Prof. Dr. Daniel Huybrechts
Dr. René Mboro

Retry exam solutions: Commutative Algebra (V3A1, Algebra I)

Exercise A. (Points: 3)
Let M be an A-module and $\mathfrak{a} \subset A$ an ideal such that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{a} \subset \mathfrak{m} \subset A$. Show that then $M = \mathfrak{a}M$.
Solution:
Let $N := M/\mathfrak{a}M$ and consider the exact sequence $0 \to \mathfrak{a}M \to M \to N \to 0$. For a maximal ideal $\mathfrak{m}$, tensoring the exact sequence with $A_{\mathfrak{m}}$, by exactness of localization, we get the exact sequence $0 \to \mathfrak{a}_{\mathfrak{m}}M_{\mathfrak{m}} \to M_{\mathfrak{m}} \to N_{\mathfrak{m}} \to 0$ (right exactness is sufficient for what follows).**(0.5)** If $\mathfrak{a} \not\subset \mathfrak{m}$ then $\mathfrak{a} \cap A \backslash \mathfrak{m} \neq \emptyset$ so that $\mathfrak{a}_{\mathfrak{m}} = (1) \subset A_{\mathfrak{m}}$. As $1 \in \mathfrak{a}_{\mathfrak{m}}$, the first homomorphism of the above sequence is surjective. Hence $N_{\mathfrak{m}} = 0$. **(1)**
If $\mathfrak{a} \subset \mathfrak{m}$ then $N_{\mathfrak{m}} = 0$ as quotient of the trivial (by assumption) module $M_{\mathfrak{m}} = 0$. **(0.5)** As a conclusion, $N_{\mathfrak{m}} = 0$ for any $\mathfrak{m} \in \mathrm{MaxSpec}(A)$, which yields $N = 0$ i.e. $M = \mathfrak{a}M$. **(1)** Alternatively, one can consider N as a module over $A/\mathfrak{a}$ and use the natural bijection between $V(\mathfrak{a}) \cap \mathrm{MaxSpec}(A)$ and $\mathrm{MaxSpec}(A/\mathfrak{a})$.

Exercise B. (Points: 3)
Show that a finitely generated ideal $\mathfrak{a} \subset A$ is a principal ideal and generated by an idempotent element if and only if $\mathfrak{a}^2 = \mathfrak{a}$.
Solution:
If $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}^2 = \mathfrak{a}$ then by Nakayama lemma, there is a $b = 1 - a$ with $a \in \mathfrak{a}$ such that $b\mathfrak{a} = 0$. **(1)** Hence for any $x \in \mathfrak{a}$, $(1-a)x = 0$, which can be written $x = ax$. So we get $\mathfrak{a} \subset (a)$. The converse inclusion is clear as $a \in \mathfrak{a}$. Moreover, we get $a = a \cdot a = a^2$. So a is idempotent. **(1)**
Conversely, if $\mathfrak{a} = (e)$ with $e^2 = e$, we have $\mathfrak{a}^2 = (e^2) = (e) = \mathfrak{a}$. **(1)**

Exercise C. (Points: 5)
Consider the ring $A := k[x,y,z]/(xy, z^2 - (x+y))$. Describe all irreducible components of $\mathrm{Spec}(A)$, i.e. the maximal closed irreducible subsets, and decide which of them have a non-empty intersection with $\mathrm{Spec}(A_x)$.
Solution:
We have **(1.5)**

$$\begin{aligned} V(xy, z^2 - (x+y)) &= V(xy) \cap V(z^2 - (x+y)) = (V(x) \cup V(y)) \cap V(z^2 - (x+y)) \\ &= V(x, z^2 - (x+y)) \cup V(y, z^2 - (x+y)) = V(x, z^2 - y) \cup V(y, z^2 - x). \end{aligned}$$

Moreover, $k[x,y,z]/(x, z^2 - y) \simeq k[z]$ is an integral domain. Hence, $(x, z^2 - y)$ is a prime ideal and $V(x, z^2 - y)$ is irreducible. **(0.5)** Likewise $k[x,y,z]/(y, z^2 - x) \simeq k[z]$, thus $(y, z^2 - x)$ is a prime ideal and $V(y, z^2 - x)$ is irreducible. **(0.5)**
Since $y \in (y, z^2 - x) \backslash (x, z^2 - y)$, $V(x, z^2 - y) \not\subset V(y, z^2 - x)$. Likewise, as $x \in (x, z^2 - y) \backslash (y, z^2 - x)$, $V(y, z^2 - x) \not\subset V(x, z^2 - y)$.**(0.5)**
Since any irreducible closed subset is of the form $V(\overline{\mathfrak{p}})$ for a $\overline{\mathfrak{p}} \in \mathrm{Spec}(A) \simeq V(x, z^2 - y) \cup V(y, z^2 - x)$ i.e. either $(x, z^2 - y) \subset \mathfrak{p}$ or $(y, z^2 - x) \subset \mathfrak{p}$, which yields $V(\mathfrak{p}) \subset V(x, z^2 - y)$ or $V(\mathfrak{p}) \subset V(y, z^2 - x)$ so $V(x, z^2 - y)$ and $V(y, z^2 - x)$ are the irreducible components of $\mathrm{Spec}(A)$. **(1)**

Recall that $\mathrm{Spec}(A_x)$ can be identified with $\{\mathfrak{p} \in \mathrm{Spec}(A),\ x \notin \mathfrak{p}\}$. As $(x) \subset (x, z^2 - y)$ we have $V(x, z^2 - y) \cap \mathrm{Spec}(A_x) = \emptyset$. **(0.5)** Clearly, $x \notin (y, z-1, x-1) \in V(y, z^2 - x)$ and, therefore, $V(y, z^2 - x) \cap \mathrm{Spec}(A_x) \neq \emptyset$.

All rings are commutative with a unit and $1 \neq 0$.

Exercise D. (Points: 2+2)
Describe explicitly a Noether normalization for the two k-algebras $k[x,y]/(x^2+y^2)$ and $k[x,y,z]/(y-z^2, xz-y^2)$.
Solution:
(i) Let us consider the natural homomorphism $i : k[x] \to k[x,y]/(x^2+y^2) = A$: if $f = \sum_{i=0}^d a_i x^i \in k[x] \subset k[x,y]$ belongs to (x^2+y^2), i.e. $i(f) = 0$, we can write $f = (x^2+y^2)g(x,y)$. Evaluating at $x = 0$, we get $a_0 = y^2 g(0,y)$, which yields $g(0,y) = 0$, for degree reason; so $a_0 = 0$ and $g(x,y) = xg_1(x,y)$. Dividing by x, we get $\sum_{i=1}^d a_i x^{i-1} = (x^2+y^2)g_1(x,y)$; evaluating at $x = 0$, $a_1 = y^2 g_1(0,y)$ which, for degree reason, yields, $g_1(0,y) = 0$; so $a_1 = 0$ and $g_1(x,y) = xg_2(x,y)$. An easy induction proves $a_i = 0$ for any $i \geq 0$ i.e. $f = 0$. So i is injective. **(1)**
Moreover, $\overline{y} \in A$ is integral over $i(k[x])$ since $\overline{y}^2 + \overline{x}^2 = 0$; so $i(k[x])[\overline{y}] \simeq A$ is finite over $i(k[x])$. **(1)**

(ii) We have $A = k[x,y,z]/(y-z^2, xz-y^2) \simeq k[x,z]/(xz-z^4)$. Consider the ring homomorphism $i\colon k[x] \to A$: if $f \in k[x] \subset k[x,z]$ belongs to $(xz-z^4)$, we can write $f = (xz-z^4)g = z(x-z^3)g$. Evaluating at $z^3 = x$, we get $f(z^3) = 0 \in k[z]$ i.e. $f = 0$ so i is injective. **(1)**
Moreover, $\overline{x} \in A$ is integral over $i(k[z])$ since $\overline{z}^4 - \overline{zx} = 0$; so $i(k[x])[\overline{z}] \simeq A$ is finite over $i(k[x])$. **(1)**

Exercise E. (Points: 2+4)
Consider the ring $A = k[x,y,z]/(xy^2 - xz^2, x^2)$ where $char(k) \neq 2$.

(i) Show that the ideals $(\bar{z} - \bar{y}) \subset A$ and $(\bar{z} + \bar{y}) \subset A$ are both primary ideals and determine their radicals.

(ii) Determine a primary decomposition of the ideal $(0) \subset A$ and decide which associated prime ideals are isolated and which are embedded.
Solution:
(i) We have $A/(\overline{z}-\overline{y}) \simeq k[x,y,z]/(z-y, x(y-z)(z+y), x^2) \simeq k[x,y,z]/(z-y,x^2) \simeq k[x,y]/(x^2)$. As $k[x,y]$ is an integral domain, the elements of $(\overline{x})$ are the only zero-divisors of $A/(\overline{z}-\overline{y})$ and they are also nilpotent since $\overline{x}^2 = 0$. So $(\overline{z}-\overline{y})$ is primary. **(0.5)**
Moreover, in $A/(\overline{z}-\overline{y})$, $\sqrt{(0)} = (\overline{x})$ so in A, $\sqrt{(\overline{z}-\overline{y})} = (\overline{z}-\overline{y}, \overline{x})$. **(0.5)**

Likewise $A/(\overline{x}+\overline{y}) \simeq k[x,y,z]/(z+y, x(y-z)(z+y), x^2) \simeq k[x,y,z]/(z+y,x^2) \simeq k[x,y]/(x^2)$ which shows that $(\overline{z}+\overline{y})$ is a primary ideal. **(0.5)** Moreover in $A/(\overline{z}+\overline{y})$, $\sqrt{(0)} = (\overline{x})$ so that, in A, $\sqrt{(\overline{z}+\overline{y})} = (\overline{z}+\overline{y}, \overline{x})$. **(0.5)**

(ii) Let us prove that $(x(y+z)(y-z), x^2) = (z-y, x^2) \cap (z+y, x^2) \cap (x)$ in $k[x,y,z]$. The inclusion '$\subset$' is clear. **(0.5)** Conversely, take a $g \in (z+y, x^2)$ and write $g = (z+y)g_1 + x^2 g_2$ for some $g_1, g_2 \in k[x,y,z]$. Then $g \in (x)$ if and only if $(z+y)g_1 \in (x)$, which yields ($k[x,y,z]$ being factorial) $x | g_1$ i.e. $g_1 = xg_3$ for some $g_3 \in k[x,y,z]$. We further have $g \in (z-y, x^2)$ if and only if $(z+y)xg_3 \in (z-y, x^2)$ i.e. if we can write $(z+y)xg_3 = (z-y)g_4 + x^2 g_5$ for some g_4, g_5. In which case, $x|g_4$ ($k[x,y,z]$ factorial); write $g_4 = xg_6$. Dividing by x, we get $(z+y)g_3 = (z-y)g_6 + xg_5$. Evaluating at $x = 0$ and $z = y$, we get $2g_3(0,y,y) = 0$ so ($char(k) \neq 2$) $g_3 = xg_7 + (z-y)g_8$ for some g_7, g_8. Putting everything together, an element g in the intersection of the three ideals, can be written $g = x(z+y)(z-y)g_8 + x^2(z+y)g_7 + x^2 g_2 \in (x(z+y)(z-y), x^2)$, proving the other inclusion. **(1.5)**
Moreover, $(z-y)(z+y) \in (z-y,x^2) \cap (z+y,x^2) \backslash (x(z+y)(z-y), x^2)$ and $x(z-y) \in (z-y,x^2) \cap (x) \backslash (x(z+y)(z-y), x^2)$ and $x(z+y) \in (z+y,x^2) \cap (x) \backslash (x(z+y)(z-y), x^2)$. So the decomposition is minimal. **(0.5)**
Since $(x(z+y)(z-y), x^2) \subset (x)$ and the later is a prime (hence primary) ideal, $(\overline{x}) \in \mathrm{Spec}(A)$.
Passing to the quotient in the above equality yields the decomposition $(0) = (\overline{x}) \cap (\overline{z}-\overline{y}) \cap (\overline{z}+\overline{y})$ which is a minimal primary decomposition by (i). **(0.5)**
So $\mathrm{Ass}((0)) = \{(\overline{z}-\overline{y}, \overline{x}), (\overline{z}+\overline{y}, \overline{x}), (\overline{x})\}$. We have $(\overline{x}) \subsetneq (\overline{z}-\overline{y}, \overline{x})$, $(\overline{x}) \subsetneq (\overline{z}+\overline{y}, \overline{x})$ so $(\overline{x})$ is an isolated associated prime and the two others are embedded associated primes. **(1)**

Exercise F. (Points: 5)
Compute $\mathrm{Ass}(M)$ and $\mathrm{Ann}(M)$ of the kernel $\ker(\psi)$ of the following A-module homomorphism $\psi\colon A^{\oplus 2} \to A$, $(a,b) \mapsto a\overline{x} + b\overline{y}$, where $A := k[x,y]/(x^2 y)$.
Solution:
Let us find generators for M: let $(\overline{a}, \overline{b}) \in \ker(\psi)$, then there is a f such that $ax + by = x^2 yf$ in $k[x,y]$.

Thus $y|a$ and $x|b$ i.e. we can write $a = ya_1$ and $b = xb_1$. Dividing by xy, we get $a_1 + b_1 = xf$. So $b_1 = -a_1 + xf$. So $(a, b) = a_1(y, -x) + f(0, x^2)$. Conversely $(\overline{y}, -\overline{x}), (0, \overline{x}^2) \in \ker(\psi)$. So those two elements form a set of generators of M. **(1.5)**
We have $\mathrm{Ann}((0, \overline{x}^2)) = (\overline{y})$ which is a prime ideal (as image of $(y) \in V(x^2y) \subset \mathrm{Spec}(k[x, y])$). **(0.5)**
We also have $(\overline{xy}, 0) = \overline{x}(\overline{y}, -\overline{x}) + (0, \overline{x}^2) \in M$ and $\mathrm{Ann}((\overline{xy}, 0)) = (\overline{x})$ which is a prime ideal. **(0.5)**
So $(\overline{x}), (\overline{y}) \in \mathrm{Ass}(M)$.

To show that those are the only ones, just observe that any prime ideal of the form $\mathfrak{p} = \mathrm{Ann}(m)$ for some $0 \neq m = (m_1, m_2) \in M$ satisfies: If $m_1 \neq 0$, then $\mathfrak{p} \subset (x)$ and if $m_2 \neq 0$, then $\mathfrak{p} \subset (y)$. To spell this out in detail, write $0 \neq m = \overline{\alpha}(\overline{y}, -\overline{x}) + \overline{\beta}(0, \overline{x}^2) \in M$, and $\overline{a} \in \mathrm{Ann}(m)$, then we can write $a\alpha y = x^2yf$ and $a(\beta x^2 - \alpha x) = x^2yg$ for some $f, g \in k[x, y]$. Then either $x^2|\alpha$ or $x|a$. In the later case, we have $a \in (x)$ that we already know to be an associated prime. In the first case, write $\alpha = x^2\alpha_1$, we get $\overline{\alpha y} = 0$ and $a(\beta - x\alpha_1) = yg$ so either $y|a$ (in which case, $a \in (y)$ that we already know to be an associated prime) or $y|(\beta - x\alpha_1)$; in the later case write we can write $\beta = x\alpha_1 + y\beta_1$ so $m = (0, -\overline{x}^3\alpha_1 + \overline{x}^3\alpha_1 + \overline{x}^2\overline{y}\beta_1) = (0, 0)$, contradiction. So in the case $x^2|\alpha$, we must have $a \in (y)$. In the other case $a \in (x)$. Hence $\mathrm{Ass}(M) = \{(\overline{x}), (\overline{y})\}$. **(1.5)**
If $\overline{a} \in \mathrm{Ann}(M)$, we have in particular that $\overline{a}(\overline{y}, -\overline{x}) = 0$ i.e. $ay, ax \in (x^2y)$ in $k[x, y]$. So $x^2|a$ and $y|a$ so $a \in (x^2y)$ i.e. $\overline{a} = 0$; hence $\mathrm{Ann}(M) = 0$. **(1)**

Exercise G. (Points: 4+4)
Consider $A = k[x, y, z]/(xyz, z^2)$ as a graded ring with $\deg(\overline{x}) = \deg(\overline{y}) = \deg(\overline{z}) = 1$.

(i) Compute the Poincaré series $P(A, t)$ and determine the dimension of A.

(ii) Is $A_{(x,y,z)}$ regular or Cohen–Macaulay?

Solution:
(i) We have the exact sequence $0 \to \mathfrak{a} \to k[x, y, z] \to A \to 0$ with $\mathfrak{a} = (xyz, z^2)$ a homogeneous ideal. So to compute $\dim_k(A_n)$ it is sufficient to compute $\dim_k(\mathfrak{a}_n)$ and the monomials of degree n form a basis of those spaces. We have $\dim_k(\mathfrak{a}_0) = 0 = \dim_k(\mathfrak{a}_1)$ and $\mathfrak{a}_2 = \mathrm{Span}(z^2)$ and $\mathfrak{a}_3 = \mathrm{Span}(xyz, xz^2, yz^2, z^3)$. For $n \geq 4$, the monomials of degree n which are in (xyz) are of the form $xyz \times$monomial of $\deg = n-3$; the monomials of degree n which are in (z^2) are of the form $z^2 \times$ monomial of $\deg = n - 2$. Those monomials belong to both ideals if they can be written $xyz^2 \times$ monomial of $\deg = n - 4$. Hence

$$\begin{aligned} \dim_k(\mathfrak{a}_n) &= \binom{2+n-3}{2} + \binom{2+n-2}{2} - \binom{2+n-4}{2} \\ &= \frac{(n-1)n}{2} + \frac{(n-2)(n-1)}{2} - \frac{(n-3)(n-2)}{2} \\ &= \frac{n^2+n-4}{2} \end{aligned}$$

so that $\dim_k(A_n) = \binom{n+2}{2} - \frac{n^2+n-4}{2} = n + 3$. **(2)**
So

$$\begin{aligned} P(A, t) &= 1 + 3t + \sum_{n\geq 2}(n+3)t^n = 1 + 3t + \sum_{n\geq 2}(n+1)t^n + 2\sum_{n\geq 2} t^n \\ &= \frac{-t^3+t+1}{(1-t)^2} \textbf{(0.5)} \end{aligned}$$

Localizing at $(\overline{x}, \overline{y}, \overline{z})$ we can form the graded ring

$$\mathfrak{gr}_{(\overline{x},\overline{y},\overline{z})}(A_{(\overline{x},\overline{y},\overline{z})}) = \oplus_{n\geq 0}\overline{(x, y, z)}^n / \overline{(x, y, z)}^{n+1} \simeq \oplus_{n\geq 0}(x, y, z)^n/(\mathfrak{a}_n + (x, y, z)^{n+1})$$

from which we see that the polynomial that has been computed is also $P(A_{\overline{(x,y,z)}}, t)$. **(0.5)** As 1 is not a root of the numerator, we get that the degree of the Hilbert-Samuel polynomial of $(A_{\overline{(x,y,z)}}, \overline{(x, y, z)})$, which is equal to the dimension of $A_{\overline{(x,y,z)}}$, is 2. **(0.5)**
Since any maximal ideal $\overline{\mathfrak{m}}$ of A is induced by a maximal ideal $\mathfrak{m}$ of $k[x, y, z]$ and the later (and its localization) is an integral domain and $\frac{xyz}{1} \neq 0$, we get that

$$\dim(A_{\overline{\mathfrak{m}}}) \leq \dim(k[x, y, z]_{\mathfrak{m}}/(\frac{xyz}{1})) = \dim(k[x, y, z]_{\mathfrak{m}}) - 1 = 2.$$

Hence $\dim(A) = 2$. **(0.5)** Alternatively, $\operatorname{Spec}(A) \cong V(xyz, z^2) = V(x,z) \cup V(y,z) \cup V(z)$, which immediately yields $\dim(A) = 2$.

(ii) We have $\overline{(x,y,z)}/\overline{(x,y,z)}^2 \simeq (x,y,z)/(xyz,z^2) + (x,y,z)^2$. But since $(xyz, z^2) \subset (x,y,z)^2$, we get $\dim_k(\overline{(x,y,z)}/\overline{(x,y,z)}^2) = \dim_k((x,y,z)/(x,y,z)^2) = 3 > \dim(A_{(x,y,z)}) = 2$. Therefore $A_{(x,y,z)}$ is not regular. **(1)**

We know that $\operatorname{depth}(A) \leq \dim(A/\mathfrak{p})$ for every associated prime ideal $\mathfrak{p}$. Consider $yz \in A$ and its annihilator $\operatorname{Ann}(yz) = (x,z)$, which is a prime ideal. As $A/(x,z) = k[y]$, one has $\operatorname{depth}(A) \leq 1 < 2 = \dim(A)$ and hence A is not CM.