1. Let $M = (M, \omega)$ be a closed symplectic manifold and let J be a compatible almost-complex structure.

   (i) Suppose that $A \in H_2(M; \mathbb{Z})$ has the property that $[\omega](A) < 0$ prove that $\overline{\mathcal{M}}^A_{g,n}(M, J) = \emptyset$ for any g, n.

   (ii) Convince yourself that there is a canonical bijection of sets $\overline{\mathcal{M}}^A_{g,n}(M, J) = \overline{\mathcal{M}}_{g,n} \times M$. (In fact, this is also an isomorphism of topological spaces).

2. Let Σ_g denote a closed connected Riemann surface of genus $g \geq 0$.

   (i) If $\mathbb{CP}^1 \to \Sigma_g$ is a non-constant holomorphic map, prove that $g = 0$. (Hint: look up the Riemann–Hurwicz formula).

   (ii) Deduce that a (closed, connected) symplectic surface of positive genus has no nontrivial genus zero Gromov–Witten invariants.

3. A *Kähler manifold* is a Riemannian manifold (M, g) equipped with a compatible almost-complex structure I satisfying the following conditions:

   (i) I is covariantly constant (with respect to the Riemannian connection)

   (ii) $\omega := g(I-, -)$ is symplectic

   We also say that an almost-complex structure I satisfying (i) and (ii) is Kähler. One would get an equivalent notion by replacing (i) with the condition that I is integrable.

   A *hyperkähler manifold* is a Riemannian manifold (M, g) equipped with a triple of compatible almost-complex structures I, J, K satisfying the following conditions:

   (i) I, J, K are Kähler

   (ii) I, J, K satisfy the *quaternionic relations*: $I^2 = J^2 = K^2 = IJK = -id$

   Let us now fix a hyperkähler manifold (M, g, I, J, K).

   (i) verify that M has real dimension divisible by 4 *Hint: the tangent space at any point is a module over the division algebra* $\mathbb{H}$.

   Given a point $t = (t_x, t_y, t_z) \in S^2 \subset \mathbb{R}^3$ (i.e. $t_x^2 + t_y^2 + t_z^2 = 1$), we write $J_t := t_x I + t_y J + t_z J$. Similarly we write $\omega_t = t_x \omega_I + t_y \omega_J + t_z \omega_K$.

(ii) Verify that J_t is a covariantly constant almost-complex structure.

(iii) Verify that ω_t is symplectic and J_t is an ω_t-compatible almost-complex structure.

(iv) Prove that all nontrivial Gromov–Witten invariants of (M, ω_t) vanish. In particular, the quantum cup product agrees with the ordinary cup product in cohomology.

In real dimension 4*, all Kähler manifolds are automatically hyperkähler. The only compact examples are* 4*-tori and* $K3$ *surfaces. Beauville discovered that the Hilbert scheme of* k *points on a (compact)* 4*-dimensional hyperkähler is a* $4k$*-dimensional (compact) hyperkähler. Many interesting examples of non-compact hyperkähler manifolds arise from geometric representation theory (such as hypertoric varieties, Nakajima quiver varieties, moduli spaces of Higgs bundles, ...)*

4. Let $M = (M, \omega)$ be closed symplectic with compatible almost-complex structure J.

   Given classes $a, b, c \in H^*(M)$ with $|a| + |b| + |c| = \operatorname{vdim} \overline{\mathcal{M}}^A_{0,3}(M, J)$, recall that $(a * b)_A \in H^{|a|+|b|-2c_1(A)}(M)$ is the unique class satisfying

$$\langle (a * b)_A \cup c, [M] \rangle = \mathrm{GW}^A_{0,3}(a, b, c) \tag{0.1}$$

   (i) Give a geometric interpretation for the cycle

$$(\mathrm{ev}_0^* \, a \cup \mathrm{ev}_1^* \, b) \cap [\overline{\mathcal{M}}^A_{0,3}(M, J)] \in H_{|c|}(\overline{\mathcal{M}}^A_{0,3}(M, J))$$

   (ii) Explain why $(ev_\infty)_*((\mathrm{ev}_0^* \, a \cup \mathrm{ev}_1^* \, b) \cap \mathrm{ev}_*[\overline{\mathcal{M}}^A_{0,3}(M, J)]) = \mathrm{PD}((a*b)_A)$

5. The purpose of this exercise is to compute the quantum cohomology of $\mathbb{CP}^n$. We will work over the Novikov field $\Lambda_{\mathbb{C}}$ and take the "pseudo-theorem" from class as an axiom.

   The first step is to discuss the ordinary cohomology ring. Let $c \in H_2(\mathbb{CP}^n; \Lambda)$ be the Poincaré dual to the hyperplane class $H := [\mathbb{CP}^{n-1}] \in H^{2n-2}(\mathbb{CP}^n; \Lambda)$. Let $L = [\mathbb{CP}^1] \in H_2(\mathbb{CP}^n; \Lambda)$.

   (i) Verify that $c_1(\mathbb{CP}^n) = (n+1)c$

   (ii) Verify that $\mathrm{GW}^{\ell L}_{0,3}(a, b, c) = 0$ if $\ell \notin \{0, 1\}$. *Hint: recall the virtual dimension formula for* $\overline{\mathcal{M}}^{\ell L}_{0,3}(\mathbb{CP}^n, j)$.

In class we saw that $(c^i * c^j)_0 = c^i \cup c^j$. It therefore remains to compute $(c^i * c^j)_L$. Recall from class (or the previous exercise...) that

$$\begin{aligned}PD((c^i * c^j)_L) &= (ev_\infty)_*(([c]^i \cup [c]^j) \cap [\overline{\mathcal{M}}^A_{0,3}(\mathbb{CP}^n, j)]) \\ &= (\pi_\infty)_*(PD(\pi_0^*[c]^i) \cap PD(\pi_1^*[c]^j) \cap ev_*[\overline{\mathcal{M}}^A_{0,3}(\mathbb{CP}^n, j)])\end{aligned}$$

Using this formula:

(iii) Convince yourself that $(\pi_\infty)_*(PD(\pi_0^*[c]^i) \cap PD(\pi_1^*[c]^j) \cap ev_*[\overline{\mathcal{M}}^A_{0,3}(\mathbb{CP}^n, j)]) = [\mathbb{CP}^{n-i+n-j+1}]$. *Hint: represent $PD(\pi_0^*[c]^i)$ by (the closure of) an $(n-i)$-dimension affine linear subspace of $\mathbb{C}^n \subset \mathbb{CP}^n$ and $PD(\pi_0[c]^j)$ by (the closure of) an $(n-j)$-dimensional affine linear subspace. The space of lines intersecting these two subspaces is an $n-i+n-j+1$ dimensional affine linear subspace whose closure is precisely the cycle which you need to compute.*

(iv) Conclude that $\mathrm{GW}^L_{0,3}(c^i, c^j, c^k) = 1$ if $i + j + l = 2n + 1$ and 0 else.

(v) Conclude that $QH^*(\mathbb{CP}^n) = \Lambda_{\mathbb{C}}[x, q]/(x^{n+1} - q)$.