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Exam solutions: Commutative Algebra (V3A1, Algebra I)

Exercise A. (Points: 3+2)
Assume A is a commutative ring such that for every element $a \in A$ there exists an integer $n(a) > 1$ such that $a^{n(a)} = a$.

(i) Show that $\dim(A) = 0$.

(ii) Describe an explicit example of such a ring that is not a field.

Solution:

(i) Let $\mathfrak{p} \in \operatorname{Spec}(A)$. Then for any $\bar{a} \in A/\mathfrak{p}$, we have $\bar{a}^{n(a)} = \bar{a}$, i.e. $\bar{a} \cdot (\bar{a}^{n(a)-1} - 1) = 0$. Thus, as $A/\mathfrak{p}$ is an integral domain, $\bar{a} = 0$ or $\bar{a} \cdot \bar{a}^{n(a)-2} = \bar{a}^{n(a)-1} = 1$. Hence, any non-zero element in $A/\mathfrak{p}$ is invertible, i.e. $A/\mathfrak{p}$ is a field and $\mathfrak{p}$ is a maximal ideal. **(2)** Hence, any chain of prime ideals in A can contain only one element, so $\dim(A) = 0$. **(1)**

(ii) Consider $A \coloneqq \mathbb{Z}/(2) \times \mathbb{Z}/(2)$ which consists of four elements. Note $(1,1)^2 = (1,1)$, $(1,0)^2 = (1,0)$, $(0,1)^2 = (0,1)$, and $(0,0)^2 = (0,0)$. Hence, $a^2 = a$ for all $a \in A$ and A is not a field, it is not even an integral domain, as $(1,0) \cdot (0,1) = (0,0)$. **(2)**

Exercise B. (Points: 5)
Consider the ring $A \coloneqq k[x,y]/(x(y+1), x(y+x^2))$ with $\operatorname{char}(k) \neq 2$. Describe all connected components of $\operatorname{Spec}(A)$, decide which ones consist of just one closed point and which ones have a non-empty intersection with $\operatorname{Spec}(A_{x+y})$.

Solution:
We have **(1.5)**

$$\begin{aligned} V((x(y+1), x(y+x^2))) &= V((x) \cdot (y+1, y+x^2)) = V(x) \cup V(y+1, x^2-1) \\ &= V(x) \cup (V(y+1) \cap (V(x-1) \cup V(x+1))) \\ &= V(x) \cup V(y+1, x-1) \cup V(y+1, x+1). \end{aligned}$$

The ideal $(x) \subset k[x,y]$ is prime and, therefore, $V(x)$ is irreducible and in particular connected. **(0.5)** The ideals $(y+1, x-1)$ and $(y+1, x+1)$ are maximal ideals so that $V(y+1, x-1)$ and $V(y+1, x+1)$ are closed points **(1)** which are not contained in $V(x)$ (as $x \notin (y+1, x-1)$, $(y+1, x+1)$). **(0.5)** Thus, the connected components of $\operatorname{Spec}(A)$ are $V(x)$, $V(y+1, x-1)$ and $V(y+1, x+1)$.

Recall that $\operatorname{Spec}(A_{x+y})$ can be identified with $\{\mathfrak{p} \in \operatorname{Spec}(A),\ x+y \notin \mathfrak{p}\}$. As $x+y \in (y+1, x-1)$, we have $V(y+1, x-1) \cap \operatorname{Spec}(A_{x+y}) = \emptyset$. **(0.5)** Suppose $x+y \in (y+1, x+1)$. Then one can write $x+y = (y+1)f + (x+1)g$, which by evaluating at $x = -1 = y$ yields $-2 = 0$ contradicting $\operatorname{char}(k) \neq 2$. Hence, $V(y+1, x+1) = \{(y+1, x+1)\} \subset \operatorname{Spec}(A_{x+y})$. **(0.5)** Finally, we have $(x) \subset (x, y+1)$ so that the maximal ideal $(x, y+1)$ belongs to $V(x)$. As above, one checks that $x+y \notin (x, y+1)$ (evaluate the corresponding equality at $x = 0,\ y = -1$) and, therefore, $V(x) \cap \operatorname{Spec}(A_{x+y}) \neq \emptyset$. **(0.5)**

Exercise C. (Points: 2+4)
Consider the ring $A = k[x,y,z]/(xyz, y^2)$.

(i) Show that the ideals $(\bar{x}) \subset A$ and $(\bar{z}) \subset A$ are both primary ideals and determine their radicals.

(ii) Determine a primary decomposition of the zero ideal in A and decide which associated prime ideals are isolated and which are embedded.

Solution:
(i) We have $A/(\bar{x}) \simeq k[x,y,z]/(x, xyz, y^2) \simeq k[x,y,z]/(x,y^2) \simeq k[y,z]/(y^2)$. **(0.5)** As $k[y,z]$ is an integral domain, the only zero divisors in $A/(\bar{x})$ are the elements of the ideal generated by $\bar{y}$, which are nilpotent as $\bar{y}^2 = \bar{0}$. So $(\bar{x})$ is a primary ideal. **(0.5)** Moreover, the nilradical of $A/(\bar{x})$ is generated by $\bar{y}$ and, therefore, $\sqrt{(\bar{x})} = (\bar{x}, \bar{y})$ in A. **(0.5)** Analogously, $(\bar{z}) \subset A$ is a primary ideal with radical $(\bar{y}, \bar{z})$. **(0.5)**

(ii) Let us prove that $(xyz, y^2) = (x, y^2) \cap (z, y^2) \cap (y)$ in $k[x,y,z]$. The inclusion '$\subset$' is clear. **(0.5)** Conversely, take $g \in (x, y^2)$ and write $g = xf_1 + y^2 f_2$ for some polynomials f_1, f_2. Then $g \in (y)$ if and only if $xf_1 \in (y)$ which means that $f_1 = yf_3$ (as $k[x,y,z]$ is factorial). Now $g \in (z, y^2)$ if and only if $xyf_3 \in (z, y^2)$, i.e. $xyf_3 = zh_1 + y^2 h_2$ for some $h_i \in k[x,y,z]$. Hence, $y | zh_1$ and, thus, $h_1 = yh_3$. Dividing by y yields $xf_3 = zh_3 + yh_2$. Evaluating the later at $y = 0 = z$ yields $f_3(x,0,0) = 0$, which shows that we can write $f_3 = yf_4 + zf_5$. Hence, $g = xy^2 f_4 + xyzf_5 + y^2 f_2 \in (xyz, y^2)$, proving the other inclusion. **(1.5)**
Moreover, the decomposition is minimal, since $xy \in (x,y^2) \cap (y) \setminus (xyz, y^2)$, $zy \in (z, y^2) \cap (y) \setminus (xyz, y^2)$ and $xz \in (x, y^2) \cap (z, y^2) \setminus (xyz, y^2)$. **(0.5)**
Passing to the quotient (notice that (y) is a prime ideal containing (xyz, y^2) so $(\bar{y})$ is a prime hence primary ideal) we get $(\bar{0}) = (\bar{x}) \cap (\bar{z}) \cap (\bar{y})$ in A, which is a minimal primary decomposition by (i). **(0.5)**
Hence, $\mathrm{Ass}((\bar{0})) = \{(\bar{x}, \bar{y}), (\bar{y}, \bar{z}), (\bar{y})\}$. We have $(\bar{y}) \subsetneq (\bar{x}, \bar{y})$ and $(\bar{y}) \subsetneq (\bar{y}, \bar{z})$ so that $(\bar{y})$ is an isolated associated prime and the two others are embedded. **(1)**

Exercise D. (Points: 4+4)
Consider $A = k[x,y,z]/(xy, xz)$ as a graded ring with $\deg(\bar{x}) = \deg(\bar{y}) = \deg(\bar{z}) = 1$.

(i) Compute the Poincaré series $P(A,t)$ and determine the dimension of A.

(ii) Is $A_{(x,y,z)}$ regular or Cohen–Macaulay?

Solution:
(i) We have the exact sequence $0 \to \mathfrak{a} \to k[x,y,z] \to A \to 0$ with $\mathfrak{a} := (xy, xz)$ a homogeneous ideal. So to compute $\dim_k(A_n)$ it is sufficient to compute $\dim_k(\mathfrak{a}_n)$ and the monomials contained in $\mathfrak{a}_n$ form a basis of $\mathfrak{a}_n$. We have $\dim_k(\mathfrak{a}_0) = 0 = \dim_k(\mathfrak{a}_1)$ and $\mathfrak{a}_2 = \langle xy, xz \rangle$.
For $n \geq 3$, the monomials of degree n which are in (xy) are of the form $xy\times$monomial of deg $n-2$. Likewise, the monomials of degree n which are in (xz) are of the form $xz\times$monomial of deg $n-2$. Moreover, a monomial of degree n is contained in $(xy) \cap (xz)$ if and only if it can be written $xyz \times$ monomial of deg $n-3$. As a consequence, for $n \geq 3$

$$\dim_k(\mathfrak{a}_n) = 2 \cdot \binom{2+n-2}{2} - \binom{2+n-3}{2} = (n-1) \cdot \left(n - \frac{n-2}{2}\right) = \frac{(n-1)(n+2)}{2}$$

and hence

$$\dim_k(A_n) = \binom{2+n}{2} - \frac{(n-1)(n+2)}{2} = n+2. \ (\mathbf{2})$$

Thus

$$\begin{aligned}P(A,t) &= 1+3t+\sum_{n=2}^{\infty}(n+2)t^n = 1+3t+\sum_{n=2}^{\infty}(n+1)t^n+\sum_{n=2}^{\infty}t^n \\ &= 1+3t+\left(\frac{1}{(1-t)^2}-2t-1\right)+\left(\frac{1}{(1-t)}-t-1\right) \\ &= \frac{1+t-t^2}{(1-t)^2}. \quad \mathbf{(0.5)}\end{aligned}$$

Localizing at $(\overline{x},\overline{y},\overline{z})$ we can form the graded ring

$$\mathfrak{gr}_{(\overline{x},\overline{y},\overline{z})}(A_{(\overline{x},\overline{y},\overline{z})}) = \bigoplus_{n\geq 0}\overline{(x,y,z)}^n/\overline{(x,y,z)}^{n+1} \simeq \bigoplus_{n\geq 0}(x,y,z)^n/(\mathfrak{a}_n+(x,y,z)^{n+1})$$

from which we see that the polynomial that have been computed is also $P(A_{\overline{(x,y,z)}},t)$. **(0.5)**
As 1 is not a root of the numerator, we get that the degree of the Hilbert–Samuel polynomial of $(A_{\overline{(x,y,z)}},\overline{(x,y,z)})$, which is equal to the dimension of $A_{\overline{(x,y,z)}}$, is 2. **(0.5)**
Since any maximal ideal $\overline{\mathfrak{m}}$ of A is induced by a maximal ideal $\mathfrak{m}$ of $k[x,y,z]$ and the latter (and its localization) is an integral domain and $\frac{xy}{1}\neq 0$, we get that

$$\dim(A_{\overline{\mathfrak{m}}}) \leq \dim(k[x,y,z]_{\mathfrak{m}}/\left(\frac{xy}{1}\right)) = \dim(k[x,y,z]_{\mathfrak{m}})-1 = 2.$$

Hence, $\dim(A)=2$. **(0.5)**

(ii) We have $\overline{(x,y,z)}/\overline{(x,y,z)}^2 \simeq (x,y,z)/((xy,xz)+(x,y,z)^2)$. But since $(xy,xz)\subset(x,y,z)^2$, we get $\dim_k(\overline{(x,y,z)}/\overline{(x,y,z)}^2) = \dim_k((x,y,z)/(x,y,z)^2) = 3 > \dim(A_{(x,y,z)}) = 2$. Therefore, $A_{(x,y,z)}$ and hence A is not regular. **(1)**
We claim that $\frac{\bar{x}+\bar{y}}{1}$ is not a zero divisor in $A_{(x,y,z)}$. Indeed, if $\frac{x+y}{1}\frac{f}{g} = \frac{xy}{1}\frac{f_1}{g_1}+\frac{xz}{1}\frac{f_2}{g_2}$ in $k[x,y,z]_{(x,y,z)}$, then

$$(x+y)g_1g_2f = xygf_1g_2+xzgf_2g_1$$

in $k[x,y,z]$. Note that $g(0,0,0)\neq 0\neq g_i(0,0,0)$. Thus, x divides $(x+y)g_1g_2f$. However, since the g_i have non-zero constant term, x divides $(x+y)f$ and hence f, i.e. $f=xh$. Dividing by x, we get $(x+y)g_1g_2h = ygf_1g_2+zgf_2g_1$. Evaluating at $y=0=z$, we get $xg_1(x,0,0)g_2(x,0,0)h(x,0,0)=0$. Thus, using again $g_1(0,0,0)\neq 0$ and $g_2(0,0,0)\neq 0$, one finds $h(x,0,0)=0$, i.e. $h=yh_1+zh_2$. Hence, $f=xyh_1+xzh_2$, i.e. $\overline{\frac{f}{g}}=0$ in $A_{(x,y,z)}$. **(1)**

Now $(A/(\overline{x}+\overline{y}))_{\overline{(x,y,z)}} \simeq (k[x,y,z]/(x+y,xy,xz))_{(x,y,z)}$ and let us show that in this ring any element of $\overline{(x,y,z)}_{\overline{(x,y,z)}}$ is a zero divisor. Consider $\frac{f}{g}\in\overline{(x,y,z)}_{\overline{(x,y,z)}}$ and write $\frac{f}{g} = \bar{x}\frac{f_1}{g_1}+\overline{y}\frac{f_2}{g_2}+\bar{z}\frac{f_3}{g_3}$. Taking the product with $\bar{x}\neq 0$, we get $\overline{x}\frac{f}{g}=\bar{x}^2\frac{f_1}{g_1}$. However, $x^2\in(x+y,xy,xz)$ and hence $\bar{x}\frac{f}{g}=0$. **(1)** As any regular sequence can be extended to a regular sequence of maximal length $\operatorname{depth}(A_{(x,y,z)})$ **(0.5)** and $(\overline{x}+\overline{y})\subset A_{(x,y,z)}$ cannot be further extended, we get $\operatorname{depth}(A_{(x,y,z)}) = 1 < 2 = \dim(A_{(x,y,z)})$. Hence, $A_{(x,y,z)}$ is not Cohen–Macaulay. **(0.5)**

Exercise E. (Points: 4)
Consider the ring $A := k[x]$ and the A-module $M := \operatorname{coker}(\psi)$, where $\psi\colon A^{\oplus 2}\to A^{\oplus 2}$ is given by the matrix $\psi = \begin{pmatrix} x-1 & 1-x \\ 1-x & x-1\end{pmatrix}$. Determine $\operatorname{Ass}(M)$ and $\operatorname{Supp}(M)$.

Solution:
Let us calculate the image of the canonical basis under ψ:

$\psi(e_1) = (x-1)e_1 + (1-x)e_2 = (x-1)(e_1 - e_2)$ and $\psi(e_2) = (1-x)e_1 + (x-1)e_2 = -(x-1)(e_1 - e_2)$.

So that writing $A^{\oplus 2} \simeq A(e_1 - e_2) \oplus A(e_1 + e_2)$, we get

$$M = \operatorname{coker}(\psi) \simeq k[x]/(x-1) \oplus k[x]. \ (\mathbf{1.5})$$

Hence,

$$\operatorname{Ass}(M) = \operatorname{Ass}(k[x]/(x-1)) \cup \operatorname{Ass}(k[x]) = \{(x-1)\} \cup \{(0)\}. \ (\mathbf{1.5})$$

As M is a finite A-module,

$$\operatorname{Supp}(M) = \overline{\operatorname{Ass}(M)} = \overline{\{(0), (x-1)\}} = V((0)) = \operatorname{Spec}(A). \ (\mathbf{1}).$$

Exercise F. (Points: 2+2)
Describe explicitly Noether normalization for the k-algebras $k[x,y,z]/(xy)$ and $k[x, x^{-1}]$.

Solution:
(i) Assume $A = k[x,y,z]/(xy)$ and consider the change of variables $x = u + v$ and $y = u - v$. Then $k[x,y,z] \simeq k[u,v,z]$ and $A \simeq k[u,v,z]/(u^2 - v^2)$. Consider the natural ring homomorphism $i\colon k[u,z] \to A$. We claim it is injective. Indeed, if $f \in k[u,z]$ is contained in $(u^2 - v^2)$, then $f(u,z) = (u^2 - v^2) \cdot g(u,v,z)$ and evaluating at $v = u$, we get $f(u,z) = 0$. **(1)** Furthermore, $\bar{v} \in A$ is integral over $i(k[u,z])$, since $\bar{v}^2 - \bar{u}^2 = 0$. Thus, $i(k[u,z])[\bar{v}] \simeq A$ is finite over $i(k[u,z])$, which proves that $k[u,z] \hookrightarrow A$ is a Noether normalization of A. **(1)**

(ii) For $A = k[x, x^{-1}] \simeq k[x,y]/(xy-1)$: Consider the change of variables $x = u+v$, $y = u-v$ to get $A \simeq k[u,v]/(u^2 - v^2 - 1)$. Consider the natural ring homomorphism $i\colon k[u] \to A$. We claim it is injective. If $f = \sum_{i=0}^{d} a_i u^i \in k[u]$ is contained in $(u^2 - v^2 - 1)$, we can write $f = (u^2 - v^2 - 1) \cdot g(u,v)$. Evaluating at $u = 0$, we get $a_0 = -(v^2+1) \cdot g(0,v)$, which, for degree reason, yields $g(0,v) = 0$. Hence, $a_0 = 0$ and $g(u,v) = u \cdot g_1(u,v)$. Dividing by u, we get $\sum_{i=1}^{d} a_i u^{i-1} = (u^2 - v^2 - 1) g_1(u,v)$. Again evaluating at $u = 0$, we get $a_1 = 0$ and $g_1 = u g_2$. By induction we get $f = 0$, i.e. i is injective. **(1)** Furthermore, $\bar{v} \in A$ is integral over $i(k[u])$ since $\bar{v}^2 + (1 - \bar{u}^2) = 0$; thus $i(k[u])[\bar{v}] \simeq A$, which proves that $k[u] \hookrightarrow A$ is a Noether normalization of A. **(1)**

Exercise G. (Points: 3)
Let $\mathfrak{a} \subset A$ be an ideal and $f\colon M \to N$ an A-module homomorphism such that the induced $A/\mathfrak{a}$-module homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective. Assume that N is a finite A-module and show that there exists an $a \in \mathfrak{a}$ for which $M_b \to N_b$ is surjective, where $b = 1 + a$.

Solution:
Let $P \coloneqq \operatorname{coker}(f)$ and consider the exact sequence $M \xrightarrow{f} N \to P \to 0$. Tensoring with $A/\mathfrak{a}$ yields the exact sequence $M/\mathfrak{a}M \xrightarrow{\bar{f}} N/\mathfrak{a}N \to P/\mathfrak{a}P \to 0$, i.e. $P/\mathfrak{a}P$ is isomorphic to the cokernel of $\overline{f}\colon M/\mathfrak{a}M \to N/\mathfrak{a}N$, which is trivial by assumption. **(1)** Thus, $P = \mathfrak{a}P$. Hence, by Nakayama lemma, there is a $b = 1 + a$, with $a \in \mathfrak{a}$ such that $bP = 0$. **(1)** Now, localizing the first exact sequence with respect to b yields the exact sequence $M_b \xrightarrow{f_b} N_b \to P_b \to 0$. However, since $\frac{b}{1}$ is a unit in A_b, the vanishing $bP = 0$ implies $P_b = 0$, proving surjectivity of $M_b \to N_b$. **(1)**