb{Z}[x,y]\left[\frac{1}{\Delta}\right]/(y^2 - p(x))$ where $p(x)$ has no repreated roots over $\mathbb{C}$, and consider $X = \operatorname{Spec} A$. Then we have $X(\mathbb{C}) = \{(x,y) \in \mathbb{C}^2 | y^2 = p(x)\}$ and this is homeomorphic to the surface of genus $g$ after we add a point at infinity, also called a hyperelliptic curve.

**Theorem 9.8.** *Given $X$ as above, we have*

$$\left|\#\overline{X}(\mathbb{F}_p) - (p+1)\right| \leq 2g\sqrt{p},$$

*which is sharp in some sense.*

Weil's observation was that $2g$ was exactly the rank of the first cohomology group $H^1(\overline{X}(\mathbb{C}), \mathbb{Z})$ of $\overline{X}(\mathbb{C})$ with $\mathbb{Z}$-coefficients (which we all know from topology anyway), and Weil gave a generalisation to a general (proper and smooth) $\overline{X}$. This related $\overline{X}(\mathbb{F}_p)$ to the $i$-th cohomology (sheaf cohomology!) of $X(\mathbb{C})$.

[16] i.e. $\operatorname{End}(E) = \mathbb{Z}$.

## 10 Projective Space - 22/11/2016

Today we are going to talk about a scheme $\mathbb{P}^n$ which severly generalises topological projective space $\mathbb{RP}^n$ and $\mathbb{CP}^n$.

**Example 10.1.** Define the $n$th complex projective space $\mathbb{CP}^n$ as follows,

$$\mathbb{CP}^n = \mathbb{C}^{n+1} - 0 \Big/ (x_0, \dots, x_n) \sim \lambda(x_0, \dots, x_n), \forall \lambda \in \mathbb{C}^\times .$$

We denote points in $\mathbb{CP}^n$ by homogenous coordinates, $(x_0 : \dots : x_n)$ where $x_i \in \mathbb{C}$ and not all $x_i$ are zero. Notice that each $x_i$ is not well-defined, because we have this equivalence relation by a non-zero complex number, however the ratios $\frac{x_i}{x_j}$ are well-defined whenever $x_j \neq 0$. The standard cover for $\mathbb{CP}^n$ is by $(n+1)$-many copies of $\mathbb{C}^n$ (hence $\mathbb{CP}^n$ is an $n$-dimensional complex manifold) defined by

$$\begin{aligned} U_i = \{(x_0 : \dots : x_n) \in \mathbb{CP}^n \mid x_i \neq 0\} &\longrightarrow \mathbb{C}^n \\ (x_0 : \dots : x_n) &\longmapsto \left(\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i}\right). \end{aligned}$$

For all $i \neq j$, we have $U_i \cap U_j \cong \mathbb{C}^{n-1} \times \mathbb{C}^\times$ which we define as $\{(X_{i,k})_{k=0,\dots,n,k\neq i} \,|\, X_{i,j} \neq 0\}$. We will use these types of sets when we start talking about projective space as schemes. Another way to construct $\mathbb{CP}^n$ would be to glue together all of these $U_i$ along these $U_i \cap U_j = U_{i,j}$.

**Example 10.2.** More generally, for any field $k$ we can define

$$\mathbb{P}^n(k) = k^{n+1} - \{0\} \,/\, k^\times .$$

Our goal now is to construct a scheme $\mathbb{P}^n$ such that the $k$-valued points of $\mathbb{P}^n$ are given exactly by $\mathbb{P}^n(k)$. There are three ways to go about doing this. We are going to do this super explicitly. We could also generalise the functor Spec to a functor called Proj[17], and then define $\mathbb{P}^n = \mathrm{Proj}(\mathbb{Z}[x_0, \dots, x_n])$. Another thing we could do is the functor of points approach, i.e. write down $R \to \mathbb{P}^n(R)$ for all rings, and show this is in the essential image of the fully faithful functor $\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set})$. We are going to do this explicitly, so don't worry.

*Caution!* It will usually *not* be the case that $\mathbb{P}^n(R) = R^{n+1} - 0 \,/\, R^\times$, for a general ring $R$.

**Example 10.3** (Construction of $\mathbb{P}^n$)**.** For any $i = 0, \dots, n$, let

$$U_i = \mathrm{Spec}\,\mathbb{Z}[(X_{i,j})_{j=0,\dots,n,i\neq j}] \cong \mathbb{A}^n_{\mathbb{Z}}.$$

This of $X_{i,j}$ as being the fraction "$\frac{X_j}{X_i}$". For each $i \neq j$ we have

$$U_{i,j} = D(X_{i,j}) \subseteq U_i,$$

so $U_{i,j} \cong \mathrm{Spec}\,\mathbb{Z}[(X_{i,k})_{k\neq i,j}, (X_{i,j})^{\pm 1}]$. We have an isomorphism between $U_{i,j}$ and $U_{j,i}$ denoted as

$$\begin{gathered} \alpha_{i,j} : U_{i,j} \to U_{j,i} \\ X_{i,k}, k \neq i \mapsto X_{j,k} \cdot X_{j,i}^{-1} \end{gathered}$$

The inverse of this map is simply $X_{j,k}, k \neq j \mapsto X_{i,k} \cdot X_{i,j}^{-1}$.[18] There is a lemma which we are yet to prove, but which tells us we can glue schemes together, so long as the pieces slot together coherently.

[17] Which takes graded rings to schemes.

[18] Heuristically, we have the following,

$$X_{k,i} \text{``}=\text{''} \frac{x_i}{x_k} = \frac{x_j}{x_k}\frac{x_i}{x_j} = \frac{x_j}{x_k}\left(\frac{x_j}{x_i}\right)^{-1} \text{``}=\text{''} X_{k,j} X_{i,j}^{-1}.$$

.

**Lemma 10.4.** *Let $\mathcal{I}$ be a set and $U_i$ for all $i \in \mathcal{I}$ be schemes. For $i, j$ we have $U_{i,j} \subseteq U_i = U_{i,i}$ is an open subscheme and the isomorphisms $\alpha_{i,j} : U_{i,j} \to U_{j,i}$ which satisfy the cocycle condition, so $\alpha_{i,k} = \alpha_{j,k} \circ \alpha_{i,j}$ for all $i, j, k \in \mathcal{I}$ on $U_{i,j,k} := U_{i,j} \cap U_{j,k}$. Then we have a scheme*

$$X = \bigcup_{i \in \mathcal{I}} U_i,$$

*i.e. $X$ admits an open covering $X = \cup V_i$ with $\beta_i : V_i \cong U_i$ such that $\beta_i : V_i \cap V_j \cong U_{i,j}$ and $\beta_j : V_i \cap V_j \cong U_{j,i}$, and $\alpha_{i,j} = \beta_j \circ \beta_i^{-1}$.*

Noticing that in our case we have $\alpha_{i,k} = \alpha_{j,k} \circ \alpha_{i,j}$, we apply this gluing lemma and obtain the scheme,

$$\mathbb{P}^n = \bigcup_{i_0}^{n} U_i = \bigcup_{i=0}^{n} \operatorname{Spec} \mathbb{Z}[(X_{i,j})_{j=0,\dots,n, i \neq j}] = \bigcup_{i=0}^{n} \mathbb{A}^n_{\mathbb{Z}}.$$

In particular, for all fields $k$, we have

$$\mathbb{P}^n(k) = \bigcup_{i_0}^{n} U_i(k) = \bigcup_{i=0}^{n} k^n = \mathbb{P}^n(k).$$

Although this line seems tautological, the first $\mathbb{P}^n(k)$ is the $k$-valued points of a scheme, and the latter is our old definition of projective space of a field.

**Remark 10.5.** There is a theorem of Chow from the 1950's which states if a complex manifold $X$ admits a closed immersion $X \hookrightarrow \mathbb{CP}^n$, then there exists a finite set of homogenous polynomials $F_1, \dots, F_m \in \mathbb{C}[X_0, \dots, X_m]$, such that $X \cong V(F_1, \dots, F_m) \subseteq \mathbb{CP}^n$, where we define,

$$V(F_1, \dots, F_m) := \{(x_0 : \cdots : x_n) \in \mathbb{CP}^n \mid F_i(x_0, \dots, x_n) = 0, 0 \leq i \leq n\}.$$

Notice this condition makes sense as $F$ is homogeneous. This implies there exists a closed subscheme (we'll talk about this later) $X^{alg} \subseteq \mathbb{P}^n_{\mathbb{C}} := \mathbb{P}^n \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{C}$ such that $X^{alg}(\mathbb{C}) = X$. Note that all compact Riemann surfaces admit an embedding into $\mathbb{CP}^3$, hence they are all algebraic.

Now we come to a theorem which will occupy the rest of this lecture, but first a quick definition.

**Definition 10.6.** *An $R$-module $M$ is invertible if the endofunctor $L \otimes_R -$ on the category of $R$-modules is an equivalence of categories.*

This is equivalent to the existence of an $R$-module $L'$ such that $L \otimes_R L' \cong R$ (this is simple to prove, one should try it). From exercise sheet 6 problem 4(iii)[19] we also know this is equivalent to the existence of a cover of $\operatorname{Spec} R$ by $D(f_j)$ such that $L[f_j^{-1}] \cong R[f_j^{-1}]$ is locally free of rank 1.

**Theorem 10.7.** *For all rings $R$, there is a natural (functorial in $R$) bijection from $\mathbb{P}^n(R)$ to the set of all surjections $R^{n+1} \xrightarrow{p} L$ where $L$ is an invertible $R$-module, modulo the equivalence relation that $p : R^{n+1} \to L$ is equivalent to $p' : R^{n+1} \to L'$ if and only if there exists an isomorphism $\alpha : L \to L'$ such that $p' = \alpha \circ p$.*

**Example 10.8.** If $R = k$ is a field, then any invertible $R$-module $L$ is isomorphic to $k$, and surjections $\alpha : k^{n+1} \twoheadrightarrow k$ are in bijection with $(x_0, \dots, x_n) \in k^{n+1} - \{0\}$ simply by sending $\alpha$ to $(\alpha(e_0) : \cdots : \alpha(e_n))$. We then notice that $p(x_0, \dots, x_n) \sim p(x'_0, \dots, x'_n)$ in the theorem above if and only if there is an isomorphism $\lambda : k \to k$ which is simply multiplication by $\lambda \in k^\times$. This is equivalent to the existence of $\lambda \in k^\times$ such that $x'_i = \lambda x_i$ for all $i = 0, \dots, n$. In other words, this theorem states what we would like it to state about fields.

[19] Prove that a module $M$ is invertible if and only if it is locally free of rank 1.

*Proof of Theorem 10.7.* To prove we have a bijection, we will give maps in both directions. Let $\phi : \operatorname{Spec} R \to \mathbb{P}^n$ be a map of schemes, and let $\phi^{-1}(U_i)$ be covered by $D(f_{i,k})$ for some $f_{i,k} \in R$, where $U_i$ is a standard affine open of $\mathbb{P}^n$. In particular, we now have maps $\phi_{i,k} : \operatorname{Spec} R[f_{i,k}^{-1}] \to U_i$, which are given by maps of rings,

$$\mathbb{Z}[(X_{i,j})_{i \neq j}] \to R[f_{i,k}^{-1}],$$

where we now set $x_{i,j,k} = \phi_{i,k}(X_{i,j}) \in R[f_{i,k}^{-1}]$. Consider the sujection,

$$p_{i,k} : R[f_{i,k}^{-1}]^{n+1} \longrightarrow R[f_{i,k}^{-1}],$$

which sends $e_j \mapsto x_{i,j,k}$ for $i \neq j$ and $e_i$ to 1. Then we want to glue these $p_{i,k}$'s together to obtain a surjection $p : R^{n+1} \twoheadrightarrow L$. For simplicity of notation, let $\phi^{-1}(U_i) = D(f_i)$ for some $f_i \in R$. The more general case has the same proof, but needs messier notation. We have surjections,

$$p_i : R[f_i^{-1}]^{n+1} \longrightarrow R[f_i^{-1}].$$

For $i \neq i'$ we then have $\phi^{-1}(U_i \cap U_{i'}) = D(f_i f_{i'} = \operatorname{Spec} R[(f_i f_{i'})^{-1}] \subseteq \operatorname{Spec} R$. We also have the following commutative diagram, where the vertical map is multiplication by $X_{i,i'}^{-1}$, and the other two maps are $p_i$ and $p_{i'}$ localised at $p_i$, respecitively $p_{i'}$.

$$\begin{tikzcd}
& R[(f_i f_{i'})^{-1}] \arrow[dd, "\cdot x_{i,i'}^{-1}"] & \\
R[(f_i f_{i'})^{-1}]^{n+1} \arrow[ur, "p_i"] \arrow[dr, "p_{i'}"'] & & . \\
& R[(f_i f_{i'})^{-1}] &
\end{tikzcd} \tag{10.9}$$

The coordinate transformation $X_{i',j} = X_{i,j} \cdot X_{i',i}$ ensures that the above diagram commutes. We then need the following lemma, which we will prove later as a corollary to a fundamental equivalence of categories.

**Lemma 10.10.** *The functor $M \mapsto M_i = M[f_i^{-1}]$ from the category of $A$-modules to the category of collections of $A[f^{-1}]$-modules $M_i$ and isomorphisms $\alpha_{ij} : M_i[f_j^{-1}] \to M_j[f_i^{-1}]$ which satisfy the cocycle condition, is an equivalence of categories.*

*Proof.* This will be proved as Corollary 11.10 later. □

By applying this to $L_i = R[f_i^{-1}]$, with maps $\alpha_{i,i'} : L_i[(f_i f_{i'})^{-1}] \cong L_{i'}[(f_i f_{i'})^{-1}]$ given by multiplication by $X_{i,i'}^{-1}$, just like Diagram 10.9, we have glued together an $R$-module $L$ such that localising $L$ is isomorphic to a localisation of $R$, i.e. $L$ is an invertible $R$-module. Also the maps $p_i$ glue together to form a surjective maps $p : R^{n+1} \to L$ as desired. The equivalence relation stated in Theorem 10.7 comes from Diagram 10.9.

Conversely, assume we have a surjection $p : R^{n+1} \twoheadrightarrow L$ onto an invertible $R$-module $L$, then we want a map $\phi : \operatorname{Spec} R \to \mathbb{P}^n$. It is enough to construct $\phi$ locally in a coherent way, and this is a condition not a datum. Thus we can assume that $L \cong R$ so that $p : R^{n+1} \to R$ is given by $e_i \mapsto x_i$ which by surjectivity of $p$ tells us this collection of $x_i$'s generate $R$ as an ideal. Hence $\operatorname{Spec} R = \bigcup_{i=0}^n D(x_i)$. We can assume that some $x_i$ is invertible, since we are working locally, so $p : R^{n+1} \to R$ sending $e_j$ to $x_j$ is equivalent to $p' : R^{n+1} \to R$ sending $e_j$ to $x_j/x_i$ under our equivalence relation. Hence we define $\phi : \operatorname{Spec} R \to \mathbb{P}^n$ as $\phi : \operatorname{Spec} R \to U_i \subseteq \mathbb{P}^n$ by,

$$\mathbb{Z}[(X_{i,j})_{i \neq j}] \longrightarrow R,$$

which sends $X_{i,j} \mapsto x_j/x_i$. Technically we should now check that these constructions are mutual inverses, but this is left to the reader. □

# 11 Quasi-Coherant Sheaves and Closed Immersion 24/11/2016

We have already seen that schemes are geometrisations of rings, and today we'll see that quasi-coherent sheaves on schemes are geometrisations of modules.

**Definition 11.1.** *Given a ring $A$ and an $A$-module $M$. Then we define a presheaf $\widetilde{M}$ on $X = \operatorname{Spec} A$ defined on the basis of principal opens by*

$$\widetilde{M}(D(f)) = M[f^{-1}], f \in A.$$

On exercise sheet 3 problem 3 we exactly showed this defines a sheaf on this basis of principal opens, which we can extend uniquely to a sheaf $\widetilde{M}$ on all of $X$ by problem 2 of the same sheet. The expected thing happens on stalks of $\widetilde{M}$ too.

**Proposition 11.2.** *Let $x \in X = \operatorname{Spec} A$, and let $\mathfrak{p} \subseteq A$ be the corresponding prime ideal in $A$, then $\widetilde{M}_x = M_{\mathfrak{p}}$.*

*Proof.*

$$\widetilde{M}_x = \operatorname{colim}_{D(f) \ni x} M[f^{-1}] = \operatorname{colim}_{f \notin \mathfrak{p}} M[f^{-1}] = M_{\mathfrak{p}}.$$

□

This was easy. We now have to make the obvious definition to extend this idea to general schemes.

**Definition 11.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space, then a sheaf of $\mathcal{O}_X$-modules is a sheaf of abelian groups $\mathcal{M}$ together with a map $\mathcal{O}_X \times \mathcal{M} \to \mathcal{M}$ of sheaves such that $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module for all open subsets $U \subseteq X$.*

The fact that we ask the action map $\mathcal{O}_X \times \mathcal{M} \to \mathcal{M}$ to be a map of sheaves assures us that our restriction maps respect this module structure.

**Proposition 11.4.** *Given an $A$-module $M$ and $X = \operatorname{Spec} A$, then $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules.*

*Proof.* Taken $U = D(f) \subseteq X$ and $f \in A$ we have to give an action map $\mathcal{O}_X(U) \times \widetilde{M}(U) \to \widetilde{M}(U)$, but this can simply be the $A[f^{-1}]$-module structure,

$$A[f^{-1}] \times M[f^{-1}] \longrightarrow M[f^{-1}].$$

This clearly commutes with restrction maps etc. □

We now have the techincal theorem which drives the types of results we desire.

**Theorem 11.5.**

1. *The functor from $A$-modules to sheaves of $\mathcal{O}_X$-modules defined by $M \mapsto \widetilde{M}$ is fully faithful.*
2. *Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-modules, and assume there exists a cover of $X$ by open affines $D(f_i)$, $f_i \in A$ such that $\mathcal{M}|_{D(f_i)} \cong \widetilde{M_i}$ for some $A[f_i^{-1}]$-module $M_i$, then there exists an $A$-modules $M$ such that $\mathcal{M} \cong \widetilde{M}$.*

Necessarily we have $M = \mathcal{M}(X)$ as $M = \widetilde{M}(X)$, but we still don't really know $M$ exists yet. Assuming this theorem is true just for a little bit, we make the following definition.

**Definition 11.6.** *Let $X$ be some scheme, then a quasi-coherent sheaf on $X$ is a sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ such that there exists a covering $\{U_i = \operatorname{Spec} A_i\}$ of $X$ by open affines, and $A_i$-modules $M_i$ such that $\mathcal{M}|_{U_i} \cong \widetilde{M_i}$. A morphism of quasi-coherent sheaves is simply a morphism of $\mathcal{O}_X$-modules.*

We then have the following corollary of Theorem 11.5.

**Corollary 11.7.** Given an affine scheme $X = \operatorname{Spec} A$, then we have an equivalence of categories between $A$-modules and quasi-coherent sheaves on $X$, given by $M \mapsto \widetilde{M}$ with inverse $\mathcal{M} \mapsto \mathcal{M}(X)$.

In particular, this implies that given a scheme $X$, an open affine $\operatorname{Spec} A = U \subseteq X$, then any quasi-coherent sheaf $\mathcal{M}$ over $X$ restricts to $\widetilde{M}$ on $U$, where $M$ is an $A$-module.

*Proof of Corollary.* From the above definitions we have the following commutative diagram.

$$\begin{tikzcd}
\{A\text{-modules}\} \arrow[r, "F_1"] \arrow[dr, "F_2"'] & \{\text{Quasi-coherent sheaves on } X\} \arrow[d, hook, "\text{full inc.}"] \\
 & \{\text{Sheaves of } \mathcal{O}_X\text{-modules}\}
\end{tikzcd}$$

Theorem 11.5 part 1 says that $F_2$ is fully faithful, and this implies that $F_1$ is fully faithful since the inclusion above is an inclusion of a full subcategory. Theorem 11.5 part 2 then says that $F_1$ is essentially surjective, and we have our equivalence of categories. $\square$

*Proof of Theorem 11.5.* First let's deal with part 1. Similar to how Theorem 7.3 implies Theorem 7.1, we are actually going to prove a more general version of Theorem 11.5, which will restrict to what we want. In this case, let $M$ be an $A$-module and $\mathcal{N}$ an sheaf of $\mathcal{O}_X$-modules, then we claim that

$$\operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{N}) \longrightarrow \operatorname{Hom}_A(M, \mathcal{N}(X)), \tag{11.8}$$

is a bijection. If we set $\mathcal{N} = \widetilde{N}$ for some $A$-module $N$, then we have our desired part 1 of our theorem (using $\mathcal{N}(X) = \widetilde{N}(X) = N$). Category theoretically, this implies an adjunction between the category of $A$-modules and the category of sheaves of $\mathcal{O}_X$-modules, with left adjoint $M \mapsto \widetilde{M}$ and right adjoint $\mathcal{M} \mapsto \mathcal{M}(X)$. First we'll check the injectivity of the map in Equation 11.8. Let $\phi, \phi' : \widetilde{M} \to \mathcal{N}$ be two maps of sheaves of $\mathcal{O}_X$-modules with $\psi : \phi(X) = \phi'(X) : M \to \mathcal{N}(X)$, then for each open subset $U = D(f) \subseteq X$, for some $f \in A$ we have the following commutative diagram.

$$\begin{tikzcd}
M \arrow[r, "res_U^X"] \arrow[d, "\psi"] & M[f^{-1}] \arrow[d, shift right, "\phi(U)"'] \arrow[d, shift left, "\phi'(U)"] \\
\mathcal{N}(X) \arrow[r, "res_U^X"] & \mathcal{N}(U)
\end{tikzcd}$$

In general though, if $M$ is an $A$-module and $N$ is an $A[f^{-1}]$-module (here $N = \mathcal{N}(U)$), then a map $M \to N$ factor in exactly one way over $M[f^{-1}] = M \otimes_A A[f^{-1}]$, hence we have $\phi(U) = \phi'(U)$ for all $U \subseteq X$ open, so $\phi = \phi'$. Hence the map of Equation 11.8 is injective. This factorisation is part of a much more general change of rings isomorphism, which we'll leave below for the reader to digest.

**Proposition 11.9.** *Given a map $g : A \to B$ of rings, an $A$-module $M$ and a $B$-module $N$, then we have the following adjunction,*

$$Hom_A(M, N) \cong Hom_B(M \otimes_A B, N),$$

*where $N$ on the left hand side is given the $A$-module structure via $g$.*

For the sujrectivity of the map in Equation 11.8, take some $\psi : M \to \mathcal{N}(X)$, a map of $A$-modules. Now take some $U = D(f)$ for a $f \in A$, then we have the following commutative diagram.

$$\begin{tikzcd}
M \arrow[r, "res_U^X"] \arrow[d, "\psi"] & M[f^{-1}] \arrow[d, dashed, "\psi_U"] \\
\mathcal{N}(X) \arrow[r, "res_U^X"] & \mathcal{N}(U)
\end{tikzcd}$$

Notice that $\mathcal{N}(U)$ is an $A[f^{-1}]$-module, so using the change of rings adjunction above we get a unique dotted map $\psi_U : M[f^{-1}] \to \mathcal{N}(U)$ of $A[f^{-1}]$-modules such that the above diagram commutes. These maps $\psi_U$ all assemble uniquely into a map of sheaves of $\mathcal{O}_X$-modules $\widetilde{M} \to \mathcal{N}$ on a basis of principal opens of $X$. This of course extends uniquely to a map of $\mathcal{O}_X$-modules on $X$. For part 2, let $M = \mathcal{M}(X)$, then by the proof of part 1 we know that $M \mapsto \mathcal{M}(X)$ has adjoint map $\phi : \widetilde{M} \to \mathcal{M}$. It suffices to show that for all $g \in A$, we have an isomorphism $\phi(D(g)) : M[g^{-1}] \to \mathcal{M}(D(g))$. Take the cover $X$ by the open affines $U_i = D(f_i)$, which gives us $D(f_i) \cap D(g) = D(f_i g)$ as a cover of $D(g)$. We only need finitely many of them since $X$ is quasi-compact. Using the fact that $\mathcal{M}(D(f_i g)) = M_i[g^{-1}]$, which comes from $\mathcal{M}|_{D(f_i)} = \widetilde{M_i}$, we start to re-write $M$.

$$M = eq\left( \bigoplus_i \mathcal{M}(D(f_i)) \rightrightarrows \bigoplus_{i,j} \mathcal{M}(D(f_i f_j)) \right) = eq\left( \bigoplus_i M_i \rightrightarrows \bigoplus_{i,j} M_{i,j} \right)$$

Localisation is exact, so we have

$$M[g^{-1}] = eq\left( \bigoplus_i \mathcal{M}(D(f_i g)) \rightrightarrows \bigoplus_{i,j} \mathcal{M}(D(f_i f_j g)) \right) = eq\left( \bigoplus_i M_i \rightrightarrows \bigoplus_{i,j} M_{i,j} \right) = \mathcal{M}(D(g)).$$

For the last equality we used the fact that $\mathcal{M}$ is a sheaf. $\square$

We now have a corollary which is super useful. This corollary would have been so painful to prove explicitly with modules too, but now it's basically trivial.

**Corollary 11.10.** [Gluing Modules] The functor $M \mapsto M_i = M[f_i^{-1}]$ from the category of $A$-modules to the category of collections of $A[f^{-1}]$-modules $M_i$ and isomorphisms $\alpha_{ij} : M_i[f_j^{-1}] \to M_j[f_i^{-1}]$ which satisfy the cocycle condition, is an equivalence of categories.

*Proof.* We have equivalences of categories,

$$\{A\text{-modules}\} \cong \{\text{quasi-coherent sheafs on } \operatorname{Spec} A\}$$

$$\cong \{\text{quasi-coherent sheaves on } D(f_i) + \text{ gluing data}\} \cong \{\text{collections } (M_i, \alpha_{i,j}) \text{ as above}\}.$$

$\square$

Now we can make some new definitions, and really get into some more scheme stuff.

**Definition 11.11.** *A map $f : Z \to X$ of schemes is called a closed immersion if the induced map on topological spaces is a closed immersion (a homeomophism onto a closed subset) and $f^\flat : \mathcal{O}_X \to f_*\mathcal{O}_Z$ is a surjective map of sheaves.*

Open and closed immersions are probably our favourite maps of schemes.

**Proposition 11.12.** *Let $f : Z \to X$ be a map of schemes, then the following are equivalent.*

1. *$f$ is a closed immersion.*
2. *For all open subsets $U \subseteq X$ with $U = \operatorname{Spec} A$, then $f^{-1}(U) = \operatorname{Spec} B \subseteq Z$ is open affine, and $A \to B$ is surjective.*
3. *There exists an open cover of $X$ by affine schemes which satisfy the property of part 2.*

We willl prove this shortly, but let's make a remark first.

**Remark 11.13.** In particular, for $X = \operatorname{Spec} A$ affine, then closed immersions are in bijection with surjections $A \to B$.

For a morphism $f : X \to Y$ of scheme, the sheaf $f_*\mathcal{O}_X$ is not always quasi-coherent, but it will be in the case from a proposition we will see shortly, but first we need a quick definition.

**Definition 11.14.** *A space $X$ is called quasi-compact if every open cover has a finite subcover, and quasi-separated if given two quasi-compact open subsets $U$ and $V$, then the intersection $U \cap V$ is also a quasi-compact open subset. A map of schemes is called quasi-compact (resp. quasi-separated) if the inverse images of quasi-compact (resp. quasi-separated) open subsets of $X$ are quasi-compact (resp. quasi-separated).*

Notice that affine schemes are quasi-separated, since a basis of principal opens are quasi-compact, and their intersection is too. In general, most schemes will be quasi-compact and quasi-separated, and many maps too.

**Proposition 11.15.** *Let $f : Y \to X$ be a map of schemes which is quasi-compact and quasi-separated, and let $\mathcal{M}$ be a quasi-coherent sheaf on $Y$, then $f_*\mathcal{M}$ is a quasi-coherent sheaf on $X$ with its natural $\mathcal{O}_X$-structure.*

*Proof of Proposition 11.15.* We may assume that $X$ if affine, so $X = \operatorname{Spec} A$ is quasi-compact and quasi-separated, and by assumption $Y$ is also quasi-compact and quasi-separated so $Y = \bigcup_i \operatorname{Spec} B_i$ for some finite collection of rings $B_i$. For all $i$ and $j$ we notice that $\operatorname{Spec} B_i \cap \operatorname{Spec} B_j \subseteq Y$ can be written as a finite union,

$$\bigcup_{k \in J_{ij}} \operatorname{Spec} B_{i,j,k}.$$

Let $M = (f_*\mathcal{N})(X) = \mathcal{N}(Y)$, from which we get a map $\phi : \widetilde{M} \to f_*\mathcal{N}$, which we want to recognise as an isomorphism. For all $g \in A$, we have,

$$\phi(D(g)) : M[g^{-1}] \longrightarrow (f_*\mathcal{N})(D(g)),$$

is an isomorphism. Then, similar to the proof of part 2 of Theorem 11.5, we have

$$M = \mathcal{N}(Y) = eq\left( \prod_i \mathcal{N}(\operatorname{Spec} B_i) \rightrightarrows \prod_{i,j,k \in J_{ij}} \mathcal{N}(\operatorname{Spec} B_{i,j,k}) \right).$$

Since $J_{ij}$ is a finite set, these products are isomorphic to direct sums. Now localisation is exact, so we obtain,

$$M[g^{-1}] = eq\left( \prod_i \mathcal{N}(\operatorname{Spec} B_i)[g^{-1}] \rightrightarrows \prod_{i,j,k \in J_{ij}} \mathcal{N}(\operatorname{Spec} B_{i,j,k})[g^{-1}] \right).$$

Then we use the fact that $\mathcal{N}$ is quasi-coherent and that $\mathcal{N}$ is a sheaf to obtain,

$$eq\left( \prod_i \mathcal{N}(\operatorname{Spec} B_i)[g^{-1}] \rightrightarrows \prod_{i,j,k \in J_{ij}} \mathcal{N}(\operatorname{Spec} B_{i,j,k})[g^{-1}] \right)$$

$$= eq\left( \prod_i \mathcal{N}(\operatorname{Spec} B_i[g^{-1}]) \rightrightarrows \prod_{i,j,k \in J_{ij}} \mathcal{N}(\operatorname{Spec} B_{i,j,k}[g^{-1}]) \right) = \mathcal{N}(f^{-1}(D(g))) = (f_*\mathcal{N})(D(g)).$$

Hence $\widetilde{M} \to f_*\mathcal{N}$ is an isomorphism. □

*Proof of Proposition 11.12.* Part 2 implies part 3 is clear, so we'll show part 3 implies part 1. We can check this locally, since check surjections of sheaves on stalks and closed maps of spaces can be checked on a cover too, so assume $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} B$, and let $p : A \twoheadrightarrow B$ surject onto $B$. Let $I \subseteq A$ be the kernel of this surjection $p$, then $|\operatorname{Spec} B| = V(I) \subseteq |\operatorname{Spec} A|$ where $V(I)$ is closed in $\operatorname{Spec} A$. Also, for all $g \in A$ we have,

$$A[g^{-1}] = \mathcal{O}_X(D(g)) \to (f_*\mathcal{O}_Z)(D(g)) = \mathcal{O}_{\operatorname{Spec} B}(D(p(g))) = B[p(g))^{-1}].$$

Since localisation is exact, then the above map $A[g^{-1}] \to B[p^{-1}]$ is a surjection. To show part 1 implies part 2, we can again work locally, so let $X = \operatorname{Spec} A$, then $|Z| \subseteq |X|$ is closed which implies that $Z$ is also quasi-compact and quasi-separated (since $X$ is). Proposition 11.15 now tells us that $f_*\mathcal{O}_Z$ is quasi-coherent so $f_*\mathcal{O}_Z \cong \widetilde{B}$ for the $A$-module $B = (f_*\mathcal{O}_Z)(X) = \mathcal{O}_Z(Z)$. Where $B = \mathcal{O}_Z(Z)$ is an $A$-algebra. We want to show that $Z = \operatorname{Spec} B$. We have a surjection $\mathcal{O}_X \to f_*\mathcal{O}_Z$, and exercise sheet 6 problem 3 tells us that a sequence of $\mathcal{O}_X$-modules is exact if and only if the sequence of global sections is exact as $A$-modules, where $X = \operatorname{Spec} A$. In our case this tells us that we have a surjection $A \to B$, so the map $B \to \mathcal{O}_Z(Z)$ of $A$-algebras induces a map $\phi : Z \to \operatorname{Spec} B$ as schemes over $\operatorname{Spec} A$. It remains to see that $\phi$ is an isomorphism.

If $x \in \operatorname{Spec} B \backslash Z$, then $0 = (f_*\mathcal{O}_Z)_x = \widetilde{B}_x \neq 0$, so $|Z| = |\operatorname{Spec} B|$. It remains to check on our sheaves, but we know that if we evaluate,

$$\mathcal{O}_Z(D(g) \cap Z) = (f_*\mathcal{O}_Z)(D(g)) = \widetilde{B}(D(g)) = B[g^{-1}] = \mathcal{O}_{\operatorname{Spec} B}(D(g) \cap Z).$$

Hence $\mathcal{O}_Z = \mathcal{O}_{\operatorname{Spec} B}$. $\square$

## 12 Vector Bundles and the Picard Group 29/11/2016

We begin today with a short remark.

**Remark 12.1.** Let $X$ be a ringed space. Similar to $\mathcal{O}_X$-modules we can define $\mathcal{O}_X$-algebras. Namely as a sheaf of commutative rings $\mathcal{A}$ with an $\mathcal{O}_X$-module structure, or equivalently the structure of a sheaf of rings morphism $\mathcal{O}_X \to \mathcal{A}$.

**Definition 12.2.** *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-modules, then we define $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ as the sheafification of $U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)$.*

If $\mathcal{M} = \mathcal{A}$ is an $\mathcal{O}_X$-algebra, then the tensor product $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{N}$ is an $\mathcal{A}$-module and if $\mathcal{N}$ is also an $\mathcal{O}_X$-algebra, then the tensor product is an $\mathcal{A}$-algebra too. Notice that if $X = \operatorname{Spec} A$ and $\mathcal{M} = \widetilde{M}$ and $\mathcal{N} = \widetilde{N}$ are quasi-coherent $\mathcal{O}_X$-modules, then $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} = \widetilde{M \otimes_A N}$. Indeed, for a $V = D(f) \subseteq X$ with $f \in A$ we have

$$\mathcal{M}(D(f)) \otimes_{\mathcal{O}_X(D(f))} \mathcal{N}(D(f)) = M[f^{-1}] \otimes_{A[f^{-1}]} N[f^{-1}] = (M \otimes_A N)[f^{-1}] = \widetilde{M \otimes_A N}(D(f)).$$

This presheaf is a sheaf in this case, so no sheafification is necessary!

**Corollary 12.3.** If $X$ is a scheme, $\mathcal{M}$ and $\mathcal{N}$ quasi-coherent sheaves, then $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is quasi-coherent.

*Proof.* We can check this locally on open affines, where this follows from the discussion above. $\square$

We've seen some of the next definition before, but we get to see the rest of it now! The reasons why we only see this now will become obvious.

**Definition 12.4.** *Let $(f, f^\flat)$ be a map of ringed spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$.*

1. *If $\mathcal{N}$ is an $\mathcal{O}_Y$-module, then the pushforward $f_*\mathcal{N}$ is the $\mathcal{O}_X$-module with the structure morphism*

$$\mathcal{O}_X \times f_*\mathcal{N} \xrightarrow{f^\flat \times \mathrm{id}_{f_*\mathcal{N}}} f_*\mathcal{O}_Y \times f_*\mathcal{N} = f_*(\mathcal{O}_Y \times \mathcal{N}) \xrightarrow{f_*(-)} f_*\mathcal{N} .$$

2. *If $\mathcal{M}$ is an $\mathcal{O}_X$-module, then $f^{-1}\mathcal{M}$ is a sheaf of $f^{-1}\mathcal{O}_X$-modules via*

$$f^{-1}\mathcal{O}_X \times f^{-1}\mathcal{M} = f^{-1}(\mathcal{O}_X \times \mathcal{M}) \longrightarrow f^{-1}(\mathcal{M}).$$

   *We now define the pullback $f^*\mathcal{M}$ as the $\mathcal{O}_Y$-module*

$$f^*\mathcal{M} = f^{-1}\mathcal{M} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y.$$

See, we couldn't have defined that before today. Notice that sometimes people write $f^*$ when they really mean $f^{-1}$, but that is usually just in the case when they are talking about ringed spaces in general. In the language of schemes, and our algebraic geometry class, we really reserve the symbols $f^*$ to mean the pullback of $\mathcal{O}_X$-modules we defined above.

**Proposition 12.5.** *There is an adjunction with left adjoint $f^*$ and right adjoint $f_*$. In other words, for each $\mathcal{O}_X$-module $\mathcal{M}$, and each $\mathcal{O}_Y$-module $\mathcal{N}$, we have the following natural identification,*

$$\mathit{Hom}_{\mathcal{O}_Y}(f^*\mathcal{M}, \mathcal{N}) \cong \mathit{Hom}_{\mathcal{O}_X}(\mathcal{M}, f_*\mathcal{N}).$$

*Sketch of a Proof.* We already have the adjunction

$$\operatorname{Hom}(f^{-1}\mathcal{M}, \mathcal{N}) \cong \operatorname{Hom}(\mathcal{M}, f_*\mathcal{N}) \supseteq \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, f_*\mathcal{N}),$$

and a subset on the right hand side. There is a subset on the left hand side corresponding to the $\mathcal{O}_X$-linear maps $\mathcal{M} \to f_*\mathcal{N}$, and that is $\mathrm{Hom}_{f^{-1}\mathcal{O}_X}(f^{-1}\mathcal{M}, \mathcal{N})$, so we (have to check we) have an adjunction between these subsets,

$$\mathrm{Hom}_{f^{-1}\mathcal{O}_X}(f^{-1}\mathcal{M}, \mathcal{N}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, f_*\mathcal{N}).$$

We can now use a change of rings isomorphism to change the left hand side to,

$$\mathrm{Hom}_{f^{-1}\mathcal{O}_X}(f^{-1}\mathcal{M}, \mathcal{N}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(f^{-1}\mathcal{M} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y, \mathcal{N}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(f^*\mathcal{M}, \mathcal{N}).$$

□

Last time we saw Proposition 11.15 which needed some hypothesis about a map $f : X \to Y$ to assure we could pushforward our quasi-coherent module structures. Remember that we needed these hypotheses, because Proposition 11.15 is not true in general[20]. Now we'll see an analogous proof about the pullback, but we don't need any topological nicities at all!

**Proposition 12.6.** *1. Let $f : Y \to X$ be (any!) map of schemes, and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module, then $f^*\mathcal{M}$ is a quasi-coherent $\mathcal{O}_Y$-module.*

*2. If $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, then $\mathcal{M} \cong \widetilde{M}$ for some $A$-module $M$, and then we have $f^*\mathcal{M} \cong \widetilde{M \otimes_A B}$.*

*Proof.* For part 1, we notice that we can cover $Y$ by open affines $V = \operatorname{Spec} B \subseteq Y$ mapping into an open affine $U = \operatorname{Spec} A \subseteq X$, then let $g : V \to U$ be the restriction of $f$ to $V$, then $(f^*\mathcal{M})|_V = g^*(\mathcal{M}|_V)$. To check the quasi-coherentness of $f^*\mathcal{M}$, it suffices to check the quasi-coherentness of $g^*(\mathcal{M}|_V)$. This implies that we can replace $Y$ by $\operatorname{Spec} B$ and $X$ by $\operatorname{Spec} X$, thus we only have to prove part 2. In that case $\mathcal{M} = \widetilde{M}$ and for all $\mathcal{O}_Y$-modules $\mathcal{N}$ we have the following series of isomorphisms.

$$\mathrm{Hom}_{\mathcal{O}_Y}(f^*\mathcal{M}, \mathcal{N}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, f_*\mathcal{N}) \cong \mathrm{Hom}_A(M, f_*\mathcal{N}(X)) = \mathrm{Hom}_A(M, \mathcal{N}(Y))$$

$$\cong \mathrm{Hom}_B(M \otimes_A B, \mathcal{N}(Y)) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\widetilde{M \otimes_A B}, \mathcal{N}).$$

The Yoneda lemma now tells us that $f^*\mathcal{M} \cong \widetilde{M \otimes_A B}$. □

**Example 12.7.** Recall our characterisation of $\mathbb{P}^n_{\mathbb{Z}}(R)$ for all rings $R$ from Theorem 10.7. We can now extend that result to arbitrary schemes!

**Definition 12.8.** *If $X$ is a scheme, then an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is a quasi-coherent $\mathcal{O}_X$-module such that there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{N}$ with $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N} \cong \mathcal{O}_X$.*

**Remark 12.9.** 1. If $X = \operatorname{Spec} A$ then $\mathcal{L} \cong \widetilde{L}$ for some $A$-module $L$, and $\mathcal{L}$ is invertible if and only if $L$ is invertible.

2. We can make the same definitions without the words quasi-coherent above at all, and then $\mathcal{L}$ would still turn out to be a quasi-coherent sheaf. A sketch of this fact is to prove that any such invertible $\mathcal{O}_X$-module is locally on $X$ a direct summand of $\mathcal{O}_X^n$, for some $n$, analogous to the case of $A$-modules.

**Corollary 12.10.** For any scheme $X$,

$$\mathbb{P}^n_{\mathbb{Z}}(X) \cong \{\mathcal{O}_X^{n+1} \longrightarrow \mathcal{L} \mid \text{surjection, with } \mathcal{L} \text{ an invertible } \mathcal{O}_X\text{-module}\}/\sim.$$

[20]It was stated however that these exceptions are a pathology, and indeed some algebraic geometry books assume that all maps are quasi-compact and quasi-separated without losing our favourite examples. A counterexample can be constructed using the morphism $\prod_{pprime} \operatorname{Spec} \mathbb{Z}_{(p)} \to \operatorname{Spec} \mathbb{Z}$.

*Proof.* Given $U \subseteq X$ an open subset, we make two definitions $U \mapsto \mathbb{P}^n(U)$ and $U \mapsto \{\mathcal{O}_U^{n+1} \to \mathcal{L}\}/\sim$. Both of these sheaves agree on a cover of open affines, so they must be equal. In fact, these exists a natural morphism between both which is an isomorphism when evaluated on affines. $\square$

In particular, if $X = \mathbb{P}^n_{\mathbb{Z}}$, then we have $\mathrm{id} \in \mathbb{P}^n_{\mathbb{Z}}(\mathbb{P}^n_{\mathbb{Z}})$ and we get a canonical ("tautological") surjection $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}^{n+1} \to \mathcal{L}$ where $\mathcal{L}$ is an invertible $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}$-module.

**Definition 12.11.** *We define $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) := \mathcal{L}$. For $m \geq 1$ we have $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(m) := \mathcal{L}^{\otimes m}$, and for $m < 0$ we have*

$$\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(m) := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}}(\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(-m), \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}).$$

The definition of the sheaf $\mathcal{H}om$ is coming in lecture 17.

**Definition 12.12.** *For any scheme $X$, define the Picard group of $X$ as*

$$\mathrm{Pic}(X) = \{\textit{invertible } \mathcal{O}_X\textit{-modules}\}/\cong.$$

This is an abelian group because if $\mathcal{L}$ and $\mathcal{L}'$ are invertible $\mathcal{O}_X$-modules, then so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$, and each $\mathcal{L}$ has an inverse (since it is invertible!). This group is abelian since the tensor product is commutative up to isomorphism (our equivalence relation on the Picard group). The following is a theorem that we will see in exercise sheet 9 problem 2.

**Theorem 12.13.** *For any field $k$, we can compute $\mathbb{Z} \cong \mathrm{Pic}(\mathbb{P}^n_k)$ by sending $m \mapsto \mathcal{O}_{\mathbb{P}^n_k}(m)$.*

We have written something down in that theorem which we have yet to define though.

**Definition 12.14.** *Given a ring $R$, we define $\mathbb{P}^n_R$ simply as*

$$\mathbb{P}^n_R = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, R.$$

Usually invertible $\mathcal{O}_X$-modules are simply called line bundles (Geradenbündel auf Deutsch), which is a good name, justified by the following theorem.

**Theorem 12.15.** *Let $X$ be a scheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, then $\mathcal{L}$ defines a functor $\mathbb{V}(\mathcal{L})$ on all schemes $f : Y \to X$ over $X$ by*

$$\mathbb{V}(\mathcal{L})(Y) := (f^*\mathcal{L})(Y) = \Gamma(Y, f^*\mathcal{L}).$$

*This functor is representable by a schemes also denoted by $\mathbb{V}(\mathcal{L})$ over $X$, such that there is a cover $X = \bigcup_i U_i$ such that*

$$\mathbb{V}(\mathcal{L}) \times_X U_i = \mathbb{V}(\mathcal{L})|_{U_i} \cong U_i \times \mathbb{A}^1.$$

The scheme $\mathbb{V}(\mathcal{L})$ over $X$ can then be thought of as a line bundle.

*Proof.* It's enough to prove this result locally on $X$, by general gluing lemmas[21]. So assume that $X$ is affine, $X = \mathrm{Spec}\, A$, so that $\mathcal{L} = \widetilde{L}$, then $L$ is locally free on $\mathrm{Spec}\, A$ of rank 1. In this case we can actually assume that $L = A$, then

$$\mathbb{V}(\mathcal{L})(Y) = f^*\mathcal{L}(Y) = f^*(\mathcal{O}_X)(Y) \cong \mathcal{O}_Y \cong \mathrm{Hom}_{Ring}(\mathbb{Z}[T], \mathcal{O}_Y(Y)).$$

We then use the universal property of fibre products to obtain,

$$\mathbb{V}(\mathcal{L})(Y) = \mathrm{Hom}_{Ring}(\mathbb{Z}[T], \mathcal{O}_Y(Y)) \cong \mathrm{Hom}_{\mathrm{Sch}}(Y, \mathbb{A}^1) \cong \mathrm{Hom}_{\mathrm{Sch}\,/X}(Y, \mathbb{A}^1 \times X).$$

In this way we can see that $\mathbb{V}(\mathcal{L})$ is represented by $X \times \mathbb{A}^1$, locally. $\square$

More generally, we want to define what a vector bundle means to us.

[21] See [4][Tag 01JF, Lemma 25.15.4]

**Definition 12.16.** *Given a scheme $X$, then a vector bundle $\xi$ is a sheaf of $\mathcal{O}_X$-modules that is locally free of finite rank, i.e. there exists a cover $X = \bigcup_i U_i$ such that $\xi|_{U_i} \cong \mathcal{O}_{U_i}^{n_i}$, for some $n_i \geq 0$. Note that $n_i$ does not have to be constant for all $i$. If $n_i = n$ for all $i$, then $\xi$ is called a vector bundle of rank $n$.*

**Remark 12.17.** If $\xi$ is a vector bundle, then it is in fact a quasi-coherent sheaf. If a vector bundle has rank 1, then it is a line bundle and precisely an invertible $\mathcal{O}_X$-module.

**Proposition 12.18.** *Let $\xi$ be a vector bundle on $X$, then $\mathbb{V}(\xi)(Y) = (f^*\xi)(Y)$ is representable by a scheme $\mathbb{V}(\xi)$ over $X$. There exists a cover $X = \bigcup_i U_i$ such that $\mathbb{V}(\xi)|_{U_i} \cong U_i \times \mathbb{A}^{n_i}$.*

*Proof.* The proof here is the same as the proof of Theorem 12.15, but we use

$$\mathcal{O}_Y(Y)^{n_i} \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[T_1, \ldots, T_{n_i}], \mathcal{O}_Y(Y)).$$

□

Although $\mathbb{V}(\mathcal{M})$ is defined for all quasi-coherent sheaves $\mathcal{M}$ (maybe even more general objects), it is not in general representable like vector bundles. This makes the study of vector bundles somewhat special. We need one more definition (so that we can do the exercises for this week).

**Definition 12.19.** *A quasi-coherent sheaf $\mathcal{O}_X$-module $\mathcal{M}$ is of finite type if there exists an open cover $X = \bigcup_i U_i$ with $U_i = \operatorname{Spec} A_i$ such that $\mathcal{M}(U_i)$ is a finitely generated $A_i$-module.*

Notice that any vector bundle is of finite type, obviously.

**Proposition 12.20.** *If $X = \operatorname{Spec} A$ and $\mathcal{M} = \widetilde{M}$, then $\mathcal{M}$ is of finite type if and only if $M$ is finitely generated.*

*Proof.* If $M$ is finitely generated then $\mathcal{M}$ is clearly of finite type. Conversely, let $X = \bigcup_i D(f_i)$ for some finite collection $f_i \in A$, then we have $\mathcal{M}(D(f_i)) = M[f^{-1}]$ are finitely generated as $A[f^{-1}]$-modules. For each $i$ we choose a finite set of generators $m_{i_j}/f_i^{n_i}$ with $m_{i_j} \in M$ and $n_j \geq 0$, and of course the collection of all $i_j$'s is finite. We then claim that these $m_{i_j}$ generate $M$. We can see this because the map $A^N \to M$ induced by $m_{i_j}$ (with $N$ the sum of all $i$'s and $j$'s) is surjective after inverting $f_i$, to $A^N \to M$ is surjective. Hence $M$ is finitely generated. □

# 13 Finiteness Conditions and Dimension 01/12/2016

Today we start the goal for the rest of our course. We want to talk about projective curves over an algebraically closed field. For this we need to define what a curve is (dimension), what projective means (proper, separated), and what the adjective smooth means in an algebraic geometry context (smoothness and normality are equivalent for curves). We do quickly need some more definitions though, to help us with problem 4 in exercise sheet 7.

**Definition 13.1.** *Let $X$ be a scheme, $\mathcal{M}$ a quasi-coherent sheaf, then $\mathcal{M}$ is globally generated if it is generated by global sections, i.e. if there exists sections $s_i \in \mathcal{M}(X)$ with $i \in \mathcal{I}$ some set, such that the map*

$$\bigoplus_{i \in \mathcal{I}} \mathcal{O}_X \to \mathcal{M},$$

*sending $e_i \mapsto s_i$ is surjective.*

**Definition 13.2.** *We defined the twisting sheaves on $\mathbb{P}^n_R$ last time, now for any quasi-coherent sheaf $\mathcal{M}$ on $\mathbb{P}^n_R$, we define $\mathcal{M}(m) := \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{P}^n_R}} \mathcal{O}(m)$ for any $m \in \mathbb{Z}$.*

Back to today. We want to discuss finiteness conditions inside Sch, as well as a definition of dimension.

**Definition 13.3.** *A scheme $X$ is noetherian if it has a finite open cover by $U_i = \operatorname{Spec} A_i$ where each $A_i$ is a noetherian ring.*

Notice that we demand this cover to be *finite*.

**Proposition 13.4.** *The following are equivalent for a scheme $X$.*

1. *$X$ is a noetherian scheme.*
2. *$X$ is quasi-compact and for all open affines $U = \operatorname{Spec} A \subseteq X$ the ring $A$ is noetherian.*

This proposition is a classic case of demanding some property on an open affine cover of a scheme $X$, which then decends to a property on all affine open subschemes of $X$.

*Proof.* The fact that 2 implies 1 is clear. Assume now that $X$ is noetherian, then $X$ has a finite cover by quasi-compact opens, so $X$ itself is quasi-compact. Let's write this cover of $X$ as $X = \bigcup_i U_i$ with $U_i = \operatorname{Spec} A_i$ open affines and $U_i$ noetherian. Let's show that our candiate open set $U = \operatorname{Spec} A \subseteq X$ is noetherian. Since it is an affine scheme it is quasi-compact, so we have

$$U \cap U_i = \bigcup_{j \in \mathcal{J}_i} D_{U_i}(f_{ij}), \qquad f_{ij} \in A_i.$$

Now $D_{U_i}(f_{ij}) = \operatorname{Spec} A_i[f_{ij}^{-1}]$, and we know that $A_i$ is noetherian, which implies that any localisation is as well. Hence $U$ can be covered by affine opens which are noetherian, and the quasi-compactness gives us finitely many of these. Hence $U = \operatorname{Spec} A$ is a noetherian scheme. Hence we may assume that $X = U = \operatorname{Spec} A$. By a similar argument, we may assume that $U_i = D(f_i)$ for some $f_i \in A$. We need to see that if $A$ is a ring with cover $X = \operatorname{Spec} A = \bigcup_i D(f_i)$ with $A[f_i^{-1}]$ noetherian for all $i$, then $A$ is noetherian. Let $I \subseteq A$ be an ideal in $A$, then $\mathcal{I} = \widetilde{I} \subseteq \mathcal{O}_X = \widetilde{A}$ is a quasi-coherent sheaf in $\operatorname{Spec} A$, and $\mathcal{I}$ is of finite type, as $\mathcal{I}(D(f_i) = I[f_i^{-1}]$ is a finitely generated $A[f_i^{-1}]$-module. Proposition 12.20 now tells us that $I$ is finitely generated. □

If $X$ is a noetherian scheme, then the space $|X|$ has some extra properties.

**Definition 13.5.** *Let $T$ be a topological space, then $T$ is noetherian if every decreasing sequence of closed subsets of $T$ stabilises.*

**Remark 13.6.** If $T$ is a noetherian space, then $T$ is quasi-compact. Given $T = \bigcup_i U_i$ an open cover without a finite subcover, then we can choose a sequence $(x_j)_j$ of points in $T$ such that for all $j$ we have $x_j \notin \bigcup_{j'<j} U_{j'}$, but $x_j \in U_j$. In this case we take $Z_j = T - \bigcup_{j'<j} U_{j'}$ and we find an nested chain of closed subsets $T \supsetneqq Z_0 \supsetneqq Z_1 \supsetneqq \cdots$ which is strictly decreasing. Since $T$ is noetherian, this stabilitses, which implies that the $U_i$ were indexed by finite set.

**Remark 13.7.** If $T$ is a noetherian space, then any open subset $U \subseteq T$ is also noetherian. If we have some chain of nested closed subsets $Z_i$ in $U$, then we simply look at $Z_i' = Z_i \cup (T - U)$, which is closed in $T$ and is a nested chain of subsets. In particular, any open subset of $T$ is quasi-compact.

In fact, any subspace $A$ of a noetherian space $X$ is noetherian, since closed subsets $Z_i \subseteq A$ come from closed subsets $\overline{Z_i} \subseteq X$, which stabilise by assumption.

**Remark 13.8.** If $T$ is noetherian, this also implies that $T$ is quasi-separated. Given $U_1, U_2 \subseteq T$ which are quasi-compact opens, then $U_1 \cap U_2$ is also quasi-compact, since it is open in $T$. In partular, any noetherian space $T$ is quasi-compact and quasi-separated.

**Proposition 13.9.** *Let $X$ be a notherian scheme, then $|X|$ is a noetherian space.*

*Proof.* Let $X = \bigcup_i U_i$ be an open cover with $U_i = \operatorname{Spec} A_i$, where $A_i$ are all noetherian. Let

$$|X| \supseteq Z_0 \supseteq Z_1 \supseteq \cdots$$

be a decreasing sequence of closed subsets. Since our cover is finite, it's enough to check

$$U_i \supseteq U_i \cap Z_0 \supseteq U_i \cap Z_1 \supseteq \cdots$$

for all $i$. Hence we can assume that $X = \operatorname{Spec} A$, where $A$ is a noetherian ring. Then we have $Z_i = V(I_i)$ for some radical ideal $I_i \subseteq A$, which gives us a chain of ideals

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$$

Since $A$ is noetherian, then there exists an $N$ such that $I_i = I_{i+1}$ for all $i > N$. $\square$

The converse of this statement is false! Given a scheme $X$ where $|X|$ is a noetherian space, then $X$ is not necessarily a noetherian scheme. The problem in the above proof comes in when we consider only radical ideals. Let's see an example of this.

**Example 13.10.** Let $R$ be a discrete valuation ring with quotient field $K$ (for example $R = k[[t]]$ for some algebraically closed field $k$ of characteristic zero, and $K = k((t))$). Let $\overline{K}$ be the algebraic closure of $K$ (which in our example is $\overline{K} = k((t))(\sqrt[n]{t}, n > 1)$). Let $\overline{R}$ be the integral closure of $R$ in $\overline{K}$ (for us this is $\overline{R} = k[[t]][\sqrt[n]{t}, n > 1] = \bigcup_{n\geq 1} k[[t]][\sqrt[n]{t}] = k[[\sqrt[n]{t}]]$). Then $\overline{R}$ is *not* noetherian (for example because $(t) \subsetneqq (\sqrt{t}) \subsetneqq (\sqrt[4]{t}) \subsetneqq$) but $\operatorname{Spec} \overline{R}$ is. In our example, this is because $\operatorname{Spec} \overline{R} = \lim_n \operatorname{Spec} k[[\sqrt[n]{t}]]$, and each $k[[\sqrt[n]{t}]]$ is a discrete valuation ring so it constists of exactly two points, corresponding to the fraction field at the maximal ideal. All the transition maps are homeomorphisms too, so $\operatorname{Spec} \overline{R} = * \sqcup *$.

This is somehow the 'most natually occuring example' of a non-noetherian ring. In practice, all classical algebraic geometry is done over noetherian rings. We only really see slight variants of $\operatorname{Spec} A$ where $A$ is a finitely generated $k$-algebra, which is noetherian by Hilbert's basis theorem.

**Definition 13.11.** *Let $X$ be a noetherian scheme, A a coherent $\mathcal{O}_X$-module is a quasi-coherent $\mathcal{O}_X$-module of finite type.*

There is a more general definition of a coherent $\mathcal{O}_X$-module on any scheme, but this definition is not very useful. For example, $\mathcal{O}_X$ itself may not be a coherent $\mathcal{O}_X$-module in this case. So we'll only ever talk about coherent $\mathcal{O}_X$-modules when $X$ is a noetherian scheme.

**Theorem 13.12.** *Let $X$ be a scheme. The category of quasi-coherent sheaves is abelian, and the forgetful functor to $\mathcal{O}_X$-modules is exact.*

*Proof.* This is problem 3 on exercise sheet 8. $\square$

**Definition 13.13.** *A morphism $f : Y \to X$ of schemes is of finite type if $f$ is quasi-compact, and there is an open cover of $Y$ by $\operatorname{Spec} B_i$, such that $f|_{\operatorname{Spec} B_i}$ factors over some $\operatorname{Spec} A_i \subseteq X$, and $B_i$ is a finitely generated $A_i$-algebra through the corresponding map of rings.*

**Proposition 13.14.** *If $f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ is a morphism of finite type, then $B$ is a finitely generated $A$-algebra.*

*Proof.* The localisations $A \to A[f^{-1}]$ are finitely generated, so we really have to prove that if $Y = \bigcup_i D(g_i)$ is a finite open cover of $B$ for $g_i \in B$, such that for each $i$ we have $B[g_i^{-1}]$ is a finitely generated $A$-algebra, then $B$ is a finitely generated $A$-algebra. Fix algebra generators $\frac{b_{ij}}{g_{ij}^{n_{ij}}} \in B[g_i^{-1}]$, of which there are finitely many for each $i$. We want to check now that the collection $\{g_i, b_{ij}\}$ generate $B$ as an $A$-algebra. Let $A[G_i, B_{ij}] \to B$ be the map sending $G_i \mapsto g_i$ and $B_{ij} \mapsto b_{ij}$, then for all $i$ we have that $A[G_i, B_{ij}][G_i^{-1}] \to B[G_i^{-1}] = B[g_i^{-1}]$ is surjective by assumption. It is enough now to see that for all $x \in \operatorname{Spec} A[G_i, B_{ij}]$ we have surjectivity on the localisations,

$$A[G_i, B_{ij}]_{\mathfrak{p}_x} \longrightarrow B_{\mathfrak{p}_x}.$$

If $x \in \bigcup_i D(G_i)$, then this claim is obvious by our assumptions above. If $x \notin \bigcup_i D(G_i)$ then $B_{\mathfrak{p}_x} = 0$, as $\operatorname{Spec} B = \bigcup D(g_i)$, and we have surjectivity for free. $\square$

There are some obvious connections between noetherian-ness and morphisms of finite type, and they both fall under our general theme today of finiteness. The next proposition solidifies this connection.

**Proposition 13.15.** *If $f : Y \to X$ is a morphism of finite type and $X$ is noetherian, then $Y$ is noetherian.*

*Proof.* We know $X$ is quasi-compact, and the fact that $f$ is of finite type means $f$ is quasi-compact, so $Y$ is quasi-compact. Given any $V = \operatorname{Spec} B \subseteq Y$ which maps to some $U = \operatorname{Spec} A \subseteq X$, then $B$ is a finitely generated $A$-algebra. Now $A$ is noetherian, so by Hilbert's basis theorem $B$ is also noetherian. $\square$

**Remark 13.16.** If $k$ is an algebraically closed field, then the classical notation of varieties over $k$ is essentially (up to being separated and irreducible as well) the same as a scheme of finite type over $\operatorname{Spec} k$ (we'll talk about this next lecture).

Next we are going to talk about dimension, and the classical commutative algebra notation of Krull dimension.

**Definition 13.17.** *Let $T$ be a locally spectral space, then the (Krull) dimension of $T$ is defined as the supremum minus one of the length of all chains of specialisations of points in $T$, i.e.*

$$\dim T = \sup_n \{x_0 \succ x_1 \succ x_2 \succ \cdots \succ x_n \mid x_i \in T, x_i \neq x_j, \forall i \neq j\}$$

Let's make some remarks about this definition. Firstly, we write $x \succ y$ to mean $x$ specialises to $y$, which means that for all open sets $U \ni y$ we have $x \in U$. We also say that $y$ generalises $x$ and we write this as $y \prec x$. Secondly, this definition can be made for any topological space $X$, but it really doesn't belong there. For example, we can easily check that any Hausdorff space is zero dimensional with this definitions, which means all manifolds are zero dimensional, which is obviously not what we want. This is a definition that belongs in algebraic geometry, since it comes from the Krull dimension of rings.

**Definition 13.18.** *If $X$ is a scheme, then* $\dim X = \dim |X|$.

**Example 13.19.** Let $k$ be an algebraically closed field, and let $X = \mathbb{A}^1_k$ then $|X|$ looks like $k$ but with a generic point. The generic point specialises to all closed points. This gives us specialisations of length 1, which implies $\dim X = 1$.

**Example 13.20.** If $k$ is still algebraically closed and $X = \mathbb{A}^2_k$ then the points of $X$ are closed points, irreducible curves, and the generic point. We then have a specialisation of length 2, so at least $\dim X \geq 2$. We'll make a re-interpretation of this dimension definition in the next lecture (see Remark 14.8), and see that $\dim X = 2$.

**Lemma 13.21.** *If $X = \bigcup_i U_i$ is an open cover of a scheme $X$, then* $\dim X = \sup_i \dim U_i$.

*Proof.* If we have a chain of specialisations in $X$, then this whole chain belongs in one fixed $U_i$ for some $U_i$, by the definition of a specialisation. $\square$

Next lecture we'll see the real theorem we want to see with this definition of dimension. We'll show that the dimension of $X$ can be measured using the Noether normalisation of a finitely generated $k$-algebra, for a field $k$.

## 14 Krull Dimension and (Pre)Varieties 06/12/2016

The theorem we want to prove first today is the following reality check about our current definition of dimension.

**Theorem 14.1.** *Let $X = \operatorname{Spec} A$, where $A$ is a finitely generated $k$-algebra for some field $k$, and given a map*

$$k[X_1, \ldots, X_n] \hookrightarrow A,$$

*from Noether normalisation, i.e. a finite injective map. Then $\dim X = n$.*

To prove this theorem, we're going to make a few definitions, and prove some lemmas. In the end, the proof of this theorem will be seemingly trivial commutative algebra.

**Definition 14.2.** *A map $\phi : A \to B$ is integral if for all $b \in B$ there exists an $m \in \mathbb{N}$ and $a_0, \ldots, a_{m-1} \in A$ such that $b^m + \phi(a_{m-1})b^{m-1} + \cdots + \phi(a_0) = 0$.*

**Lemma 14.3.** *Let $\phi : A \to B$ be an injective integral map of rings, then $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$.*

*Proof.* The proof is by the so called "Going-Up theorem" from commutative algebra.

**Theorem 14.4.** *[Going-Up] Let $\phi : A \to B$ be an integral map of rings.*

1. *If $\mathfrak{q} \subseteq B$ is a prime ideal, and $p = \phi^{-1}(\mathfrak{q})$, then given $\mathfrak{p} \subseteq \mathfrak{p}' \subseteq A$ there exists a prime ideal $\mathfrak{q}' \supseteq \mathfrak{q}$ of $B$ such that $\mathfrak{p}' = \phi^{-1}(\mathfrak{q}')$.*
2. *If $\mathfrak{q}_1, \mathfrak{q}_2$ are two prime ideals of $B$, and $\phi^{-1}(\mathfrak{q}_1) = \phi^{-1}(\mathfrak{q}_2)$, then $\mathfrak{q}_1 \not\subsetneq \mathfrak{q}_2$ and $\mathfrak{q}_2 \not\subsetneq \mathfrak{q}_1$.*
3. *If $\phi$ is injective, then the induced map of spectra is surjective.*

*Proof.* A proof can be found in Atiyah-Macdonald [1][Theorem 5.11 p.62]. ☐

We note that part 3 follows from: firstly we know that the induced map is dominant (all minimal prime ideals of $A$ are contractions from $B$), and part 1 tells us that all prime ideals are contractions from $B$.

Back to the proof, we are first going to show that $\dim \operatorname{Spec} B \leq \dim \operatorname{Spec} A$. If $\mathfrak{q}_0 \subsetneqq \mathfrak{q}_1 \subsetneqq \cdots \subsetneqq \mathfrak{q}_n$ is any chain in $\operatorname{Spec} B$, then we simply let $\mathfrak{p}_i = \phi^{-1}\mathfrak{q}_i$, and we find a chain in $\operatorname{Spec} A$,

$$\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n.$$

This containment is in fact strict, since we have part 2 of Theorem 14.4 above, so $\dim \operatorname{Spec} B \leq \dim \operatorname{Spec} A$. To show the converse direction, let $\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n$ be a chain in $\operatorname{Spec} A$, then part 3 of Theorem 14.4 tells us that there exists a $\mathfrak{q}_0 \in B$ with $\phi^{-1}(\mathfrak{q}_0) = \mathfrak{p}_0$. Part 1 of Theorem 14.4 gives us a whole chain $\mathfrak{q}_0 \subsetneqq \mathfrak{q}_1 \subsetneqq \cdots \subsetneqq \mathfrak{q}_n$ in $\operatorname{Spec} B$ with $\phi^{-1}(\mathfrak{q}_i) = \mathfrak{p}_i$, and part 2 again gives us strict containments. ☐

*Proof of Theorem 14.1.* By the lemma we have just proved, we can assume that $A = k[X_1, \ldots, X_n]$. Applying the lemma one more time to the induced inclusion

$$k[X_1, \ldots, X_n] \hookrightarrow \overline{k}[X_1, \ldots, X_n],$$

into the algebraic closure of $k$ means we can assume that $k$ is also algebraically closed. It remains to show now that $\dim \mathbb{A}^n_k = n$, which we have always hoped to be true. This, like much of the lecture before now, will come as a consequence of the following definitions and theorems.

**Definition 14.5.** *A ring $A$ is called catenary if for all prime ideals $\mathfrak{p} \subseteq \mathfrak{q} \subseteq A$, then any maximal chain $\mathfrak{p} \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{q}$ has the same length.*

In practice, almost every noetherian ring is catenary. Counter-examples do exist, but it actually took mathematicians quite a while to find a counter-example.

**Theorem 14.6.** *If $A$ is finitely generated over a field or a Dedekind domain, then $A$ is catenary. All localisations of catenary rings are catenary.*

If $A$ is a catenary ring, then $\dim A$ is the maximum over all such $n = n(\mathfrak{p}, \mathfrak{q})$ where $\mathfrak{p}$ is a minimal prime ideal and $\mathfrak{q}$ is a maximal ideal. Applying this observation to $A = k[X_1, \dots, X_n]$ where $k$ is algebraically closed, then all the maximal ideals are of the form $(X_1 - a_1, \dots, X_n - a_n)$, $a_i \in k$ by Hilbert's Nullstellensatz. Up to a change of coordinates, we may assume that all $a_i = 0$, so we construct the chain,

$$(0) \subsetneqq (X_1) \subsetneqq \cdots, \subsetneqq (X_1, \dots, X_n).$$

To show this chain is maximal, assume we can add some prime ideal $\mathfrak{p} \subseteq A$ into the chain, so

$$(X_1, \dots, X_{i-1}) \subsetneqq \mathfrak{p} \subsetneqq (X_1, \dots, X_i).$$

We can replace $A$ by $A/(X_1, \dots, X_{i-1}) = k[X_i, \dots, X_n]$, so we may assume that $i = 1$ above. This implies that $\mathfrak{p} \subseteq k[X_1, \dots, X_n]$ is a prime ideal contained in $(X_1)$. If $f \in \mathfrak{p}$ then we can write $f = X_1^n g$, where $g \notin (X_1)$. Since $X_1 \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime, we see that $g \in \mathfrak{p} \subseteq (X_1)$, which implies $g = 0$, a contradiction. $\square$

We move on now to a more geometrically flavoured part of the material. Let $k$ be an algebraically closed field for the rest of the lecture.

**Definition 14.7.** *A prevariety over $k$ is a reduced (Definition 8.1), irreducible (Definition 3.7) scheme of finite type over $k$. A variety is a prevariety that is also separated*[22].

**Remark 14.8.** Let's make some preliminary remarks about these definitions.

1. This makes it sound like varieties are some very peculiar case of schemes, whose precise definitions is artificial. This is true in a way, and we won't focus on varieties anyway.

2. The category of irreducible affine algebraic sets over $k$ embeds fully faithfully into the category of (pre)varieties over $k$, by sending

$$V(I) \subseteq \mathbb{A}_k^n \mapsto \operatorname{Spec}(k[X_1, \dots, X_n]/I),$$

   where $I \subseteq k[X_1, \dots, X_n]$ is a radical ideal.

3. Let $X$ be a prevariety, then the dimension of $X$ can be redefined as the maximum length of chains of irreducible closed subsets $\{x\} \subsetneqq Z_1 \subsetneqq \cdots, \subsetneqq Z_n = X$. We can reduce this to the case then $X = \operatorname{Spec} A$, then this follows from the fact that points of $\operatorname{Spec} A$ are prime ideals of $A$, which are irreducible closed subsets of $\operatorname{Spec} A$, which we proved in Proposition 3.10.

**Definition 14.9.** *A scheme $X$ is integral, if $\mathcal{O}_X(U)$ is an integral domain for all $U \neq \varnothing$ open in $X$.*

This is really just a new word for when a scheme is reduced and irreducible.

**Proposition 14.10.** *A scheme $X$ is reduced and irreducible if and only if $X$ is integral.*

**Remark 14.11.** This is *not* a local condition. Let $A = k \times k$, then we see that $A$ is not an integral domain, but $\operatorname{Spec} A = \operatorname{Spec} k \sqcup \operatorname{Spec} k$, and $k$ is an integral domain.

[22] We'll see the definition of separated next lecture in Definition 18.2.

*Proof.* Assume first that for all nonempty $U = \operatorname{Spec} A \subseteq X$ we have $A$ is an integral domain. Then all such $A$ are reduced, so $A$ is a reduced domain. If $X$ was not irreducible, then there would exist $U, V \subseteq X$, a pair of non-empty open subsets with $U \cap V = \varnothing$. Without loss of generality we can take $U$ and $V$ to be affine, but then $U \cup V = U \sqcup V$ is affine, with $\mathcal{O}_X(U \sqcup V) = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$. This has zero divisors, which is a contradiction. Conversely, assume that $X$ is reduced and irreducible, and consider $\operatorname{Spec} A \subseteq X$ a non-empty open subset. These hypotheses tell us that $\operatorname{Spec} A$ is reduced and irreducible. Without loss of generality (again), we may take $X = \operatorname{Spec} A$, so then $X$ being reduced implies that $A$ is a reduced ring. Now consider $f, g \in A$ with $fg = 0$, then $V(f) \cap V(g) = V(fg) = \operatorname{Spec} A$, so $D(f) \cap D(g) = \varnothing$. The fact that $X$ is irreducible implies that $D(f) = \varnothing$ or $D(g) = \varnothing$. Hence either $f$ or $g$ are nilpotent, but $A$ is reduced, so $f = 0$ or $g = 0$. $\square$

What are the advantages of (pre)varieties vs. schemes?

**Remark 14.12** (Advantages of (Pre)Varieties)**.** Given a prevariety $X$ over $k$ (recall $k$ is algebraically closed), then $X(k) \subseteq X$ is the subset of closed points, and the functor from prevarieties over $k$ to Set sending $X \mapsto X(k)$ is faithful. For affine prevarieties, we saw this in lecture 1, and this can fail for schemes. For examples, let $Y = \operatorname{Spec} k[t])$ and $X = \operatorname{Spec} k[\epsilon]/\epsilon^2$, then we have two morphisms $X \to Y$ given by $t \mapsto \epsilon$ and $t \mapsto 0$ respectively. On $k$-valued points, $X(k) = * \to Y(k) = k$ both maps send $*$ to $0 \in k$. The first morphism somehow recalls the tangent direction of the points $*$, where as the second map does not.

**Remark 14.13** (Advantages of Schemes)**.** Schemes have some better categorical properties though. For example fibre products do not exist for prevarieties in general. Schemes are better than (pre)varieties at tracking interesting information related to degenerate behaviour, such as the case when schemes are not reduced. We will often come across properties of schemes that we can not talk about with varieties, such as the valuation theorems of Lectures 18 and 19.

**Example 14.14.** Consider $\mathbb{A}^1_k \cong \operatorname{Spec} k[x,t]/(x^2 - t) \to \operatorname{Spec} k[t] \cong \mathbb{A}^1_k$ where $k$ is an algebraically closed field with characteristic not equal to 2. The fibre over $t \mapsto a \neq 0 \in k$ is $\operatorname{Spec} k[x]/(x^2 - a) = \operatorname{Spec} k \sqcup \operatorname{Spec} k$, which represent two points $x = \pm\sqrt{a}$. The fibre over $t \mapsto 0 \in k$ is $\operatorname{Spec} k[x]/(x^2)$, which is a single point (recall a single point *and* a tangent direction), but notice that $\dim_k k[x]/x^2 = 2$. The fibre over $t = 0$ still remembers that generically there are two points in the fibre, it remembers multiplicity.

This multiplicity statement is given in very nice terms with the following classical theorem.

**Theorem 14.15** (Bézout's Theorem)**.** *Let $f, g \in k[x,y,z]$ be homogenous polynomials of degree $d$ and degree $e$ with no common irreducible factors, then the intersection multiplicity of $f$ and $g$ is $de$.*

In scheme theoretic terms we can say that $f \in \mathcal{O}_{\mathbb{P}^2_k}(d)(\mathbb{P}^2_k)$ and $g \in \mathcal{O}_{\mathbb{P}^2_k}(e)(\mathbb{P}^2_k)$, and we have closed subschemes $V(f)$ and $V(g) \subseteq \mathbb{P}^2_k$. Here we notice that $V(f)(k) = \{[x,y,z] \in \mathbb{P}^2_k | f(x,y,z) = 0\}$. Then $V(f), V(g) \in \mathbb{P}^2_k$ are both one dimensional, and $V(f) \cap V(g)$ is zero dimensional (in general).

**Lemma 14.16.** *If a scheme $X$ of finite type over $k$ is 0-dimensional, then $X = \operatorname{Spec} A$ where $A$ is a finite-dimensional $k$-algebra.*

We will see a proof of this lemma in lecture 20! Now let $V(f) \cap V(g) = \operatorname{Spec} A$. The scheme theoretic version of Bézout's theorem then says that $\dim_k A = de$.

**Example 14.17.** One should always consider some low dimensional examples here. For example two lines will usually intersect each other only once (or once at $\infty$ if they are parallel), and a line and a conic will have 2 generic intersections, up to multiplicity.

## 15 Separated Schemes and Locally Closed Immersions 08/12/2016

This lecture was given by our tutor Isabell Große-Brauckmann. Let $k$ be a field.

**Definition 15.1.** *A scheme $X$ over a field $k$ is called projective if it is isomorphic to a closed subscheme of $\mathbb{P}^n_k$ for some $n \geq 0$.*

Notice that a projective scheme is always of finite type over $k$. Let $X \subseteq \mathbb{P}^n_k = \bigcup_{i=0}^n U_i$, with the standard open affines $U_i$, then $X$ is covered by $X \cap U_i$, which are each closed in $U_i$. Since $U_i$ is affine, then $X \cap U_i = V(\mathfrak{a})$ where $\mathfrak{a} \subseteq \mathcal{O}_{U_i}(U_i)$, and this is of finite type. We proved in exercise sheet 7 problem 4(ii) that all closed subschemes of $\mathbb{P}^n_R$ are generated by homogeneous equations where $R$ is noetherian. Let's review a shorter version of that proof when $R = k$ is a field. Recall first the global sections of Serre's twisted sheaves,

$$\mathcal{O}(d)(\mathbb{P}^n_R) = k[X_0, \dots, X_n]_d := \{f \in k[X_0, \dots, X_n] \mid f \text{ homogeneous of degree } d\}.$$

**Proposition 15.2.** *Let $X \subseteq \mathbb{P}^n_k$ be a closed subscheme, then $X \cong V(f_1, \dots, f_n) \subseteq \mathbb{P}^n_k$ for some homogenous polynomials $f_1, \dots, f_m \in k[X_0, \dots, X_n]$, where $V(f_1, \dots, f_m)$. Given a scheme $S$, a vector bundle $\mathcal{E}$ and a global section $s \in \Gamma(S, \mathcal{E})$ we define $V(s)$ to be the closed subscheme of $S$ associated to the ideal sheaf $\mathcal{I}$*[23]*, which is the image of $s^\vee$, where $s^\vee : \mathcal{E}^\vee \to \mathcal{O}_X^\vee \cong \mathcal{O}_X$ is the dual map to our section $s : \mathcal{O}_X \to \mathcal{E}$.*

*Proof.* Let $i : X \to \mathbb{P}^n_X$ be the closed immersion and set $\mathcal{I} = \ker(i^\flat : \mathcal{O}_{\mathbb{P}^n_k} \to i_*\mathcal{O}_X)$, which is a coherant sheaf from Propsoition 11.15. Now problem 4 on exercise sheet 7[24] (sometimes called a theorem of Serre), says there exists $d \in \mathbb{Z}$ such that

$$\mathcal{I}(d) := \mathcal{I} \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathcal{O}_{\mathbb{P}^n_k}(d),$$

is generated by finitely many global sections. Hence we obtain a surjection,

$$\bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^n_k} \xrightarrow{\phi} \mathcal{I}(d) \subseteq \mathcal{O}(d).$$

Let $f_i = \phi(e_i)$, then $\mathcal{I}(d)$ is actually just the image of $f_1, \dots, f_m$ in $\mathcal{O}(d)$. We claim that $X = V(s)$ where $s = (f_1, \dots, f_m)$. It is enough to show that $X \cap U_i = V(s) \cap U_i$ for all $i = 0, \dots, n$. Without loss of generality, let $i = 0$, such that we have $U_0 = \operatorname{Spec} k[X_1, \dots, X_n]$, then the restriction map on $\mathcal{O}(d)$ simply sends $f_j$ to $f_j(1, X_1, \dots, X_n)$. Hence,

$$V(f_1, \dots, f_m) \cap U_i = V(f_1(1, X_1, \dots, X_n), \dots, f_m(1, X_1, \dots, X_m)) \subseteq U_i.$$

We also know that $\mathcal{I}(U_i)$ is generated by $f_j(1, X_1, \dots, X_n)$ for $j = 1, \dots, m$, which implies,

$$X \cap U_i = \operatorname{Spec}\left(\mathcal{O}_X(U_i) \big/ \mathcal{I}(U_i)\right) = \operatorname{Spec}\left(k[X_1, \dots, X_n] \big/ (f_1(1, x_1, \dots, x_n), \dots, f_m(1, x_1, \dots, x_n))\right).$$

This is simply $V(f_1, \dots, f_m) \cap U_i$. □

**Example 15.3.** Consider the following affine curve,

$$X_0 = \operatorname{Spec}\left(k[x, y] \big/ (y^2 - x^3 - ax - b)\right) \subseteq \mathbb{A}^2_k.$$

[23] Closed subschemes $Z$ of a scheme $X$ are in bijective correspondence with quasi-coherent ideal subsheaves of $\mathcal{O}_X$. We send the closed subscheme $i : Z \to X$ to the kernel of $i^\flat : \mathcal{O}_X \to i_*\mathcal{O}_Z$, and the ideal subsheaf $\mathcal{I}$ to the support of $\mathcal{O}_X/\mathcal{I}$. See Hartshorne [3, p.116, Proposition II.5.9] for a proof.

[24] Given a ring $R$ and $X = \mathbb{P}^n_R$, and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module of finite type. Show that there exists an $d \in \mathbb{Z}$ such that for every $m \geq d$ the $\mathcal{O}_X$-module $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)$ is globally generated.

This can be "compactified" to

$$X = V\left(Y^2 Z - X^3 - aXZ^2 - bZ^3\right) \subseteq \mathbb{P}^2_k.$$

Notice that $X \cap U_2 = X_0$, so we can recover our uncompactified guy. It is know that we cannot embed a normal compactified hyperelliptic curve $y^2 = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ for $n \geq 4$ inside $\mathbb{P}^2_k$, we need at least $\mathbb{P}^3_k$.

What do we actually mean by "compactification" here? Because our schemes have a finite cover of affines, they are quasi-compact as spaces, but we never want to impose Hausdorffness on our schemes because this isn't natural[25]. We want something like: a scheme $X$ of finite type over $\mathbb{C}$ is compact (or maybe we need a new word like "proper") if $X(\mathbb{C})$ is compact Hausdorff (with the analytic topology). For example $\mathbb{A}^n_{\mathbb{C}}$ is not compact as $\mathbb{A}^n_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}^n$, but $\mathbb{P}^n_{\mathbb{C}}$ should be compact as $\mathbb{P}^n_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}P^n$, which is compact. We are going to start with a weaker notion today, and work up from here.

**Definition 15.4** ((Pre)Definition)**.** *A scheme $X$ over $\mathbb{C}$ is called separated if $X(\mathbb{C})$ is Hausdorff.*

An example of a non-separated scheme would be the classical affine line with two origins. To generalise this notation, recall what it means for a topological space to be Hausdorff.

**Proposition 15.5.** *A space $T$ is Hausdorff if and only if the diagonal $\Delta \subseteq T \times T$ is closed.*

*Proof.* Given $t \neq t' \in T$, a neightbourhood of $(t, t')$ in $T \times T$ is given by $U \times V$ with $t \in U$ and $t' \in V$. In this case $(U \times V) \cap \Delta = U \cap V \subseteq T$. We can find an open neighbourhood of $(t, t')$ in $T \times T$ if and only if $t, t'$ admit open neightbourhoods $U$ and $V$ respectively with $U \cap V \neq 0$. ☐

We now have an idea for a better definition of a separated scheme.

**Definition 15.6.** *A morphism of schemes $f : X \to S$ is called separated if the diagonal $\Delta_{X/S} : X \to X \times_S X; x \mapsto (x, x)$ is a closed immersion. A scheme $X$ is called separated if $X \to \operatorname{Spec} \mathbb{Z}$ is separated.*

Recall that we have previously seen that $|X \times_S X| \neq |X| \times_{|S|} |X|$, so a separated scheme is not necessarily a Hausdorff space. This is nice, because if this were true, it would be a super restrictive criteria. Recall that a closed immersion $i : X \to Y$ is a closed embedding of topological spaces such that $i^\flat : \mathcal{O}_Y \to i_*\mathcal{O}_X$ is surjective. When we check if the diagonal map is a closed immersion, the fact that $\Delta^\flat$ is an epimorphism is automatic, so we need only check the topological condtion.

**Definition 15.7.** *A morphism $i : X \to Y$ is called a locally closed immersion if*

1. *$|i| : |X| \to |Y|$ is a locally closed immersion, and $|X|$ is open in its closure.*
2. *$i^\# : i^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is surjective.*

This is only a slight change, but notice that $i^\#$ is a morphism of sheaves over the 'smaller' space $X$, so we expect this condition to be weaker to that of a closed immersion.

**Proposition 15.8.** *Let $i : X \to Y$ be a morphism of schemes. Then $i$ is a locally closed immersion if and only if $i$ can be written as a closed immersion followed by an open immersion.*

*Proof.* Both open and closed immersions are locally closed immerions, and composites of locally closed immersions are locally closed immersions, so we have one direction. Conversely, take $i$ to be a locally closed immersion, then $|X| \subseteq |Y|$ is a locally closed map of spaces, as there exists an open set $|V| \subseteq |Y|$ such that $|X| \subseteq |V|$ is closed. Then there exists a unique open subscheme $V \subseteq Y$ with underlying space $|V|$, and $X \to V$ is a closed immersion. ☐

**Proposition 15.9.** *Let $f : X \to S$ be an morphism of schemes, then the diagonal map $\Delta_{X/S} : X \to X \times_S X$ is a locally closed immersion. If $X$ and $S$ are affine, then this is in fact just a closed immersion.*

[25] A spectral space which is Hausdorff is automatically profinite and in particular 0-dimensional.

*Proof.* First assume that $X$ and $S$ are affine, so let $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$, then we have,

$$X \times_S X \cong \operatorname{Spec}(A \otimes_R A),$$

and the map $\Delta_{X_S} : X \to X \times_S X$ corresponds to the multiplication map $A \otimes_R A \to A$, which is surjective. Hence $\Delta_{X/S}$ is a closed immersion in this case. In general, for any $x \in X$, we choose some open affine $U$ with $x \in U = \operatorname{Spec} A \subseteq X$ mapping to $\operatorname{Spec} R \subseteq S$. Then again we have,

$$W = \operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} A \cong \operatorname{Spec}(A \otimes_R A),$$

which is an open neighbourhood of $\Delta(U)$. Also, $\Delta(X) \cap W = \Delta(X) \cap (\operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} A) = \operatorname{Spec} A$ is closed in $W$ by the affine case, hence $\Delta_{X/S}$ is a locally closed immersion. □

Note that the proof does not show that $\Delta$ is closed. Indeed, the open sets $U \times_S U \subseteq X \times_S X$ for $U \subseteq X$ affine open need not even cover $X \times_S X$. There are some immediate corollaries of this proposition.

**Corollary 15.10.** If $X$ is an affine scheme over an affine scheme $S$, $X$ is separated inside of $\operatorname{Sch}/S$.

The following corollary is what we have been wanted to prove this whole time.

**Corollary 15.11.** The scheme over $S$, $f : X \to S$ is separated if and only if $|\Delta_{X/S}|(|X|) \subseteq |X \times_S X|$ is a closed subspace.

*Proof.* The only condition which differs a locally closed immersion and a closed immersion is that closed immersions also have $|\Delta_{X/S}|(|X|) \subseteq |X \times_S X|$ is a closed subspace. Indeed, if $|\Delta_{X/S}|(|X|) \subseteq |X \times_S X|$ is closed, then we want the following map to be a surjection for all $y \in X \times_S X$,

$$\Delta_y^\flat : \mathcal{O}_{X \times_S X, y} \longrightarrow (i_* \mathcal{O}_X)_y = \left\{ \begin{array}{rl} 0 & y \notin |\Delta|(|X|) \\ \mathcal{O}_{X,y} & y \in |\Delta|(|X|) \end{array} \right. .$$

However, we always have $\Delta^* : \mathcal{O}_{X \times_S X, y} \to \mathcal{O}_{X,x}$ is surjective, since we have the following factorisation,

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r, "\mathrm{pr}_i^*"] \arrow[dr, "\mathrm{id}_X^*"'] & \mathcal{O}_{X \times_S X, x} \arrow[d, "\Delta_{X/S}"] \\ & \mathcal{O}_{X,x} \end{tikzcd} .$$

□

We now have a quick proposition about some permanence properties of separated morphisms.

**Proposition 15.12.** *Consider the following commutative diagram of schemes.*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

1. *If $f$ and $g$ are separated, then so is $h$.*
2. *If $h$ is separated, then so is $f$.*

*Proof.* For part 1, factor the diagonal map $\Delta_{X/Z}$ using other diagonal maps as follows,

$$\begin{tikzcd} X \arrow[r, "\Delta_{X/Y}"] \arrow[dr, "\Delta_{X/Z}"'] & X \times_Y X \cong (X \times_Z X) \times_{Y \times_Z Y} Y \arrow[d, "(X \times_Z X) \times_{Y \times_Z Y} \Delta_{Y/Z}"] \\ & X \times_Z X \cong (X \times_Z X) \times_{Y \times_Z Y} Y \times_Z Y \end{tikzcd} .$$

Here $\Delta_{X/Y}$ is a closed immersion, and the second map is simply the base change of a closed immersion, hence a closed immersion. So $\Delta_{X/Z}$ is a closed immersion. Part 2 comes from the following diagram.

$$\begin{tikzcd}
 & {|X \times_Y X|} \arrow[dd, "i_3"] \\
{|X|} \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\
 & {|X \times_Z X|}
\end{tikzcd}$$

Now if $i_2(|X|)$ is closed, then $i_1(|X|) = i_3^{-1}(i_2(|X|))$ is closed, and we're done. $\square$

# 16 Proper Maps of Schemes 13/12/2016

Recall Defintion 15.1, which can be equivalently stated as: a scheme $X$ is projective over $k$ if there is a closed immersion $i : X \to \mathbb{P}^n_k$ for some $n \geq 0$. The Segre embedding that we explored in our problem 2 on exercise sheet 6[26] shows that the product of two projective spaces is projective. We now see a short corollary of this.

**Corollary 16.1.** If $X$ and $Y$ are two projective schemes of $k$, then $X \times_k Y$ is projective.

*Proof.* We can choose closed immersions $i_X : X \to \mathbb{P}^n_k$ and $i_y : Y \to \mathbb{P}^m_k$ for some $n, m \geq 0$, and then the following composite is a closed immersion by Proposition 16.3,

$$X \times_k Y \xrightarrow{i_X \times i_Y} \mathbb{P}^n_k \times_k \mathbb{P}^m_k \xrightarrow{\text{Segre}} \mathbb{P}^{nm+n+m}_k \ .$$

□

**Definition 16.2.** *Let $P$ be a property of maps of schemes (eg. of finite type, open/closed/locally closed immersions, separated, ...), then we say $P$ is*

*COMP compatible with composition if the composite of two maps in $P$ is in $P$.*

*BC compatible with base change if for all $f : X \to S$ and $g : S' \to S$, then if $f$ has $P$ we have $f \times_S S' : X \times_S S' \to S'$ is in $P$.*

*PROD compatible with products if given $f : X \to Y$ and $f' : X' \to Y'$ in $P$, we have $f \times f' : X \times X' \to Y \times Y'$ is in $P$, where the fibre product is taken over* $\operatorname{Spec} \mathbb{Z}$.[27]

*LOCT local on target if for all $f : X \to S$ if there is an open cover $U_i$ of $S$ such that $f \times_S U_i : X \times_S U_i \to U_i$ is in $P$, then $f$ is in $P$.*

*LOCS local on source if for all $f : X \to S$, if there is an open cover $V_i$ of $X$ such that $f|_{V_i}$ is in $P$ for all $i$, then $f$ is in $P$.*

We will definitely not see the proof of the next proposition, but there is a proposition in Görtz and Wedhorn [2, p.573-7] which gives proofs or at least directions of proofs. They are all relatively straightforward.

**Proposition 16.3.** *Let $P$ be the property of a closed immersion, open immersion, locally closed immersion, or finite type, then $P$ satisfies COMP, BC, PROD and LOCT.*

It was suggested in lectures that locally closed immersions also satisfies LOCS, but a clear counterexample of this fact is the map $f : \operatorname{Spec} k \sqcup \operatorname{Spec} k \to \operatorname{Spec} k$ for some field $k$.

*Example Proof.* Consider whether or not closed immersions are closed under base change. Let the following be a pullback square of schemes,

$$\begin{array}{ccc} X' = X \times_Y Y' & \xrightarrow{i'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{i} & Y \end{array} \ .$$

Assuming that $i$ is a closed immersion, we have to show now that $i'$ is a closed immersion. By the definition of a closed immersion, we notice that $P$ in this case is LOCT, so we can work locally on

[26]Here we constructed a morphism $\mathbb{P}^m_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^n \to \mathbb{P}^{nm+n+m}_{\mathbb{Z}}$ (which turns out to be a closed immersion) using the fact that we understand scheme valued points of projective space.

[27]PROD and BC together imply that product over arbitrary schemes $S$ are in $P$.

$Y'$ and $Y$, so let $Y' = \operatorname{Spec} B'$ and $Y = \operatorname{Spec} B$. As $i$ is a closed immersion then $X = \operatorname{Spec} A$ where $\widetilde{i} : B \to A$ induced by $i$ is a surjective ring homomorphism. Then we have $X' \cong \operatorname{Spec}(B' \otimes_B A)$, and a surjective map $B' \to B' \otimes_B A$, since base change of rings preserves surjections. Hence $i'$ is a closed immersion. □

Recall Definition 15.6 about separated morphisms.

**Example 16.4.** If a scheme $X$ over $\mathbb{C}$ is separated, then $X(\mathbb{C})$ with the complex topology is Hausdorff. An example of a non-separated scheme is the affine line with doubled origin.

This now gives us all the adjectives we need to define varieties in scheme theoretic language.

**Definition 16.5.** *A variety over an algebraic closed field $k$ is an integral separated scheme of finite type over $k$, i.e. a separated prevariety over $k$.*

Historically, when Grothendieck first defined schemes, he called schemes preschemes, and separated preschemes were schemes. To him, this separatedness was so important is was built into the definition of a scheme.

**Proposition 16.6.** *If $X$ is separated and $U, V \subseteq X$ are open affine subsets, then $U \cap V$ is open affine.*

*Proof.* We can re-write $U \cap V$ as,

$$U \cap V = (U \times V) \times_{X \times X} X \subseteq U \times V.$$

Since $\Delta_X$ is closed, and using the base change property of closed immersions, we can conclude that $U \cap V$ is closed inside the affine schemes $U \times V$, hence it is also affine. □

**Remark 16.7.** This fails for non-separated schemes. For example we have the affine place with doubled origin, then $\mathbb{A}^2_k - \{0\}$ is the intersection of two copies of $\mathbb{A}^2_k$, yet it is not affine (see Example 7.9).

The following displays why the adjective separated is a good word choice.

**Corollary 16.8.** If $X$ is separated, then $|X|$ is quasi-separated.

*Proof.* Given $U, V \subseteq X$ which are quasi-compact opens, we want to see that $U \cap V$ is a quasi-compact open. If cover $U$ and $V$ with finite open affine covers $U_i$ and $V_i$, we then notice that $U \cap V$ is a finite union of $U_i \cap V_j$, which are open affines by Proposition 16.6, hence $U \cap V$ is quasi-compact. □

Notice how the above fails to be an equivalence in logic, we only have an implication. Remark 16.7 explains this.

**Proposition 16.9.** *Any affine scheme is separated.*

*Proof.* If $X = \operatorname{Spec} A$ then the diagonal map is induced by the surjective map $A \otimes_{\mathbb{Z}} A \to A$. □

**Proposition 16.10.** *Let $f : X \to S$ be a morphism of schemes, where $S$ is separated, then $f$ is separated if and only if $X$ is separated (over $\operatorname{Spec} \mathbb{Z}$).*

*Proof.* This is just Proposition 15.12 from last time. □

**Proposition 16.11.** *When $P$ is the class of separated morphisms, we have COMP, BC, PROD and LOCT.*

Separatedness is not LOCS, since all affine schemes are separated.

*Proof.* This reduces because diagonal maps behave well with respect to pullbacks, to similar properties for closed immersions by definition. □

**Definition 16.12.** *A map $f : X \to S$ of topological spaces is universally closed if for each map $S' \to S$ of topological spaces the map $X \times_S S' \to S'$ is a closed map (the image of closed sets are closed). In otherwords, a map $f : X \to S$ is universally closed if it is closed under arbitrary base change. A map of schemes $f : X \to S$ is universally closed if for all maps $S' \to S$ the map $X \times_S S' \to S'$ is closed.*

Notice in the scheme case we are *not* asking $|X| \times_{|S|} |S'| \to |S'|$ to be closed. Some motivation for this definition is the following statement in point-set topology: Given a compact Hausdorff space $X$, and $T$ any topological space, then the projection map $X \times T \to T$ is always a closed map.

**Definition 16.13.** *A morphism $f : X \to S$ is proper (eigentlich in Deutsch) if it is separated, of finite type and universally closed. A scheme over $S$ is proper if the structure map $X \to S$ is proper.*

**Example 16.14.** Some intuition for this definition is the following: a scheme $X$ over $\mathbb{C}$ is proper if and only if $X(\mathbb{C})$ is compact Hausdorff. For example notice that $\mathbb{A}^n_{\mathbb{C}}$ is not proper, as $\mathbb{A}^n_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}^n$ which is not compact. On the other hand, $\mathbb{P}^n_{\mathbb{C}}$ is proper, as $\mathbb{P}^n_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{CP}^n$, which is compact Hausdorff.

A big theorem we will prove using the valuation criterion of properness in lecture 19 is the following.

**Theorem 16.15.** *The map $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is proper.*

This amounts to saying that if $S$ is any scheme, and $Z \subseteq |\mathbb{P}^n_{\mathbb{Z}} \times S|$ is a Zariski closed subset, then the image of $|Z|$ in $S$ is Zariski closed. Before we prove this, let's see some corollaries.

**Corollary 16.16.** If $k$ is a field and $X$ is projective over $k$, then $X$ is proper over $k$.

*Proof.* Choose a closed immersion $i : X \to \mathbb{P}^n_k$. We now use the easy statement that the class of proper maps satisfies COMP, BC, PROD and LOCT. This is obvious, since we only need to check these things for universally closed maps (finite type and separated are mentioned in Proposition 16.3). For example $f : X \to Y$ and $g : Y \to Z$ are proper maps, then $g \circ f : X \to Z$ is separated and of finite type, and for any $Z' \to Z$, we have $|X \times_Z Z'| \to |Z'|$ identifies with the composition,

$$|X \times_Z Z'| = |X \times_Y (Y \times_Z Z')| \xrightarrow{f \times (Y \times_Z Z')} |Y \times_Z Z'| \overset{g}{|} Z'| \ .$$

Since $f$ and $g$ are universally closed then we see that $g \circ f$ is also now universally closed, and hence also proper. Then Theorem 16.15 we see that $\mathbb{P}^n_k \to \operatorname{Spec} k$ is proper. Also all closed immersions are proper (see Proposition 16.17), thus $X \to \mathbb{P}^n_k \to \operatorname{Spec} k$ is proper by COMP and BC. □

**Proposition 16.17.** *Let $i : Z \to X$ be a closed immersion, then $i$ is proper.*

*Proof.* Being proper is LOCT so we can assume that $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A/I$ for some ideal $I \subseteq A$. In particular $Z$ is affine, so $i$ is separated, and $A/I$ is a finitely generated $A$-algebra, so $i$ is of finite type. Also, for any $X' \to X$, we have $i' : Z' = Z \times_X X' \to X'$ is a closed immersion, using that closed immersions are preserved by base change. Which implies that $|i'| : |Z'| \to |X'|$ is a closed immersion, which in particular is closed. □

# 17 Internal Hom Sheaves and Affine Morphisms 15/12/2016

**Proposition 17.1.** *Let $X$ be a topological space and $\mathcal{F}$ and $\mathcal{G}$ be sheaves on $X$, then the assignment*

$$U \mapsto \mathrm{Hom}_{Sh(U)}\left(\mathcal{F}|_U, \mathcal{G}|_U\right),$$

*is a presheaf, called $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.*

**Remark 17.2** (Caution)**.** The simply and perhaps more 'natural' assignment one might think of is simply

$$U \mapsto \mathrm{Hom}(\mathcal{F}(U), \mathcal{G}(U)),$$

but this is not even a presheaf. We have no way to define restriction maps.

*Proof.* If $U$ is covered by opens $U_i$ and we have morphisms $\phi_i : \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ which agree on overlaps, then we need to find a unique map $\phi : \mathcal{F}|_U \to \mathcal{G}|_U$ which restricts to these $\phi_i$. For all $V \subseteq U$, we can cover $V$ by $V \cap U_i$, and then $\mathcal{F}(V)$ and $\mathcal{G}(V)$ are simply the equalisers corresponding to this covering. We have maps $\phi_i(V \cap U_i)$ between both equaliser diagrams, so we obtain a unique induced map $\phi_V : \mathcal{F}(V) \to \mathcal{G}(V)$. These maps assemble to the desired maps of sheaves $\phi$. □

The above proposition specialises to the case of ringed spaces and $\mathcal{O}_X$-modules.

**Proposition 17.3.** *Let $(X, \mathcal{O}_X)$ is a ringed space, $\mathcal{F}, \mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules, then the assignment*

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X}\left(\mathcal{F}|_U, \mathcal{G}|_U\right),$$

*defines a sheaf of $\mathcal{O}_X$-modules denoted by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.*

*Proof.* The proof here is similar to the above proof, but just noticing at each step we have the additional structure of an $\mathcal{O}_X$-module is respected. □

**Example 17.4.** Let $X$ be a scheme and $\mathcal{F}, \mathcal{G}$ two quasi-coherent sheaves, then for all open affines $U = \operatorname{Spec} A \subseteq X$ we have

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \cong \mathrm{Hom}_A(\mathcal{F}(U), \mathcal{G}(U)),$$

despite the warning above. This happens precisely because quasi-coherent sheaves on affine schemes are just $A$-modules. We really need quasi-coherent sheaves and affine schemes for this to work!.

**Remark 17.5** (Caution)**.** If $X$ is a scheme, and $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent sheaves on $X$, then in general the internal hom sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is *not* quasi-coherent. We do have the following proposition though, which comes after a necessary definition.

**Definition 17.6.** *A quasi-coherent sheaf $\mathcal{F}$ is finitely presented if there is a cover of $X$ by affine opens $\{U_i = \operatorname{Spec} A_i\}$ such that $\mathcal{F}(U_i)$ is a finitely presented $A_i$-module.*

**Proposition 17.7.** *If $\mathcal{F}$ is a finitely presented quasi-coherent sheaf on $X$, then for all open affines $U = \operatorname{Spec} A \subseteq X$, $\mathcal{F}(U)$ is a finitely presented $A$-module.*

*Proof.* This is the same argument as in the finitely generated case. The key step is the following: If $X = \operatorname{Spec} A$, then we already know that $M = \mathcal{F}(X)$ is a finitely generated $A$-module, so by choosing a surjection $A^n \to M$ for some finite $n$ we obtain a surjection $\mathcal{O}_X^n \to \mathcal{F}$. If we let $\mathcal{G} = \ker(\mathcal{O}_X^n \to \mathcal{F})$, then $\mathcal{G}$ is still an $\mathcal{O}_X$-module of finite type, so $\mathcal{G}(X)$ is finitely generated and we have $M = A^n/\mathcal{G}(X)$ is finitely presented. □

**Proposition 17.8.** *If $X$ is a scheme, $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent sheaves with $\mathcal{F}$ being finitely presented, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent.*

Note that if $X$ is noetherian, then an $\mathcal{O}_X$-module of finite type is necessary a finitely presented $\mathcal{O}_X$-module, and vice-versa. Hence coherent sheaves pass this test.

*Proof.* To show this, we need to show that given $U = \operatorname{Spec} A \subseteq X$, and any $f \in A$, then the following map,

$$\mathcal{H}om_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U) \xrightarrow{\cong} \mathcal{H}om_{\mathcal{O}_X|_{D(f)}}(\mathcal{F}|_{D(f)}, \mathcal{G}|_{D(f)}),$$

is an isomorphism, where the left hand side is an $A$-module and the right is an $A[f^{-1}]$-module. This is a local problem, so for notation, let $X = U = \operatorname{Spec} A$, then we have $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$ for some $A$-modules $M, N$. Whence we have the following equivalent problem, of showing that

$$\operatorname{Hom}_A(M, N)[f^{-1}] \xrightarrow{\cong} \operatorname{Hom}_{A[f^{-1}]}(M[f^{-1}], N[f^{-1}]),$$

is an isomorphism. Since $\mathcal{F}$ is finitely presented, we choose an exact sequence $A^m \to A^n \to M$. Of course localising is exact, so we also have,

$$A^m[f^{-1}] \longrightarrow A^n[f^{-1}] \longrightarrow M[f^{-1}] \longrightarrow 0.$$

Taking $\operatorname{Hom}_A(-, N)$, and $\operatorname{Hom}_{A[f^{-1}]}(-, N[f^{-1}])$, and recalling these are left exact, we obtain three more exact sequences, the last one from the first by localisation.

$$0 \longrightarrow \operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_A(A^n, N) \cong N^n \longrightarrow \operatorname{Hom}_A(A^m, N) \cong N^m$$

$$0 \longrightarrow \operatorname{Hom}_{A[f^{-1}]}(M[f^{-1}], N[f^{-1}]) \longrightarrow \operatorname{Hom}_{A[f^{-1}]}(A^n[f^{-1}], N[f^{-1}]) \longrightarrow \operatorname{Hom}_{A[f^{-1}]}(A^m[f^{-1}], N[f^{-1}])$$

$$0 \longrightarrow \operatorname{Hom}_A(M, N)[f^{-1}] \longrightarrow \operatorname{Hom}_A(A^n, N)[f^{-1}] \longrightarrow \operatorname{Hom}_A(A^m, N)[f^{-1}]$$

We notice that the latter two of these exact sequences falls into the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(M, N)[f^{-1}] \arrow[r] \arrow[d] & N^n[f^{-1}] \arrow[r] \arrow[d] & N^m[f^{-1}] \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_{A[f^{-1}]}(M[f^{-1}], N[f^{-1}]) \arrow[r] & \operatorname{Hom}_{A[f^{-1}]}(A^n[f^{-1}], N[f^{-1}]) \arrow[r] & \operatorname{Hom}_{A[f^{-1}]}(M'[f^{-1}], N[f^{-1}])
\end{tikzcd}$$

Since the localisation commutes with direct sums the middle map is an isomorphism, and the right map is injective since $M'$ is finitely generated. This implies the left horizontal map is an isomorphism, and we're done. ☐

**Example 17.9.** If $\mathcal{L}$ is a line bundle on $X$, then $\mathcal{L}^\vee$ (the dual line bundle) which gives us $\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X$ is given by,

$$\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X),$$

which is quasi-coherent since $\mathcal{L}$ is clearly finitely presented. More generally, given any vector bundle $\mathcal{E}$ on $X$, then $\mathcal{E}$ is finitely presented and $\mathcal{E}^\vee$ is the sensible definition.

We now move on a little to a new section, which comes with some new definitions.

**Definition 17.10.** *A morphism $f : X \to S$ of schemes is affine if there exists a cover of $S$ by open affines $U_i$ such that $f^{-1}(U_i)$ is an affine scheme in $X$ for all $i$.*

First we have to prove the expected proposition which comes after such a definition.

**Proposition 17.11.** *If $f : X \to S$ is affine, then for all $U \subseteq S$ open affine, the inverse image $f^{-1}(U)$ is affine in $X$. Moreover, for a fixed $S$, the functor from the category of affine morphisms into $S$ to the opposite category of quasi-coherent $\mathcal{O}_S$-algebras, defined by $(f : Y \to S) \mapsto f_*\mathcal{O}_Y$ is an equivalence of categories.*

The inverse functor is denoted as $\mathcal{A} \mapsto \underline{\operatorname{Spec}}_{\mathcal{O}_S} \mathcal{A}$, the relative spectrum of $\mathcal{A}$. We proved in problem 1 on exercise sheet 7 that this scheme represents a functor (see Remark 17.14).

*Proof.* First we let $S = \operatorname{Spec} A$. We need to check that $f$ is both quasi-compact and quasi-separated. This is true since both properties are LOCT and affine schemes are quasi-compact and quasi-separated. This implies that $f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_S$-algebra, and hence in this case $f_*\mathcal{O}_X = \widetilde{B}$ for some $A$-algebra $B$. We claim now that $X = \operatorname{Spec} B$. Notice that,

$$B = \Gamma(S, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X),$$

so we have maps of schemes,

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr] & & \operatorname{Spec} B \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}.$$

To check that $\phi$ is an isomorphism, it suffices to work locally on $S$, but locally this is true by assumption. Now, if $S = \operatorname{Spec} A$ is affine, then affine maps $Y \to S$ are mapped via an equivalence of categories to the $A$-algebra $\Gamma(Y, \mathcal{O}_Y)$, then we arrive at the chain of equivalences,

$$\{\text{affine maps } f : X \to S\} \cong \{A\text{-algebras }\} \cong \{\text{quasi-coherent } \mathcal{O}_S\text{-algebras}\}.$$

From this we obtain $\widetilde{\Gamma(Y, \mathcal{O}_Y)} = f_*\mathcal{O}_Y$, which comes from the fact that $f_*\mathcal{O}_Y$ is quasi-coherent. Hence the composite functor is an equivalence of categories. In general, the equivalence from affine maps $Y \to S$ to quasi-coherent $\mathcal{O}_S$-algebras via $Y \mapsto f_*\mathcal{O}_Y$ comes by general gluing lemmas[28]. □

**Proposition 17.12.** *The class of affine morphisms satisfies COMP, BC, PROD and LOCT.*

*Proof.* All of these proofs are straight forward, see Grötz and Wedhorn [2][Proposition 12.3, p.321] for details. □

Under base change, if $\phi : S' \to S$ is a morphism of schemes, then an affine map $Y \to S$ is mapped to the affine map $Y \times_S S' \to S'$. The corrosponding map of quasi-coherent $\mathcal{O}_S$-algebras sends $\mathcal{A}$ to $\phi^*\mathcal{A}$.

**Proposition 17.13.** *Given an affine map $f : X \to S$, then $f$ is separated.*

*Proof.* Both properties are LOCT so we can assume $S$ is affine. In this case $X$ is also affine and maps of affine schemes are separated. □

**Remark 17.14.** If $S$ is a scheme and $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_S$-algebra, then $\underline{\operatorname{Spec}}_{\mathcal{O}_S}\mathcal{A}$ (the relative scheme to $\mathcal{A}$) represents the functor of scheme over $S$,

$$(f : Y \to S) \mapsto \operatorname{Hom}_{\mathcal{O}_S-\mathrm{alg}}(\mathcal{A}, f_*\mathcal{O}_Y).$$

We proved this is problem 1 on exercise sheet 7[29].

**Example 17.15.** If $\mathcal{E}$ is a vector bundle over $S$, then we have a total space vector bundle $\mathbb{V}(\mathcal{E})$, which is a scheme over $S$, whose structure map is affine since locally $\mathbb{V}(\mathcal{E})$ is just $\mathbb{A}^n \times S \to S$. In fact, we can explicitly write $\mathbb{V}(\mathcal{E})$ as,

$$\mathbb{V}(\mathcal{E}) \cong \underline{\operatorname{Spec}}_{\mathcal{O}_S}\operatorname{Sym}^\bullet\mathcal{E}^\vee.$$

The symmetric product of the free $A$-module $A^n$ is a free $A$-algebra over $M$, i.e. $\operatorname{Sym}^\bullet M \cong A[X_1, \dots, X_n]$. In general we have the universal property,

$$\operatorname{Hom}_{A\text{-mod}}(M, B) \cong \operatorname{Hom}_{A\text{-alg}}(\operatorname{Sym}^\bullet M, B),$$

[28] See [4][Tag 01JF, Lemma 25.15.4]

[29] Given a scheme $S$ and a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$, prove that the functor,

$$(f : Y \to S) \longmapsto \operatorname{Hom}_{\mathcal{O}_S-\mathrm{alg.}}(\mathcal{A}, f_*\mathcal{O}_Y),$$

on the category of schemes over $S$ is representable.

for all $A$-algebras $B$. So Sym is the left adjoint free functor to the forgetful functor from $A$-algebras to $A$-modules. The proof that $\mathbb{V}(\mathcal{E}) \cong \underline{\mathrm{Spec}}_{\mathcal{O}_S} \mathrm{Sym}^\bullet \mathcal{E}^\vee$ can be given by manipulating the two functors which either scheme represents, recalling that $\mathbb{V}(\mathcal{E})$ represents the functor $(f : Y \to S) \mapsto \Gamma(Y, f^*\mathcal{E})$.

$$\Gamma(Y, f^*\mathcal{E}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(f^*\mathcal{E}^\vee, \mathcal{O}_Y) \cong \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{E}^\vee, f_*\mathcal{O}_Y) \cong \mathrm{Hom}_{\mathcal{O}_S\text{-alg}}(\mathrm{Sym}^\bullet \mathcal{E}^\vee, f_*\mathcal{O}_Y)$$

This is going to seem a little off topic now, but we are going to slightly motivate the next lecture. In the next lecture we are going to talk about the valuation crierion for properness. Recall a proper map is separated, of finite type, and universally closed. For (non-)example, consider the following.

**Example 17.16** (Non-Example)**.** Let $\mathbb{A}^1_k \to \mathrm{Spec}\, k$ be the canonical map, where $k$ is a field. This map is not universally closed, and hence it is not proper. To see this, let $S' = \mathbb{A}^1_k$, then we have $X \times_S S' \cong \mathbb{A}^2_k \to \mathbb{A}^1_k = S'$. If we let $Z = V(xy - 1) \subseteq \mathbb{A}^2_k = \mathrm{Spec}\, k[x, y]$, then the image of $Z$ in $\mathbb{A}^1_k$ is not closed, it is exactly $\mathbb{A}^1_k \backslash \{0\}$. To show that $\mathbb{P}^n_{\mathbb{Z}} \to \mathrm{Spec}\, \mathbb{Z}$ is closed we need to show that if $\mathbb{A}^1 \backslash \{0\}$ is in the image, then so is $\{0\}$, or something along these lines.

For this, we need the generic points of schemes. This approach would never work and totally fails for varieties. In the case of schemes we can equivalently say something like, if the generic point of $\mathbb{A}^1_k$ is in the image, then so is zero. The advantage of this scheme theoretic case is that we only have to talk about 2 points on our schemes, as opposed to many more. In fact, we can boil this theorem to a statement about $(\mathbb{A}^1_k)_{(0)} = \mathrm{Spec}\, k[t]_{(t)}$, which is a discrete valuation ring.

## 18 Valuations and Valuation Rings 20/12/2016

We want to see the following theorem.

**Theorem 18.1.** *The canonical map $\mathbb{P}^n \to \mathbb{Z}$ is proper.*

To prove this map is of finite type is very doable, and even to prove this map is separated is doable by hand. To show this canonical map is universally closed is tough though, we we're going to use the following two theorems to help us.

**Theorem 18.2.** *[Valuation Criterion for Separatedness] A morphism $f : X \to S$ of schemes is separated if and only if $f$ is quasi-separated and for any valuation ring $V$ with fraction field $K$, then any diagram of the following form,*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} V \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

*has at most one lift (the dotted arrow) such that the above diagram commutes.*

**Theorem 18.3.** *[Valuation Criterion for Properness] A morphism $f : X \to S$ of schemes is proper if and only if $f$ is of finite type (which implies quasi-compact) and quasi-separated, and for any valuation ring $V$ with fraction field $K$, then any diagram of the following form,*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} V \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

*there exists exactly one lift such that the above diagram commutes.*

**Remark 18.4.** If $S$ is noetherian, and $f$ is of finite type (which is often the case in practical circumstances), then quasi-separatedness is automatic, and we only need to test special kinds of valuation rings (discrete valuation rings). If $V$ is a valuation ring, then $V$ is noetherian if and only if $V$ is as discrete valuation ring.

**Example 18.5** (Intuition)**.** Assume that $S = \operatorname{Spec} \mathbb{Z}$ for simplicity, then $X$ is separated if and only if $X$ is quasi-separated and for every valuation ring $V$ with fraction field $K$, the map $X(V) \to X(K)$ is injective. It is a little exercise to translate Theorem 18.3 into this language.

**Example 18.6** ((Non)-Example)**.** If $X$ is the affine line $\mathbb{A}^1_k$ with doubled origin over some field $k$, then we can consider $V = k[t]_{(t)}$ with fraction field $K = k(t)$, then we see that $\operatorname{Spec} V$ is simply a closed point and a generic point, and $\operatorname{Spec} K$ is simply a generic point. Now we can map our generic point of $\operatorname{Spec} K$ to the generic point of $X$, and then we have two different lifts of this, either sending the closed point of $\operatorname{Spec} V$ to either one of the 'origins'. This failiure is captured by different specialisations of points.

Now we haven't actually defined a discrete valuation ring, or a valuation ring, or even a valuation. We are going to spend the rest of this lecture discussing these things. Our basic intuition should be that fields are points on our schemes, and valuation rings are somehow chains of specialisations of points of our schemes (this analogy is not 100% clean).

**Remark 18.7** (Historic)**.** Valuation rings were prominant in algebraic geometry in the times of Zariski, Krull and Nagata. However Grothendieck didn't like them. In fact EGA does not mention valuation rings in an important way, and Grothendieck supposedly asking to remove the section in Bourbaki on valuations rings. Recently, valuation rings have become much more prominant, for example in "$p$-adic analytic geometry". Also Nagata's compactification theorem, which states that a separated morphism of finite type can be factored as a proper morphism and an open immersion, is proved using valuation rings.

**Definition 18.8.** *A totally order abelian group is an abelian group $\Gamma$ with total order $\leq$ on $\Gamma$ (so $x \leq y$ and $y \leq x$ implies that $x = y$), such that if $x \leq y$ and $x' \leq y'$ we have $x + x' \leq y + y'$.*

**Example 18.9.** Examples include $\mathbb{Z}$ and $\mathbb{R}$ with the usual ordering, and $\mathbb{R} \oplus \mathbb{R}$ with the lexicographical ordering[30], which we will write as $\mathbb{R} \oplus \mathbb{R}\epsilon$, to imply that $\epsilon$ is some infinitesimal, i.e. $\epsilon < r$ for all $r \in \mathbb{R}_{>0}$.

**Definition 18.10.** *Given a ring $R$, then a valuation on $R$ is a map $v : R \to \Gamma \cup \{\infty\}$ for some totally ordered abelian group $\Gamma$, such that*

1. $v(0) = \infty$ *and* $v(1) = 0$.
2. $v(xy) = v(x) + v(y)$, *with the convention that* $\gamma + \infty = \infty + \gamma = \infty + \infty = \infty$.
3. $v(x + y) \geq \min(v(x), v(y))$.

*If $\Gamma \cong \mathbb{Z}$, then $v$ is a discrete valuation.*

**Example 18.11.** Valuations with $\Gamma = \{0\}$ are in one-to-one correspondence with prime ideals in $R$. The bijection is given by $v \mapsto v^{-1}(\infty)$. It's a good exercise on the definitions to see that $v^{-1}(\infty) \subseteq R$ is always a prime ideal. In this sense valuations generalise prime ideals.

**Example 18.12.** If $R = \mathbb{Z}$, then we can take $v_p$ to be the $p$-adic valuation, for some prime $p$. This is defined as,

$$v_p(n) = r, \qquad \text{if } n = p^r m, \quad p \not| m,$$

where we set $v_p(0) = \infty$ (naturally).

**Example 18.13.** For $R = k[T]$ we can take $v = v_T$ to be the vanishing order of $T = 0$. If $f = \sum_{i=0}^{n} a_i T^i$, then $v_T(f) = \inf\{i \mid a_i \neq 0\} \cup \{\infty\}$.

**Example 18.14.** An example of a higher rank valuation (where the group $\Gamma$ does not embed inside $\mathbb{R}$), is given by $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\epsilon$ on the ring $R = k[X, Y]$, and $v$ is simply taken to be the lexicographical vanishing of a polynomial in $R$. So we look at the largest power of $X$ to occur, followed by the largest power of $Y$ if $X$ does not divide our polynomial, so

$$v\left(\sum a_{ij} X^i Y^j\right) = \inf\{i + \epsilon j | a_{ij} \neq 0\}.$$

**Definition 18.15.** *A valuation ring is an integral domain $V$ with fraction field $K$ such that for all $x \in K^\times$ one of either $x$ or $x^{-1}$ is in $V$.*

The above definition seems so unmotivated, since it a priori has absolutely nothing to do with valuations, but we have the next proposition to remedy this.

**Proposition 18.16.** *Let $V$ be a ring, then $V$ is a valuation ring if and only if there exists a valuation $v : V \to \Gamma \cup \{\infty\}$ such that for all $x, y \in V$, $v(x) \geq v(y)$ if and only if $y|x$ in $V$.*

In particular this last condition implies that all $x \in V$ with $v(x) = 0$ are units in $V$.

*Proof.* Assume there exists such a valuation $v$, then $V$ is an integral domain. This is because $v(x) = \infty$ implies $0|x$, i.e. $x = 0$. We then see $\{0\} = v^{-1}(\infty)$ is a prime ideal, hence $V$ is an integral domain. Moreover, let $K = \mathrm{Frac}(V)$ then there exists a unique valuation $\widetilde{v} : K \to \Gamma \cup \{\infty\}$ which extends $v$, which is given by $\widetilde{v}(x/y) = v(x) - v(y)$. We notice that $y \neq 0$ which implies that $v(y) \neq \infty$ so this is well-defined. We claim now that $V = \{x \in K \mid \widetilde{v}(x) \geq 0\}$. To prove this, suppse $x = y/z$ for $z \neq 0$, then $\widetilde{v}(x) \geq 0$ if and only if $v(y) \geq v(z)$ which occurs if and only if $z|y$ in $V$ by our hypothesis on $v$, so $x \in V$. Now for all $x \in K^\times$, we have $0 = \widetilde{v}(1) = \widetilde{v}(x) + \widetilde{v}(x^{-1})$ which implies that either $x$ or $x^{-1}$ are in $V$.

[30] Concretely this means $x + x' \leq y + y'$ if and only if $x \leq y$, or $x = y$ and $x' \leq y'$, i.e. the first summand is in control.

Conversely, assume that $V$ is a valuation ring. Let $\Gamma = K^\times/V^\times$, be our abelian group, and let $x'$ and $y'$ be the images of $x, y \in K^\times$. We define our total order on $\Gamma$ to be $x' \geq y'$ if and only if $x/y \in V$. One should prove this is independent of our choice of representatives $x, y$ for $x'$ and $y'$, but multiplication by a unit in $V$ doesn't change this condition. This defines $\Gamma$ as a totally ordered abelian group, because for all $x', y' \in \Gamma$ we have $x' \geq y'$ or $y' \geq x'$, as one of $x/y$ or $y/x$ lies in $V$. We can define a valuation $\widetilde{v} : K \to \Gamma \cup \{\infty\}$ by setting $x \mapsto \overline{x}$ and $0 \mapsto \infty$. Now we take $V = \{x \in K \mid \widetilde{v}(x) \geq 0\}$ and $v : V \to \Gamma \cup \{\infty\}$ to simply be the restriction of $\widetilde{v}$ to $V \subseteq K$. This satisfies all our desired properties. □

**Corollary 18.17** (Of the above Proof)**.** A ring $V$ is a valuation ring if and only if there exists a valuation $\widetilde{v} : K \to \Gamma \cup \{\infty\}$ for a field $K$ such that $V = \{x \in K | \widetilde{v}(x) \geq 0\}$.

**Definition 18.18.** *A discrete valuation ring is a valuation ring $V$ with $K^\times/V^\times \cong \mathbb{Z}$.*

This if $V$ is a discrete valuation ring and $\pi \in V$ is a uniformiser[31], then any $f \in V - \{0\}$ is of the form $f = \pi^n g$ for $g \in V^\times$ and $n = v(f)$.

**Lemma 18.19.** *Let $R$ be a ring, $v : R \to \Gamma \cup \{\infty\}$ be a valuation such that for all $x \in R$, $v(x) \geq 0$. Then there exists a valuation ring $V$ with fraction field $K$, a map $R \to V$ and a commutative diagram,*

$$\begin{array}{ccc} R & \xrightarrow{v} & \Gamma \cup \{\infty\} \\ \downarrow & & \uparrow \\ V & \xrightarrow{v_{can}} & \Gamma_V \cup \{\infty\} \end{array},$$

*where $v_{can} : V \to \Gamma_V \cup \{\infty\}$ and $\Gamma_V = K^\times/V^\times$ is a canonical valuation of $V$, and the inclusion of totally order abelian groups is a map of totally ordered abelian groups.*

*Proof.* Let $\mathfrak{p} = v^{-1}(\infty) \subseteq R$, then $v$ factors as $R \to R/\mathfrak{p} \xrightarrow{\overline{v}} \Gamma \cup \{\infty\}$ so replacing $R$ with $R/\mathfrak{p}$, we may assume that $\mathfrak{p} = (0)$. In particular, $R$ is an integral domain, so we pass to the fraction field $K = \mathrm{Frac}(R)$ and define $\widetilde{v} : K \to \Gamma \cup \{\infty\}$ to send $x/y$ to $v(x) - v(y)$, as we've done before. This makes sense because we checked that $R$ in an integral domain. Let $V = \{x \in K \mid \widetilde{v}(x) \geq 0\}$, then we have $R \to K$ factoring through $V$, as $v(x) \geq 0$ for all $x \in R$. We now set $\Gamma_V = K^\times/V^\times$ and we see that $\widetilde{v}$ induces an injection $\Gamma_V \to \Gamma$, so we have our commutative diagram. □

Back to our favourite examples now.

**Example 18.20.** If $R = \mathbb{Z}$ and $v = v_p$, then $V = \mathbb{Z}_{(p)}$. If $R = k[T]$ with $v = v_T$ then $V = k[T]_{(T)}$. If $R = k[X, Y]$, then $\Gamma_V = \mathbb{Z} \oplus \mathbb{Z}\epsilon$, and $V$ is actually kind of hard to write down.

**Lemma 18.21.** *Let $V$ be a valuation ring, and let $v = v_{can} : V \to \Gamma \cup \{\infty\}$ be the associated valuation, then*

1. *The ideals of $V$ are in one-to-one correspondence with subsets $S \subset \Gamma_{\geq 0}$ such that $\gamma \in S$ and $\gamma' \geq \gamma$, then $\gamma' \in S$, through the map $S \mapsto v^{-1}(S \cup \{\infty\})$.*
2. *$V$ is a local ring, with maximal ideal $v^{-1}(\Gamma_{>0} \cup \{\infty\})$.*
3. *For all ideals $I, I' \subseteq V$ we have $I \subseteq I'$ or $I' \subseteq I$.*

*Proof.* For part 1, let $I \subset V$ be an ideal, $S = v(I) - \{\infty\} \subseteq \Gamma$, then clearly $I \subseteq v^{-1}(S \cup \{\infty\})$, and we claim this is in fact an equality. Assume that $x \in V, y \in I$, with $v(x) = v(y)$, and $y|x$, so $x \in I$. This implies part 1. Part 3 follows by setting $I = v^{-1}(S \cup \{\infty\})$ and $I' = v^{-1}(S' \cup \{\infty\})$ using part 1. For a contradiction, take $x \in S - S'$ and $y \in S' - S$, then if $x \geq y$ we have $y \in S'$ so $x \in S'$, but otherwise we have $y \geq x$ and $x \in S$ implies $y \in S$, a contradiction. Part 2 is obvious. □

[31]This means that $v(\pi) = 1 \in \mathbb{Z} \cong K^\times/V^\times$

The previous proposition gives us a geometric understanding of valuation rings.

**Corollary 18.22.** If $V$ is a valuation ring, the $\operatorname{Spec} V$ is a totally ordered chain of specialisations.

*Proof.* This is simply part 3 of Lemma 18.21. $\square$

# 19 Separated and Properness Criterions 22/12/2016

**Theorem 19.1.** *Let $K$ be a field, and $A \subset K$ with $\operatorname{Frac} A = K$ such that $A$ is a local ring. Then the following are equivalent.*

1. *$A$ is a valuation ring of $K$, so for all $x \in K \backslash \{0\}$ either $x$ or $x^{-1}$ lies in $A$.*
2. *There exists a valuation $v : K \to \Gamma \cup \{\infty\}$ such that $A$ is $\{x \in K \mid v(x) \geq 0\}$.*
3. *For all local rings $A \subseteq B \subseteq K$ where $A \hookrightarrow B$ is a local map, $A = B$.*

*Proof.* We proved that 1 was equivalent to 2 last lecture (Proposition 18.16). The fact that 1 is equivalent to 3 can be found as Proposition B.66 in [2][p.562]. $\square$

**Corollary 19.2.** If $K$ is a field and $A \subseteq K$ is a local subring, then there eixsts a valuation ring $V$ such that $A \subseteq V \subseteq K$ with $A \to V$ a local map.

*Proof.* This just uses Zorn's lemma and characteristation 3 in Theorem 19.1 above. $\square$

*Proof of Theorems 18.2 and 18.3.* First we assume that $f$ is separated, and consider the following diagram,

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} V \arrow[r] \arrow[ur, shift left, "a"] \arrow[ur, shift right, "b"'] & S \end{tikzcd},$$

where $V$ is a valuation ring, $K$ is its fraction field, and $a, b : \operatorname{Spec} V \to X$ are two maps. As the diagonal map $\Delta_f$ is a closed immersion, and closed immersions are stable under base change, then the maps $\Delta'_a, \Delta'_b : \operatorname{Spec} V \times_{X \times_S X} X \to \operatorname{Spec} V$ are closed immersions, as a map of schemes under $\operatorname{Spec} K$. This implies $\operatorname{Spec} V \times_{X \times_S X} X = \operatorname{Spec}(V/I)$ for some ideal $I \subseteq V$, from which we obtain the following little diagram, since we are working with maps of schemes under $\operatorname{Spec} K$.

$$\begin{tikzcd} V \arrow[r] \arrow[dr, hook] & V/I \arrow[d, hook] \\ & K \end{tikzcd}$$

the above commutative diagram implies that $V \to V/I$ is an injection, so $I = (0)$ is the zero ideal and $V/I = V$. This implies that $\operatorname{Spec} V \times_{X \times_S X} X = \operatorname{Spec} V$, so our maps $\Delta'_a, \Delta'_b : \operatorname{Spec} V \to X \xrightarrow{\Delta_f} X \times_S X$ are equal. Assuming now that $f$ is proper, then we have to show the unique existence of our lift. Given a diagram,

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "a"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} V \arrow[r] & S \end{tikzcd}.$$

We can replace $S$ by $\operatorname{Spec} V$, and $X$ by $X \times_S \operatorname{Spec} V$, so without loss of generality $S = \operatorname{Spec} V$. Let $x \in X$ be the image of $a$, and let $Z = \overline{\{x\}} \subseteq X$, then $Z \subseteq X$ is closed. The fact that $|f|$ is closed then implies that the image of $Z$ in $\operatorname{Spec} V$ is closed and non-empty. Thus, the closed point of $\operatorname{Spec} V$ lies in the image of $Z$. Let $z \in Z \subseteq X$ map to the closed point of $\operatorname{Spec} V$. As $z \in \overline{\{x\}}$, there is a natural maps $\mathcal{O}_{X,z} \to \mathcal{O}_{X,x}$, since all open subsets of $z$ also contain $x$. Hence we obtain a chain of maps,

$$V \to \mathcal{O}_{X,z} \to \mathcal{O}_{X,x} \to K,$$

whose composition is injective since $V \to K$ is simply the natural inclusion. Thus we have $V \subseteq B \subseteq K$, where $B$ is the image of $\mathcal{O}_{X,z}$ inside $K$, and $B$ is a local ring, as it is a quotient of a local ring, and

$V \to B$ is a local map, as $z$ maps to the closed point of $\operatorname{Spec} V$. By part 3 of Theorem 19.1, we see that $B = V$. Then the data $(z, \mathcal{O}_{X,z} \to B = V)$ defines a $V$-point of $X$ over $S = \operatorname{Spec} V$, from exercise sheet 5 problem 1[32].

We have now seen one full direction of Theorems 18.2 and 18.3. For the reverse implication, we need to see $|\Delta_f| \to |X \times_S X|$ is closed, for which we need a few lemmata.

**Lemma 19.3.** *Let $g : Y' \to Y$ be a quasi-compact morphism of schemes. Then the topological image of $g$ is closed if and only if it is stable under specialisations.*

*Proof.* It is clear that if $g(|Y'|)$ is closed then it is also stable under specialisations. Conversely, closed maps are local on the target so without loss of generality we can take $Y = \operatorname{Spec} B$. In this case $Y$ is quasi-compact, so $Y'$ has a finite affine cover by $\operatorname{Spec} B_i$'s, then by replacing $Y'$ by,

$$\coprod_i \operatorname{Spec} B_i = \operatorname{Spec}\left(\prod_i B_i\right),$$

we may assume that $Y' = \operatorname{Spec} B$ is also affine. Say now we have $x \notin g(|Y'|)$, then the localisation $Y_x$ does not meet the topological image of $g$, as this image is closed under specialisations. Then $\varnothing Y_x \times_Y Y' = \operatorname{Spec}(B_{\mathfrak{p}} \otimes_B B')$, so $0 = 1$ inside the ring $B_{\mathfrak{p}} \otimes_B B'$. If we write this as,

$$B_{\mathfrak{p}} \otimes_B B' = \lim\nolimits_{f \notin \mathfrak{p}} B[f^{-1}] \otimes_B B',$$

we see there exists $f \notin \mathfrak{p}$ such that $B[f^{-1}] \otimes_B B' = 0$, which implies that $D(f) \times_Y Y' = \varnothing$, which implies that $D(f) \cap g(|Y'|) = \varnothing$. This implies that the complement of the topological image of $g$ is open, hence the image is closed. □

**Lemma 19.4.** *Let $X$ be a scheme, and $x \succ y$ a specialisation of points in $X$ with $K = k(x)$ the residue field at $x$, then there exists a valuation subring $V \subseteq K = \operatorname{Frac}(V)$ and a map $\operatorname{Spec} V \to X$ mapping the closed point of $\operatorname{Spec} V$ to $y$.*

Remark that if $X$ is notherian, then $V$ can be chosen to be a discrete valuation ring.

*Proof.* We may replace $X$ by the localisation $X_y = \operatorname{Spec} \mathcal{O}_{X,y}$, so without loss of generality $X = \operatorname{Spec} A$, where $A$ is a local ring and $y$ is its unique closed point. Now $x \in X$ corresponds to some prime ideal $\mathfrak{p} \subseteq A$. We may further replace $A$ by $A/\mathfrak{p}$, so in addition, our ring $A$ can be assumed to be an integral domain and $x$ corresponds to the generic point, the zero ideal in $A$. In particular, $K = k(x) = \operatorname{Frac} A$. We can now use Corollary 19.2 to obtain a valuation ring $V$, such that $A \subseteq V \subseteq K$ and $\operatorname{Spec} V \to \operatorname{Spec} A = X$ maps closed points to the closed point $y$. □

Back to the proof of Theorems 18.2 and 18.3. Assume we have a map $f : X \to S$ which is quasi-separated and satisfies the valuation criterion for separatedness. Then, because $f$ is quasi-separated we know that $\Delta_f$ is quasi-compact by definition, so by Lemma 19.3 it is enough to show the image of $\Delta_f$ is closed under specialisations. If $x \succ y$ is a specialisation in $X \times_S X$ with $x \in \Delta_f(|X|)$, then by Lemma 19.4 we obtain a diagram of the form,

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "="] \arrow[d] & \operatorname{Spec} k(x) \arrow[r] \arrow[d] & X \arrow[dl, "\Delta_f"] \\
\operatorname{Spec} V \arrow[r] & X \times_S X &
\end{tikzcd},$$

[32]Let $R$ be a local ring, $S = \operatorname{Spec} R$ and $s \in S$ be the unique closed point. Prove that for every scheme $X$ the map,

$$\operatorname{Hom}(S, X) \longrightarrow \{(x \in X, \phi : \mathcal{O}_{X,x} \to R) | \phi \text{ is a local ring homomorphism}\},$$

sending a morphism $f : S \to X$ of schemes to the pair $(f(s), \mathcal{O}_{X,f(s)} \xrightarrow{f^\#} \mathcal{O}_{S,s} \cong R)$ is a bijection.

which maps the closed point of $\operatorname{Spec} V$ to $y$. This precisely corresponds to a diagram of the form,

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} V \arrow[r] \arrow[ur, shift left] \arrow[ur, shift right] & S
\end{tikzcd}.$$

The valuation criterion for separatedness now tells us that these two maps are equal. In other words, $\operatorname{Spec} V \to X \xrightarrow{\Delta_f} X \times_S X$ implies that $y \in \Delta_f(|Y|)$. Hence we now only need to worry about properness. Assume now that $f : X \to S$ is quasi-separated, of finite type and satisfies the valuation cirterion for properness. Then $f$ is separated by Theorem 18.2 which we have now proved, and of finite type remains, so we only need to show universally closed now. Given some $S' \to S$, then we can replace $S$ by $S'$ and $X$ by $X \times_S S'$, so without loss of generality we can take $S = S'$, so we need to see that $|X| \to |S|$ is closed. Let $Z \subseteq |X|$ be a closed subset, then we can endow $Z$ with the reduced subscheme structure (see Proposition 19.5). Hence $i : Z \to X$ is a closed subscheme, hence $i$ is proper so it satisfies the valuation criterion for properness. Then, also the composite satisfies the valuation criterion for properness, so without loss of generality we can take $X = Z$.

We need to see $f(|X|) \subseteq |S|$ is closed, but $f$ is of finite type, so it is quasi-compact, so by Lemma 19.3 implies it is enough to see this image is stable under specialisations. Let $x \succ y$ be point of $S$ with $x \in f(|X|)$. Lemma 19.4 gives us a valuation ring $V$ with $\operatorname{Frac} V = k(x)$, and a diagram,

$$\begin{tikzcd}
\operatorname{Spec} k(x) \arrow[r] \arrow[d] & S \\
\operatorname{Spec} V \arrow[ur] &
\end{tikzcd},$$

mapping the closed point of $\operatorname{Spec} V$ to $y$. Now $X \times_S \operatorname{Spec} k(x)$ is of finite type and non-empty over $k(x)$, so we now let $K' = k(x)$. There exists a finite field extension $K'/K$, such that,

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[r] \arrow[dr] & X \times_S \operatorname{Spec} K \arrow[d] \\
& \operatorname{Spec} K
\end{tikzcd}.$$

We can also find a valuation ring $V' \subseteq K'$ such that $\operatorname{Spec} V' \to \operatorname{Spec} V$ is a local map. We now finally have the following diagram,

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} V \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd},$$

The valuation criterion gives us the dotted map, and as $y$ is in the image of $|\operatorname{Spec} V'| \to |S'|$ we see that $y$ is in the image of $|X| \to |S|$. $\square$

Let us just wrap things up with a little statement we have been able to prove and digest for a while now.

**Proposition 19.5.** *Given a scheme $X$, then there is a bijection between closed reduced subschemes of $X$ and closed subsets of $|X|$.*

*Proof.* Assume that $X = \operatorname{Spec} A$ then reduced closed subschemes are simply $\operatorname{Spec} A/I$ where $A/I$ is reduced, which are the same as ideals $I \subseteq A$ where $I$ is a radical ideal, which by definition give us our closed subsets $V(I)$. In general we just simply glue. We do observe though that the localisation of a reduced subscheme is still reduced. $\square$

We can now easily prove the theorem we have been waiting for (recall Theorem 16.15).

**Theorem 19.6.** *The morphism* $f : \mathbb{P}^n \to \operatorname{Spec} \mathbb{Z}$ *is proper.*

*Proof.* We know that our map $f$ is quasi-separated and of finite type over $\operatorname{Spec}\mathbb{Z}$, so we need to see that for all valuation rings $V$ and $K = \operatorname{Frac}(V)$, we have a unique lift of the following diagram,

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathbb{P}^n \arrow[d] \\ \operatorname{Spec} V \arrow[r] \arrow[ur, dashed, "l"] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}.$$

One way to show this would be that $\mathbb{P}^n(V) \to \mathbb{P}^n(K)$ is a bijection, and we can explicitly do this because we know both sides. Assume that $(x_0 : \cdots : x_n), (x'_0 : \cdots : x'_n) \in \mathbb{P}^n(V)$ map to the same point of $\mathbb{P}^n(K)$, then we have some $\lambda \in K^\times$ such that $x_i = \lambda x'_i$ for all $i$. As sub-$V$-modules of $K$, we have,

$$V = V \cdot x_0 + \cdots + V \cdot x_n = V \cdot \lambda x'_0 + \cdots + V \cdot \lambda x'_n = \lambda(V \cdot x'_0 + \cdots + V \cdot x'_n) = \lambda \cdot V,$$

so $\lambda \in V^\times$ and we have injectivity. Now if $(x_0 : \cdots : x_n) \in \mathbb{P}^n(K)$ is any point, then we choose $i$ such that $v(x_i) \le v(x_j)$ for all other $j$. Then we have,

$$(x_0 : \cdots : x_n) = \left(\frac{x_0}{x_i} : \cdots : \frac{x_n}{x_i}\right),$$

where $x_j/x_i \in V$ since $v(x_j/x_i) \ge 0$. Together they generate $V$ since $x_i/x_i = 1$, so we have surjectivity too. $\square$

# 20 Normal Schemes and Curves 10/01/2017

We start this lecture by discussing the various definitions we have seen so far. We know what schemes are, and we know some topological properties, global properties, local properties, and finiteness properties. This game of making more definitions is an endless story in algebraic geometry, but a problem is that we need so many words to obtain the examples and objects that we really want to study. What we want to define today is a normal scheme, which is equivalent to a smooth scheme if the dimension of our scheme is less than or equal to 1.

**Proposition 20.1.** *Let $X$ be a scheme of finite type over a field $k$ with dimension 0. Then $X = \operatorname{Spec} A$ for some finite dimensional $k$-algebra. Conversely, if $A$ is a finitely dimensional $k$-algebra, then $X = \operatorname{Spec} A$ is a scheme of finite type over $k$ with dimension 0.*

To attack this proposition, we are going to use some more general facts one noetherian topological spaces.

**Lemma 20.2.** *If $T$ is a noetherian topological space, then $T$ is a finite union $T = \bigcup_{i=1}^{n} T_i$ where $T_i \subseteq T$ are irreducible closed subspaces of $T$.*

*Proof.* Assume this is not the case, then we can define $\Phi$ to be the set of all closed subspaces $Z \subseteq T$ which violate the statement of the lemma. Then because $T$ is noetherian $\Phi$ has a minimal element. If not we have an infinite descending chain in $\Phi$, which contradicts that $T$ is noetherian. This means $Z$ is not irreducible, so $Z = Z_1 \cup Z_2$ where $Z_i \subsetneq Z$ are proper non-empty closed subspaces of $Z$. Thus $Z_i \notin \Phi$, since that would contradict the minimality of $Z$, so $Z_i = \bigcup_i Z_{i,j}$ some finite union of irreducible closed subsets. This means we can write $Z$ as the finite union $\bigcup_{i,j} Z_{i,j}$, which means that $Z \notin \Phi$, a contradiction. $\square$

**Lemma 20.3.** *Let $T$ be a noetherian spectral topological space with $d = \dim T < \infty$, then $T = \bigcup_{i=1}^{n} T_i$ for $T_i \subseteq T$ pointwise distinct closed irreducible subsets, which can be seen as $T_i = \overline{\{\eta_i\}}$ where $\eta \in T_i$ is the unique generic point of $T_i$. Moreover, if $T_i \neq T_j$, then $\dim(T_i \cap T_j) < d$.*

**Example 20.4.** Let's have a look at the case when $d = 0$, then $T$ has no specialisations so $T_i = \overline{\{\eta_i\}} = \{\eta_i\}$, so $T = \bigcup_{i=1}^{n} T_i$ is a discrete finite space.

**Example 20.5.** For $d = 1$, then our $T$ might be a finite collection of curves and points, and two curves always intersect each other at finitely many point, one dimension lower than curves.

*Proof of Lemma 20.3.* The only thing we really have to prove is that $\dim(T_i \cap T_j) < d$. Let $x_0 \prec \cdots \prec x_m$ be a chain of generalisations in $T_i \cap T_j$, then $x_m \in T_i \cap T_j \subseteq T_i, T_j$. We must have $T_i \cap T_j \neq T_i$ or $T_j$, so without loss of generality take $T_i \cap T_j \neq T_i$. Then we have $\eta_i \notin T_i \cap T_j$, and $x_m \prec \eta_i$ in $T_i$, thus we have a new chain of generalisations $x_0 \prec \cdots \prec x_m \prec \eta_i$ in $T$, which were not available in $T_i \cap T_j$. Hence $m + 1 \leq d$ so $m < d$ for all chains of generalisations of length $m$. $\square$

Let's see a quick consequence of these two lemmas.

**Corollary 20.6.** Something like $\{\sin x = y\} \cap \{y = 0\}$ can not appear in algebraic geometry.

This silly corollary just reminds us there are very strict finiteness conditions placed on algebraic geometry that are absent in real analysis and other areas of mathematics. We can now classify zero dimensional schemes of finite type over a field $k$.

*Proof of Proposition 20.1.* Notice that the underlying space of our scheme $X$ is a noetherian spectral topological space of $\dim X = 0$, so $|X|$ is simply a finite set of points, $X = \coprod_{i=1}^{n} X_i$ for some $|X_i| = *$ a point. Then all $X_i = \operatorname{Spec} A_i$ are affine, and $\coprod_{i=1}^{n} X_i = \operatorname{Spec} \prod_{i=1}^{n} A_i$, so $X = \operatorname{Spec} A$ is affine. Noether normalisation gives us an injection $k[x_1, \ldots, x_n] \hookrightarrow A$ which is finite, but we have seen in Theorem 14.1 this implies that $0 = \dim X = \dim \operatorname{Spec} A = n$. Hence we have an injection $k \to A$ which is finite, so $A$ is a finite dimensional (as a $k$-vector space) $k$-algebra. Conversely, if $A$ is any finite dimensional $k$-algebra, then it is obviously a finitely generated $k$-algebra, and $A = \prod_{i=1}^n A_i$ where $A_i$ is a local Artinian ring, which implies that $\operatorname{Spec} A = \coprod_{i=1}^n \operatorname{Spec} A_i$, where $\operatorname{Spec} A_i$ are a collection of points. □

**Corollary 20.7.** If $X$ is as in Proposition 20.1, and $X$ is connected, reduced, then $X = \operatorname{Spec} k'$ where $k'/k$ is a finite field extension.

*Proof.* $X$ is connected implies that $X = \operatorname{Spec} A$ where $A$ is a local Artinian ring, and $X$ reduced implies that $A$ is reduced. But a reduced Artinian ring is a field. □

**Remark 20.8.** Peter believes it is still not understood how to classify all possible non-reduced structures (such as in Proposition 20.1). If $k$ is algebraically closed, $A$ is a local Artinian finite dimensional $k$-algebra, then $\dim_k A = 2$ implies that $A \cong k[x]/(x^2)$, and $\dim_k A = 3$ implies that $A \cong k[x]/x^3$ or $k[x, y]/(x^2, xy, y^2)$, but these calculations get harder. When $k$ is not algebraically closed this is a really hard question to ask.

We will now assume that $\dim X = 1$, and for the rest of this lecture it will always be of finite type over a field $k$. We will also assume that $X$ is reduced, since otherwise our classification problem becomes an open one. In general we have $X_{red} \subseteq X$ and often problems on $X$ can be reduced to problems on $X_{red}$, so this assumption doesn't do too much harm.

We have seen that $X = \bigcup_{i=1}^n X_i$ where $X_i \subseteq X$ is an irreducible closed subset, which reduced closed subscheme structures, and $\dim X_i \leq 1$ and $\dim(X_i \cap X_j) = 0$ for all $X_i \neq X_j$. In this way we can in some sense build a general $X$ from irreducible and reduced 1-dimensional $X_i$'s and 0-dimensional schemes. From now on, we are going to assume that $X$ is also irreducible, thus $X$ is also integral (recall Definitions 8.1 and 14.9 and Proposition 14.10).

**Definition 20.9.** *Let $k$ be a field. A curve over $k$ is a reduced and irreducible 1-dimensional scheme $C$ of finite type over $k$.*

There are still lots of singular curves, such as the nodal and cuspidal cubics. Again, it is basically impossible to classify curves if we consider these singularites, so we need another definition to get what we want. This is where we would like to say smooth, but we are only working with "curves", so it suffices to define normal.

**Definition 20.10.** *A scheme $X$ is normal if for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a normal integral domain.*

Recall that an integral domain $A$ is normal if $A$ is integrally closed in $K = \operatorname{Frac}(A)$, i.e. if $x \in K$ satisfies $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ for some $a_i \in A$, then $x \in A$.

**Example 20.11** (Non-example)**.** Let $A = k[x, y]/(y^2 - x^3)$, the cuspidal cubic. We claim that $A$ is not normal. To see this, embed $A$ into $k[t]$ by sending $x \mapsto t^2$ and $y \mapsto t^3$, then $A$ is identified with $f = \sum_{i=0}^n r_i t^i$ with $r_i \in k$ and $r_1 = 0$ inside $k[t]$. In particular, $A$ is an integral domain, since it injects into $k[t]$, and $t = y/x \in K = Frac(A) = Frac(k[t])$, and $t$ satisfies the equation $t^2 - x = 0$. By $t \notin A$, so $A$ is not normal.

This normality condition does kill some examples we don't like. Let's see some properties of normal schemes.

**Proposition 20.12.** *If $X$ is normal, then $X$ is reduced. If $X$ is integral, then $X$ is normal if and only if for all $U = \operatorname{Spec} A \subseteq X$ an open affine subset, $A$ is a normal integral domain.*

Recall that $X$ is integral if it is reduced and irreducible, so by the first part of this proposition we only need to assume that $X$ is irreducible to imply that all open $U = \operatorname{Spec} A \subseteq X$ have $A$ as a normal integral domain.

**Remark 20.13.** If $X$ is normal, but not irreducible, it can be that for some $U = \operatorname{Spec} A \subseteq X$ we have that $A$ is not an integral domain, for example $X = \operatorname{Spec} k \sqcup \operatorname{Spec} k = \operatorname{Spec}(k \times k)$ for some field $k$. The definition of normality is designed to be a local property of $X$. One can show that if $X$ is noetherian and normal, then $X = \coprod_{i=1}^{n} X_i$ with $X_i$ irreducible, normal and noetherian.

*Proof of Proposition 20.12.* First, for all $x \in X$ the fact that $\mathcal{O}_{X,x}$ is reduced implies for all $U = \operatorname{Spec} A \subseteq X$ open in $X$ we have an injection $A \to \prod_{x \in U} \mathcal{O}_{X,x}$. Since the right hand side of this injection is reduced, then so is $A$.[33] Now let $X$ be an integral normal scheme, then for all $U = \operatorname{Spec} A \subseteq X$ open in $X$, $A$ is an integral domain, so we let $K = \operatorname{Frac}(A)$. For all $x \in U$ we then have $\mathcal{O}_{X,x} = A_{\mathfrak{p}_x}$ is integrally closed in $\operatorname{Frac}(A_{\mathfrak{p}_x}) = \operatorname{Frac}(A) = K$. Assume that $f \in K$ is integral over $A$, then for all $x \in U$, $f$ is integral over $A_{\mathfrak{p}_x}$, so $f \in A_{\mathfrak{p}_x}$. This specifically says that there exists $g \notin \mathfrak{p}_x$ such that $f \in A[g^{-1}]$, since $A_{\mathfrak{p}_x}$ is a direct limit of these localisations. Thus, $f$ defines a section of $\mathcal{O}_X$ locally on $U$. These sections glue to a section of $\mathcal{O}_X(U) = A$, thus $f \in A$.

Conversely, it is enough to show that if $A$ is a normal integral domain, $\mathfrak{p} \subseteq A$ a prime ideal then $A_{\mathfrak{p}}$ is a normal domain. This follows from a general statement in commutative algebra, that localisation preserves normality (Lemma 20.14). □

**Lemma 20.14.** *Let $A$ be a normal integral domain, $K = Frac(A)$, $S \subseteq A$ be a multiplicatively closed subset of $A$, then $A[S^{-1}]$ is a normal integral domain.*

*Proof.* We have an inclusion $A[S^{-1}] \to K$ so $A[S^{-1}]$ is an integral domain. Assume then that $f \in K$ is integral over $A[S^{-1}]$, so

$$f^n + \frac{a_{n-1}}{s_{n-1}} f^{n-1} + \cdots + \frac{a_0}{s_0} = 0,$$

for $a_i \in A$ and $s_i \in S$. Let $s = \prod_i s_i$, and multiply the above equation by $s^n$ to obtain,

$$(fs)^n + s\frac{a_{n-1}}{s_{n-1}}(fs)^{n-1} + \cdots + s^n \frac{a_0}{s_0} = 0.$$

In this way all the coefficients $s\frac{a_{n-1}}{s_{n-1}}, \ldots, s^n \frac{a_0}{s_0}$ are in $A$. Since $fs$ is integral over $A$ and $A$ is normal then $fs \in A$ which means that $f = fs/s$ is in $A[S^{-1}]$. □

[33]A scheme if reduced if and only if the stalks $\mathcal{O}_{X,x}$ are reduced for all $x \in X$.

# 21 Normalisations 12/01/2017

Recall Proposition 20.12, then notice also that if $A$ is a normal integral domain, then $\operatorname{Spec} A$ is normal. We will now begin the systematic process of taking an irreducible scheme, and producing a canonical irreducible normal scheme.

**Proposition 21.1.** *Let $X$ be an irreducible scheme, then there exists a normal and irreducible scheme $\widetilde{X}$ and a dominant map*[34] *$f : \widetilde{X} \to X$ which is universal in the sense that for all other normal and irreducible schemes $Y$ with a dominant map $g : Y \to X$, then $g$ factors through $f$. Moreover, the morphism $f$ is affine, and for all $U = \operatorname{Spec} A \subseteq X$ open in $X$, we have $\widetilde{U} = f^{-1}(U) = \operatorname{Spec} \widetilde{A}$, where $\widetilde{A}$ is the integral closure of $A_{red}$ in $K = \operatorname{Frac}(A_{red})$.*

Notice that if $X$ is reduced, then $A$ is an integral domain, then $\widetilde{A}$ is simply the integral closure of $A$ in $K = Frac(A)$.

*Proof.* We may assume that $X$ is reduced. In general we have a closed reduced subscheme $X_{red} \subseteq X$ and $\widetilde{X_{red}} = \widetilde{X}$ by the universal property of normalisation, since all normal schemes are reduced. Without loss of generality we may then take $X$ to be reduced, which means that $X$ is integral. Let $\eta \in X$ be the unique generic point, and write $K = k(\eta)$, the residue field at $\eta$. Concretely, for all $U = \operatorname{Spec} A \subseteq X$ we have $K = \operatorname{Frac}(A)$. We claim that $U \mapsto \widetilde{\mathcal{O}}_X(U) = \widetilde{A}$ define a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. To see this we need to notice that for all $a \in A$, the following canonical map is an isomorphism,

$$\widetilde{A}[a^{-1}] \longrightarrow \widetilde{A[a^{-1}]}.$$

The idea is that once we have this, we get a quasi-coherent sheaf on principal opens on $\operatorname{Spec} A$, which gives us a quasi-coherent sheaf on all of $\operatorname{Spec} A$, and these all glue together to a quasi-coherent sheaf on $X$ for varying $U = \operatorname{Spec} A \subseteq X$. We have the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, hook] & \widetilde{A} \arrow[r, hook] & \operatorname{Frac}(A) \arrow[d, "\mathrm{id}"] \\
A[a^{-1}] \arrow[r, hook] & \widetilde{A}[a^{-1}] \arrow[r, hook] & \operatorname{Frac}(A[a^{-1}]) = \operatorname{Frac}(A)
\end{tikzcd}$$

We thus obtain an induced injection $\widetilde{A}[a^{-1}] \hookrightarrow \widetilde{A[a^{-1}]}$ by the universal property of localisation. However $\widetilde{A}$ is normal, so $\widetilde{A}[a^{-1}]$ is normal (Lemma 20.14), so $\widetilde{A[a^{-1}]} \subseteq \widetilde{A}[a^{-1}]$ which implies that $\widetilde{A}[a^{-1}] = \widetilde{A[a^{-1}]}$.

Define $\widetilde{X} = \underline{\operatorname{Spec}}_{\mathcal{O}_X}(\widetilde{\mathcal{O}}_X)$, then $f : \widetilde{X} \to X$ is affine, and has the desired explicit description (from the properties of the relative spectrum). To check the universal property, we have a dominant map $g : Y \to X$, where $Y$ is normal and irreducible, then we can construct a unique factorisation over $\widetilde{X}$, and we can work this out locally on $Y$. As usual then, we let $Y = \operatorname{Spec} B$ and $g : Y \to U = \operatorname{Spec} A \subseteq X$. Let $L = \operatorname{Frac}(B)$, then because $g$ is dominant we obtain the following commutative diagram, by mapping global sections to stalks at the unique generic point.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\
K \arrow[r] & L
\end{tikzcd}$$

[34] Recall that a dominant map sends generic points to generic points.

Note that $B$ is integrally closed in $L$, which implies that $\phi : A \to B$ extends uniquely to $\widetilde{\phi} : \widetilde{A} \to B$.[35] We then obtain the unique map $g$ such that the following diagram commutes,

$$\begin{tikzcd}
Y = \operatorname{Spec}(B) \arrow[r, "\widetilde{g}"] \arrow[dr] & \operatorname{Spec} \widetilde{A} = \widetilde{U} \arrow[r, hook] \arrow[d] & \widetilde{X} \arrow[d] \\
& \operatorname{Spec} A = U \arrow[r, hook] & X
\end{tikzcd}.$$

This is the desired universal property. $\square$

**Example 21.2.** Let $X = \operatorname{Spec} A$ with $A = k[x,y]/(y^2 - x^3)$ be our cuspidal cubic, then we have $(y/x)^2 = x$ inside $\operatorname{Frac}(A) = K$. Let $t = y/x$, then we see that $t \in \widetilde{A} \subseteq K$, such that $x = t^2$ and $y = tx = t^3$, thus $\widetilde{A} \supseteq k[t]$, but $k[t]$ is normal so $\widetilde{A} = k[t]$. Let

$$\widetilde{X} = \operatorname{Spec} \widetilde{A} = \mathbb{A}^1_k \to \operatorname{Spec} A = V(y^2 - x^3) \subseteq \mathbb{A}^2_k,$$

be defined by $t \mapsto (t^2, t^3)$, then geometrically this map sends the vertical line $\mathbb{A}^1_k$ homeomorphically (not isomorphically as schemes) to the cuspidal cubic.

**Example 21.3.** Our second favourite example is when $A = k[x,y]/(y^2 - x^3 - x^2)$ be our nodal cubic. Then similarly we have $(y/x)^2 = x + 1$ inside $\operatorname{Frac}(A) = K$, and we let $t = y/x \in \widetilde{A} \subseteq K$. Then we have $x = t^2 - 1$ and $y = tx = t(t^2 - 1)$, so $A \subsetneq k[t] \subseteq \widetilde{A} \subseteq K$, so again $\widetilde{A} = k[t]$. Again, we have a map $\operatorname{Spec} \widetilde{A} \to \operatorname{Spec} A$, which is a local scheme isomorphism away from the singularity. Notice this map identifies $t = -1$ and $t = 1$, but at all other points we can recover $t = t(t^2-1)/(t^2-1) = y/x$.

**Remark 21.4.** Notice that in these examples $\widetilde{X}$ is again of finite type over $K$, which actually happens in arbitrary dimensions. Notice also that $\widetilde{X}$ has no singularities, which unfortunately we can only guerentee in this low-dimensional case (see problem 3(ii) on exercise sheet 11[36]).

**Theorem 21.5.** *Assume $A$ is a finitely generated $k$-algebra (where $k$ is a field, but could also be a Dedeking ring, like $\mathbb{Z}$), and $A$ is an integral domain. Then the normalisation $\widetilde{A}$ is a finitely generated $A$-module and in particular $\widetilde{A}$ is a finitely generated $k$-algebra.*

*Proof.* This is a difficult, but classical theorem from commutative algebra. $\square$

**Remark 21.6.** This is true even more generally if $A$ is excellent. However most noetherian rings are excellent. We have to try really hard to find a noetherian ring which is not excellent. Excellent rings are a subtle part of commutative algebra. For example it was only proved in 2015-16 that completions of excellent rings are (quasi)-excellent, which is well beyond the scope of this course.

Recall now that a discrete valuation ring is a valuation ring such that $\Gamma \cong \mathbb{Z}$.

**Proposition 21.7.** *Given a discrete valuation ring $A$, then $A$ is a local noetherian domain with* $\dim \operatorname{Spec} A = 1$.

*Proof.* To see this, notice that $A$ is a valuation ring so $A$ is local and an integral domain, and also recall that ideals of $A$ correspond to subsets $S \subseteq \mathbb{Z}_{\geq 0}$ such that $s \in S$ and $s \leq s'$ implies that $s' \in S$. In our case these subsets $S$ correspond to $\mathbb{Z}_{\geq 0} \cup \{\infty\}$, where $\infty$ corresponds to the empty set. A natural number $n$ is mapped to the ideal $\{x \in A | v(x) \geq n\}$. This implies that $A$ is noetherian, since descending chains of non-negative integers stabilise.

[35] If $x/y \in \widetilde{A}$ with $x, y \in A$ and $y \neq 0$, then $\phi(x)/\phi(y)$ is integral over $B$, which implies it is in $B$ and we define $\widetilde{\phi}(x/y) = \phi(x)/\phi(y)$.

[36] Let $k$ be a field of characteristic $\neq 2$. Prove that the cone $X := V(f) \subseteq \mathbb{A}^3_k$ with $f = xy - z^2$ is a normal domain and that the point $(0,0,0) \in X(k)$ is not smooth.

Let $\pi \in A$ such that $v(\pi) = 1$, then $v(x) \geq n$ is equivalent to $v(x) \geq v(\pi^n)$ which is equivalent to $\pi^n | x$. Thus $\{v(x) \geq n\} = (\pi^n)$, so all ideals are principal. If $n < \infty$, then $\sqrt{(\pi)^n} = (\pi)$ so the only radical ideals of $A$ are $(0)$ and $(\pi)$. Moreover, $(\pi)$ is really a prime ideal. If $x, y \in A$ such that $xy \in (\pi)$, then $v(x) + v(y) = v(xy) \geq 1$, so one of $v(x), v(y) \geq 1$, in other words $x$ or $y$ are in $(\pi)$. This implies that $\operatorname{Spec} A = \{(0), (\pi)\}$ where $(0) = \eta$ is the generic point and $(\pi) = s$ is the unique closed point. □

We now have a theorem which generalise the previous proposition.

**Theorem 21.8.** *Let $A$ be a local noetherian integral domain of $\dim \operatorname{Spec} A = 1$, then $\operatorname{Spec} A = \{(0), \mathfrak{m}\}$. More over $A$ is normal if and only if $\mathfrak{m} \subseteq A$ is a principal ideal if and only if $A$ is a discrete valuation ring.*

*Proof.* The zero ideal is the unique generic point of $\operatorname{Spec} A$ since $A$ is a domain, and $\mathfrak{m}$ is the unique closed point since $A$ is local. Assume there exists $x \in X$ with $\mathfrak{m} \prec x \prec (0)$, but we know that $\dim \operatorname{Spec} A = 1$, so $x = \mathfrak{m}$ or $(0)$.

**($A$ is normal $\Rightarrow$ $\mathfrak{m} \subseteq A$ is a principal ideal)** Let $a \neq 0$ be an element of $A$, and $a \notin A^\times$, then $V(a) = \{\mathfrak{m}\} \subseteq A$. This implies that the radical of $(a)$ is equal to $\mathfrak{m}$, so there exists $n$ with $(a) \supseteq \mathfrak{m}^n$. Let $n$ be the minimal such $n$, and let $b \in \mathfrak{m}^{n-1} \backslash (a)$. Let $x = a/b \in K = \operatorname{Frac} A$, then $x^{-1} \notin A$, as $b \in (a)$. As $A$ is normal, then $x^{-1}$ is not integral, so $A[x^{-1}] \subseteq K$ is not a finitely generated $A$-module, since if it was then there would exist an $m$ such that it is generated by $1, x^{-1}, \ldots, x^{-m+1}$, and

$$x^{-m} = a_{m-1} x^{-m+1} + \cdots + a_0, \qquad a_i \in A,$$

contradicting the fact that $x^{-1}$ is not integral. This implies $x^{-1}\mathfrak{m} \subsetneqq \mathfrak{m}$ as otherwise $A[x^{-1}] \hookrightarrow \operatorname{End}_A(\mathfrak{m})$, the later of which is finitely generated over $A$. But $x^{-1}\mathfrak{m} \subseteq A$ as $y \in \mathfrak{m}$, and

$$x^{-1}\mathfrak{m} = \frac{b\mathfrak{m}}{a} \subseteq \frac{\mathfrak{m}^n}{a} \subseteq A.$$

Thus, $x^{-1}\mathfrak{m} \subseteq A$ is a sub-$A$-module such that $x^{-1}\mathfrak{m} \subsetneqq \mathfrak{m}$, so $x^{-1}\mathfrak{m} = A$, so $\mathfrak{m} = Ax$, hence $x \in \mathfrak{m} \subseteq A$ and $\mathfrak{m} = (x)$, a principal ideal.

**($\mathfrak{m} \subseteq A$ is a principal ideal $\Rightarrow$ $A$ is a discrete valuation ring)** Let $\pi \in \mathfrak{m}$ be a generator of $\mathfrak{m}$. We claim that for all $a \neq 0$ in $A$ is of the form $a = \pi^n a_0$ where $n \geq 0$ and $a_0 \in A^\times$. Again we have the radical of $(a)$ is equal to $\mathfrak{m} = (\pi)$, so there exists $n$ such that $\pi^n \in (a)$. Then we choose a maximal $m \leq n$ such that $a \in (\pi^m)$. Then $a = \pi^m a_0$, and $a_0 \in A\backslash(\pi) = A\backslash\mathfrak{m} = A^\times$. This shows existence, so for uniqueness take $\pi^n a_0 = \pi^m b_0$, for $a_0, b_0 \in A^\times$, and without loss of generality take $n \geq m$. The fact $A$ is integral implies $\pi^{n-m} a_0 = b_0 \in A^\times$ but $\pi^{n-m} a_0 \in \mathfrak{m}$, a contradiction unless $n = m$, but in this case we see $a_0 = b_0$ as well. We then define $A \to \mathbb{Z}_{\geq n} \cup \{\infty\}$ by mapping,

$$0 \neq a_0 \pi^n = a \mapsto n, \qquad 0 \mapsto \infty.$$

Easy to check this is indeed a valuation, and as $v^{-1}(\infty) = \{0\}$, it extends to $v : K = \operatorname{Frac}(A) \to \mathbb{Z} \cup \{\infty\}$ mapping,

$$0 \neq x\pi^n = a \mapsto n \in \mathbb{Z}, \qquad 0 \mapsto \infty.$$

Then we check that $A = \{x \in K \mid v(x) \geq 0\}$.

**($A$ is a valuation ring $\Rightarrow$ $A$ is a normal)** We can actually show that any valuation ring is normal. Assume $v : K \to \Gamma \cup \{\infty\}$ is a valuation with $V = \{x \in K \mid v(x) \geq 0\}$, and $x \in K$ with,

$$x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0,$$

all $a_i \in V$. If $x \notin V$, then $v(x) < 0$ and $v(x^{-1}) > 0$, $x^{-1} \in V$. We now obtain,

$$x = -a_{n-1} - a_{n-2} x^{-1} - \cdots - a_0 x^{1-n},$$

so $x \in V$, a contradiction. Hence $V = A$ is normal. □

**Corollary 21.9.** Let $C$ be a curve over a field $k$, then $C$ is normal if and only if for each non-generic point $x \in C$, $\mathcal{O}_{C,x}$ is a discrete valuation ring.

*Proof.* Take a nongeneric point $x \in C$, then the local ring $\mathcal{O}_{C,x}$ is a local noetherian integral domain of Krull dimension 1, so by Theorem 21.8 we see $\mathcal{O}_{C,x}$ is normal if and only if it is a discrete valuation ring. The converse is simple, but we do have to notice $\mathcal{O}_{C,\eta} = k(\eta)$ is a field which is normal. $\square$

## 22 Function Fields 17/01/2017

We start todays lecture with the definition of a Dedekind domain.

**Definition 22.1.** *A Dedekind domain is a normal, noetherian, integral domain such that* $\dim \operatorname{Spec} A = 1$.

Notice this implies that $\operatorname{Spec} A$ has a generic point and a closed point, and for all $x \in \operatorname{Spec} A$ not the generic point we have $A_x$ is a discrete valuation ring.

Also, if $C$ is a normal curve over a field $k$, then for all $U = \operatorname{Spec} A \subseteq C$, the ring $A$ is a Dedekind domain.

**Proposition 22.2.** *If $K$ is a finite extension of $\mathbb{Q}$ and the integral closure of $\mathbb{Z}$ in $K$ is $\mathcal{O}_K$, then $\mathcal{O}_K$ is a Dedekind domain.*

*Proof.* This is classical commutative algebra, and we won't really use this result also. □

There is an important analogy in arithmetic geometry, between the number field case ($\operatorname{Spec} \mathcal{O}_K$ where $K$ is a number field), and the function field case (curves over a field). Using number theory like the Langlands programs, we can take information from the number field case and make statements about the function field case, and the same can be done the other way around.

**Definition 22.3.** *Let $X$ is a reduced and irreducible scheme with some generic point $\eta \in X$, then the function field of $X$ is $K(X) = k(\eta)$ is the residue field of $X$ at $\eta$.*

**Remark 22.4.** If $X$ is integral (the above case), then $|X|$ has a unique generic point, so this is a good definition. This is because $|X|$ is $T_0$, so any generic point is unique, and to construct a generic point, we need only look at an affine open $\varnothing \neq V = \operatorname{Spec} A \subseteq X$ open and affine, then $A$ is an integral domain with unique generic point $\eta_V$, which infact lifts over all of $X$ to a unique generic point. The uniqueness argument shows this construction is independent on the choice of $V$. The comment also implies that $K(X) = Frac(A)$ for any such $\varnothing \neq V = \operatorname{Spec} A \subseteq X$.

**Proposition 22.5.** *Let $X$ be an integral scheme of finite type over a field $k$, with function field $K = K(X)$. Then $\dim(X)$ is equal to the transcendence degree of $K/k$. In particular, for all non-empty opens $U \subseteq X$, $\dim U = \dim X$, since $\eta$ exists in all such opens.*

**Remark 22.6.** Less of a remark, and more of a recall. Given any field extension $k \subseteq K$, one can find a transcendence basis $x_i \in K, i \in I$ such that $k[x_i | i \in I]$ injects into $K$, and $k(x_i | i \in I) \hookrightarrow K$ is a normal field extension. Moreover, the cardinality of $I$ is independent of our choice of transcedence basis, and we then set,

$$\operatorname{trdeg}(K/k) = |I|.$$

*Proof of Proposition 22.5.* Let $U$ be a non-empty open affine of $X$, so $U = \operatorname{Spec} A$. Using Noether normalisation choose a finite injective map $k[X_1, \ldots, X_n] \to A$, then $\dim \operatorname{Spec} A = n$ and $k(X_1, \ldots, X_n) \to \operatorname{Frac} A = K$, so the transcendence degree of $K$ over $k$ is equal to $n = \dim \operatorname{Spec} A$. Thus, $\dim X = \sup_{\varnothing \neq U \subseteq X} (\dim U)$ which is equal to the supremum of the transcendence degrees $K/k$ for which we have nomore $U$ dependence, so $\dim X$ is the transcendence degree of $K/k$,

$$\dim X = \sup_{\varnothing \neq U \subseteq X} \dim U = \sup_{\varnothing \neq U \subseteq X} \operatorname{trdeg}(K/k) = \operatorname{trdeg}(K/k).$$

□

In in the converse direction we have the following result.

**Proposition 22.7.** *Given a finitely generated field extension $k \subseteq K$ (so there exists a finite set in $K$ such that $K$ is generated by $k$ and this finite set as a field, so $K$ is equal to the fraction field of $k$ adjoint these elements), then there exists an integral scheme of finite type over $k$ such that $K(X) = K$.*

*Proof.* We choose generators $x_i \in K$ and let $A$ be the image of $k[x_i \mid i \in I] \to K$, then $A$ is finitely generated over $k$, an integral domain, and $\operatorname{Frac} A = K$, so we take $X = \operatorname{Spec} A$. $\square$

Talking about curves again, we have the following immediate corollary.

**Corollary 22.8.** A field extension $K/k$ is the function field of a curve if and only if $K/k$ is finitely generated and $\operatorname{trdeg}(K/k) = 1$.

We want to describe the curve $C$ through it's function field $K(C)$. We want to reduce the geometry of $C$ to the algebra of $K(C)/k$.

**Proposition 22.9.** *Let $C$ be a normal curve over a field $k$ and $K = K(C)$. There is a natural map $\nu : |C| \to \operatorname{Spa}(K, k)$, where the latter is the set of all valuation rings $V$ with $k \subseteq V \subseteq K$ and $\operatorname{Frac}(V) = K$ (the adic spectrum), defined by $x \mapsto \mathcal{O}_{C,x}$.*

*Proof.* There is not really anything to prove here. $\square$

Notice that if $x = \eta$ we have $\mathcal{O}_{C,\eta} = K$. Later we are going to topologise $\operatorname{Spa}(K, k)$ and prove some interesting things about it, but for now we just need to map into it.

**Example 22.10.** Given an algebraically closed field $k$ and $C = \mathbb{P}^1_k \supseteq U_0 = \mathbb{A}^1_k = \operatorname{Spec} k[t]$, then $K(C) = k(t)$. What is $\nu$ in this case? Well $|C|$ consists of a generic point $\eta$, a point at infinity $\infty$, and $\mathbb{A}^1_k(k) = k$. The codomain, $\operatorname{Spa}(K, k)$ of valuation rings with certain properties is equivalent to valuations $v : K \to \mathbb{Z} \cup \{\infty\}$, where $v(k^\times) = 0$. The map $\nu$ then sends $\eta \mapsto K$, and $x \in k$ to the (discrete) valuation $v_x : k(t) \to \mathbb{Z} \cup \{\infty\}$. We define this valuation on $f \in k[t]$ as the order of zeros of $f$ at $t = x$ (here we use the fact that we can factorise $f$ into linear factors over $k$). The natural map $\nu$ then sends $\infty$ to $-\deg f$. The intuition for this is that to evaluate $f \in k[t]$ at $\infty$, we need to make the substitution $u = 1/t$, and we then obtain $f(u) = u^{-n} + a_{n-1}u^{-n+1} + \cdots + a_0$, which has a pole of order $n$ at $u = 0$ which corresponds to $t = \infty$. Hence $v_\infty(f) = -n$.

**Proposition 22.11** (Product Formula)**.** *Let $k$ be an algebraically closed field. For all $0 \neq f \in K(X)$ with $X = \mathbb{P}^1_k$, we have $\sum_{x \neq \eta \in X} v_x(f) = 0$.*

First note that this sum is finite since $v_x(f)$ is zero almost everywhere, since $f$ only has finitely many zeros. Why is this called the product formula? There are only sums involved? We'll explain this in a remark shortly after the proof.

*Proof.* If $f = g/h$ with $g, h \in k[t]$ and $g, h \neq 0$, then for all $x \in X$ we have $v_x(f) = v_x(g) - v_x(h)$, so it is enough to prove this for $f \in k[t] \subseteq K$. Write $f$ as $c(t - x_1)^{n_1} \cdots (t - x_k)^{n_k}$. In this case we have two contributions in the following sum, from $x \in k$ and $x = \infty$,

$$\sum_{x \neq \eta \in X} v_x(f) = \sum_{i=1}^{k} n_i - \deg f = n - n = 0.$$

$\square$

**Definition 22.12.** *A multiplicative seminorm on a ring $A$ is a map $|-| : A \to \mathbb{R}_{\geq 0}$ such that $|1| = 1$, $|0| = 0$, $|xy| = |x||y|$, and $|x + y| \leq |x| + |y|$. This is called a norm if in addition we have $|x| = 0$ implies that $x = 0$. A seminorm is nonarchimedean if the stronger triangle inequality $|x + y| \leq \max(|x|, |y|)$ holds.*

**Remark 22.13.** Given a ring $A$, and a discrete valuation $v : A \to \Gamma \cup \{\infty\}$, then we have a map $\Gamma \to \mathbb{R}_{>0}$ of abelian groups reversing the partial order. For example $\mathbb{Z} \to \mathbb{R}_{>0}$ by the map $n \mapsto c^n$ for some $c \in (0,1)$, then the composition $A \to \Gamma \cup \{\infty\} \to \mathbb{R}_{\geq 0}$ is a nonarchimedean seminorm. In our specific case, each $v_x$ gives rise to a nonarchimedean seminorm $K \to \mathbb{R}_{\geq 0}$ defined by $f \mapsto e^{-v_x(f)}$. In this way Proposition 22.11 becomes,

$$\prod_{x \neq \eta \in X} |f|_x = 1,$$

for all $0 \neq f \in K$.

There is an analogy between number fields and function fields, through a theorem by Ostrowski.

**Theorem 22.14.** *The norms on $\mathbb{Q}$ are, up to equivalence given by:*

1. *The trivial norm, with $|0| = 0$ and $|x| = 1$ for all $x \in \mathbb{Q}^\times$.*
2. *The real norm, $|x|_\mathbb{R} = x$ if $x \geq 0$ and $|x|_\mathbb{R} = -x$ if $x < 0$.*
3. *The p-adic norm, so given a prime $p$ we define $|x|_p = p^{-v_p(x)}$, where $v_p(x)$ is the number of times $p$ divides $x$.*

Again we have a product formula, so for all $0 \neq x \in \mathbb{Q}$ we have,

$$\prod_{p \text{ prime}} |x|_p \cdot |x|_\mathbb{R} = 1.$$

A proof of this formula is simply to write $x = \pm \prod_p p^{v_p(x)}$, then we see that

$$\prod_{p \text{ prime}} |x|_p \cdot |x|_\mathbb{R} = \prod_p p^{-v_p(x)} \cdot \prod_p p^{v_p(x)} = 1.$$

Okay, enough talk, now we have some more important mathematics to work through.

**Proposition 22.15.** *A curve $C$ is separated if and only if $\nu$ from Proposition 22.9 is injective. A curve $C$ is proper (over $k$) if and only if $\nu$ is bijective.*

We will prove this next lecture, using the valuation criterion. A consequence of this is an analogue of Ostrowski's theorem for $\mathbb{P}^1_k$.

**Corollary 22.16.** For an algebraically closed field $k$, the valuations on $k(t)$ such that $v(k^\times) = 0$ are, up to equivalence given by,

1. $v(0) = \infty$, $v(f) = 0$ for all $f \in k(t)$.
2. $v_\infty$ described in Example 22.10.
3. $v_x$ for all $x \in k$ described in Example 22.10.

*Proof.* The curve $\mathbb{P}^1_k$ is proper, so the space of valuations $\mathrm{Spa}(k(t), k)$ is given by $|\mathbb{P}^1_k| = \{\eta, \infty\} \cup k$, by Proposition 22.15. The is discussed more in Example 22.10. □

## 23 Topology on the Adic Spectrum 19/01/2017

Our next goal in this course is to prove the theorem which states that the category of proper normal curves over a field $k$, and dominant maps is equivalent to the category of finitely generated field extensions $K/k$ with transcendence degree 1. The functor from left to right takes a curve $C$ and sends it to its function field $K(C)$. Today though, we are going to first prove Proposition 22.15, for which we need a quick lemma.

**Lemma 23.1.** *Let $K/k$ be a finitely generated field extension of transcendence degree 1, and let $k \subseteq V \subseteq K$ be a valuation ring with $\operatorname{Frac} V = K$, then either $V = K$ or $V$ is a discrete valuation ring.*

*Proof.* Let $A \subseteq V$ be a finitely generated $k$-algebra such that $\operatorname{Frac} A = K$, then $V$ is normal implies that without loss of generality we can take $A$ to be normal. Hence $\dim \operatorname{Spec} A = 1$ since this is the transcendence degree of our field extension, so $\operatorname{Spec} A$ is a normal curve over $k$. The inclusion $A \subseteq V$ gives us a map $\operatorname{Spec} V \to \operatorname{Spec} A$, from which we let $x \in \operatorname{Spec} A$ be the image of the closed point $s \in \operatorname{Spec} V$. We then obtain a map $A_x \to V$ by factorising our localisation, and this is a local map. We already know $A_x$ is a discrete valuation ring. Recall then that if $W \subseteq K$ is a local ring, then $W$ is a valuation ring if and only if for each $W \subseteq W' \subseteq K$ with $W'$ a local ring and $W \subseteq W'$ a local map, we have $W = W'$ (part 3 of Theorem 19.1). This implies that $V = A_x$. $\square$

*Proof of Proposition 22.15.* Assume that $x, x' \in C$ with $\mathcal{O}_{C,x} = \mathcal{O}_{C,x'} =: V \subseteq K$, then we obtain two maps $\operatorname{Spec} V \to C$ sending the closed point of $\operatorname{Spec} V$ to $x$ and $x'$ respectively.[37] Hence we obtain the following commutative diagram.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & C \arrow[d] \\ \operatorname{Spec} V \arrow[r] \arrow[ur] & \operatorname{Spec} k \end{tikzcd}$$

The valuation criterion of separatedness say that our two maps must be equal, so $x = x'$. If we now assume that $C$ is proper over $k$, then we already know that $\nu$ is injective, so for surjectivity, let $k \subseteq V \subseteq K$ be a valuation ring with $Frac(V) = K$, then we have the following commutative diagram.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & C \arrow[d] \\ \operatorname{Spec} V \arrow[r] \arrow[ur, dotted, "l"] & \operatorname{Spec} k \end{tikzcd}$$

The valuation criterion of properness gets us the unique lift $l : \operatorname{Spec} V \to C$, which corresponds to $x \in C$ and a local map $\mathcal{O}_{C,x} \to V$ within the fraction field $K$. Using the dominance theorem for valuation rings, we see that this implies that $V = \mathcal{O}_{C,x}$, which implies that $V$ is in the image of $\nu$, as required.

Conversely, we're again going to use the valuation criterion, so consider the diagrams of the form,

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & C \arrow[d] \\ \operatorname{Spec} V' \arrow[r] \arrow[ur, dotted, "l"] & \operatorname{Spec} k \end{tikzcd},$$

and we want to know when we have a lift $l$, and how many such $l$'s exist. Let $x \in |C|$ be the image of $|\operatorname{Spec} K'| \to |C|$. We claim that if $x \neq \eta$, then there is always a unique lift $l$. If $x \neq \eta$, then $x$ is a closed point (as the dimension of $C = 1$), so let $Z \subseteq C$ be the reduced closed subscheme with

[37] Recall that if $A$ is a local ring, and $X$ is a scheme, then $X(A) = \{(x, \alpha) \mid x \in X, \mathcal{O}_{X,x} \to A \text{ local ring homomorphism}\}$.

$|Z| = \{x\} \subseteq |C|$. Then $Z$ is 0-dimensional, reduced, connected, so $Z = \operatorname{Spec} k'$ where $k'$ is a finite field extension of $k$. The map $\operatorname{Spec} K' \to C$ factors through $\operatorname{Spec} k' = Z$ set theoretically, so it does scheme theoretically since $K'$ is reduced. Also, any extension $\operatorname{Spec} V' \to C$ must also factor over $Z$, this time because $\operatorname{Spec} K' \subseteq \operatorname{Spec} V'$ is dense and $Z$ is closed, and $V'$ is also reduced. Hence we need to check there is a unique dotted arrow in the following diagram.

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} k' \arrow[d] \\
\operatorname{Spec} V' \arrow[r] \arrow[ur, dotted] & \operatorname{Spec} k
\end{tikzcd}$$

Since we have affine schemes, this is equivalent to,

$$\begin{tikzcd}
k' \arrow[r] \arrow[dr] & K' \\
k \arrow[r] \arrow[u] & V' \arrow[u]
\end{tikzcd}.$$

At most one map exists if and only if $k' \subseteq V'$, but $k'/k$ is a finite extension so $k'/k$ is algebraic, and hence integral. Notice as $V'$ is normal, $k \subseteq V'$ so we have $k' \subseteq V'$ as desired.

We now consider the case when $x = \eta$, so $k(x) = k(\eta) = K$. We want to find $f$ in the following diagram.

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[r] \arrow[d] & C \arrow[d] \\
\operatorname{Spec} V' \arrow[r] \arrow[ur, dotted, "f"] & \operatorname{Spec} k
\end{tikzcd}$$

Any extension $f$ is given by a map $\operatorname{Spec} V' \to C$, which is equivalent to give $y \in C$ and a local map $\mathcal{O}_{C,y} \to V'$. Let $V = V' \cap K \subseteq K$, then $V$ is again a valuation ring with $Frac(V) = K$. Indeed, for all $a \in K^\times \subseteq (K')^\times$, one of $a, a^{-1} \in V'$, which implies that one of $a, a^{-1} \in V$. This gives us a local map $\mathcal{O}_{C,y} \to V$ of local rings, thus the set of all $f$'s,

$$\{f\} = \{y \in C \mid \mathcal{O}_{C,y} = V = V' \cap K \subseteq K'\}.$$

Thus, if $\nu$ is injective we have at most one $f$, which implies that $C$ is separated. If $\nu$ is bijective, then we have exactly one $f$, which means $C$ is proper over $k$. $\square$

In particular, if $C$ is a proper normal curve over $k$, we have $|C| \cong \operatorname{Spa}(K, k)$. We could define the topology on $\operatorname{Spa}(K, k)$ by simply taking the topology induced by $\nu$, but there is a more independent definition.

**Definition 23.2.** *We give* $\operatorname{Spa}(K, k)$ *the topology generated by open sets of the form,*

$$\{V \subseteq K \mid x_1, \ldots, x_n \in V\},$$

*for some varying collection of* $x_i \in K$.

This can be done for any field $k$, and it is called the Zariski-Riemann space. A non-trivial theorem states that $\operatorname{Spa}(K, k)$ is homeomorphic to the inverse limit of $|X|$ where $X$ varies over all integral, projective schemes over $k$ with $K(X) = K$. If we look at this inverse limit in higher dimensions ($\neq 1$) then this inverse limit is crazy huge (c.f. blow ups). If $n = 1$, then the natural map $\nu$ is the homeomorphism from $\operatorname{Spa}(K, k)$ to this inverse limit.

**Lemma 23.3.** *Assume $K/k$ is a finitely generated field extension, and the transcendence degree is equal to 1. Then* $\mathrm{Spa}(K,k)$ *is a spectral space*[38] *of dimensional one with a unique generic point $\eta$, and all other points closed. In fact, a non-empty open set of* $\mathrm{Spa}(K,k)$ *is equivalent to $\eta \in U$ and* $\mathrm{Spa}(K,k)\backslash U$ *is finite.*

Some of these facts hold in more generality, like the transcendence degree of $K/k$ is equal to the dimension of $\mathrm{Spa}(K,k)$.

*Proof.* We will begin by showing $\mathrm{Spa}(K,k)$ has a unique generic point.

**(Unique Generic Point)** Given $U \neq \varnothing$ open in $\mathrm{Spa}(K,k)$, then $U$ is a union of generating

$$\{V \subseteq K \mid x_1,\dots,x_n \in V\},$$

and these open subsets all contain $\{K\}$, our generic point.

**(All Other Points are Closed)** Let $W \subsetneq K$ be a valuation ring over $k$ with $\mathrm{Frac}\, W = K$, then consider,

$$Z = \{V \mid \forall x \in K\backslash W, x \notin V\} \subseteq \mathrm{Spa}(K,k),$$

which is closed. Assume $V \in Z$, then by the construction of $Z$, $V \subseteq W$. Moreover, $V$ is a valuation ring by Lemma 23.1. We have the nested inclusions $V \subseteq W \subseteq K$ and this gives us,

$$\mathrm{Spec}\, K \to \mathrm{Spec}\, W \to \mathrm{Spec}\, V.$$

Since $V \subseteq K$ is the canonical inclusion, we see that one of the above maps is an equality, i.e. a local map. Thus either $V = W$ or $W = K$, but we assumed $W \neq K$ so $W = V$ and $Z = \{W\}$ (since clearly $\{W\} \subseteq Z$).

**(Characterisation of Open Sets)** From the previous part, we see that if $U \subseteq \mathrm{Spa}(K,k)$ with $\eta \in U$ and $\mathrm{Spa}(K,k)\backslash U$ finite, then $U$ is open, as the complement of $U$ is a finite union of closed points. Conversely, we need to see that if $U$ is a non-empty open subset of $\mathrm{Spa}(K,k)$, then the complement of $U$ is finite. We may assume,

$$U = \{V \mid x_1,\dots,x_n \in V\} = \bigcap_{i=1}^{n} \{V | x_i \in V\},$$

for some $x_1,\dots,x_n \in K$, so we may reduce to the case where $U = \{V | x \in V\}$, for some $x \in K$. Then we set $Z = \mathrm{Spa}(K,k)\backslash U$. Then $Z = \{V | x^{-1} \in \mathfrak{m}_V\}$, since $x \notin V$ so $v_V(x) < 0$, which implies $v_V(x^{-1}) > 0$ so $x^{-1} \in \mathfrak{m}_V$. Let $A = k[x^{-1}] \subseteq K$, and let $\widetilde{A}$ be the normalisation of $A$ in $K$. For all $V \in Z$, $A = k[x^{-1}] \subseteq V$ which implies that $\widetilde{A} \subseteq V$, since $V$ is normal as all valuations rings are. From this we obtain a map $\mathrm{Spec}\, V \to \mathrm{Spec}\, \widetilde{A}$ into a normal curve over $k$. This is essentially given by $s \in \mathrm{Spec}\, \widetilde{A}$, from which we obtain a local map $\widetilde{A}_s \to V$. Since $\widetilde{A}_s$ is a valuation ring, then $V = \widetilde{A}_s$, which means that $x^{-1} \in \mathfrak{m}_V$ implies that $s \in V(x^{-1})$, the vanishing locus of $x^{-1}$ inside $\mathrm{Spec}\, \widetilde{A}$. This discussion implies the map $V(x^{-1}) \to Z$ is surjective, and the domain finite, hence $Z$ is finite. This finishes our characterisation of open subsets.

**(Spectral Space)** From the characterisation of open subsets, we see that $\mathrm{Spa}(K,k)$ is quasi-compact, since each non-empty open subset has almost all the points of $\mathrm{Spa}(K,k)$. In fact, any open subset is quasi-compact by this same argument. Recall that to be spectral means to be quasi-compact, have a quasi-compact basis closed under intersections, and every irreducible closed subset has a unique generic point. The only thing left to prove is this final piece. However, the irreducible closed subsets of $\mathrm{Spa}(K,k)$ are itself, which has a generic point $\eta$, and points, which are their own generic points. $\square$

[38] Recall the definition of a spectral space from Definition 3.12

Next is a lemma that we need to prove, otherwise all of our $\mathrm{Spa}(K,k)$ and $\nu$ would not be connected.

**Lemma 23.4.** *If $C$ is a normal curve over a field $k$ with $K = K(C)$, then $\nu$ is continuous and open.*

In particular it is an open embedding if $C$ is separable and a homeomorphism if $C$ is proper.

*Proof.* These questions are all local, so we assume that $C = \operatorname{Spec} A$.

**(Continuity)** For continuity, check the preimage of closed sets are closed. We've seen though that a non-trivial closed subset of $\mathrm{Spa}(K,k)$ is a finite union of closed points, so without loss of generality let $Z = \{V\}$ with $V \neq K$. Then the preimage is at most one point of $C = \operatorname{Spec} A$ since $C$ is separated and $\nu$ is injective (recall Proposition 22.15), which is not the generic point, thus closed.

**(Openness)** Given $x_1, \ldots, x_n \in A$ which generate the $k$-algebra $A$, then the image of $\nu$ is

$$\{V \mid x_1, \ldots, x_n \in V\}.$$

Indeed, if $V$ is in the image of $\nu$, then $V = \mathcal{O}_{C,s}$ for some $s \in C$, so $A \ni x_1, \ldots, x_n$. Conversely, if $x_1, \ldots, x_n \in V$ we obtain a map $A \to V$ which factors through $A_s \to V$ for some $s \in C = \operatorname{Spec} A$. We then get that $V = A_s$ by the domination of valuation rings, so $V$ is in the image of $\nu$. Hence the image of $\nu$ is open. $\square$

# 24 Proper Normal Curves are Field Extensions 24/01/2017

Let $k$ be a field, then we want to prove:

**Theorem 24.1.** *The category of proper normal curves over $k$ with dominant morphisms, and finitely generated field extensions $K$ of transcendence degree 1 are equivalent. The functor from left to right sends a curve $C$ to its function field $K(C)$.*

The following is a quick proposition to tell us that only considering dominant maps is not that restrictive.

**Proposition 24.2.** *Given a map $f : C_1 \to C_2$ of curves, then either $f$ is dominant or there is some $x \in C_2$ such that $f$ factors through $\operatorname{Spec} k(x) \subseteq C_2$.*

*Proof.* Assume $f$ is not dominant, so $\eta_2$ is not in the image of $f$. Then the image of $f$ is contained in $C_2 \backslash \{\eta_2\}$, hence the image of $f$ is a quasi-compact subscheme of dimension 0, so the image of $f$ is just a discrete set of points (by Lemma 24.3 below). However $C_1$ is connected as curves are irreducible, so the image of $f$ is just a point $x \in C_2$. Let $Z = \operatorname{Spec} k(x) \subseteq C_2$ be the corresponding reduced closed subscheme, then $C_1$ is reduced and $f : C_1 \to C_2$ factors over $\{x\} \subseteq C_2$ topologically. Lemma 24.4 then tells us that $f$ factors over $Z$ as schemes. □

**Lemma 24.3.** *Let $f : Y \to X$ be a quasi-compact map of schemes, where $X$ is irreducible, such that $\eta \notin im(f)$, where $\eta$ is the generic point of $X$. Then there exists an open dense subset $U$ of $X$ such that $im(f) \cap U = \varnothing$.*

*Proof.* We may assume $X = \operatorname{Spec} A$ is affine and reduced (for topological reasons). Then $Y = \bigcup_i \operatorname{Spec} B_i$ for some finite collections of open affines, where $B_i$ are $A$-algebras. Let

$$f_i = f|_{\operatorname{Spec} B_i} : \operatorname{Spec} B_i \to \operatorname{Spec} A,$$

then if we can find a dense open $U_i \subseteq \operatorname{Spec} A$ such that $U_i \cap im(f_i) = \varnothing$, then $U = U_1 \cap \cdots \cap U_n \subseteq \operatorname{Spec} A$ is still a dense open and $U \cap im(f) = \varnothing$. Hence we may restrict to each $f_i$, hence we may assume that $Y = \operatorname{Spec} B$, for some $A$-algebra $B$. Let $K = \operatorname{Frac} A$, then $\eta \notin im(f)$ if and only if

$$\operatorname{Spec}(B \otimes_A K) = \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} K = \varnothing,$$

which is equivalent to $B \otimes_A K = 0$, which is equivalent to $1 = 0$ inside $B \otimes_A K = B[(A \backslash \{0\})^{-1}]$. This condition is the same as there exists $a \in A$ and $a \neq 0$ with $a = 0$ in $B$, in other words $0 = B[a^{-1}]$. In this case, we obtain $\operatorname{Spec} B \times_{\operatorname{Spec} A} D(a) = \varnothing$, where $D(a)$ is a dense open subset of $\operatorname{Spec} A$. This implies $im(f) \cap D(a) = \varnothing$. □

**Lemma 24.4.** *Let $X$ be any scheme, $Z \subseteq X$ some reduced closed subscheme, then if $Y$ is any reduced scheme with a map $f : Y \to X$, then $f$ factors over $Z$ if and only if $|f|$ factors over $|Z|$.*

*Proof.* This can be checked locally in $Y$, so we may assume $Y = \operatorname{Spec} B \to U = \operatorname{Spec} A \subseteq X$, in which case $Z \cap U = \operatorname{Spec} A/I$ for some radical ideal $I$ of $A$. It is clear that if $f$ factors over $Z$, then we have the topological statement. Conversely, if $|f|$ factors over $|Z|$, then for all $a \in I$, $a$ vanishes at all points of $im(f)$, so the image of $a$ in $B$ vanishes everywhere. Since $B$ is reduced this implies that $a$ is sent to 0 in $B$, so $I \mapsto 0$ in $B$. Hence $A \to B$ factors over $A/I$ which is what we wanted to prove. □

The following proof will require a few intermediate lemmas, which we will include in the body of the proof.

*Proof of Theorem 24.1.* We want to construct an inverse functor, so given some $K$, we need to construct a locally ringed topological space $C = (|C|, \mathcal{O}_C)$ such that $C$ is a proper normal 1-dimensional integral scheme of finite type over $k$. We do that by letting $|C| = \operatorname{Spa}(K, k)$ which we defined last lecture. Last time we also showed this was a spectral space, it was 1-dimensional and irreducible (amoungst other things). Now we have to define $\mathcal{O}_C$. For any non-empty open subset $U \subseteq |C| = \mathrm{Spa}(K,k)$, we define,

$$\mathcal{O}_C(U) = \{a \in K \mid \forall x \in U, \quad x = V \subseteq K, \ a \in V\},$$

so $V \in U$.

**Lemma 24.5.** *$\mathcal{O}_C$ is a sheaf of $k$-algebras. For all $x \in |C|$ (so $V \subseteq K$) we have $\mathcal{O}_{C,x} = V$.*

*Proof of Lemma 24.5.* To show that $\mathcal{O}_C$ is a sheaf, we take $U = U_1 \cup \cdots \cup U_n$, and check the required equaliser.

$$\textstyle\prod_{i=1}^n \mathcal{O}_C(U_i) \rightrightarrows \prod_{i,j=1}^n \mathcal{O}_C(U_i \cap U_j)$$

There is an injection from $\prod_{i,j} O_C(U_i \cap U_j)$ into $\prod_{i,j} K$, so the equaliser above is equal to,

$$\{(a_1, \dots, a_n) \in K^n \mid \forall x \in U_i, \ x = V_i \subseteq K, \ a_i \in V_i, \text{ and } \forall i, j, a_i = a_j\},$$

where the last condition $a_i = a_j$ for all $i, j$ comes from our injection mentioned above. This is equal to,

$$\{a \in K \mid \forall V \subseteq K, a \in V\} = \mathcal{O}_C(U).$$

To check this property of the stalks, we take $x \in |C|$ which corresponds to a $V \subseteq K$, then $\mathcal{O}_{C,x} \subseteq V$ from the definition. Conversely, let $a \in V$, then $U = \{W \mid a \in W\} \subseteq |C|$ is an open subset and contains $x$, and $a \in \mathcal{O}_C(U)$ so we have $a \in \mathcal{O}_{C,x}$. $\square$

In particular, all $\mathcal{O}_{C,x}$ are local rings, so $C = (|C|, \mathcal{O}_C)$ is a locally ringed space.

**Lemma 24.6.** *Let $C'$ be a normal curve over $k$ with $K(C') = K$, then there is a unique map of locally ringed spaces $f : C' \to C$ such that the following diagram commutes.*

$$\begin{tikzcd} C' \arrow[r, "f"] & C \\ \operatorname{Spec} k \arrow[u] \arrow[ur] & \end{tikzcd}$$

*Proof of Lemma 24.6.* We have a map $\nu : |C'| \to \mathrm{Spa}(K,k) = |C|$ which is continuous. Any $f$ needs to topologically be $\nu$. If $x \in |C'|$, then $f_x^\# : \mathcal{O}_{C,f(x)} \to \mathcal{O}_{C',x}$ is a local map over $K$, and $\mathcal{O}_{C,f(x)}$ is a valuation ring, so by our usual argument we have $\mathcal{O}_{C,f(x)} = \mathcal{O}_{C',x}$, hence $f(x)$ corresponds to $\mathcal{O}_{C',x} \subseteq K$. To define $f^\# : f^{-1}\mathcal{O}_C \to \mathcal{O}_{C'}$, we need to define a map $\mathcal{O}_C(U) \to \mathcal{O}_{C'}(U')$ where $U'$ is the preimage of some open subset $U \subseteq C$ under $\nu$.

We may assume that $U = \operatorname{Spec} A$ and $U' = \operatorname{Spec} A'$ are some affines. In this case we have $\operatorname{Frac} A' = K(C') = K$ and $A' = \{a \in K \mid \forall x \in \operatorname{Spec} A', a \in A'_x\}$, which holds for any integral domain by the sheaf property of $\mathcal{O}_{\operatorname{Spec} A}$. Now we see,

$$A \supseteq \{a \in K \mid \forall y \in U, \ x = V \subseteq K, \ a \in V\} = \mathcal{O}_C(U),$$

which gives us our desired map. $\square$

**Remark 24.7.** Moreover, if the map $U' \to U$ from the proof of Lemma 24.6 is bijective, then $A' = \mathcal{O}_{C'}(U') = \mathcal{O}_C(U)$, which immediately implies the following lemma.

**Lemma 24.8.** *If in the situation of Lemma 24.6, the curve $C'$ is also separated, then $f : C' \to C$ is an open immersion.*

*Proof of Lemma 24.8.* The map $f$ is an open immersion if and only if it is topologically an open immersion and $f^\# : f^{-1}\mathcal{O}_C \to \mathcal{O}_{C'}$ is an isomorphism. Since we know that $f$ is $\nu$, then we have this first condition from Lemma 23.4, and the second part is from Remark 24.7. $\square$

**Lemma 24.9.** *The $C$ we have defined is a proper, normal curve over $k$.*

*Proof of Lemma 24.9.* Choose any normal finitely generated $k$-algebra $A \subseteq K$ with $Frac(A) = K$, then $C' = \operatorname{Spec} A$ is a separated normal curve over $k$ with $K(C') = K$, so by Lemma 24.8 we obtain an open immersion $f : C' \to C$, with $im(f) = \{V | A \subseteq V\}$ (we saw this last lecture). These open subsets $C'$ actually cover $C$, as given $x \in \operatorname{Spa}(K, k)$, so some $V \subseteq K$, we can find an $A$ such that $A \subseteq V$.[39] This implies that $C$ is a scheme of finite type over $k$. It is 1-dimensional and irreducible as this is true for the topological space $|C|$. It is reduced as all $A$ are reduced. The object $C$ is normal from Lemma 24.5 as $\mathcal{O}_{C,x} = V$ are all local valuations rings, hence normal. It is proper as by construction $\nu$ is a bijection, so we then apply Proposition 22.15. □

We thus have a functor $K \mapsto C(K) = \left(\operatorname{Spa}(K, k), \mathcal{O}_{\operatorname{Spa}(K,k)}\right)$. This functor and $C \mapsto K(C)$ are inverse equivalences, since $K(C(K)) = K$ by construction, and $C \to C(K(C))$ is an open immersion by Lemma 24.8, so we only need to check it is an isomorphism of spaces. However $C$ is proper so $\nu : |C| \to \operatorname{Spa}(K(C), k) = |C(K(C))|$ is a homeomorphism. □

**Remark 24.10.** We have constructed a scheme $C(K)$ completely abstractly. We have only used algebra to do this, rather than writing down local equations and then gluing.

This big theorem we have just proved says that any open normal curve knows how it wants to compactitfy (embed into something proper), namely there is a unique proper normal curve containing it as a dense open subset.

**Theorem 24.11.** *Let $C$ be a proper curve over a field $k$, then $C$ is projective. In fact we have $C \hookrightarrow \mathbb{P}^3_k$, definitely when $k$ is algebraically closed, but Peter also believes this holds for all fields $k$.*

**Example 24.12** (Elliptic Curves)**.** Let $k$ be algebraically closed and characteristic not equal to 2, and consider $f = y^2 - x^3 - ax - b$ for $a, b \in k$ and $\Delta = 4a^3 + 27b^2 \neq 0$.

**Proposition 24.13.** *For the curve $C = \operatorname{Spec} A$ for $A = k[x, y]/(f)$, then $C$ is normal if and only if $\Delta \neq 0$.*

*Proof.* We notice that $A$ is normal is equivalent to $C$ being smooth, so we apply the Jacobian criterion[40] for smoothness and complete our proof. □

In this case the normal compactification of $C$ is given by $V\left(Y^2Z - X^3 - aXZ^2 - bZ^3\right) \subseteq \mathbb{P}^2_k$.

**Example 24.14** (Hyperelliptic Curves)**.** In the hyperelliptic case, $x = x^{2n+1} + \cdots + a_1x + a_0$ for $n \geq 2$, then the normal compactification is not the homogenisation like we have done above, but rather something else which we have constructed before by hand in Example 9.4 and refered to in Example 15.3.

[39] To do this we take $a_1, \ldots, a_n \in K$ we generate $K$ as a field over $k$, and we may assume that $a_i \in V$ by replacing $a_i$ by $a_i^{-1}$. We then let $A$ be the normalisation of $k[a_1, \ldots, a_n] \subseteq K$.

[40] Problem 4(ii) on exercise sheet 11 was the following: Assume $X = V(f_1, \ldots, f_r) \subseteq \mathbb{A}^n_k$ and write $x = (x_1, \ldots, x_n) \in X(k)$. We define the Jacobi matrix $J_x \in k^{r\times n}$ as we do in real analysis, where we take partial derivatives of $f \in k[X_1, \ldots, X_n]$ in the obvious way. Let $d$ be the Krull dimension of the local ring $\mathcal{O}_{X,x}$. Show that $x$ is a smooth point of $X$ if and only if $J_x$ has rank $n - d$.

# 25 An Example and Ample Line Bundles 26/01/2017

**Example 25.1.** Let $k = \mathbb{C}$ and $K = \operatorname{Frac}(\mathbb{C}[x,y]/(y^2 - x^n + 1))$ where $n \geq 2$. We can construct $C = C(K)$ as follows. Let $A = \mathbb{C}[x,y]/(y^2 - x^n + 1)$, which is a noetherian integral domain of Krull dimension 1. In fact, $A$ is normal. To show this, we check that all the $\mathbb{C}$-valued points are smooth. The non-smooth points satisfy $f(x,y) = 0$, $\frac{\partial f}{\partial x}(x,y) = nx^{n-1} = 0$, and $\frac{\partial f}{\partial y}(x,y) = 2y = 0$, where $f(x,y) = y^2 - x^n + 1$, but we can easily check no such $x, y \in \mathbb{C}$ exist. This $\operatorname{Spec} A \hookrightarrow C$ is an open immersion. We will try now to compactify $\operatorname{Spec} A$. First, we might try a compactification inside $\mathbb{P}^2_{\mathbb{C}}$, by looking at the scheme theoretic image of $\operatorname{Spec} A \subseteq \mathbb{A}^2_{\mathbb{C}} \subseteq \mathbb{P}^2_{\mathbb{C}}$, and this is precisely,

$$C' := V\left(Y^2 Z^{n-2} - X^n + Z^n\right) \subseteq \mathbb{P}^2_{\mathbb{C}}.$$

Then $\operatorname{Spec} A = C' \cap \mathbb{A}^2_{\mathbb{C}} = C' \cap U_Z \subseteq C' \subseteq \mathbb{P}^2_{\mathbb{C}}$, with both $\mathbb{P}^2_{\mathbb{C}}$ and $C'$ proper over $\operatorname{Spec} \mathbb{C}$. What is $C' \backslash \operatorname{Spec} A$?

$$C'(\mathbb{C}) = \{(X : Y : Z) \in \mathbb{C}^3 \backslash \{(0,0,0)\} \mid Y^2 Z^{n-2} - X^n + Z^n = 0\}/\mathbb{C}^\times,$$

then either $Z \neq 0$ or $Z = 0$ which would imply $X = 0$ and $Y = 1$ (up to scaling). Hence $(C' \backslash \operatorname{Spec} A)(\mathbb{C}) = \{(0 : 1 : 0)\}$. This extra point lies in $U_Y$, so

$$C' \cap U_Y = \operatorname{Spec} \mathbb{C}[x', z']/((z')^{n-2} - (x')^n + (z')^n).$$

If $C'$ was normal, then $C'$ is proper and normal with $K(C') = K$, so necessarily $C = C'$. However, in general $C'$ is not normal at $(0 : 1 : 0)$, this extra point we have added. Hence we need to compute the normalisation of $B = \mathbb{C}[x', z']/((z')^{n-2} - (x')^n + (z')^n)$. We will now do this. If $n = 2m + 2$ is even, then we have,

$$\left(\frac{x'^{m+1}}{z'^m}\right)^2 = (z')^2 + 1.$$

If we let $u = \frac{x'^{m+1}}{z'^m}$ and $v = \frac{z'}{x'}$. We now set $u = x'\left(\frac{x'}{z'}\right)^m = x'(v^{-1})^m$ which implies that $x' = uv^m$ and $z' = vx' = uv^{m+1}$. The defining equation of $B$ now becomes,

$$(uv^{m+1})^{2m} - (uv^m)^{2m+2} + (uv^{m+1})^{2m+2} = 0.$$

This is still not normal, so we take out some $u$ and $v$ factors, and obtain the equation,

$$g(u,v) = 1 - u^2 + u^2 v^{2m+2} = 0.$$

This gives a normal curve, which we can again check by checking the conditions $g(u,v) = 0$, $\frac{\partial g}{\partial u} = 0$ and $\frac{\partial g}{\partial v} = 0$. Thus we have,

$$C = \operatorname{Spec} A \cup \operatorname{Spec} \widetilde{B},$$

where $\widetilde{B}$ is the normalisation of $B$ we just calculated. What is interesting is,

$$C \backslash \operatorname{Spec} A = V_{\operatorname{Spec} \widetilde{B}}(v) = \operatorname{Spec} \mathbb{C}[u,v]/(v, 1 - u^2 - u^2 v^{2m+2}) = \operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C}.$$

Hence this compactification has two points, corresponding to $u = 1$ and $u = -1$, which we will call $\infty_1$ and $\infty_2$. When we draw the real valued points of this we have two symmetric curves either side of the $y$-axis, with $\infty_1$ obtained in the top right corner, and $\infty_2$ obtained in the bottom right corner. If $m$ if odd then $\infty_1$ is obtained in the top left corver, otherwise $\infty_2$ is obtained in the top left corner. Conversely, if $m$ is odd then $\infty_2$ is obtained in the bottom left corner, otherwise $\infty_1$ is obtained in the bottom right corner. Hence the parity of $m$ changes how our natural compactifcation tie up the loose ends of our original curve.

When $n = 2m + 1$ is odd, then there is only one point at infinty, which was already discussed in the hyperelliptic case. Both cases when $n$ is even and $n$ is odd are called hyper elliptic even though their compactifications differ.

Recall Theorem 10.7 and its extension to Corollary 12.10 which we've used in our exercise sheets a number of times. This gives us an idea that perhaps we can classify closed embedding into projective space (whether a scheme is projective or not) through the types of line bundles on a scheme.

**Definition 25.2.** *Let $X$ be a scheme of finite type over a field $k$, then $X$ is quasi-projectve if there exists a locally closed immersion into $\mathbb{P}^n_k$ for some $n \geq 0$.*

**Proposition 25.3.** *Let $X$ be a scheme of finite type over a field $k$, then projective implies $X$ is proper and quasi-projective, proper implies separated, quasi-projective implies separated, and affine implies quasi-projective. Moreover, $X$ is projective if and only if $X$ is proper and quasi-projective.*

In general none of these implications are reversable.

*Proof.* We've seen before that projective implies proper implies separated, and projective implies quasi-projective is also clear. Consider now that $X$ is quasi-projective, then we have a map $X \to \mathbb{P}^n_k$ which is a locally closed immersion. Since $\mathbb{P}^n_k$ is proper, and hence separated, and the map $X \to \mathbb{P}^n_k$ is separated, then the map $X \to \operatorname{Spec} k$ is separated by composition. If $X$ is affine, then $X = \operatorname{Spec} A$ and we have a surjection $k[X_1, \dots, X_n] \twoheadrightarrow A$, so $X$ is a closed subscheme of $\mathbb{A}^n_k \subseteq \mathbb{P}^n_k$, so $X \hookrightarrow \mathbb{P}^n_k$ is a locally closed immersion.

Proving the last statement now, we take $X$ to be proper and quasi-projective, then $f : X \hookrightarrow \mathbb{P}^n_k$ is a locally closed immersion. We claim that $f$ is a closed immersion.[41] To see this we can show that $f(X)$ is closed inside $\mathbb{P}^n_k$ simply as a topological space. We will show that the image $f(X)$ is stable under specialisations, so take $y \succ x$ inside $\mathbb{P}^n_k$ such that $x \in f(X)$. We then consider the following diagram,

$$\begin{tikzcd}
\operatorname{Spec} k(x) \arrow[r, "x"] \arrow[d] & \mathbb{P}^n_k \arrow[d] \\
\operatorname{Spec} V \arrow[r] \arrow[ur, dashed, "g"] & \operatorname{Spec} k
\end{tikzcd},$$

such that the map $x$ represents inclusion of the point $x$ and the map $g$ shows that $y$ is a specialisation of $x$ in $\mathbb{P}^n_k$. We obtain the lift since $\mathbb{P}^n_k$ is proper over $\operatorname{Spec} k$. Now we consider the following diagram,

$$\begin{tikzcd}
\operatorname{Spec} k(x) \arrow[r, "x"] \arrow[d] & X \arrow[r, "f"] \arrow[d] & \mathbb{P}^n_k \arrow[dl] \\
\operatorname{Spec} V \arrow[r] \arrow[ur, dashed, "h"] & \operatorname{Spec} k &
\end{tikzcd}.$$

Since $f$ and the structure map for $\mathbb{P}^n_k$ are proper, then $X \to \operatorname{Spec} k$ is proper, and we obtain a unique lift $h$. Since $g$ was the unique map in the previous diagram, then $g = f \circ h$, hence $y \in X$. $\square$

**Definition 25.4.** *Let $X$ be a scheme, $\mathcal{L}$ a line bundle on $X$, then $\mathcal{L}$ is ample if $X$ is quasi-compact and for every $x \in X$ there exists $m \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that $D(s) = X \backslash V(s)$ is affine and $x \in D(s)$.*

In other words, $\mathcal{L}$ is ample if and only if there exists an $m > 0$ and $s_1, \dots, s_n \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that all $D(s_i)$ are affine and $X$ is covered by these $D(s_i)$. We warn the reader to distinguish between $D(s)$ and $D(f)$ when $s$ is a section of a line bundle over $X$ and $f \in A$ where $X = \operatorname{Spec} A$.

It is clear that this second interpretation implies the definition, using that $X$ is a finite union of quasi-compact opens implies that $X$ is quasi-compact. Conversely, by quasi-compactness, we can find integers $m_1, \dots, m_n$ and sections $s_1, \dots, s_n \in \Gamma(X, \mathcal{L}^{\otimes m_i})$ such that $D(s_i)$ are all affine and $X$ is covered by these $D(s_i)$'s. We then take $m = m_1 \cdots m_n$ and $s_i' = s_i^{m/m_i}$, such that all $s_i' \in \Gamma(X, \mathcal{L}^{\otimes m})$. Then we also have $D(s_i') = D(s_i)$ for all $i = 1, \dots, n$ so all $D(s_i')$ are affine with union $X$.

[41] The following argument can be easily generalised. If $f : X \to Y$ is a locally closed immersion where $X$ is proper and $Y$ is separated, then $f$ is a closed immersion.

**Example 25.5** (The Most Important Example)**.** The line bundle $\mathcal{O}(1)$ over $\mathbb{P}^n_k$ is ample. Indeed, $\mathcal{O}(1)$ comes with canonical sections $X_0, \dots, X_n \in \Gamma(\mathbb{P}^n_k, \mathcal{O}(1))$ with $U_i = D(X_i) \cong \mathbb{A}^n_k$, our standard open subset of $\mathbb{P}^n_k$. Notice this implies that $\mathcal{O}(m)$ is ample for every $m > 1$ (from Remark 26.1 to come), but $\mathcal{O}(m)$ is not ample for $m < 0$ since these sheaves have no global sections, even after taking higher tensor powers.

We'll also see that all line bundles over affine schemes are also ample (Lemma 26.2 to come).

**Remark 25.6.** On an affine scheme $\operatorname{Spec} A$ we have a basis of affine opens given by $D(a)$ for $a \in A$. Similarly, we will see that on a scheme $X$ with ample line bundle $\mathcal{L}$, we get a basis of open affines of $X$ given by $D(s)$ for $s \in \Gamma(X, \mathcal{L}^{\otimes m})$ with varying $m$.

The theorem we want to prove is the following:

**Theorem 25.7.** *Let $X$ be a scheme of finite type over a field $k$, then $X$ is quasi-projective if and only if $X$ has an ample line bundle.*

More precisely, if $f : X \to \mathbb{P}^n_k$ is a locally closed immersion then we obtain an ample line bundle $\mathcal{L} = f^*\mathcal{O}(1)$. Conversely, if $\mathcal{L}$ is ample, then there exists $m \geq 1$ and a locally closed immersion $f : X \to \mathbb{P}^n_k$ such that $\mathcal{L}^{\otimes m} \cong f^*\mathcal{O}(1)$.

**Corollary 25.8.** If $X$ is proper over $k$, then $X$ is projective if and only if $X$ has an ample line bundle.

*Proof.* Using Theorem 25.7 and the fact that a projective is equivalence to properness and quasi-projective by Proposition 25.3, we obtain this corollary. $\square$

## 26 Ample Line Bundles and Quasi-Projectiveness 31/01/2017

**Remark 26.1.** By definition, $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes m}$ is ample for some $m \geq 1$, if and only if $\mathcal{L}^{\otimes m}$ is ample for all $m \geq 1$.

Recall Theorem 25.7. We will prove this theorem by the end of today, but we will use four more general lemmas to do so.

**Lemma 26.2.** *Let $X = \operatorname{Spec} A$ and $\mathcal{L} = \widetilde{L}$ be a line bundle on $X$. Then $\mathcal{L}$ is ample. Moreover, for any $s \in \Gamma(X, \mathcal{L}^{\otimes m})$, the open set $D(s) \subseteq X$ is affine (although not necessarily a principal open).*

*Proof.* First we check that for each $s \in \Gamma(X, \mathcal{L}^{\otimes m}) = L^{\otimes m}$, $D(s)$ is affine. Equivalently, we can show the inclusion $j : D(s) \hookrightarrow X$ is an affine map. Affine maps can be checked locally on $X$, hence we can assume our line bundle because trivial, so we may assume $\mathcal{L} \cong \mathcal{O}_X$ so $L \cong A$. Then $s \in L^{\otimes m} = A$, and $s$ is simply a function $f \in A$, and $D(s) = D(f)$, which is simply a principal open, and thus affine. To see $\mathcal{L}$ is ample we need to show that we can cover $X$ by $D(s)$. This is clear as for $x \in X$ a closed point we can lift a generator of $\mathcal{L} \otimes k(x)$ to $A$. $\square$

**Remark 26.3.** There is another way to prove the above lemma, which we will general even further shortly. We can define a ring $A[s^{-1}]$, which isn't strictly the localisation of $A$ at $s \in \Gamma(\operatorname{Spec} A, \mathcal{L}^{\otimes m})$, since this doesn't make sense, but rather the colimit of the diagram,

$$A \xrightarrow{\cdot s} L \xrightarrow{\cdot s} L^{\otimes 2} \xrightarrow{\cdot s} \cdots .$$

This ring as the obvious addition and multiplication induced by the maps $L^{\otimes i} \otimes L^{\otimes j} \to L^{\otimes (i+j)}$. Then $D(s) = \operatorname{Spec} A[s^{-1}]$. If $L = A$ then we have seen a result like this already, since the above colimit just specialises to the localisation of $A$ at the element $s \in A$.

**Lemma 26.4.** *Let $X$ be a scheme with an ample line bundle $\mathcal{L}$. Then $X$ is quasi-compact and quasi-separated.*

In fact, we will prove in an exercise sheet[42] that $X$ is separated, using the valuation criterion for separatedness.

*Proof.* The fact $X$ is quasi-compact is in the definition of having an ample line bundle $\mathcal{L}$, and quasi-separated can be shown as follows. Take $m \geq 1$ and $s_1, \ldots, s_n \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that $U_i = D(s_i)$ are affine and cover $X$. Then it follows that,

$$U_i \cap U_j = D(s_i) \cap D(s_j) = X \backslash (V(s_i) \cup V(s_j)) = X \backslash (V(s_i s_j)) = D(s_i s_j).$$

Then $D(s_i s_j) \subseteq D(s_i) = \operatorname{Spec} A_i$, so by Lemma 26.2 we see $D(s_i s_j)$ are affine and hence quasi-compact. $\square$

Now we have the lemma which expands on Remark 26.3.

**Lemma 26.5.** *Given a quasi-compact, quasi-separated scheme $X$, and a line bundle $\mathcal{L}$ on $X$, $s \in \Gamma(X, \mathcal{L})$, and $\mathcal{M}$ a quasi-coherent sheaf on $X$. Then $\Gamma(D(s), \mathcal{M})$ can be computed as the colimit of the following diagram,*

$$\Gamma(X, \mathcal{M}) \xrightarrow{\cdot s} \Gamma(X, \mathcal{M} \otimes \mathcal{L}) \xrightarrow{\cdot s} \Gamma(X, \mathcal{M} \otimes \mathcal{L}^{\otimes 2}) \xrightarrow{\cdot s} \cdots .$$

**Remark 26.6.** Notice the map $\mathcal{O}_{D(s)} \to \mathcal{L}|_{D(s)}$ defined by multiplication by $s$ is an isomorphism since $s$ is a unit on $D(s)$. Also, if $X = \operatorname{Spec} A$, $\mathcal{L} = \mathcal{O}_X$, $f \in A = \Gamma(X, \mathcal{L})$ and $\mathcal{M} \cong \widetilde{M}$, then Lemma 26.5 just tells us that $\Gamma(D(f), \mathcal{M}) = M[f^{-1}]$ which we already know.

[42] Problem 4 on exercise sheet 14 asks us to show that a scheme with an ample line bundle is separated.

*Proof.* The proof is essentially formal sheaf theory. First we cover $X$ by finitely many open affines $U_i = \operatorname{Spec} A_i$, with $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$ and $U_i \cap U_j = \bigcup U_{i,j,k}$ is a finite union. Then our equaliser diagram can be replaced with direct sums, and we obtain,

$$\Gamma(D(s), \mathcal{M}) = eq\left( \bigoplus_i \Gamma(D(s) \cap U_i, \mathcal{M}) \rightrightarrows \bigoplus_{i,j,k} \Gamma(D(s) \cap U_{i,j,k}, \mathcal{M}) \right).$$

We can substitute each $\Gamma(D(s) \cap V, M)$ with a colimit along multiplication by $s$ as in Remark 26.6, since we understand the affine case. We then use the fact that this sequential colimit is exact to obtain,

$$\operatorname{colim}_{m,\cdot s} eq\left( \bigoplus_i \Gamma(D(s) \cap U_i, \mathcal{M} \otimes \mathcal{L}^{\otimes m}) \rightrightarrows \bigoplus_{i,j,k} \Gamma(D(s) \cap U_{i,j,k}, \mathcal{M} \otimes \mathcal{L}^{\otimes m}) \right).$$

By the sheaf property this is simpily the colimit of $\Gamma(X, \mathcal{M} \otimes \mathcal{L}^{\otimes m})$ along multiplication by $s$, as desired. ☐

**Lemma 26.7.** *Let $X$ be a quasi-compact scheme and $\mathcal{L}$ a line bundle on $X$. Then $\mathcal{L}$ is ample if and only if the open sets $D(s)$ for varying $s \in \Gamma(X, \mathcal{L}^{\otimes m})$ for varying $m \geq 1$ form a basis of the topology for $X$.*

*Proof.* Let $\mathcal{L}$ be ample, and choose $s_1, \dots, s_n \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that $U_i = D(s_i) = \operatorname{Spec} A_i$ are affine opens covering $X$. For simplicity, and without loss of generality, take $m = 1$. Take any open $U \subseteq X$, then $U$ is covered by principal open subsets $D_{U_i}(f_i)$ for varying $i$, and varying $f_i \in A_i$, so without loss of generality $U = D_{U_i}(f_i)$ for some $i$ and some $f_i \in A_i$. Now,

$$f_i \in \Gamma(D(s_i), \mathcal{O}_X) = \operatorname{colim}_{\cdot s_i} \Gamma(X, \mathcal{L}^{\otimes m}),$$

by Lemma 26.6, so there exists $m \geq 1$ and $\widetilde{f_i} \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that,

$$f_i = \frac{\widetilde{f_i}|_{U_i}}{s_i^m}.$$

Then we see that,

$$U_i = D(s_i) \supseteq D(s_i \widetilde{f_i}) = D_{U_i}(s_i \widetilde{f_i}|_{U_i}) = D_{U_i}(f_i) = U,$$

since $s_i$ is invertible on $U_i$. This shows one direction. Conversely, we have to see that the open affine subsets of the form $D(s)$ still cover $X$. Let $x \in X$, then $x \in U = \operatorname{Spec} A \subseteq X$. Our hypotheses imply that there exists $m \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that $D(s) \subseteq U$ and $x \in D(s)$. Then,

$$D(s) = D_U(s|_U) \subseteq U,$$

which is affine by Lemma 26.2. ☐

**Corollary 26.8.** If $f : X \to Y$ is a locally closed immersion, and $\mathcal{L}$ is ample on $Y$, then $f^*\mathcal{L}$ is ample on $X$.

In particular this says that if $f : X \to \mathbb{P}^N_k$ is quasi-projective, then $f^*\mathcal{O}(1)$ is ample on $X$, so we're half way to proving Theorem 25.7.

*Proof.* Let $f$ be a composite of an open followed by a closed immersion, then we handle each case separately. Let $f$ be a closed immersion, then for some $n \geq 1$ we have $s_1, \dots, s_n \in \Gamma(Y, \mathcal{L}^{\otimes m})$ such that $D(s_i)$ are all affine and cover $Y$. Then $t_i = f^*s_i \in \Gamma(X, f^*\mathcal{L}^{\otimes m})$. We then have the following pullback square of inclusions and $f$,

$$\begin{tikzcd} D(t_i) = D(s_i) \cap X \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\ D(s_i) \arrow[r, hook] & Y \end{tikzcd}.$$

Since $f$ is a closed immersion then $f$ is affine, so $D(t_i)$ is affine, and clearly $X$ is cover by $D(t_i)$ since the $D(s_i)$'s cover $X$.

If $f$ is an open immersion, then we simply apply Lemma 26.7. $\square$

We can now finally move in to proving that a scheme of finite type over a field $k$ is quasi-projective if and only if it admits an ample line bundle.

*Proof of Theorem 25.7.* If our scheme $X$ is quasi-projective then the fact that we have an ample line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ for each $n$ implies that $X$ has an ample line bundle from Corollary 26.8.

Conversely, assume that $\mathcal{L}$ is an ample line bundle, then we have sections $s_i \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that $U_i = D(s_i) = \operatorname{Spec} A_i$ are affine opens which cover $X$. As per usual, we can without loss of generality take $m = 1$. Our scheme is of finite type so all $A_i$ are finitely generated $k$-algebras, so let $x_{ij} \in A_i$ generate $A_i$ as a $k$-algebra. Lemma 26.6 gives us,

$$A_i = \operatorname{colim}_{\cdot s_i} \Gamma(X, \mathcal{L}^{\otimes m}),$$

so if $m >> 0$ we can choose $m$ independent $i$ and $j$, since both number range over some finite indexing set. We can then find sections $\widetilde{x}_{ij} \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that,

$$x_{ij} = \frac{\widetilde{x}_{ij}|_{U_i}}{s_i^m}.$$

We claim the datum,

$$(\mathcal{L}^{\otimes m}, s_1^m, \ldots, s_n^m, (\widetilde{x}_{ij})_{i,j}), \tag{26.9}$$

defines a map $f : X \to \mathbb{P}_k^{n+nN-1}$, where $i$ ranges from $1, \ldots, n$ and $j$ from $1, \ldots, N$. To show this, we need to see that $s_1^m, \ldots, s_n^m, (\widetilde{x}_{ij})_{i,j} \in \Gamma(X, \mathcal{L}^{\otimes m})$ generate $\mathcal{L}^{\otimes m}$. Actually, we already saw that $s_1^m, \ldots, s_n^m$ generate $\mathcal{L}^{\otimes m}$ as $X$ is covered by $D(s_i)$'s and $\mathcal{O}_X|_{U_i}$ is isomorphic to $\mathcal{L}|_{U_i}$ through multiplication by $s_i$. This implies the datum presented in 26.9 does in fact define $f : X \to \mathbb{P}_k^{n+nN-1}$, but it remains to check that such an $f$ is a locally closed immersion. More precisely, let $\mathbb{P}_k^{n+nN-1} = \bigcup_{\alpha=1}^{n+nN} V_\alpha$ be the standard open covering of projective space, then we claim $f$ factors as,

$$X \longrightarrow \bigcup_{\alpha=1}^{n} V_\alpha \longrightarrow \mathbb{P}^{n+nN-1}, \tag{26.10}$$

where the first map is a closed immersion and the second an open immersion. Indeed, the preimage of $V_\alpha$ in $X$ for $\alpha = 1, \ldots, n$ is $D(s_\alpha^m) = D(s_\alpha) \subseteq X$ and $X$ is covered by these open affines, so $X$ surely maps into $\bigcup_{\alpha=1}^n V_\alpha \subseteq \mathbb{P}_k^{n+nN-1}$. To see this is a closed immersion, it remains to see that $\operatorname{Spec} A_\alpha = D(s_\alpha) \to V_\alpha = \operatorname{Spec} k[(X_i)_i]$ is a closed immersion of affine schemes, in other words, that $k[(X_i)_i] \to A_\alpha$ is an epimorphism, as a map of $k$-algebras. Recall the $V_\alpha$ are the standard affine opens of $\mathbb{P}_k^{n+nN-1}$ so we could write down the generators of $k[(X_i)_i]$ if required. Part of the generators of $k[(X_i)_i]$ map to

$$x_{\alpha b} = \frac{\widetilde{x}_{\alpha b}|_{U_\alpha}}{s_\alpha^m},$$

but these generate $A_\alpha$ as a $k$-algebra. Clearly the second map of the composition 26.10 is an open immersion, so we're done. $\square$

## 27 Separated Curves are Quasi-Projective 02/02/2017

Today we would like to show the following theorem, which uses Theorem 25.7 as a key ingredient in the proof.

**Theorem 27.1.** *Let $X$ be a separated normal curve over a field $k$, then $C$ is quasi-projective. Moreover, if $C$ is proper then $C$ is projective.*

**Remark 27.2.** This statement is true in more generality, but our proof won't be. For example, any separated, 1-dimensional scheme of finite type over a field $k$ is quasi-projective. Recall also Proposition 25.3 which gives us the 'moreover' statement.

First we are going to use an alternative characterisation of ample.

**Proposition 27.3.** *Let $X$ be a noetherian scheme and $\mathcal{L}$ a line bundle on $X$, then the following are equivalent.*

1. *$\mathcal{L}$ is ample on $X$.*
2. *For all coherent sheaves $\mathcal{M}$ on $X$ there exists $m > 0$ such that $\mathcal{M} \otimes \mathcal{L}^{\otimes m}$ is globally generated.*
3. *For all coherent ideal sheaves $\mathcal{I}$ on $X$ there exists $m > 0$ such that $\mathcal{I} \otimes \mathcal{L}^{\otimes m}$ is globally generated.*
4. *For all coherent sheaves $\mathcal{M}$ on $X$ there exists $m_0 > 0$ such that for all $m \geq m_0$ the sheaf $\mathcal{M} \otimes \mathcal{L}^{\otimes m}$ is globally generated.*
5. *For all coherent ideal sheaves $\mathcal{I}$ on $X$ there exists $m_0 > 0$ such that for all $m \geq m_0$ the sheaf $\mathcal{I} \otimes \mathcal{L}^{\otimes m}$ is globally generated.*

Note that condition 4 is what Hartshorne[3, p.153] uses as the definition for ample (or highly-ample or something), but we'll see a lemma later on (Lemma 27.5) where our definition makes things amazingly simple.

*Proof.* We obviously have $2 \Rightarrow 3, 4 \Rightarrow 5, 4 \Rightarrow 2, 5 \Rightarrow 3$, so we will show $3 \Rightarrow 1$, and then $1 \Rightarrow 2$, finishing with $2 \Rightarrow 4$, at which time we'll be done.

($3 \Rightarrow 1$) We have to show that for all $x \in X$, there exists $m \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that $D(s)$ is affine and $x \in D(s)$. Here we're using the fact that $X$ is quasi-compact which comes in the definition of noetherian. Let $x \in X$ and $U = \operatorname{Spec} A \subseteq X$ be an affine open neighbourhood of $x$, then $Z = X \backslash U \subseteq X$, which is closed in $X$, so we endow it with the reduced closed subscheme structure such that $i : Z \to X$ is a closed immersion. Equivalently, we have $\mathcal{I}_Z$ which is the ideal sheaf corresponding to $Z$. Since $X$ is noetherian we know that $\mathcal{I}_Z$ is finitely generated. Our condition 3 now says that for some $m \geq 1$ we have $\mathcal{I}_Z \otimes \mathcal{L}^{\otimes m}$ is globally generated. Let $s \in \Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes m})$ be a section such that $s$ is non-zero when evaluated at $x$, so non-zero in the fibre (stalk) over $x$. The inclusion $\alpha : \mathcal{I}_Z \otimes \mathcal{L}^{\otimes m} \to \mathcal{O}_X \otimes \mathcal{L}^{\otimes m}$ is an isomorphism over $U$, so now $\alpha(s) \in \Gamma(X, \mathcal{L}^{\otimes m})$. Actually, by definition $\alpha(s)$ is in the kernel of the map,

$$\Gamma(X, \mathcal{L}^{\otimes m}) \longrightarrow \Gamma(X, i_* \mathcal{O}_Z \otimes \mathcal{L}^{\otimes m}) = \Gamma(Z, i^* \mathcal{L}^{\otimes m}).$$

Hence $\alpha(s)$ vanishes on $Z$, so $Z \subseteq V(\alpha(s))$, so $D(\alpha(s)) \subseteq U$. On the other hand, $\alpha(s)$ does not vanish at $x$ by choice of $s$ (and the fact that $\alpha$ is an isomorphism at $x$). The fact $D(\alpha(s)) \subseteq U$ implies that $D(\alpha(s)) = D_U(\alpha(s)|_U) \subseteq U$ which is affine by Lemma 26.2, and $x \in D(\alpha(s))$, which finishes this implication.

($1 \Rightarrow 2$) If $\mathcal{L}$ is ample, then we want to show for all coherent sheaves $\mathcal{M}$ there exists $m \geq 1$ such that $\mathcal{M} \otimes \mathcal{L}^{\otimes m}$ is globally generated. Take an $m$ such that $s_1, \ldots, s_n \in \Gamma(X, \mathcal{L}^{\otimes})$ are sections with

$D(s_i)$ affine and covering $X$. We may assume $m = 1$ here since we can replace $\mathcal{L}$ by $\mathcal{L}^{\otimes m}$ if necessary. Let $U_i = D(s_i) = \operatorname{Spec} A_i$, and $M_i = \Gamma(U_i, \mathcal{M})$ a finitely generated $A_i$-module, since $\mathcal{M}$ is a coherent sheaf. Choose $m_{ij} \in M_i$ with $i = 1, \dots, n$ and $j = 1, \dots, N$ which generate $M_i$ as an $A_i$-module. Lemma 26.5 tells us that $M_i = \Gamma(U_i, \mathcal{M}) = \operatorname{colim}_{\cdot s_i} \Gamma(X, \mathcal{M} \otimes \mathcal{L}^{\otimes m})$. We can choose an $m$ quite large, such that all $m_{ij}$ are in the image of,

$$\begin{tikzcd}
\Gamma(X, \mathcal{M} \otimes \mathcal{L}^{\otimes m}) \arrow[r] \arrow[dr] & \Gamma(U_i, \mathcal{M}) = M_i \arrow[d, "s_i^m"] \\
& \Gamma(U_i, \mathcal{M} \otimes \mathcal{L}^{\otimes m})
\end{tikzcd}.$$

The fact that $(\mathcal{M} \otimes \mathcal{L}^{\otimes m})|_{U_i}$ is generated by $\Gamma(X, \mathcal{M} \otimes \mathcal{L}^{\otimes m})$ for all $i$ implies that $\mathcal{M} \otimes \mathcal{L}^{\otimes m}$ is globally generated.

($2 \Rightarrow 4$) Assume now that $\mathcal{L}$ is ample and $\mathcal{M}$ is a coherent sheaf and choose $m \geq 1$ with $s_1, \dots, s_n \in \Gamma(X, \mathcal{L}^{\otimes m})$ with the collection $D(s_i)$ an affine covering of $X$, and for each $j = 0, \dots, m-1$ an $m_j$ such that,

$$(\mathcal{M} \otimes \mathcal{L}^{\otimes j}) \otimes \mathcal{L}^{\otimes m_j m},$$

is globally generated. We claim that for all $M$ greater than the maximum of $j + m_j m$, $\mathcal{M} \otimes \mathcal{L}^{\otimes M}$ is globally generated. To show this, choose $j = \{0, \dots, m-1\}$ such that $M - j$ is divisible by $m$, then $M = j + m_m m + lm$ for some $l \geq 0$. Then,

$$\mathcal{M} \otimes \mathcal{L}^{\otimes M} = \left(\mathcal{M} \otimes \mathcal{L}^{\otimes j} \otimes \mathcal{L}^{\otimes m_j m}\right) \otimes \mathcal{L}^{jm},$$

where the parenthesis on the right hand side of the equation above are globally generated by assumption. But we know that $\mathcal{L}^{\otimes m}$ is globally generated (by $s_1, \dots, s_n$ in fact), and if two coherent modules are globally generated, then so is there tensor product. We can easily see this, as surjections $\mathcal{O}_X^a \twoheadrightarrow \mathcal{M}$ and $\mathcal{O}_X^b \twoheadrightarrow \mathcal{N}$ clearly yield a surjection $\mathcal{O}_X^{ab} \twoheadrightarrow \mathcal{M} \otimes \mathcal{N}$. This finishes our proof. □

**Remark 27.4.** If $X$ is a scheme with an ample line bundle $\mathcal{L}$, we see that all coherent sheaves $\mathcal{M}$ have many sections after twisting with $\mathcal{L}$. In fact, we can describe the category of coherent sheaves on $X$ in terms of graded modules over the graded ring,

$$S = \bigoplus_{m \geq 0} \Gamma(X, \mathcal{L}^{\otimes m}),$$

by the functor $\mathcal{M} \mapsto \bigoplus_{m \geq 0} \Gamma(X, \mathcal{M} \otimes \mathcal{L}^{\otimes m})$, which is not quite an equivalence of categories, but nearly. This is the theory of Proj. We can think of Proj as a generalisation of Spec, but Peter doesn't like Proj, so he's actually avoided it this whole time. For example we proved that the closed subschemes of $\mathbb{P}^n_k$ are generated by $V(f)$ where $f$ is a homogeneous polynomial in Proposition 15.2 using one of Serre's theorems, rather than Proj and $V_+(I)$ and stuff like this.

*Proof of Theorem 27.1.* Let $K = K(C)$ be the function field of our curve $C$, then there exists a proper normal curve $\overline{C}$ over $k$ with $K(\overline{C}) = K$ and there is an open embedding $j : C \hookrightarrow \overline{C}$. Hence it is enough to prove that $\overline{C}$ is quasi-projective, or equivalently, projective. Choose an open dense affine $U = \operatorname{Spec} A \subseteq \overline{C}$ and a closed immersion $i : U \hookrightarrow \mathbb{A}^n_k$. Let $\overline{U}$ be the scheme theoretic image[43] of $U \hookrightarrow \mathbb{A}^n_k \hookrightarrow \mathbb{P}^n_k$ which is equivalent to $\overline{U} \subseteq \mathbb{P}^n_k$ a reduced closed subscheme whose underlying topological space $|\overline{U}|$ is the closure of $|U|$ inside $|\mathbb{P}^n_k|$. Then $U \hookrightarrow \overline{U}$ is an open immersion with dense image,

$$U = \overline{U} \cap \mathbb{A}^n_k \subseteq \mathbb{P}^n_k.$$

[43] The first part of problem 3(i) on exercise 10 is the following: Let $f : X \to S$ be a morphism of schemes. The schematic image (scheme theoretic image) $\operatorname{Im}(f)$ of $f$ is defined as the minimal closed subscheme $Z \subseteq S$ such that $f$ factors through the inclusion $Z \hookrightarrow S$. Prove that the schematic image $\operatorname{Im}(f)$ of $f$ exists.

In particular, $\overline{U}$ is a projective curve over $k$. We claim $\overline{C}$ is isomorphic to the normalisation of $\overline{U}$. To see this, the normalisation of $\overline{U}$ is a proper[44] normal curve over $k$ with function field $K$. Hence by our classification of proper normal curves, we see that the normalisation of $\overline{U}$ is given by $\overline{C}$, which shows our claim. Hence we obtain a map $f : \overline{C} \to \overline{U}$ which is affine, since any normalisation map is affine. We also have $\bar{i} : \overline{U} \hookrightarrow \mathbb{P}^n_k$. The fact that $\mathcal{O}(1)$ is ample implies that $f^*\bar{i}^*\mathcal{O}(1)$ is ample from Lemma 27.5 that follows. □

**Lemma 27.5.** *If $f : X' \to X$ is an affine map of schemes, and $\mathcal{L}$ is an ample line bundle on $X$, then $f^*\mathcal{L}$ is ample on $X$.*

*Proof.* We take $m \geq 1$ and some collection $s_1, \ldots, s_n \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that $U_i = D(s_i)$ is affine and open in $X$, and $X = \bigcup_i U_i$. Let $t_i = f^*(s_i) \in \Gamma(X', f^*\mathcal{L}^{\otimes m})$, then $D(t_i) = f^{-1}(D(s_i)) \subseteq X'$, which are all affine as $f$ is affine and so is $D(s_i)$. We then see,

$$\bigcup_{i=1}^n D(t_i) = f^{-1}\left(\bigcup_{i=1}^n D(s_i)\right) = f^{-1}(X) = X',$$

and we're done. □

Next week we are going to talk about line bundles over curves, and we'll see that we always have an closed immersion $C \hookrightarrow \mathbb{P}^3_k$ for every proper normal curve over $k$.

[44] We know $\overline{U}$ is proper from problem 1 on exercise sheet 13, which asks: Let $k$ be a field and let $X$ be a curve over $k$ (in particular $X$ is integral). Prove that $X$ is proper if and only if its normalisation $\widetilde{X}$ is proper over $k$.

# 28 Divisors and the Picard Group of a Curve 07/02/2017

Our last two lectures will be focused on talking about line bundles on curves. The area of vector bundles on curves is currently a very active area of research, which is related to the geometric Langlands program.

For this lecture and the next, let $C$ be a proper normal curve over a field $k$ (which will be algebraically closed).

**Definition 28.1.** 1. *A divisor on a curve $C$ is a formal sum*

$$D = \sum_{x \in C\,closed} n_i[x],$$

*where $n_x \in \mathbb{Z}$ and almost all $n_x$ are equal to 0. In other words, the (abelian) group $\mathrm{Div}(C)$ of divisors on $C$ is the free abelian group on the set of closed points of $C$. We will often not say that $x \in C$ is closed to clean up some equations, but when we are talking about divisors in these last two lectures, we always mean $x \in C$ with $x$ closed.*

2. *$D \in Div(C)$ is an effective divisor if $n_x \geq 0$ for all $x \in C$ closed. We call $\mathrm{Div}^+(C) \subseteq \mathrm{Div}(C)$ the semi-group of effective divisors.*

3. *Let $D \in Div(C)$, then we define the $\mathcal{O}_C$-module $\mathcal{O}_C(D)$ by,*

$$\Gamma(U, \mathcal{O}_C(D)) = \{f \in K \mid \forall x \in U \text{ closed}, v_x(f) \geq -n_x\},$$

*where $v_x : K \to \mathbb{Z}$ is the discrete valuation corresponding to $x$, $K = K(C)$ is the function field of $C$ and $U \subseteq C$ is a non-empty open subset.*

We can easily construct divisors, and in a few cases we can already realise $\mathcal{O}_C(D)$.

**Example 28.2.** 1. If our divisor is the unique zero divisor, $D = 0$, then $\mathcal{O}_C(D) = \mathcal{O}_C$. Indeed, for all $U = \operatorname{Spec} A \subseteq C$ we have,

$$\Gamma(U, \mathcal{O}_C) = A = \{f \in K = \operatorname{Frac} A \mid \forall x \in \operatorname{Spec}_{\max} A, f \in A_x\}$$

$$= \{f \in K \mid \forall x, v_x(f) \geq 0\} = \Gamma(U, \mathcal{O}_C(D)),$$

as $A_x$ is all $f \in K$ with $v_x(f) \geq 0$.

2. If $D = -[x]$ for some closed point $x \in C$, then $\mathcal{O}_C(D)$ is the ideal sheaf of the reduced closed subscheme $\{x\} \subseteq C$, which means,

$$\Gamma(U, \mathcal{O}_C(D)) = \{f \in \mathcal{O}_C(U) \mid f_x \in \mathfrak{m}_x \subseteq \mathcal{O}_{C,x}\}.$$

Indeed by definition we have,

$$\Gamma(U, \mathcal{O}_C(D)) = \{f \in \Gamma(U, \mathcal{O}_C(0)) = \Gamma(U, \mathcal{O}_C), \text{ and } v_x(f) \geq 1\}.$$

3. If $D = -n[x]$ for some $n \geq 0$, then $\mathcal{O}_C(D) \subseteq \mathcal{O}_C$ is the subsheaf of functions with zero of order greater than $n$ at $x$.

4. If $D = n[x]$ and $n \geq 0$ then $\mathcal{O}_C(D) \subseteq \mathcal{O}_C$ is the subsheaf of functions with a pole of order at most $n$ at $x$.

We now come to the proposition which shows us the connection between $\mathrm{Div}(C)$ and $\mathrm{Pic}(C)$.

**Proposition 28.3.** *For any divisor $D \in \mathrm{Div}(C)$, the $\mathcal{O}_C$-module $\mathcal{O}_C(D)$ is a line bundle.*

*Proof.* Let $D = \sum n_x[x]$. Fix some $x \in C$ a closed point, then we want to see there exists an open neighbourhood $U$ of $x$ such that $\mathcal{O}_C(D)|_U \cong \mathcal{O}_U$. Pick some $f \in K = \operatorname{Frac} \mathcal{O}_{C,x}$ such that $v_x(f) = -n_x$. Assume for now that $-n_x \geq 0$, then $f \in \mathcal{O}_{C,x}$. We can find an open neighbourhood $U$ of $x$ with the following three properties:

1. $f \in \Gamma(U, \mathcal{O}_C)$,
2. for all $y \neq x$ and $y \in U$ with $y$ closed, $n_y = 0$,
3. for all $y \neq x$ and $y \in U$ with $y$ closed, $v_y(f) = 0$.

Indeed, we can find $U$ as in property 1 by definition of the stalk. We can ensure property 2 by removing all $y \in U$ with $n_y \neq 0$ and $y \neq 0$ from $U$. Since such $y$ are also closed, the $U$ remains open. To ensure property 3, we note that $V(f) \subsetneq U$, so it is zero dimensional and of finite type, so it is a finite set of points. Hence we can replace $U$ by $U \backslash (V(f) \backslash \{x\})$, which gives us property 3, as then $f$ is invertible on $U \backslash \{x\}$. We now claim to have the following commutative diagram,

$$\begin{tikzcd} \mathcal{O}_U \arrow[r, "\cong"] \arrow[d, hook] & \mathcal{O}_C(D) \arrow[d, hook] \\ K \arrow[r, "f"] & K \end{tikzcd},$$

with the notated isomorphisms as above. To prove this claim, we notice that for all open subsets $V \subseteq U$ we have,

$$\mathcal{O}_U(V) = \{g \in K \mid \forall y \in V, v_y(g) \geq 0\} \xrightarrow[\cong]{\cdot f} \{g \in K \mid \forall y \in V, v_y(g) \geq v_y(f)\} .$$

Now we use the fact that we have chosen $U$ with the three properties given above to obtain,

$$\{g \in K \mid \forall y \in V, v_y(g) \geq -n_y\} = \Gamma(V, \mathcal{O}_U(D)).$$

Thus $\mathcal{O}_C(D)|_U \cong \mathcal{O}_U$ as desired. If $-n_x < 0$, we make the same argument as above except with $f^{-1} \in \mathcal{O}_{C,x}$ and the isomorphism $\mathcal{O}_U \xrightarrow{f^{-1}} \mathcal{O}_C(D)|_U$. $\square$

If we recall the group structure on both $\operatorname{Div}(C)$ and $\operatorname{Pic}(C)$, then we might hope for the following lemma.

**Lemma 28.4.** *For two divisors $D, D \in \operatorname{Div}(C)$, there is a canonical isomorphism,*

$$\mathcal{O}_C(D) \otimes_{\mathcal{O}_C} \mathcal{O}_C(D') \longrightarrow \mathcal{O}_C(D + D').$$

*Proof.* Let us first construct the correct map. We need compatible maps,

$$\Gamma(U, \mathcal{O}_C(D)) \otimes_{\Gamma(U,\mathcal{O}_C)} \Gamma(U, \mathcal{O}_C(D')) \longrightarrow \Gamma(U, \mathcal{O}_C(D + D')).$$

If $f \in \Gamma(U, \mathcal{O}_C(D)) = \{f \in K \mid \forall x \in U, v_x(f) \geq -n_x\}$ and $g \in \Gamma(U, \mathcal{O}_C(D'))$, which has a similar description, where $D = \sum n_x[x]$ and $D' = \sum n'_x[x]$, then,

$$fg \in \{h \in K \mid \forall x \in U, v_x(h) \geq -n_x - n'_x\} = \Gamma(U, \mathcal{O}_C(D + D')).$$

This isomorphism can be checked on the level of stalks, so let $x \in C$ be a closed point, then $\mathcal{O}_C(D)_x = \omega^{-n_x} \cdot \mathcal{O}_{C,x}$, where $\omega \in \mathcal{O}_{C,x}$ is the uniformiser of this valuation ring, so $v_x(\omega) = 1$. Similarly, $\mathcal{O}_C(D')_x = \omega^{-n'_x} \cdot \mathcal{O}_{C,x}$. The tensor product of stalks ignores potential sheafification so we have,

$$(\mathcal{O}_C(D) \otimes_{\mathcal{O}_C} \mathcal{O}_C(D'))_x = \mathcal{O}_C(D)_x \otimes_{\mathcal{O}_{C,x}} \mathcal{O}_C(D')_x = \omega^{-n_x - n'_x} \cdot \mathcal{O}_{C,x} = \mathcal{O}_C(D + D')_x.$$

$\square$

**Corollary 28.5.** There is a natural map of abelian groups $\mathrm{Div}(C) \to \mathrm{Pic}(C)$ which sends $D \mapsto \mathcal{O}_C(D)$.

*Proof.* Direct consequence of Lemma 28.4. $\square$

**Proposition 28.6.** *The map* $\mathrm{Div}(C) \to \mathrm{Pic}(C)$ *is surjective.*

*Proof.* Let $\mathcal{L}$ be a line bundle on $C$, $\eta \in C$ be the generic point, and let $K = K(C) = \mathcal{O}_{C,\eta}$ be the function field of $C$. Then the stalk $\mathcal{L}_\eta$ is a 1-dimensional $K$-vector space. Choose some isomorphism $\mathcal{L}_\eta \cong K$ of $K$-vector spaces, which maps $1 \in K$ to $s \in \mathcal{L}_\eta$. Take some $U \subseteq C$ a non-empty open subset, such that $s \in \Gamma(U, \mathcal{L}) \subseteq \mathcal{L}_\eta$. For all $x \in C$ which are closed we have an injection,

$$\mathcal{L}_x \hookrightarrow \mathcal{L}_\eta \cong K = \mathrm{Frac}\, \mathcal{O}_{C,x},$$

where $\mathcal{L}_x$ is free of rank 1 over $\mathcal{O}_{C,x}$. This implies there exists some $n_x \in \mathbb{Z}$ such that $\mathcal{L}_x = \{f \in K | v_x(f) \geq -n_x\}$. Explicitly, take the generator $f_x \in \mathcal{L}_x \to \mathcal{L}_\eta \cong K$ and set $-n_x = v_x(f_x)$. We claim that $n_x = 0$ for almost all $x$. To see this, we take $U$ as above such that $s \in \Gamma(U, \mathcal{L})$, and let $U' = U \backslash V(s) \subseteq U$ be a non-empty proper open subset. It is enough to show $\forall x \in U'$ we have $n_x = 0$. However for $x \in U'$, $s_x \in \mathcal{L}_x$ is a generator as $x \notin V(s)$, thus $\mathcal{L}_x \hookrightarrow \mathcal{L}_\eta \cong K$ maps $s_x \mapsto s = s_\eta$ corresponds to the unit of 1. Hence $n_x = v_x(1) = 0$. We construct our divisor then as $D = \sum n_x[x] \in \mathrm{Div}(C)$. Let $\underline{K}$ be the constant sheaf with $\Gamma(V, \underline{K}) = K$ for all non-empty open subsets of $C$. Both $\mathcal{O}_C(D)$ and $\mathcal{L}$ then include into $\underline{K}$, the latter through the map $\mathcal{L}_\eta \cong K$. We have two sub-bundles of $\underline{K}$ with the same stalks, hence $\mathcal{O}_C(D) = \mathcal{L}$. $\square$

Given any $f \in K^\times$, we can construct the divisor,

$$\mathrm{div}(f) = \sum_{x \in C \text{closed}} v_x(f)[x].$$

Note that if there exists $U, U'$ such that $f \in \Gamma(U, \mathcal{O}_C)$ and $f^{-1} \in \Gamma(U', \mathcal{O}_C)$, then we have $v_x(f) = 0$ for all $x \in U \cap U'$.

We now come to the theorem which tells us how to calculate the Picard group of a curve $C$, if we know about its function field and divisors.

**Theorem 28.7.** *The follow sequence is exact:*

$$0 \longrightarrow k^\times \longrightarrow K^\times \longrightarrow \mathrm{Div}(C) \longrightarrow \mathrm{Pic}(C) \longrightarrow 0.$$

*Proof.* First we will see all compositions of two adjacent maps in the above sequence are zero. Given $f \in k^\times \subseteq K^\times$, then $v_x(f) = 0$, for all $x \in C$, so we see that $\mathrm{div}(f) = 0$.

For $f \in K^\times$ and $D = \mathrm{div}(f)$, then we want to see $\mathcal{O}_C(D) \cong \mathcal{O}_C$. However, for all $U \subseteq C$,

$$\Gamma(U, \mathcal{O}_C(D)) = \{g \in K \mid \forall x \in U, v_x(g) \geq -v_x(f)\} \xrightarrow[\cong]{\cdot f} \{g \in K \mid \forall x \in U, v_x(g) \geq 0\} = \Gamma(U, \mathcal{O}_C) \ .$$

Hence $\mathcal{O}_C(\mathrm{div}(f)) \cong \mathcal{O}_C$ as desired. Now for exactness.

It is clear that $k^\times \hookrightarrow K^\times$ is injective as it is a field extension.

We want to show now that if $f \in K^\times$ with $\mathrm{div}(f) = 0$, then $f \in k^\times$. If $div(f) = 0$, then $v_x(f) = 0$ for all closed points $x \in C$. Hence,

$$f \in \Gamma(C, \mathcal{O}_C) = \{f \in K \mid \forall x \in C, v_x(f) \geq 0\},$$

but as $C$ is proper and integral, and $k$ algebraically closed, then exercise sheet 12 problem 4(ii)[45] tells us $\Gamma(C, \mathcal{O}_C) = k$. Hence $f \in k^\times$.

[45] Let $k$ be an algebraically closed field and $X$ be a reduced and connected scheme over $k$. Deduce that $\Gamma(X, \mathcal{O}_X) = k$.

Given $D \in \mathrm{Div}(C)$ with $\mathcal{O}_C(D) \cong \mathcal{O}_C$, we would like to find $f \in K^\times$ such that $div(f) = D$. Assume $\mathcal{O}_C(D) \cong \mathcal{O}_C$, then by localising at $\eta$, we have canonical isomorphisms $\mathcal{O}_C(D)_\eta \cong K$ and $\mathcal{O}_{C,\eta} \cong K$, so we have an isomorphism $\mathcal{O}_C(D)_\eta \cong \mathcal{O}_{C,\eta}$ of $K$-vector spaces given by multiplication by $f \in K^\times$. We claim this is the $f$ we are looking for, i.e. $\mathrm{div}(f) = D$. To see this, as always, we have a look at stalks for all closed points $x \in C$, where we find,

$$\mathcal{O}_C(D)_x = \{g \in K | v_x(g) \geq -n_x\} \xrightarrow[\cong]{\cdot f} \mathcal{O}_{C,x} = \{g \in K \ | \ v_x(g) \geq 0\} \ .$$

Hence $v_x(f) = n_x$, so $\mathrm{div}(f) = D$.

The final check of exactness, i.e. surjectivity at $\mathrm{Pic}(C)$ is simply Proposition 28.6. $\square$

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Let us consider an example of the exact sequence from Theorem 28.7.

**Example 29.1.** Let $C = \mathbb{P}^1_k$ for some algebraically closed field $k$. Then we claim the degree map $Div(C) \to \mathbb{Z}$ which sends a divisor $D = \sum n_x[x] \mapsto \sum n_x$ has the following properties.

1. For all $f \in K^\times$ we have $\deg(\operatorname{div}(f)) = 0$.
2. Given $D \in Div(C)$ such that $\deg(D) = 0$ then there exists $f \in K^\times$ such that $D = \operatorname{div}(f)$.

These two facts imply that $\operatorname{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z}$.[46]

*Proof.* For part 1 we notice that for all $f \in K^\times$ we have $\sum v_x(f) = 0$, since this is simply the product formula from Proposition 22.11. For part 2, we look at $D = \sum n_x[x]$ such that $\sum n_x = 0$. For $\operatorname{Spec} k[t] = \mathbb{A}^1_k \subseteq \mathbb{P}^1_k$, so $K = k(t)$. We then define,

$$f = \prod_{x \in \mathbb{A}^1_k(k) = k} (t - x)^{n_x} \subseteq k(t).$$

By definition we have $v_x(f) = n_x$ for all $x \in \mathbb{A}^1_k(k)$ and $v_\infty(f) = N_\infty$ by the product formula again and the assumption that $\sum n_x = 0$. Hence $\operatorname{div}(f) = D$ and $\operatorname{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z}$. $\square$

For other curves we notice that the Picard group is much harder to calculate, but we always have this degree map.

**Proposition 29.2.** *[Generalised Product Formula] For all $f \in K^\times$ we have $\deg(\operatorname{div}(f)) = 0$.*

For the rest of this lecture we will not prove anything properly, but almost everything we say we will be able to prove by next semester.

**Corollary 29.3.** The degree map factors through a unique map $\deg : \operatorname{Pic}(C) \to \mathbb{Z}$.

*Proof.* This is a direct consequence of Proposition 29.2 and our exact sequence. $\square$

**Theorem 29.4.** *A line bundle $\mathcal{L} \in \operatorname{Pic}(C)$ is ample if and only if $\deg(\mathcal{L}) > 0$.*

*Proof.* We need the fact that there are many non-zero sections in $\Gamma(C, \mathcal{L}^{\otimes})$ if $m >> 0$, but for this we usually use the Riemann-Roch theorem. $\square$

To state the Riemann-Roch theorem, we need to define canonical bundles.

**Definition 29.5** (Kähler Differentials)**.** *Recall, if $A$ is a $k$-algebra (over any ring $k$), we have an $A$-module of Kähler differentials $\Omega^1_{A/k}$, which by definition is the universal $A$-module $M$ equipped with a $k$-linear derivation $d : A \to M$, so,*

$$d(\lambda a + \eta b) = \lambda d(a) + \eta d(b), \quad d(ab) = ad(b) + bd(a), \qquad a, b \in A, \lambda, \eta \in k.$$

*It is universal as an intial $A$-module with such a derivation $d$ as above.*

**Example 29.6.** For example if $k$ is a field and $A = k[t]$ then we have,

$$\Omega^1_{k[t]/k} = k[t] \cdot dt,$$

is simply free of rank 1, and when $d : k[t] \to \Omega^1_{k[t]/k]}$ is what one would expect,

$$d\left(\sum \lambda_n t^n\right) = \sum n\lambda_n t^{n-1} dt.$$

We note that $d(t^2) = d(t \cdot t) = 2tdt$, and by induction we obtain $d(t^n) = nt^{n-1}dt$.

[46]Recall in problem 2 on exercise sheet 9 we proved that $\operatorname{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$ for all $n \geq 1$.

**Example 29.7.** Similarly we have,

$$\Omega^1_{k[t_1,\ldots,t_n]/k} = \oplus_{i=1}^n k[t_1,\ldots,t_n]\cdot dt_i.$$

Where $f \in k[t_1,\ldots,t_n]$ is mapped to,

$$df = \sum_{i=1}^n \frac{\partial f}{\partial t_i} dt_i.$$

**Remark 29.8.** In characteristic $p > 0$, $df = 0$ does not imply that $f$ is a constant, because clearly we have things such as $d(t^p) = pt^{p-1} = 0$.

**Proposition 29.9.** *The map $A \to \Omega^1_{A/k}$ commutes with localisation, i.e.*

$$\Omega^1_{A[S^{-1}]/k} = \Omega^1_{A/k} \otimes_A A[S^{-1}].$$

Once we have a result on localisations, then we can make a definition of a sheaf on a basis of principal opens in the topology of an affine scheme, which we glue into the following corollary.

**Corollary 29.10.** If $X$ is a scheme over $k$, then we obtain a quasi-coherent sheaf $\Omega^1_{X/k}$ such that for all open affine $U = \operatorname{Spec} A \subseteq X$ we have,

$$\Gamma(U, \Omega^1_{X,k}) = \Omega^1_{A/k}.$$

**Proposition 29.11.** *If $C$ is a proper normal curve over a field $k$, then $\Omega^1_{C/k}$ is a line bundle. This is called the canonical line bundle, or confusingly as the canonical divisor. This is confusing since no canonical divisor gives us this line bundle for general curves.*

**Definition 29.12.** *Let $C$ be a proper normal curve over an algebraically closed field $k$, then the genus $g$ (Geschlecht auf Deutsch) of $C$ is defined as the dimension of $\Gamma(C, \Omega^1_{C/k})$ as a $k$-vector space. This is necessarily finite.*

We can now state one of the fundamental theorems in the theory of algebraic curves, one which we will see and prove next semester. The usual proof uses the cohomology of coherent sheaves.

**Theorem 29.13** (Riemann-Roch Theorem)**.** *For all line bundles $\mathcal{L}$ on a proper normal curve $C$ over an algebraically closed field $k$, we have,*

$$\dim_k \Gamma(C, \mathcal{L}) - \dim_k \Gamma(C, \Omega^1_{C/k} \otimes \mathcal{L}^\vee) = \deg \mathcal{L} + 1 - g.$$

An explanation of the genus of a curve is the usual topological notion. If $k = \mathbb{C}$, then $g$ is simply the number of holes in $C(\mathbb{C})$ which is a compact Riemann surface, for a proper normal curve $C$. If $g = 0$ this implies $C = \mathbb{P}^1_k$, and $g = 1$ implies that $C$ is an elliptic curve. Let us see the Riemann-Roch Theorem in some examples.

**Example 29.14.** If $\mathcal{L} = \mathcal{O}_C$ then we have,

$$\dim_k \Gamma(C, \mathcal{O}_C) - \dim_k \Gamma(C, \Omega^1_{C/k}) = 1 - g,$$

which is a tautology, since we can actually calculate the left hand side from the definition of the genus and exercise sheet 12, problem 4(ii)[47].

**Example 29.15.** Let $\mathcal{L} = \Omega^1_{C/k}$, then we obtain,

$$\dim_k \Gamma(C, \Omega^1_{C/k} - \dim_k \Gamma(C, \mathcal{O}_C) = \deg \Omega^1_{C,k} + 1 - g.$$

Again, we can calculate the left hand side by hand, so this implies that the $\deg \Omega^1_{C/k} = 2g - 2$.

[47] Let $k$ be an algebraically closed field and $X$ be a reduced and connected scheme over $k$. Deduce that $\Gamma(X, \mathcal{O}_X) = k$.

A corollary of the Riemann-Roch Theorem tells us that,

$$\dim_k \Gamma(C, \mathcal{L}) \geq \deg \mathcal{L} + 1 - g,$$

with equality if and only if $\Gamma(C, \Omega^1_{C/k} \otimes \mathcal{L}^\vee) = 0$. This happens if $\deg \mathcal{L} > 2g - 2$, as then,

$$\deg(\Omega^1_{C/k} \otimes \mathcal{L}^\vee) = \deg(\Omega^1_{C/k}) - \deg \mathcal{L} < 0,$$

and if $\mathcal{L}'$ is a line bundle of degree less than zero, then $\mathcal{L}'$ has no global sections. Indeed, if $0 \neq s \in \Gamma(C, \mathcal{L}')$ then $\mathcal{L}' \cong \mathcal{O}_C(D)$ where $D = \sum v_x(s)[x]$. Using this, one can easily prove that any line bundle of positive degree is ample, and the converse.

*Sketch of a Proof.* First assume $\mathcal{L}$ is ample, then for some line bundle $\mathcal{L}'$ of $\deg \mathcal{L}' < 0$ we take some $m >> 0$ such that $\mathcal{L}' \otimes \mathcal{L}^{\otimes m}$ is globally generated. If $\deg \mathcal{L} \leq 0$, then the degree of $\mathcal{L}' \otimes \mathcal{L}^{\otimes m}$ will be less than zero which implies $\mathcal{L}' \otimes \mathcal{L}^{\otimes m}$ has no global sections, a contradiction.

Now take a line bundle $\mathcal{L}$ with positive degree. Let $\mathcal{I} \subseteq \mathcal{O}_C$ be an ideal sheaf. To see there exists $m >> 0$ such that $\mathcal{I} \otimes \mathcal{L}^{\otimes m}$ is globally generated, it is enough that if $\mathcal{L}'$ is a line bundle of degree greater than or equal to $2g$, then $\mathcal{L}'$ is globally generated. To see this, fix $x \in C$ a closed point, then $\mathcal{L}' \otimes \mathcal{O}_C(-[x]) \subseteq \mathcal{L}'$ is the subsheaf of sections that vanish at $x$. We calculate the degree of this as the degree of $\mathcal{L}'$ minus 1, which is greater than $2g - 2$, and we then use the Riemann-Roch Theorem to see that,

$$\dim_k \Gamma(C, \mathcal{L}') = \deg \mathcal{L} + 1 - g,$$

which implies that,

$$\dim_k \Gamma(C, \mathcal{L}' \otimes \mathcal{O}(-[x])) = \deg \mathcal{L}' - g.$$

This implies we have some section $s \in \Gamma(C, \mathcal{L}')$ not in the image of $\Gamma(C, \mathcal{L}' \otimes \mathcal{O}(-[x]))$, i.e. the section $s$ does not vanish at $x$. □

**Example 29.16.** Let $E$ be an elliptic curve over a field $k$ given by the compactifiation of the affine equation $x^2 = x^3 + ax + b$ with $\Delta = 4a^3 + 27b^2 \neq 0$ inside $\mathbb{P}^2_k$, so $E = V(Y^2Z - X^3 - aXZ^2 - bZ^3)$. This curve has a point at $\infty \in E(k)$ given by $(0:1:0)$. A corollary of the Riemann-Roch Theorem says that $E(k) \to \operatorname{Pic}^1(E)$ which maps $x \mapsto \mathcal{O}([x])$, or equivalently, $E(k) \xrightarrow{\cong} \operatorname{Pic}^0(E)$ mapping $x \mapsto \mathcal{O}([x] - [\infty])$ is bijective, so $\operatorname{Pic} E = E(k) \times \mathbb{Z}$. In particular $\operatorname{Pic}^0 E$ admits the structure of a variety. There is in fact an abelian group structure on $E(k)$.

To prove this bijection above, let $\mathcal{L} \in \operatorname{Pic}^1 E$, and we have to see there exists a unique $x \in E(k)$ such that $\mathcal{L} = \mathcal{O}([x])$, but $\deg \mathcal{L} = 1 > 2g - 2 = 0$. The Riemann-Roch Theorem then says,

$$\dim_k \Gamma(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g = 1,$$

so up to a factor of $k^\times$, there exists a unique $0 \neq s \in \Gamma(C, \mathcal{L})$, so $\mathcal{L} \cong \mathcal{O}(D)$ where $D = \sum v_x(s)[x]$. However $\sum v_x(s) = \deg(D) = \deg \mathcal{L} = 1$ so $D = [x]$. The uniqueness of $x$ follows from the uniqueness of $s$, and always we have an injection $\mathcal{O}_C \to \mathcal{O}_C([x]) = \mathcal{L}$.

Let's think about this group structure on $E(k)$. Take $P, Q$ and $R$ all in $E(k)$ and colinear, and $i : E \to \mathbb{P}^2_k$. Then,

$$\mathcal{O}([P] + [Q] + [R]) = i^* \mathcal{O}_{\mathbb{P}^2_k}(1),$$

which is independent of our chosen line. We then look at $\mathcal{O}([R] + [\overline{R}] + [\infty])$, where $\overline{R}$ is colinear to $R$ and $\infty$, from which we see,

$$\mathcal{O}([P] + [Q] - 2[\infty]) \cong \mathcal{O}([P] - [\infty]) \cong \mathcal{O}([R] - [\infty]),$$

in other words $P + Q - \overline{R}$ is in $E(k)$.

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