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Summer term 2020

Exercises, Algebra I (Commutative Algebra) – Week 13

Exercise 65. (Dimension)
Compute the dimension of the following local rings and compare it with the minimal number of generators of their maximal ideals: $k[x,y]_{(x,y)}/(x^2-y^3)$; $k[x,y]_{(x,y)}/(x^2-y)$; $k[x,y]_{(x,y)}/(x^2,y^3)$; $k[x,y,z]_{(x,y,z)}/(x^2+y^2+z^n)$ for $n \geq 1$

Exercise 66. (Height and dimension)
Recall that in Proposition 18.28 we showed the formula $\mathrm{ht}(\mathfrak{p}) + \dim(A/\mathfrak{p}) = \dim(A)$ for all finite type k-algebras which are integral domains A and arbitrary prime ideals $\mathfrak{p} \subset A$. Does this still hold true for the ring $k[x,y,z]_{(x,y,z)}/(xy,xz)$?

Exercise 67. (Fibre dimension)
Consider the ring inclusion $A := k[x,y] \subset B := k[x,y,z]/(yz-x)$ and the prime ideals $\mathfrak{q} := (\bar{y}, \bar{z}) \subset B$ and $\mathfrak{p} := (x,y) \subset A$. Prove the following assertions:

$$\text{(i) } \mathfrak{q} \cap A = \mathfrak{p};\ \text{(ii) } \mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) = 2;\ \text{and (iii) } \dim(B_{\mathfrak{q}}/\mathfrak{p}B_q) = 1.$$

In particular,

$$\dim(B_{\mathfrak{q}}) \leq \dim(A_{\mathfrak{p}}) + \dim(B_{\mathfrak{q}} \otimes k(\mathfrak{p}))$$

is not always an equality, see lecture on thursday.

Exercise 68. (Singular points and the Jacobi criterion)
Assume $f \in k[x_1, \ldots, x_n]$ is an irreducible polynomial in n variables with coefficients in an algebraically closed field k. A closed point $(a_1, \ldots, a_n) \in k^n$, thought of as maximal ideal $\mathfrak{m} := (x_1 - a_1, \ldots, x_n - a_n) \in V(f) \subset \mathbb{A}^n_k$, is called *singular* if all partial derivatives $\partial f/\partial x_i$ have a common zero in $(a_1, \ldots, a_n)$. Show that $(a_1, \ldots, a_n) \in V(f)$ is singular if and only if $k[x_1, \ldots, x_n]_{\mathfrak{m}}/(f)$ is a not regular local ring.
Hint: Construct an explicit isomorphism $\mathfrak{m}/\mathfrak{m}^2 \simeq k^n$ and use it.

Exams:

- The exam for the class will take place on July 24^{th}, 4-6 pm in the lecture halls CP1 and CP2, Endenicher Allee 19C;
- The resit exam will take place on September 21^{st}, 2-4 pm in the lecture hall CP1, Endenicher Allee 19C;

Further information will be posted here. In particular, you will have to make sure to check where you will be seated before(!!) arriving.

You can still hand in solutions, but they will not be (necessarily) corrected anymore.