Violating the Singular Cardinals Hypothesis Without Large Cardinals

CUNY Logic Workshop, November 18, 2011

by Peter Koepke (Bonn); joint work with Moti Gitik (Tel Aviv)

Easton proved that the behavior of the exponential function 2^κ at **regular** cardinals κ is independent of the axioms of set theory except for some simple classical laws. The Singular Cardinals Hypothesis SCH implies that the Generalized Continuum Hypothesis GCH $2^\kappa = \kappa^+$ holds at a **singular** cardinal κ if GCH holds below κ. Gitik and Mitchell have determined the consistency strength of the **negation of the Singular Cardinals Hypothesis** in Zermelo Fraenkel set theory **with the axiom of choice** AC in terms of **large cardinals**.

Arthur Apter and I pursue a program of determining such consistency strengths in Zermelo Fraenkel set theory **without AC**. Moti Gitik and I showed that the following **negation of the Singular Cardinals Hypothesis** is relatively consistent with Zermelo Fraenkel set theory: GCH holds below the first uncountable limit cardinal $\aleph_\omega$ and there is a surjection from its power set $\mathcal{P}(\aleph_\omega)$ onto some arbitrarily high cardinal λ.

This leads to the conjecture that without the axiom of choice and without assuming large cardinal strength a - surjectively modified - exponential function can take rather arbitrary values at **all** infinite cardinals.

CANTOR's Continuum Hypothesis

Theorem 1. (GEORG CANTOR) *The power set* $\{x\,|\,x\subseteq\mathbb{N}\}$ *of* $\mathbb{N}$ *is* not *denumerable.*

Theorem 2. $2^{\aleph_0}\geqslant\aleph_1$.

Conjecture 3. (CANTOR's Continuum Hypothesis, CH) $2^{\aleph_0}=\aleph_1$.

KURT GÖDEL proved the consistency of CH, assuming the consistency of the ZERMELO-FRAENKEL axioms ZF, by constructing the model L of constructible sets.

Theorem 4. $L\vDash\mathrm{CH}$.

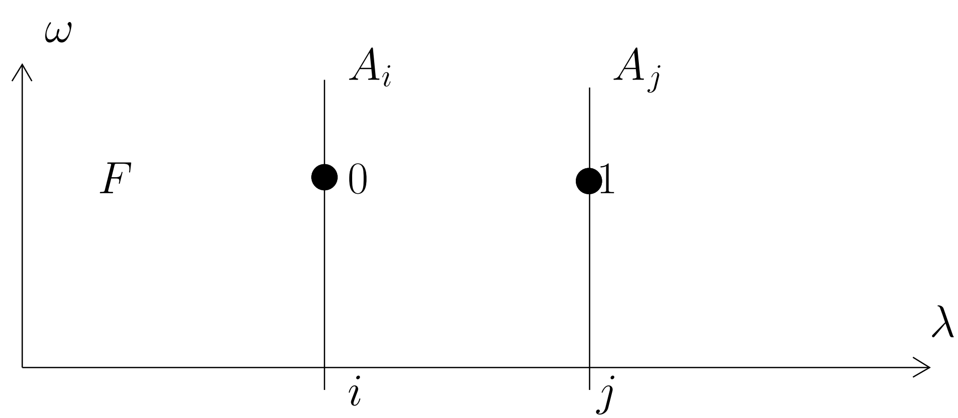
PAUL COHEN proved the opposite relative consistency

Theorem 5. *Any (countable) model* V *of* ZFC *can be extended to a model* $V[G]$ *of* $\mathrm{ZFC}+2^{\aleph_0}>\aleph_1$.

COHEN introduced the method of *forcing* to adjoin a characteristic function F to the *ground model* V satisfying

1. $F\colon \lambda \times \omega \rightarrow 2$ for some $\lambda \geqslant \aleph_2^V$
2. $\forall i < j < \lambda \;\; \lambda n.F(i,n) \neq \lambda n.F(j,n)$; set $A_i = \lambda n.F(i,n)$

$A_i \neq A_j$

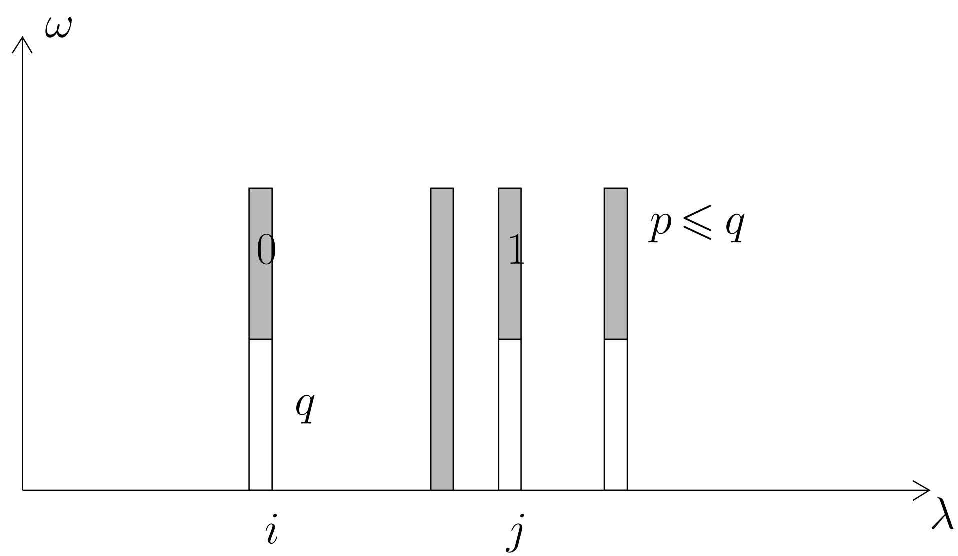


The Cohen partial order is essentially

$$P = \{(p_i)_{i<\lambda} \,|\, \exists d \in [1,\omega)\ \exists D \in [\lambda]^{<\omega}\,((\forall i \in D\ \ p_i\colon d \to 2) \wedge (\forall i \notin D\ p_i = \emptyset))\}$$

partially ordered by *reverse inclusion*:

$$p = (p_i)_{i<\lambda} \leqslant q = (q_i)_{i<\lambda}\ (p \text{ is stronger than } q)\ \text{ iff }\ \forall i < \lambda\ p_i \supseteq q_i$$



If $G \subseteq P$ is a “generic path” through P then $F = \bigcup_{p \in G, i<\lambda} i \times p_i$ is as required.

Fuzzifying the A_i

Define the symmetric difference of two functions $A, A'\colon \mathrm{dom}(A) = \mathrm{dom}(A') \rightarrow 2$ by

$$A \Delta A'(\xi) = 1 \text{ iff } A(\xi) \neq A'(\xi).$$

For $A, A'\colon \aleph_0 \rightarrow 2$ define an equivalence relation $\sim$ by

$$A \sim A' \text{ iff } A \Delta A' \in V$$

Let $\tilde{A} = \{A' | A' \sim A\}$ be the $\sim$-equivalence class of A and let

$\vec{A} = (\tilde{A}_i | i < \lambda)$ be the sequence of equivalence classes of the COHEN reals.

A symmetric submodel

The model

$$N = \mathrm{HOD}^{V[G]}(V \cup \{\vec{A}\} \cup \bigcup_{i<\lambda} \tilde{A}_i)$$

consists of all sets which, in $V[G]$, are hereditarily definable from parameters in the transitive closure of $V \cup \{\vec{A}\}$.

Lemma 6. *Every set $X \in N$ is definable in $V[G]$ in the following form: there are an $\in$-formula φ, $x \in V$, $n < \omega$, and $i_0, \ldots, i_{l-1} < \lambda$ such that*

$$X = \{u \in V[G] \mid V[G] \vDash \varphi(u, x, \vec{A}, A_{i_0}, \ldots, A_{i_{l-1}})\}.$$

Lemma 7. *N is a model of* ZF*, and there is a surjection $f: \mathcal{P}(\aleph_0) \to \lambda$ in N defined by*

$$f(z) = \begin{cases} i, \text{ if } z \in \tilde{A}_i; \\ 0, \text{ else.} \end{cases}$$

Approximating N

Lemma 8. (Approximation Lemma) *Let* $X \in N$ *and* $X \subseteq \mathrm{Ord}$*. Then there are* i_0, ..., $i_{l-1} < \lambda$ *such that*

$$X \in V[A_{i_0}, \ldots, A_{i_{l-1}}].$$

Proof. Let $X = \{u \in \mathrm{Ord} \,|\, V[G] \vDash \varphi(u, x, \vec{A}, A_{i_0}, \ldots, A_{i_{l-1}})\}$.

Define

$$\begin{aligned} X' \; = \; \{ u \in \mathrm{Ord} \;\; | \;\; & \text{there is } p = (p_i) \in P \text{ such that} \\ & p_{i_0} \subseteq A_{i_0}, \ldots, p_{i_{l-1}} \subseteq A_{i_{l-1}}, \text{ and} \\ & p \Vdash \varphi(\check{u}, \check{x}, \tau, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}) \}, \end{aligned}$$

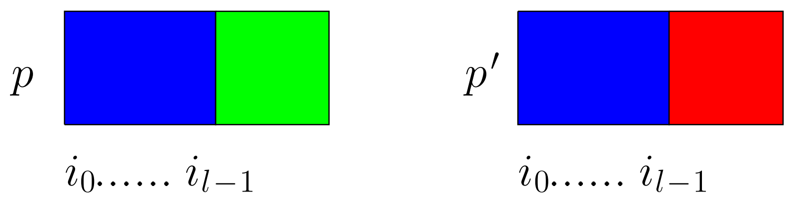
where $\tau, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}$ are canonical names for $\vec{A}, A_{i_0}, \ldots, A_{i_{l-1}}$ resp.

Then $X' \in V[A_{i_0}, \ldots, A_{i_{l-1}}]$. $X \subseteq X'$ is obvious.

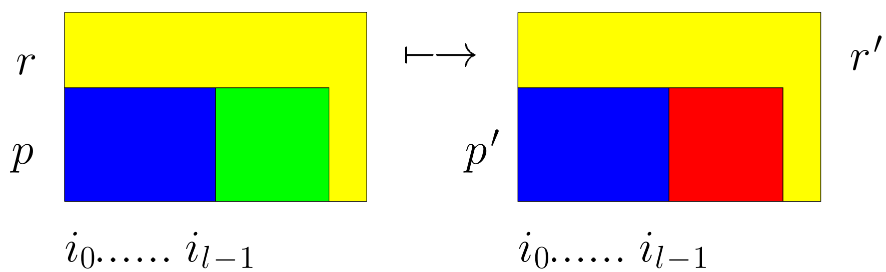
$X' \subseteq X$ uses an automorphism argument to show: whenever $p = (p_i)$ and $p' = (p_i')$ are conditions with $p_{i_0} \subseteq A_{i_0}, \ldots, p_{i_{l-1}} \subseteq A_{i_{l-1}}$ and $p_{i_0}' \subseteq A_{i_0}, \ldots, p_{i_{l-1}}' \subseteq A_{i_{l-1}}$ then we **cannot have**

$$p \Vdash \varphi(\check{u}, \check{x}, \tau, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}) \text{ and } p' \Vdash \neg\varphi(\check{u}, \check{x}, \tau, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}).$$

Wlog, p, p' have the shape:



Define an automorphism $\pi\colon \{r \in P \mid r \leqslant p\} \leftrightarrow \{r' \in P \mid r' \leqslant p'\}$:



Since the names $\check{u}$, $\check{x}$, τ, $\dot{A}_{i_0}$, ..., $\dot{A}_{i_{l-1}}$ are invariant under π, we cannot have $p \Vdash \varphi$ and $\pi(p) = p' \Vdash \neg\varphi$. $\square$

Lemma 9. *If* $\lambda \geqslant (2^{\aleph_0})^V$ *then there is* no *surjection* $\mathcal{P}(\aleph_0) \to \lambda^+$ *in* N.

Proof. By the Approximation Lemma the ground model V has λ names for elements in $\mathcal{P}^N(\aleph_0)$. A surjection $\mathcal{P}(\aleph_0) \to \lambda^+$ in N would yield a surjection $\lambda \to (\lambda^+)^N$ in $V[G]$. But cardinals are preserved between V, N and $V[G]$. □

Theorem 10. *There is a model of* ZF + ¬AC *with a surjection* $\mathcal{P}(\aleph_0) \to \aleph_\omega$ *and with no surjection* $\mathcal{P}(\aleph_0) \to \aleph_{\omega+1}$. *Hence*

$$\theta := \theta(\aleph_0) := \sup\{\xi \mid \textit{there is a surjection } \mathcal{P}(\aleph_0) \to \xi\} = \aleph_{\omega+1}.$$

Hausdorff's Generalized Continuum Hypothesis

Felix Hausdorff conjectured an extension of CH

Conjecture 11. (Generalized Continuum Hypothesis, GCH)

$$\forall\alpha\ 2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

Since GCH holds in Gödel's model L,

Theorem 12. GCH *is independent of* ZFC.

William B. Easton proved

Theorem 13. *Let* $E\colon \mathrm{Ord} \to \mathrm{Ord}$ *be a sufficiently absolute function such that*

- $E(\alpha) > \alpha$
- $\alpha < \beta \to E(\alpha) \leqslant E(\beta)$
- $\mathrm{Lim}(E(\alpha)) \to \mathrm{cof}(E(\alpha)) > \aleph_\alpha$

Then there is a model $V[G]$ *such that*

$$\forall\alpha\,(\aleph_\alpha \text{ is } \textbf{\textit{regular}} \to 2^{\aleph_\alpha} = \aleph_{E(\alpha)})$$

The Singular Cardinals Hypothesis

is / implies the statement

(SCH) if κ is a **singular** strong limit cardinal then $2^\kappa = \kappa^+$

Moti Gitik and Bill Mitchell showed

Theorem 14. *The following two theories are equiconsistent:*

- $\mathrm{ZFC} + \neg\mathrm{SCH}$
- $\mathrm{ZFC} +$ *there are "many" measurable cardinals*

SCH without the Axiom of Choice

Theorem 15. *The following theories are equiconsistent:*

- ZF
- ZF + "GCH *holds below* $\aleph_\omega$" + "*there is a surjection from* $\mathcal{P}(\aleph_\omega)$ *onto* $\aleph_\alpha$", *for some fixed big ordinal* α

This is a *surjective* failure of SCH, without requiring large cardinals. *Injective* failures possess high consistency strengths.

The forcing

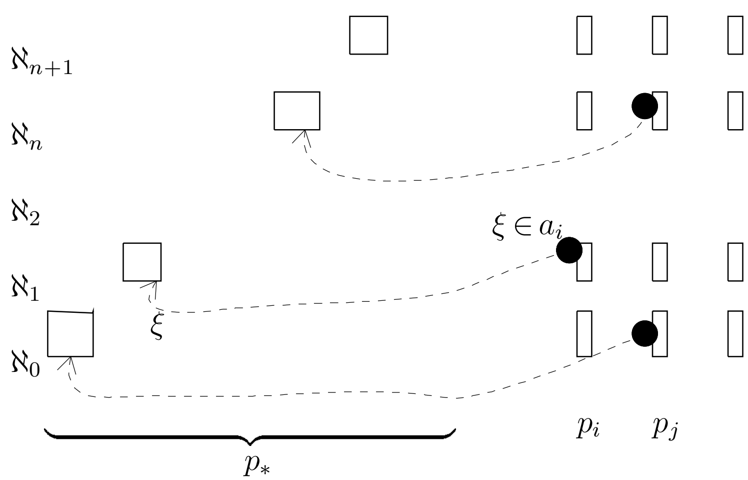
Fix a ground model V of ZFC + GCH and let $\lambda = \aleph_\alpha$ be some cardinal in V.

The forcing $P_0 = (P_0, \supseteq, \emptyset)$ adjoins one COHEN subset of $\aleph_{n+1}$ for every $n < \omega$.

$$P_0 = \{ p \,|\, \exists (\delta_n)_{n<\omega} \; (\forall n < \omega \colon \delta_n \in [\aleph_n, \aleph_{n+1}) \wedge p \colon \bigcup_{n<\omega} [\aleph_n, \delta_n) \to 2) \}.$$

The forcing $(P, \leqslant_P, \emptyset)$ is defined by

$$
\begin{aligned}
P = \{(p_*, (a_i, p_i)_{i<\lambda}) \ \mid \ & \exists (\delta_n)_{n<\omega}\, \exists D \in [\lambda]^{<\omega}\, (\forall n < \omega \colon \delta_n \in [\aleph_n, \aleph_{n+1}), \\
& p_* \colon \bigcup_{n<\omega} [\aleph_n, \delta_n)^2 \to 2, \\
& \forall i \in D\ p_i \colon \bigcup_{n<\omega} [\aleph_n, \delta_n) \to 2 \wedge p_i \neq \emptyset, \\
& \forall i \in D\ a_i \in [\aleph_\omega \setminus \aleph_0]^{<\omega} \wedge \forall n < \omega\ \operatorname{card}(a_i \cap [\aleph_n, \aleph_{n+1})) \leqslant 1, \\
& \forall i \notin D\ a_i = p_i = \emptyset)\}
\end{aligned}
$$



P is partially ordered by

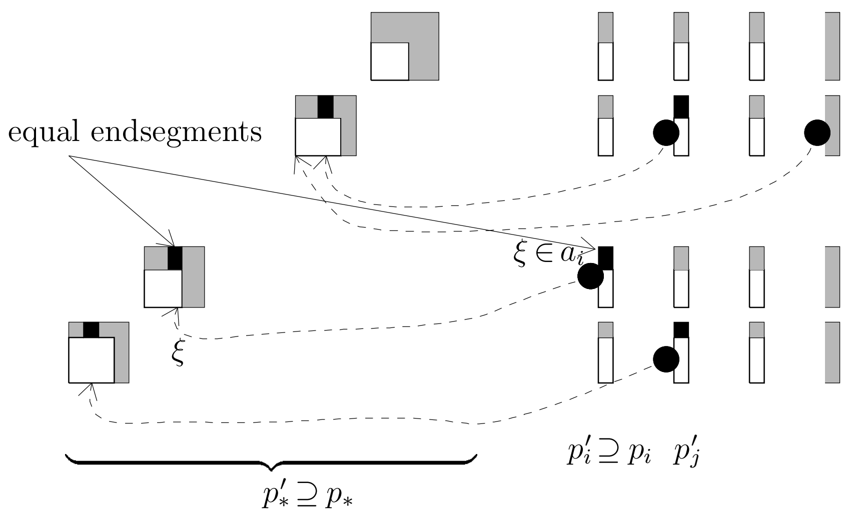
$$p' = (p'_*, (a'_i, p'_i)_{i<\lambda}) \leqslant_P (p_*, (a_i, p_i)_{i<\lambda}) = p$$

iff

a) $p'_* \supseteq p_*, \forall i < \lambda (a'_i \supseteq a_i \wedge p'_i \supseteq p_i)$,

b) $\forall i < \lambda \forall n < \omega \forall \xi \in a_i \cap [\aleph_n, \aleph_{n+1}) \forall \zeta \in \mathrm{dom}(p'_i \setminus p_i) \cap [\aleph_n, \aleph_{n+1})\ p'_i(\zeta) = p'_*(\xi)(\zeta)$, and

c) $\forall j \in \mathrm{supp}(p)\ (a'_j \setminus a_j) \cap \bigcup_{i \in \mathrm{supp}(p), i \neq j} a'_i = \emptyset$.

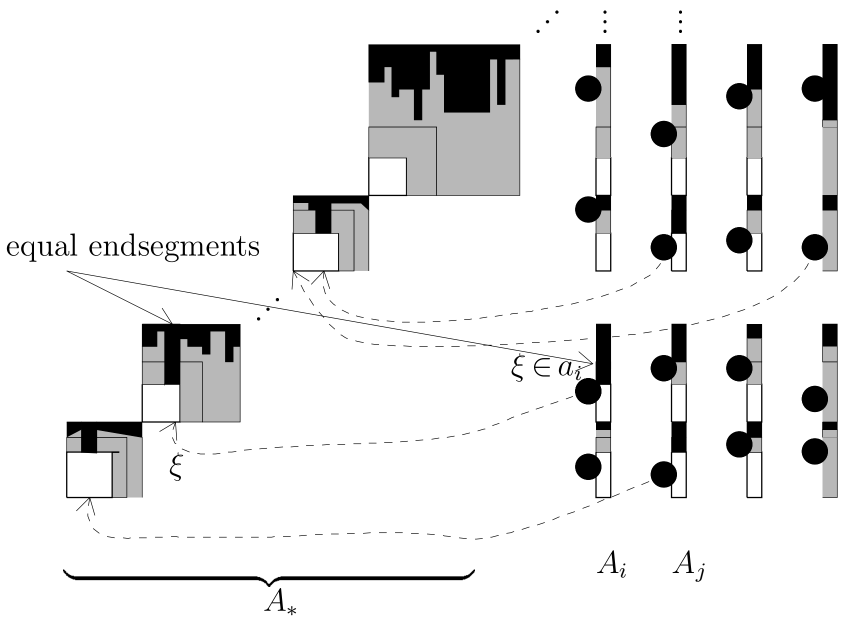


Lemma 16. *P satisfies the $\aleph_{\omega+2}$-chain condition.*

Let G be V-generic for P.

Define

$$
\begin{aligned}
G_* &= \{p_* \in P_* \,|\, (p_*, (a_i, p_i)_{i<\lambda}) \in G\} \\
A_* &= \bigcup G_* \colon \bigcup_{n<\omega} [\aleph_n, \aleph_{n+1})^2 \to 2 \\
A_*(\xi) &= \{(\zeta, A_*(\xi, \zeta)) | \zeta \in [\aleph_n, \aleph_{n+1})\} \colon [\aleph_n, \aleph_{n+1}) \to 2\,, \text{ for } \aleph_n \leqslant \xi < \aleph_{n+1} \\
A_i &= \bigcup \{p_i \,|\, (p_*, (a_j, p_j)_{j<\lambda}) \in G\} \colon [\aleph_0, \aleph_\omega) \to 2
\end{aligned}
$$



Fuzzifying the A_i

For functions $A, A' \colon (\aleph_\omega \setminus \aleph_0) \rightarrow 2$ define an equivalence relation $\sim$ by

$$A \sim A' \text{ iff } \exists n < \omega ((A \Delta A') \upharpoonright \aleph_{n+1} \in V[G_*] \wedge (A \Delta A') \upharpoonright [\aleph_{n+1}, \aleph_\omega) \in V).$$

Let $\tilde{A} = \{A' | A' \sim A\}$ be the $\sim$-equivalence class of A.

The symmetric submodel

Set

- $T_* = \mathcal{P}(<\kappa)^{V[A_*]}$, setting $\kappa = \aleph_\omega^V$;
- $\vec{A} = (\tilde{A}_i \mid i < \lambda)$.

The final model is

$$N = \mathrm{HOD}^{V[G]}(V \cup \{T_*, \vec{A}\} \cup T_* \cup \bigcup_{i<\lambda} \tilde{A}_i)$$

consisting of all sets which, in $V[G]$ are hereditarily definable from parameters in the transitive closure of $V \cup \{T_*, \vec{A}\}$.

Lemma 17. *Every set $X\in N$ is definable in $V[G]$ in the following form: there are an $\in$-formula φ, $x\in V$, $n<\omega$, and $i_0,\ldots,i_{l-1}<\lambda$ such that*

$$X=\{u\in V[G]\,|\,V[G]\vDash\varphi(u,x,T_*,\vec{A},A_*\restriction(\aleph_{n+1}^V)^2,A_{i_0},\ldots,A_{i_{l-1}})\}.$$

Lemma 18. *N is a model of* ZF*, and there is a surjection $f\colon\mathcal{P}(\kappa)\to\lambda$ in N defined by*

$$f(z)=\begin{cases} i, \text{ if } z\in\tilde{A}_i;\\ 0, \text{ else;}\end{cases}$$

Approximating N

Lemma 19. *Let $X \in N$ and $X \subseteq \mathrm{Ord}$. Then there are $n < \omega$ and $i_0, \ldots, i_{l-1} < \lambda$ such that*

$$X \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}].$$

Proof. Let

$$X = \{u \in \mathrm{Ord} \,|\, V[G] \vDash \varphi(u, x, T_*, \vec{A}, A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}})\}.$$

By taking n sufficiently large, we may assume that

$$\forall j < k < l \forall m \in [n, \omega) \forall \delta \in [\aleph_m, \aleph_{m+1}) \colon A_{i_j} \upharpoonright [\delta, \aleph_{m+1}) \neq A_{i_k} \upharpoonright [\delta, \aleph_{m+1}).$$

For $j < l$ set

$$a_{i_j}^* = \{\xi \,|\, \exists m \leqslant n \exists \delta \in [\aleph_m, \aleph_{m+1}) \colon A_{i_j} \upharpoonright [\delta, \aleph_{m+1}) = A_*(\xi) \upharpoonright [\delta, \aleph_{m+1})\}$$

where $A_*(\xi) = \{(\zeta, A_*(\xi, \zeta)) | (\xi, \zeta) \in \mathrm{dom}(A_*)\}$.

Define

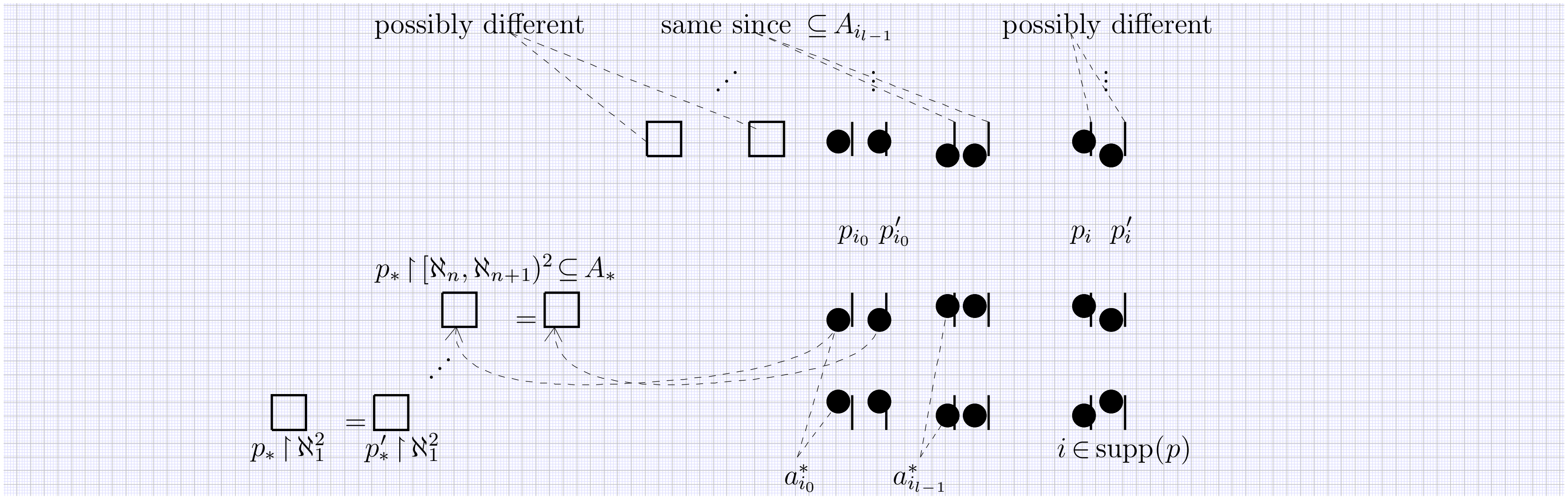
$$X' \;=\; \{\, u \in \mathrm{Ord} \;\mid\; \begin{array}{l} \text{there is } p = (p_*, (a_i, p_i)_{i<\lambda}) \in P \text{ such that} \\ p_* \restriction (\aleph_{n+1}^V)^2 \subseteq A_* \restriction (\aleph_{n+1}^V)^2, \\ a_{i_0} \supseteq a^*_{i_0}, \ldots, a_{i_{l-1}} \supseteq a^*_{i_{l-1}}, \\ p_{i_0} \subseteq A_{i_0}, \ldots, p_{i_{l-1}} \subseteq A_{i_{l-1}}, \text{ and} \\ p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \restriction (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}) \,\}, \end{array}$$

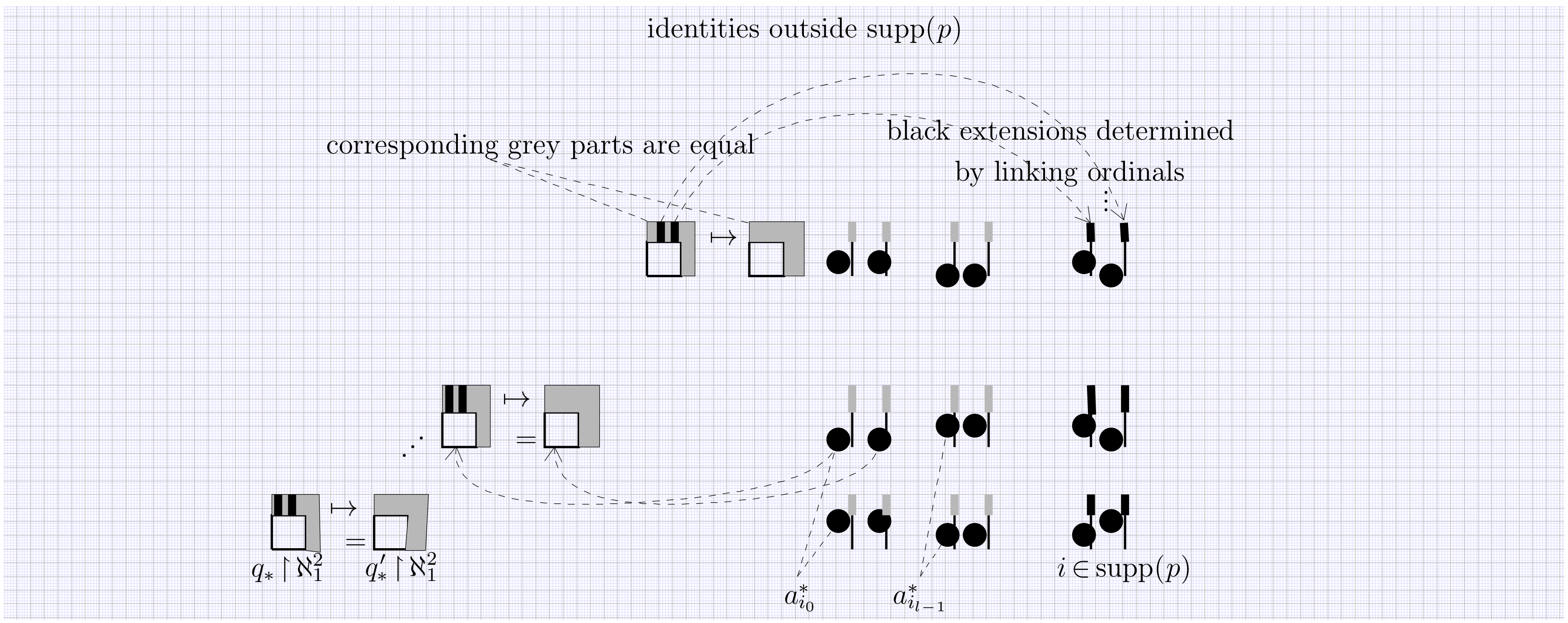
where $\sigma, \tau, \dot{A}, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}$ are canonical names for $T_*, \vec{A}, A_*, A_{i_0}, \ldots, A_{i_{l-1}}$ resp.

Then $X' \in V[A_* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}]$ and $X \subseteq X'$.

The converse direction, $X' \subseteq X$, uses an automorphism argument to show: whenever $p = (p_i)$ and $p' = (p'_i)$ are conditions as in the definition of X' then we **cannot have**

$$p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}) \text{ and } p' \Vdash \neg\varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}).$$


possibly different
same since $\subseteq A_{i_{l-1}}$
possibly different
p_{i_0} p'_{i_0}
p_i p'_i
$p_* \restriction [\aleph_n, \aleph_{n+1})^2 \subseteq A_*$
$=$
$=$
$p_* \restriction \aleph_1^2$
$p'_* \restriction \aleph_1^2$
$a^*_{i_0}$
$a^*_{i_{l-1}}$
$i \in \mathrm{supp}(p)$


identities outside supp(p)
corresponding grey parts are equal
black extensions determined
by linking ordinals
$\mapsto$
$\mapsto$
$=$
$\mapsto$
$=$
$q_* \restriction \aleph_1^2$
$q'_* \restriction \aleph_1^2$
$a^*_{i_0}$
$a^*_{i_{l-1}}$
$i \in \mathrm{supp}(p)$

Wrapping up

Lemma 20. *Let* $n<\omega$ *and* $i_0, \ldots, i_{l-1}<\lambda$ *. Then cardinals are absolute between* V *and* $V[A^* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}]$.

Lemma 21. *Cardinals are absolute between* N *and* V*, and in particular* $\kappa = \aleph_\omega^V = \aleph_\omega^N$ *.*

Proof. If not, then there is a function $f \in N$ which collapses a cardinal in V. By Lemma 19, f is an element of some model $V[A_* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}]$ as above. But this contradicts Lemma 20. $\square$

Lemma 22. GCH *holds in* N *below* $\aleph_\omega$.

Proof. If $X \subseteq \aleph_n$ and $X \in N$ then X is an element of some model $V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}]$ as above. Since $A_{i_0}, \ldots, A_{i_{l-1}}$ do not adjoin new subsets of $\aleph_n$ we have that

$$X \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2].$$

Hence $\mathcal{P}(\aleph_n^V) \cap N \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2]$. GCH holds in $V[A_* \upharpoonright (\aleph_{n+1}^V)^2]$. Hence there is a bijection $\mathcal{P}(\aleph_n^V) \cap N \leftrightarrow \aleph_{n+1}^V$ in $V[A_* \upharpoonright (\aleph_{n+1}^V)^2]$ and hence in N. □

Discussion and Remarks

To work with singular cardinals κ of *uncountable* cofinality, various finiteness properties in the construction have to be replaced by the property of being of cardinality $<\mathrm{cof}(\kappa)$. This yields choiceless violations of SILVER's theorem.

Theorem 23. *Let V be any ground model of* $\mathrm{ZFC}+\mathrm{GCH}$ *and let λ be some cardinal in V. Then there is a model $N \supseteq V$ of the theory* $\mathrm{ZF}+$ "GCH *holds below $\aleph_{\omega_1}$*" $+$ *"there is a surjection from $\mathcal{P}(\aleph_{\omega_1})$ onto λ". Moreover, the axiom of dependent choices* DC *holds in N.*

Conjecture 24. *Let* $E\colon \mathrm{Ord}\to\mathrm{Ord}$ *be a sufficiently absolute function such that*

- $E(\alpha)\geqslant\alpha+2$
- $\alpha<\beta\to E(\alpha)\leqslant E(\beta)$

Then there is a model $V[G]$ *in which for all* α

$$\theta(\aleph_\alpha) := \sup\{\xi \mid \textit{there is a surjection } \mathcal{P}(\aleph_\alpha)\to\xi\} = \aleph_{E(\alpha)}.$$

THANK YOU