SEMINAR ON CLUSTER ALGEBRAS - FALL 2021

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Talk 1: Motivational examples

a. Total positivity for $Gr(2, n)$
   - Relation to triangulations of an n-gon.
   - Mention (in one sentence) that there is also TP for G, G/P etc.

[FWZ1] Ch1, [F] 9/11

b. Somos-4 and Somos-5 sequences, Markov triples, Fermat's numbers (Define these, and pose question: why are these integral).

[FWZ1] Ex 3.4.3, Ex 3.4.1, Ex 3.4.2

c. Canonical bases

Talk 2: Quivers and quiver mutations

a. Define quivers and mutations, and prove basic properties
   - mutation is an involution.
   - mutation commutes with its opposite quiver.
   - nonadjacent mutations commute.

[FWZ1] Section 2.1

b. Mutation equivalence: play with trees and acyclic quivers. Can we prove Thm 2.6.12 after we do cluster categories?

[FWZ1] Section 2.6

c. Quivers from triangulations of an n-gon: flips correspond to quiver mutations.

[FWZ1] Section 2.2

Talk 3: Seed mutations and cluster algebras

a. Define seed and seed mutations for quivers.
   Prove that seed mutation is an involution.

[L] Section 2.1

b. Examples
   - A_1
   - A_1 with 2 frozens (Coordinate ring of SL_2)
   - A_2 (Note Ptolemy relation gives cluster variables for triangulations of 5-gon)
   - A_3 (Show a picture of the associahedron)
   - Markov
   - 2-Kronecker
   - [Sc] example 1.5

c. Define cluster algebras: compute it for A_1, A_2.

[L] Section 2.1

d. Observations
   - Define finite type, finite mutation type, acyclic type, surface type.
   - Draw Schiffler's Venn diagram.
   - Point out Laurent Phenomenon and positivity (don't write it).

[Sc] Section 1.7

TALK 4: LAURENT PHENOMENON AND POSITIVITY

a. State LP and positivity
   - Explain that it is nonobvious.
   - Give reference for positivity.

[FWZ1] Section 3.3

b. Examples:
   - Integrality for Somos-4.
   - Markov triples and Fermat numbers.

c. Prove LP for rank 2 cluster algebras.

[F] 10/30

d. Give references for general proof. Outline the general idea.

[FWZ1] Section 3.3

TALK 5: FINITE (MUTATION) TYPE CLASSIFICATIONS

a. Cluster algebras from matrices.
   - Define skew-symmetrizable matrix and seed.
   - Mention that exchange matrix of a quiver is skew symmetric, so skew-symmetrizable matrices are more general.
   - Mention how this can be viewed as a valued quiver.
   - Matrix mutation. examples

[FWZ1] Section 3.1, 3.2

b. Recall Schiffler's Venn diagram.
c. Prove finite type classification in rank 2. [F] 10/28
d. Give finite type classification. (ADE for quivers, ABCDEFG for matrices.)
e. Note FT implies FMT.
f. Give finite mutation type classification. (No proofs.) only quivers or quivers and matrices

TALK 6: FRIEZES AND CLUSTER ALGEBRAS

[P] Sections 1,2,3

a. Define frieze patterns of height n.
b. Generating frieze patterns from lightening bolts.
c. Periodicity.
d. Bijection between triangulations of an n-gon and friezes.
e. Relationship with cluster algebras. (Take a type A cluster algebra and specialize $x_1, x_2, \ldots, x_n$ to the values of the friezes in the lightening bolt.)

TALK 7: REVIEW OF QUIVER REPRESENTATIONS AND AR THEORY

[P, S]

a. Define quiver representation and morphism. Define indemcomposable representation.
b. Category $rep(Q)$ (abelian and equivalent to modules over path algebra).
c. Define AR quiver, AR translation.
d. Examples
   - A_2, A_3
   - D_4
   - 2-Kronecker
e. Mention Gabriel's theorem.

TALK 8: THE BOUNDED DERIVED CATEGORY AND $D^b(repQ)$

[K, P]

a. Define bounded derived category.
b. Explain how we can view it as a mesh for $D^b(repQ)$.

c. Examples
   – A_2, A_3
   – D_4
   – 2-Kronecker

Talk 9: Cluster categories

[K, P]

a. Define the cluster category $C_Q = D^b(repQ)/\Sigma^{-1}\tau$.
b. Compute the AR quiver for C_Q.
   – A_2, A_3
   – D_4
   – 2-Kronecker
c. Tilting and main theorem: Ext-quiver for a tilting object is cluster quiver.
d. Mutation: could mention in words.

Talk 10: The Caldero-Chapoton cluster character map

a. Define quiver grassmannians and do examples.
b. Define the CC map. Compute examples in A_1, A_2 maybe more.

[K, P]

c. The proof [KNotes]
d. Compose with Gabriel's theorem to get FT classification (assuming classification of finite rep type quivers)

Talk 11: Some applications

a. Friezes and cluster categories. [BFGST, P]
b. Cor 4 in Caldero-Keller's "From triangulated categories to cluster algebras II". [CK]
c. More depending on how much we have covered in earlier talks.

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