A NOTE ON SUPER-EXPONENTIAL ORBIT GROWTH IN REEB DYNAMICS

Abstract. The purpose of this note is to prove the following statement: given a closed contact manifold (Y, ξ) of dimension at least 3 and any sequence of natural numbers $n_1, n_2, n_3, \ldots$, there exists a non-degenerate contact form $\xi = \ker \alpha$ with the property that the number of closed Reeb orbits of length at most k is at least n_k. More informally, one can always find a contact form so that the orbit growth is as fast as one likes.

The key construction in the proof was explained to me by Dmitry Turaev. The result is plausibly known to certain experts; however, it appears to be unknown to sufficiently many experts that I thought it could be useful to write it down.

This note ends with a number of questions about super-exponential orbit growth in Reeb dynamics.

Notation. Fix polar coordinates $(r, \theta) \mapsto (r\cos 2\pi\theta, r \sin 2\pi\theta)$. We let $D_r \subset \mathbb{R}^2$ denote the closed disk of radius $r > 0$; when $r = 1$ we shall simply write D instead of D_1. For $n \geq 1$ and $\mathbf{r} = (r_1, \ldots, r_n)$, we let $\mathcal{P}^n_{\mathbf{r}} = D_{r_1} \times \cdots \times D_{r_n} \subset \mathbb{R}^{2n}$ be the polydisk of (multi)radius $\mathbf{r}$; when $\mathbf{r} = (1, \ldots, 1)$ we shall simply write $\mathcal{P}^n$.

Given a symplectic manifold M and a Hamiltonian $H : S^1 \times M \to \mathbb{R}$, we let ϕ_H be the time-1 map. More generally, for $k \geq 1$, we let $\phi^k_H = \phi_{kH}$ be the time-k map. Recall that H is said to be *strongly* non-degenerate if the fixed points of ϕ^k are non-degenerate for all $k \geq 1$.

If $U, V \subset \mathbb{R}$ are intervals, it will be convenient to abuse notation by writing expressions such as $\{U \leq r \leq V\}$ to mean "the subset of real numbers r such that there exists $u \in U, v \in V$ with $u \leq r \leq v$." Similarly $\{U < r < V\}$ is the subset of real numbers r such that for all $u \in U, v \in V$ we have $u < r < v$, etc.

Acknowledgements. This note was motivated by my joint work with Y. Barış Kartal [3]. The relevance of our work to (super)exponential growth phenomena in Reeb dynamics was partly explained to us by Semon Rezchikov.

The crucial construction of this note was explained to me by Dmitry Turaev. I would like the thank the participants of the workshop "Frontiers of quantitive symplectic and contact geometry" at Institut Mittag-Leffler for many interesting discussions.

1. The result

1.1. **Statement and summary of the proof.** Let (Y, ξ) be a closed, connected contact manifold of dimension $2n+1 \geq 3$. Fix an increasing sequence of natural numbers $n_1, n_2, n_3, \ldots$. The goal of this note is to prove[1] the following:

Theorem 1. *There exists a non-degenerate contact form $\xi = \ker \alpha$ such that the number of closed Reeb orbits of length at most k is at least n_k.*

Here is a sketch of the proof. The first and most important step is to construct a (strongly non-degenerate) Hamiltonian diffeomorphism of the polydisk $\mathcal{P}^n$ with the property that the number of orbits of the first k iterates grows faster than the sequence $\{n_k\}$. The idea of the

[1] The proof of one crucial lemma (Lemma 4) is only sketched, although it presumably can be extracted from the literature.

construction was explained to me by Dmitry Turaev.[2] The second and final step is to fix an open book decomposition for (Y, ξ) (which always exists) and to implant (the contact mapping torus of) our previously constructed Hamiltonian diffeomorphism of $\mathcal{P}^n$ into the open book.

1.2. **A Hamiltonian diffeomorphism of the disk with fast orbit growth.** We wish to prove:

Proposition 2. *There exists a (time-independent) Hamiltonian $H : D \to \mathbb{R}$ supported in the interior of D with the property that*

$$\sum_{1 \leq \ell \leq k} \#orb(\phi_H^\ell) \geq n_k, \tag{1.1}$$

and H is strongly non-degenerate on $D_{7/8} \subset D$.

The proof is conceptually quite simple but tedious to write down. To set the stage, let $\{U_i\}_{i=1}^\infty$ be a collection of open subintervals of $[1/4, 3/4]$ with pairwise disjoint closures. We also want to assume that $U_{i+1} < U_i$.[3] We let $A_i \subset D$ be the annulus $U_i \times S^1 = \{(r, \theta) \mid r \in U_i\} \subset D$.

In an ideal world, we would now implement the following argument: construct a Hamiltonian such that the k-th iterate has n_k fixed points in the annulus A_k. Do this by first constructing a Hamiltonian whose k-th iterate *is the identity* on A_k. Then iteratively perturb it to have n_k fixed points on A_k. The problem with this argument is that we that need to perturb infinitely many times (once for each k), so it's not clear that this perturbation process will converge to a smooth function.[4] To ensure that the process does converge, we need to carefully keep track of the size of the perturbations. For this reason, the construction we actually carry out is slightly different from the one we have just sketched.[5]

We now explain the details.

Fix a smooth function $f : [0, 1] \to \mathbb{R}_{\geq 0}$ with the following properties:

- f vanishes in a neighborhood of $\{0, 1\}$.
- if k is divisible by 3, then f is constant on U_k with value p_k/q_k, where p_k, q_k are relatively prime positive integers and $q_k > q_{k-1}$.
- if k is not divisible by 3, then f is constant on U_k with value in $\mathbb{R}_{\geq 0} \setminus \mathbb{Q}$.
- f is constant with value π on $[1/8, 1/4] \cup [3/4, 7/8]$.[6]

Lemma 3. *Such a function exists.*

Proof. Let $\chi : [0, 1] \to \mathbb{R}_{\geq 0}$ be a non-negative function which vanishes near $\{0, 1\}$ and takes the value π on $[1/8, 7/8]$. We iteratively construct a sequence of functions $f_k : (1/4, 3/4) \to \mathbb{R}_{\geq 0}$, along with a sequence of integer pairs $(p_k, q_k) \in \mathbb{N}^2$, having the following properties:

- the support of f_k is an interval. This interval is disjoint from $supp(f_i)$ for all $i < k$, and is also disjoint from $[1/8, 1/4] \cup [3/4, 7/8]$.
- the q_k are prime and $q_k > q_{k-1}$
- if k is divisible by 3, then $f_k + \pi$ is constant on U_k with value p_k/q_k. Otherwise $f_k + \pi$ is constant on U_k with value an arbitrary positive irrational.
- $\|f_k\|_k := \sup_{i \leq k} |\partial^i f_k| \leq 2^{-k}$

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[2] All misunderstandings are mine alone

[3] Recall that this abuse of notation just means that if $x \in U_i$ and $y \in U_{i+1}$ then $y < x$). Concretely, one can take e.g. $U_i = (1/2^{2i} + 1/4, 1/2^{2i-1} + 1/4)$.

[4] We also need to make sure that no additional degenerate orbits are created.

[5] In particular, it involves more indices.

[6] Of course, any other positive irrational would do equally well.

We leave to the reader to check that such a sequence exists.

Now set $f := \chi + \sum_k f_k$. Note that f obviously smooth away from $1/4$. To check smoothness at the point $1/4$, note that for any $\ell \in \mathbb{N}$ and any $\epsilon > 0$ there is some $\delta > 0$ so that $|\partial^\ell f_i| < \epsilon$ on $[1/4, 1/4 + \delta]$ for all $i \geq \ell$. Since the f_i have disjoint support, it follows that $\lim_{\delta \to 0} \partial^k f(1/4 + \delta) = 0$. Having established that f is smooth, it satisfies the desired properties by construction. □

We now define a symplectomorphism of the unit disk:

$$\psi_0 : D \to D$$
$$(r, \theta) \mapsto (r, \theta + f(r))$$

Observe that $\psi_0^{q_k}$ is the identity on the annulus A_k. In the annuli $[1/8, 1/4] \times S^1$ and $[3/4, 7/8] \times S^1$, ψ_0 is an irrational rotation.

Let H_0 be the Hamiltonian inducing ψ_0. Our plan is to construct a perturbation $H_0 \rightsquigarrow H$ so that

- ϕ_H fixes $D_{7/8}$ setwise.
- ϕ_H^k is strongly non-degenerate on $D_{7/8}$
- if k is divisible by 3, then $\phi_H^{q_k}$ has at least $n_{q(k+3)}$ critical points the annulus A_k.[7]

To construct the perturbation, inductively define a sequence of functions $h_k : D \to \mathbb{R}$ with the following properties:

- h_k is supported in the annulus between A_{3k+1} and A_{3k-1}.[8]
- $\|h_k\|_k < 2^{-k}$.
- $H_0 + \sum_{i \leq k} h_i$ is strongly non-degenerate on $\{(r, \theta) \mid r > A_{3k+1}\}$.
- the time q_k flow of $H + h_k$ has at least $n_{q(k+3)}$ fixed points.

Lemma 4. *Such h_k exists.*

Proof sketch. Pick a Morse function h_k which has $n_{q(k+3)}$ critical points in A_{3k}, whose support is contained between A_{3k+1} and A_{3k-1} and disjoint from that of the h_i for $i < k$, and such that $\|h_k\|_k < 2^{-k}$. We claim (but do not give a proof) that if this Morse function is chosen generically, then the strong non-degeneracy condition will also hold. □

We now consider the Hamiltonian $H_0 + \sum_k h_k$. Due to the assumption $\|h_k\|_k < 2^{-k}$, the smoothness of this Hamiltonian can be checked as in Lemma 3. By construction, it is strongly non-degenerate on the annulus $\{1/8 < r \leq U_2\} \times S^1$. Introduce a further perturbation supported in $\{U_2 \leq r < 7/8\} \times S^1$ so that it becomes non-degenerate in $\{1/8 < r < 7/8\} \times S^1$. Finally, introduce a final perturbation supported on $D_{1/4}$ to obtain the desired Hamiltonian.

This completes the proof of Proposition 2.

1.3. **Open books.** Given a compactly-supported exact symplectomorphism of a Liouville domain, we can construct a contact manifold by taking the associated open book. We briefly review this here:

Construction 5. Let (M, λ) be a Liouville domain with contact boundary $(\partial M, \lambda)$ and collar $((-1, 0] \times \partial M, e^t \lambda)$. Let $\varphi : M \to M$ be a symplectomorphism which is the identity on the collar and satisfies $\varphi^* \lambda - \lambda = dU$ for some $U : M \to \mathbb{R}_{>0}$. We let C be the value of U near

[7] If we merely required the time q_k flow to have n_{q_k} fixed points, the desired inequality $orb(\phi_H^\ell) \geq n_\ell$ would only hold when $\ell = q^k$ and k divisible by 3.

[8] More formally, I mean that if (r_0, θ_0) is in the support of h_k, then there exists $s^+ \in U_{4k+1}, s^- \in U_{4k-1}$ such that $s^+ \leq r_0 \leq s^-$).

$\partial_\infty M$. Consider the contact mapping torus $\mathbb{R} \times M/(t,x) \sim (t - U(x), \varphi(x))$. Then the contact form $dt + \lambda$ descends to the quotient. Along the collar, it looks like $(\mathbb{R}/C\mathbb{Z} \times \partial M, dt + \lambda)$, so we can glue on a torus $D^2 \times \partial_\infty Y$ exactly as in [6, p. 153] or [9]. [9]

Note that Construction 5 outputs a contact manifold along with a contact *form* (not merely a contact structure). The Reeb vector field on $\mathbb{R} \times M/\mathbb{Z}$ is ∂_t.

We have the following well-known fact:

Fact 6 (Giroux)**.** *Every contact manifold (Y, ξ) arises from Construction 5, up to contact isotopy.*

We now complete the proof of the main theorem.

By Grey stability, the following operations leave the contact manifold unchanged up to contact isotopy:

(i) changing φ by an exact symplectomorphism isotopic to the identity;
(ii) replacing M by $M \cup [0, N] \times \partial M$ and extending φ to be the identity.

By Fact 6, we can assume that (Y, ξ) arises by applying Construction 5 to some Weinstein domain (W, λ) along with an exact symplectomorphism φ. We can also assume by by Grey stability that φ is constant on some collar $((-1, 0] \times \partial M, e^t\lambda)$, and that the standard polydisk $\mathcal{P}^n$ admits a symplectic embedding into this collar.[10]

Let $H : \mathcal{P}^n \to \mathbb{R}$ be a compactly-supported, strongly non-degenerate Hamiltonian on the polydisk satisfying (1.1); the existence of H follows immediately from Proposition 2 by taking products. By construction, ϕ_H is compactly supported so extends to an exact symplectomorphism of M isotopic to the identity. Hence the open books associated to $\phi \circ \phi_H$ and ϕ are the same up to contact isotopy and both give rise to (Y, ξ).

The resulting contact form on (Y, ξ) is possibly degenerate. However, inside $\iota(D_{7/8} \times \cdots \times D_{7/8})$ all orbits are non-degenerate, since ϕ_H is strongly non-degenerate.[11] Finally, we simply perturb the contact form in the complement of $\iota(D_{7/8} \times \cdots \times D_{7/8})$ using e.g. [1, Thm. 13].

Let $T > 0$ be large enough so that the time T Reeb flow rotates the page $\{0\} \times M$ past itself.[12] Up to possibly rescaling the contact form, we can assume $T \leq 1$. Then it follows from (1.1) that

$$\#\{\text{closed Reeb orbits of length at most } k\} \geq \sum_{1 \leq \ell \leq k} \#orb(\phi_H^\ell) \geq n_k \tag{1.2}$$

This completes the proof of Theorem 1.

2. Context and questions

As far as I can tell, there is almost nothing in the literature about super-exponential growth in Reeb dynamics.[13] I would be great to have a better picture of how prevalent this type of behavior is. It would also be interesting to understand the mechanisms which cause super-exponential growth to appear. Here is a cautionary remark followed by some questions:

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[9]Strictly speaking, these constructions may require one to modify the contact form on the part of the collar; e.g. [9] modifies the form on the subcollar $[1/2, 1] \times \partial M \times S^1$.

[10]A minor technical point: for consistency with [9], we will assume that the embedding lands in $(-1, -1/2) \times \partial M \times S^1$.

[11]Here we use that a Reeb orbit in the contact mapping torus is non-degenerate iff the corresponding Hamiltonian fixed point is non-degenerate.

[12]I mean that for all $x \in \{0\} \times M$, there exists $y \in \{0\} \times M$ so that $\psi_s(x) = y$ for some $0 < s \leq T$.

[13]There is slightly more, but still not much, in the context of Hamiltonian dynamics on symplectic manifolds.

Remark 7. A useful tool for studying Reeb dynamics is the *growth rate* of Floer theoretic invariants, such as symplectic cohomology, contact homology, wrapped Floer cohomology, etc. However, it was observed in [3] that symplectic and wrapped Floer cohomology grow at most exponentially (presumably similar behavior also holds for the growth rate of other Floer-theoretic invariants). Thus other tools are needed to probe superexponential phenomena.

Some questions:

- Is there a Legendrian version of Theorem 1? More precisely, suppose that Λ is a closed Legendrian in a contact manifold (Y, ξ), prove that there exists a contact form such that the number of chords grows faster than some given sequence $n_1, n_2, \ldots$. Possibly relevant: Honda and Huang [7] proved that any such Legendrian can be placed in the page of an open book.
- Can one force super-exponential growth, e.g. via the existence of an orbit satisfying some conditions? (Note that the construction described above is essentially local, since we just implanted the mapping torus of a polydisk having super-exponential dynamics. Perhaps one can always implant such a local model around a closed orbit, under appropriate assumptions?)
- Prove/disprove, for your favorite $k \in [0, \infty]$: super-exponential growth is C^k-generic.

  Remark 8. For contact 3-manifolds, the remarkable work of Colin, Dehornoy, Hryniewicz, Rechtman [5] shows that positive entropy is C^∞-generic – hence *exponential* orbit growth is also C^∞-generic by [8]. It would be interesting to combine the methods of [4, 5] with known results about super-exponential growth of Hamiltonian diffeomorphisms of surfaces to study super-exponential orbit growth on 3-manifolds.

- A weaker notion than genericity is density. Prove/disprove, for your favorite $k \in [0, \infty]$: the space of contact forms with super-exponential growth is *dense* in the C^k-topology. (To this end, it seems that the local model constructed in Step 1 can be made C^k close to the mapping torus of the polydisk with trivial monodromy).
- Connections to KAM theory? In the case of Hamiltonian dynamics on surfaces, [2] describes a mechanism for super-exponential orbit growth based on KAM theory. This mechanism appears to be a special case of a more general mechanism described to me by Semon Rezchikov. It would be interesting to me if one could make this mechanism precise in the contact setting.

References

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