Building the Dodd-Jensen Core Model with a Simplified Fine Hierarchy

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The Dodd-Jensen core model is of the form $K = L^E$ where E is a sequence of measures. We structure the model L^E by a continuous *fine hierarchy* $(\mathcal{F}_\alpha^E)_{\alpha \in \mathrm{Ord}}$. Each $\mathcal{F}_\alpha^E$ is a structure of the form $\mathcal{F}_\alpha^E = (F_\alpha^E, \in, E, S^E, \ldots)$, which contains a Skolem function S^E and other basic constructible operations. The next level $F_{\alpha+1}^E$ is the collection of all subsets of F_α^E which are definable over the structure $\mathcal{F}_\alpha^E$ by *quantifier-free* formulas. The hierarchy satisfies condensation theorems and other finestructural laws.

The sequence E consists of measures E_α which are represented as elementary maps (*extenders*) E_α: $\mathcal{F}_\delta^E \to \mathcal{F}_\alpha^E$. Core model theory can be developed with the fine hierarchy. One can canonically define *finestructural ultrapowers* of levels $\mathcal{F}_\gamma^E$ by measures in E. If all proper initial segments of $\mathcal{F}_\gamma^E$ are finestructurally *sound* then this is inherited by finestructural ultrapowers. Iterated finestructural extensions can be used to compare structures $\mathcal{F}_\gamma^E$ and $\mathcal{F}_{\gamma'}^{E'}$. The unique predicate E defining K consists of measures for which the formation of finestructural ultrapowers can be iterated arbitrarily (*iterability*).

The use of the fine hierarchy instead of standard fine structure theory circumvents the complications of iterated projecta and reducts and simplifies the construction of finestructural ultrapowers.

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Contents

GÖDEL's constructible universe

- $L_0 = \emptyset$
- $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$
- $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for limit ordinals λ
- $L = \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha$ is the *constructible universe*

$\mathrm{Def}(X)$ is the "definable powerset" of X:

$$\begin{aligned}\mathrm{Def}(X) = \{a \subseteq X \mid & \text{ there are a first-order formula } \varphi(v, \vec{w}) \text{ and parameters} \\ & \vec{p} \in X \text{ such that } a = \{x \in X \mid (X, \in) \vDash \varphi(x, \vec{p})\}\}.\end{aligned}$$

$(L, \in)$ is a model of ZERMELO-FRAENKEL set theory ZF, of the axiom of choice AC, and of the generalized continuum hypothesis GCH.

How close is L to the set theoretic universe V?

A Corollary of JENSEN's *covering theorem*:

Let ν be a singular cardinal in V. Then

$(\nu^+)^L < \nu^+$ iff there is a nontrivial elementary embedding $e: (L, \in) \to (L, \in)$.

To approximate V one should incorporate such an e into the approximation.

Coding class-sized elementary embeddings by sets

Let $e \restriction \kappa = \mathrm{id}$ and $e(\kappa) > \kappa$, $\gamma = (\kappa^+)^L$, $\delta = e(\gamma)$.

$e \restriction L_\gamma \colon (L_\gamma, \in) \to (L_\delta, \in)$ is elementary.

$E_\delta = e \restriction L_\gamma$ is/can be chosen to be a *measure (extender)* on L:

- $\mathrm{dom}(E_\delta) = L_\gamma$ for some $\gamma < \delta$
- $E_\delta \restriction \kappa = \mathrm{id}$ and $E_\delta(\kappa) > \kappa$ for some critical point $\kappa < \gamma$
- $L_\gamma = (H_{\leqslant \kappa})^L \models \mathrm{ZFC}^-$
- $E_\delta \colon (L_\gamma, \in) \to (L_\delta, \in)$ is elementary
- $L_\delta = \mathrm{Hull}(\kappa \cup \{\kappa\})$ and $E_\delta \colon (L_\gamma, \in) \to (L_\delta, \in)$ is cofinal
- (L_δ, E_δ) is *amenable*, i.e., $\forall x \in L_\delta \; x \cap E_\delta \in L_\delta$

Ultrapowers via directed systems

Let $\mathrm{Tr}(X)$ denote the MOSTOWSKI *transitivization* of $(X, \in)$. Let $p \subseteq q$ range over finite subsets of L.

$$\begin{array}{ccccccc}
 & \mathrm{Hull}(\kappa\cup p) & \subseteq & \mathrm{Hull}(\kappa\cup q) & \subseteq & \bigcup_p \mathrm{Hull}(\kappa\cup p) = L \\
 & \updownarrow & & \updownarrow & & \| \\
L_\gamma \ni & \mathrm{Tr}(\mathrm{Hull}(\kappa\cup p)) & \xrightarrow{\sigma_{pq}} & \mathrm{Tr}(\mathrm{Hull}(\kappa\cup q)) & \xrightarrow{\sigma_q} & \operatorname{dir\,lim}_p \mathrm{Tr}(\mathrm{Hull}(\kappa\cup p)) = L \\
 & E_\delta\downarrow & & E_\delta\downarrow & & \pi_{E_\delta}\downarrow \\
 & E_\delta(\mathrm{Tr}(\mathrm{Hull}(\kappa\cup p))) & \xrightarrow{E_\delta(\sigma_{pq})} & E_\delta(\mathrm{Tr}(\mathrm{Hull}(\kappa\cup q))) & \xrightarrow{\sigma_q^*} & \operatorname{dir\,lim}_p E_\delta(\mathrm{Tr}(\mathrm{Hull}(\kappa\cup p))) \\
 & & & & & \| \\
 & & & & & \mathrm{Ult}(L, E_\delta) \\
 & & & & & \|? \\
 & & & & & L
\end{array}$$

The *ultrapower map* $\pi_{E_\delta}\colon (L, \in) \to (\mathrm{Ult}(L, E_\delta), \in^*)$ extends E_δ: $\pi_{E_\delta} \supseteq E_\delta$.

The elementarity of π_{E_δ} depends on the elementarity of the hulls.

For "algebraic" hulls, π_{E_δ} is $\forall_1$-elementary in the appropriate language.

Iterated ultrapowers

If $(\mathrm{Ult}(L, E_\delta), \in^*) = (L, \in)$ then say that L is *extendable* by E_δ.

Then the image $E^* = \bigcup \{\pi_{E_\delta}(x) | x \in L \wedge x \subseteq E_\delta\}$ is an extender on $\mathrm{Ult}(L, E_\delta)$.

L is *iterable* by E_δ iff the formation of ultrapowers by E_δ and its images can be iterated transfinitely, taking direct limits at limit ordinals:

- $M_0 = L$, $\pi_{00} = \mathrm{id}$, $E^{(0)} = E_\delta$
- $M_{\alpha+1} = \mathrm{Ult}(M_\alpha, E^{(\alpha)})$, $\pi_{\alpha,\alpha+1} = \pi_{E^{(\alpha)}}$, $E^{(\alpha+1)} = \bigcup \{\pi_{\alpha,\alpha+1}(x) \big| x \in M_\alpha \wedge x \subseteq E^{(\alpha)}\}$
- M_λ, $(\pi_{\alpha,\lambda})_{\alpha<\lambda}$ is the transitive direct limit of $(M_\alpha)_{\alpha<\lambda}$, $(\pi_{\alpha,\beta})_{\alpha\leqslant\beta<\lambda}$, $E^{(\lambda)} = \bigcup \{\pi_{0,\lambda}(x) \big| x \in M_0 \wedge x \subseteq E^{(0)}\}$

The theory of iterated ultrapowers (Kunen)

Iterated ultrapowers make structures uniform and comparable:

if $(M_\alpha,\ E^{(\alpha)})_{\alpha\in\mathrm{Ord}},\ (\pi_{\alpha,\beta})_{\alpha\leqslant\beta\in\mathrm{Ord}}$ and $(M'_\alpha,\ E'^{(\alpha)})_{\alpha\in\mathrm{Ord}},\ (\pi'_{\alpha,\beta})_{\alpha\leqslant\beta\in\mathrm{Ord}}$ are two iterations of L then $E^{(\alpha)} = E'^{(\alpha)}$ for sufficiently high α.

This implies that under the assumption of a nontrivial elementary embedding of L there is exactly one iterable extender $E_\delta\colon (L_\gamma, \in) \to (L_\delta, \in)$ on L_δ (and on L) such that L_δ is the hull of $\emptyset$ (using the function E_δ in the formation of the hull).

This E_δ is called $0^{\#}$.

Iterated sharps

$0, 0^{\#}, 0^{\#\#}, \ldots$.

Generalizing sharps: E_δ is a *measure (extender)* on L_δ^E :

- $E_\delta\colon (L_\gamma^E, \in) \to (L_\delta^E, \in)$ is elementary and cofinal for some $\gamma < \delta$
- $E_\delta \restriction \kappa = \mathrm{id}$ and $E_\delta(\kappa) > \kappa$ for some critical point $\kappa < \gamma$
- $L_\gamma^E = (H_{\leqslant\kappa})^{L_\delta^E} \models \mathrm{ZFC}^-$
- $L_\delta^E = \mathrm{Hull}(\kappa \cup \{\kappa\})$
- (L_δ^E, E_δ) is *amenable*, i.e., $\forall x \in L_\delta^E\; x \cap E_\delta \in L_\delta^E$
- .....

The Dodd-Jensen core model

- $K = L^E$
- $E_\delta \neq 0$ implies that E_δ is an iterable extender on L^E_δ
- E_δ is *not* an iterable measure on L^E
- E_δ: $L^E_\gamma \rightarrow L^E_\delta$ with critical point κ implies that there is a maximal α such that $\mathcal{P}(\kappa) \cap L^E_\alpha \subseteq L^E_\gamma$; then E_δ is an iterable extender on L^E_α

E is defined recursively. If $E \restriction \delta$ is defined and there is such an $E_\delta \neq 0$ pick it for the sequence. Otherwise set $E_\delta = 0$.

K is a model of ZFC + GCH.

The DODD-JENSEN core model theory

$$\frac{\exists\kappa\,\kappa\text{ measurable}}{K}=\frac{0^{\#}\text{ exists}}{L}$$

In particular:

- rigidity: if there is nontrivial elementary embedding $\sigma\colon(K,\in)\to(K,\in)$ then there is an inner model with a measurable cardinal

- if there is no inner model with a measurable cardinal then K covers V, e.g., for every singular cardinal ν holds $(\nu^+)^K=\nu^+$

Gödel's 1939 hulls

Gödel's proof of $2^{\omega_\mu} = \omega_{\mu+1}$ in *Consistency proof for the generalized continuum hypothesis*, Proceedings of the National Academy of Sciences forms a hull of $M_{\omega_\mu} \cup \{m\}$, where $m \subseteq M_{\omega_\mu}$; the hull uses definability for *all* formulas of the language of set theory.

"Define a set K of constructible sets, a set O of ordinals and a set F of Skolem functions by the following postulates I–VII:

I. $M_{\omega_\mu} \subseteq K$ and $m \in K$.

II. If $x \in K$, the order of x belongs to O.

III. If $x \in K$, all constants occuring in the definition of x belong to K.

IV. If $\alpha \in O$ and $\phi_\alpha(x)$ is a propositional function over M_α all of whose constants belong to K, then:

   1. The subset of M_α defined by ϕ_α belongs to K.

2. For any $y \in K \cdot M_\alpha$ the designated Skolem functions for ϕ_α and y or $\sim \phi_\alpha$ and y (according as $\phi_\alpha(y)$ or $\sim \phi_\alpha(y)$) belong to F.

V. If $f \in F$, $x_1, ..., x_n \in K$ and $(x_1, ..., x_n)$ belongs to the domain of definition of f, then $f(x_1, ..., x_n) \in K$.

VI. If $x, y \in K$ and $x - y \neq \Lambda$ the first element of $x - y$ belongs to K.

VII. No proper subsets of K, O, F satisfy I–VI.

....

Theorem 5. *There exists a one-to-one mapping* x' *of* K *on* M_η *such that* $x \in y \equiv x' \in y'$ *for* $x, y \in K$ *and* $x' = x$ *for* $x \in M_{\omega_\mu}$.

Proof: The mapping x' (....) is defined by transfinite induction on the order, ..."

Theorem 5 is the fundamental condensation property: hulls are isomorphic to levels of the hierarchy.

Hierarchies and hulls

- GÖDEL: L_α-hierarchy, and Σ_ω-hulls with respect to the $\in$-language

- JENSEN: J_α^E-hierarchy with Σ_n truth predicates, and Σ_1-hulls with respect to the $\in$-language enriched by truth predicates

- *Here*: $\mathcal{F}_\alpha^E$-hierarchy built with *quantifier-free* definability, structures enriched by certain *constructible operations*, *algebraic* hulls with respect to those operations

The fine hierarchy for L^E

The *fine hierarchy* $(\mathcal{F}^E_\alpha)_{\alpha\in\mathrm{Ord}}$ is defined by

$$\mathcal{F}^E_\alpha = (F^E_\alpha, \in, E, <^E, I^E, S^E, R^E, D^E, P^E).$$

- $F^E_0 = \emptyset$

- Assume $\mathcal{F}^E_\alpha$ is defined. For *quantifier-free* $\varphi(v_0, \ldots, v_{n-1}, v_n)$, $\vec{p} \in F^E_\alpha$ define the *interpretation*

$$I^E(F^E_\alpha, \varphi, \vec{p}) = \{v_n \in F^E_\alpha | \mathcal{F}^E_\alpha \models \varphi(\vec{p}, v_n)\} \tag{1}$$

Let

$$F^E_{\alpha+1} = \{I^E(F^E_\alpha, \varphi, \vec{p}) | \varphi(v_0, \ldots, v_{n-1}, v_n) \in \mathcal{L}_0, \vec{p} \in F^E_\alpha\}.$$

Define $I^E \upharpoonright F^E_{\alpha+1}$ to extend $I^E \upharpoonright F^E_\alpha$ and the assignments made in (1); in all other cases set $I^E(\vec{x}) = \bot$.

The *rank function*: $R^E \upharpoonright F^E_{\alpha+1} \supseteq R^E \upharpoonright F^E_\alpha$, and for $y \in F^E_{\alpha+1} \setminus F^E_\alpha$ set

$$R^E(y) = F^E_\alpha .$$

The *definition function*: $D^E \upharpoonright F^E_{\alpha+1} \supseteq D^E \upharpoonright F^E_\alpha$, and for $y \in F^E_{\alpha+1} \setminus F^E_\alpha$, $D^E(y)$ is the $<_{\mathcal{L}}$-least $\varphi \in \mathcal{L}_0$ such that

$$y = I^E(F^E_\alpha, \varphi, \vec{p})$$

for some $\vec{p} \in F^E_\alpha$;

then let the *parameter function* $P^E(y)$ be the least such $\vec{p}$ in the lexicographical wellordering induced by $<^E \upharpoonright F^E_\alpha$.

The *constructible wellorder*: $<^E \upharpoonright F^E_{\alpha+1}$ endextends $<^E \upharpoonright F^E_\alpha$ and for $y, y' \in F^E_{\alpha+1} \setminus F^E_\alpha$

$$\begin{aligned} y <^E y' \text{ iff } & D^E(y) <_{\mathcal{L}} D^E(y'), \text{ or } D^E(y) = D^E(y') \text{ and} \\ & P^E(y) \text{ is } <^E\text{-lexicographically smaller than } P^E(y'). \end{aligned}$$

The SKOLEM *function*: $S^E \restriction F^E_{\alpha+1} \supseteq S^E \restriction F^E_\alpha$ and for $\varphi(v_0, \ldots, v_{n-1}) \in \mathcal{L}_0$ and $\vec{p} \in F^E_\alpha$

$$S^E(F^E_\alpha, \varphi, \vec{p}\,) = \begin{cases} \text{the } <^E\text{-lexicographically minimal } \vec{q} \in F^E_\alpha \text{ such that} \\ \qquad \mathcal{F}^E_\alpha \vDash \varphi(\vec{p}\,, \vec{q}\,), \text{ if this exists;} \\ \bot\,, \text{ else.} \end{cases}$$

For all other arguments $\vec{x} \in F^E_{\alpha+1} \setminus F^E_\alpha$ set $S^E(\vec{x}\,) = \bot\,.$

For limit $\lambda \leqslant \infty$ take a union of structures

$$\mathcal{F}^E_\lambda = \bigcup_{\alpha<\lambda} \mathcal{F}^E_\alpha$$

Hierarchy properties

a) $\alpha \leqslant \gamma \rightarrow F_\alpha^E \subseteq F_\gamma^E$

b) $\alpha < \gamma \rightarrow F_\alpha^E \in F_\gamma^E$

c) F_γ^E is transitive

d) $F_\gamma^E \cap \mathrm{Ord} = \gamma$

e) $\bigcup_{\alpha \in \mathrm{Ord}} F_\alpha^E = L^E$

Theorem 1. *There is a theory $T^{\mathcal{F}}$ consisting of Π_1-sentences of the form $\forall \vec{x}\ \varphi$ where φ is quantifier-free, with the property: if $\mathcal{M} = (M, \in, E, <^M, I^M, S^M, R^M, D^M, P^M)$ is a transitive $\mathcal{L}$-structure then $\mathcal{M} \models T^{\mathcal{F}}$ iff $\mathcal{M} = \mathcal{F}_\alpha^E$ for some $\alpha \leqslant \infty$.*

Proof. The abbreviation $F(z)$ for $z = I(z, v_0 = v_0, \emptyset)$ expresses that z is a level of the fine hierarchy. Let $T^{\mathcal{F}}$ consist of

1. Transitivity: $x \dot{\in} y \wedge y \dot{\in} z \wedge F(z) \rightarrow x \dot{\in} z$
2. Linearity: $F(x) \wedge F(y) \rightarrow x \dot{\in} y \vee x = y \vee y \dot{\in} x$
3. $F(R(x)) \wedge \neg x \dot{\in} R(x)$
4. $R(x) \dot{\in} z \wedge F(z) \rightarrow x \dot{\in} z$
5. Interpretation: $F(x) \wedge \vec{y} \dot{\in} x \rightarrow (z \in I(x, \varphi, \vec{y}) \leftrightarrow z \dot{\in} x \wedge \varphi(\vec{y}, z))$

....

15. $\neg F(x) \vee \neg \vec{p} \dot{\in} x \rightarrow S(x, \varphi, \vec{p}) = \perp$ □

Definition 2. *A set or class $Z \subseteq L^E$ is E-*closed *if $F_\omega \subseteq Z$ and Z is closed with respect to the operations I^E, S^E, R^E, D^E and P^E. For $X \subseteq L^E$ let $\mathcal{F}^E(X)$ be the hull of X in L^E, i.e., the $\subseteq$-smallest superset of X which is E-closed. Note that all fine levels F_α^E are E-closed.*

Theorem 3. (Condensation Theorem) *Let $E \subseteq V$ be a predicate and let $Z \subseteq L^E$ be E-closed. Then there are unique $\alpha \in \mathrm{Ord}$, and $D \subseteq V$, and a unique fine isomorphism*

$$\sigma\colon \mathcal{F}_\alpha^D \cong (Z, \in, E, <^E, I^E, S^E, R^E, D^E, P^E)$$

with $D \subseteq F_\alpha^D$.

Proof. Let $\sigma\colon (M, \in) \cong (Z, \in)$ be the Mostowski transitivization. Since Π_1-theories transfer downwards, $(M, \in, \ldots)$ is a model of $T^{\mathcal{F}}$ and hence of the form $\mathcal{F}_\alpha^D$. $\square$

Fine ultrapowers

Let $E_\delta \colon (F^E_\gamma, \in) \to (F^E_\delta, \in)$ with critical point κ be a *measure on* $\mathcal{F}^E_\alpha$, i.e.,

$$\forall p \subseteq F^E_\alpha,\, p \text{ finite: } \mathrm{Tr}(\mathcal{F}^E(\kappa \cup p)) \in F^E_\gamma$$

Let $p \subseteq q$ range over finite subsets of F^E_α.

$$\begin{array}{ccccccc}
 & \mathcal{F}^E(\kappa \cup p) & \subseteq & \mathcal{F}^E(\kappa \cup q) & \subseteq & \bigcup_{p \subseteq_{\mathrm{fin}} F^E_\alpha} \mathcal{F}^E(\kappa \cup p) = \mathcal{F}^E_\alpha \\
 & \updownarrow & & \updownarrow & & \| \\
F^E_\gamma \ni & \mathrm{Tr}(\mathcal{F}^E(\kappa \cup p)) & \xrightarrow{\sigma_{pq}} & \mathrm{Tr}(\mathcal{F}^E(\kappa \cup q)) & \xrightarrow{\sigma_q} & \operatorname{dir\,lim}_p \mathrm{Tr}(\mathcal{F}^E(\kappa \cup p)) = \mathcal{F}^E_\alpha \\
 & E_\delta \downarrow & & E_\delta \downarrow & & \pi_{E_\delta} \downarrow \\
 & E_\delta(\mathrm{Tr}(\mathcal{F}^E(\kappa \cup p))) & \xrightarrow{E_\delta(\sigma_{pq})} & E_\delta(\mathrm{Tr}(\mathcal{F}^E(\kappa \cup q))) & \xrightarrow{\sigma^*_q} & \operatorname{dir\,lim}_p E_\delta(\mathrm{Tr}(\mathcal{F}^E(\kappa \cup p))) \\
 & & & & & \| \\
 & & & & & \mathrm{Ult}(\mathcal{F}^E_\alpha, E_\delta) \\
 & & & & & \|? \\
 & & & & & \mathcal{F}^{E^*}_{\alpha^*}
\end{array}$$

Fine ultrapowers

- $\pi_{E_\delta}\colon \mathcal{F}_\alpha^E \to \mathrm{Ult}(\mathcal{F}_\alpha^E, E_\delta)$ is $\forall_1$-elementary

- if $\mathcal{F}_\alpha^E$ is *extendable* by E_δ, i.e., $\mathrm{Ult}(\mathcal{F}_\alpha^E, E_\delta)$ is wellfounded, then $\mathrm{Ult}(\mathcal{F}_\alpha^E, E_\delta) = \mathcal{F}_{\alpha^*}^{E^*}$ and $\pi_{E_\delta}\colon \mathcal{F}_\alpha^E \to \mathcal{F}_{\alpha^*}^{E^*}$

- $\pi_{E_\delta} \supseteq E_\delta$, $E^* \restriction \delta + 1 = E \restriction \delta$

- $\pi_{E_\delta}\colon \mathcal{F}_\alpha^E \to \mathcal{F}_{\alpha^*}^{E^*}$ can be lifted to $\pi_{E_\delta}^+\colon \mathcal{F}_{\alpha+1}^E \to \mathcal{F}_{\alpha^*+1}^{E^*}$

Fine iterations

A commutative system $(\mathcal{F}^{E^{(i)}}_{\alpha^{(i)}}, \pi_{ij})_{i \leqslant j < \theta}$ is a *fine iteration* of $\mathcal{F}^E_\alpha$ if

- $\mathcal{F}^{E^{(0)}}_{\alpha^{(0)}} = \mathcal{F}^E_\alpha$
- $\pi_{i,i+1}\colon \mathcal{F}^{E^{(i)}}_{\tau^{(i)}} \to \mathcal{F}^{E^{(i+1)}}_{\alpha^{(i+1)}}$ is a fine ultrapower by some $E^{(i)}_\delta$, where $\tau^{(i)} \leqslant \alpha^{(i)}$ is maximal such that $E^{(i)}_\delta$ is a measure on $\mathcal{F}^{E^{(i)}}_{\tau^{(i)}}$; if $\tau^{(i)} < \alpha^{(i)}$ we say that there is a *truncation* at i
- if $\lambda < \theta$ is a limit ordinal then $\mathcal{F}^{E^{(\lambda)}}_{\alpha^{(\lambda)}}$, $(\pi_{ij})_{i \leqslant j < \lambda}$ is the transitive directed limit of $(\mathcal{F}^{E^{(i)}}_{\alpha^{(i)}}, \pi_{ij})_{i \leqslant j < \lambda}$

$\mathcal{F}^E_\alpha$ is *(finely) iterable* if every fine iteration of $\mathcal{F}^E_\alpha$ can be freely continued.

Countable completeness of measures implies iterability.

Defining K

$K = L^E = \bigcup_\alpha \mathcal{F}_\alpha^E$ is iterable, i.e., every $\mathcal{F}_\alpha^E$ is iterable.

E is defined recursively. If $E \upharpoonright \delta$ is given, choose E_δ such that there is some $\alpha \geqslant \delta$ with

- E_δ is an extender on $\mathcal{F}_\alpha^{E\upharpoonright\delta+E_\delta}$ with measurable κ but not an extender on $\mathcal{F}_{\alpha+1}^{E\upharpoonright\delta+E_\delta}$
- $\mathcal{F}_\alpha^{E\upharpoonright\delta+E_\delta}$ is finely iterable
- $\mathcal{F}_\alpha^{E\upharpoonright\delta+E_\delta} = \mathcal{F}^{E\upharpoonright\delta+E_\delta}(\kappa \cup p)$ for some finite $p \subseteq F_\alpha^{E\upharpoonright\delta+E_\delta}$
- ...

If this is not possible, set $E_\delta = \emptyset$.

Uniqueness

Theorem 4. *There is at most one such* E_δ.

Proof. Otherwise *coiterate* $\mathcal{F}_{\alpha}^{E\restriction\delta+E_\delta}$ and $\mathcal{F}_{\alpha'}^{E\restriction\delta+E'_\delta}$: let $(\mathcal{F}_{\alpha^{(i)}}^{E^{(i)}},\ \pi_{ij})_{i\leqslant j<\theta}$ and $(\mathcal{F}_{\alpha'^{(i)}}^{E'^{(i)}},$ $\pi'_{ij})_{i\leqslant j<\theta}$ be *fine iterations* of $\mathcal{F}_{\alpha}^{E\restriction\delta+E_\delta}$ and $\mathcal{F}_{\alpha'}^{E\restriction\delta+E'_\delta}$ respectively such that for all $i+1<\theta$

$$\pi_{i,i+1}\colon \mathcal{F}_{\tau^{(i)}}^{E^{(i)}} \to \mathcal{F}_{\alpha^{(i+1)}}^{E^{(i+1)}} \text{ and } \pi'_{i,i+1}\colon \mathcal{F}_{\tau'^{(i)}}^{E'^{(i)}} \to \mathcal{F}_{\alpha'^{(i+1)}}^{E'^{(i+1)}}$$

are fine extension by some $E_\delta^{(i)}$ and $E'^{(i)}_\delta$ respectively where

$$E^{(i)}\restriction\delta = E'^{(i)}\restriction\delta \text{ and } E_\delta^{(i)} \neq E'^{(i)}_\delta, \text{ if possible.}$$

This *coiteration* stops at some $\mathcal{F}_{\alpha^{(\theta-1)}}^{E^{(\theta-1)}}, \mathcal{F}_{\alpha'^{(\theta-1)}}^{E'^{(\theta-1)}}$.

If $\alpha^{(\theta-1)} < {\alpha'}^{(\theta-1)}$ then there is $a \subseteq \kappa$ such that $a \in \mathcal{F}^{E^{(\theta-1)}}_{{\alpha'}^{(\theta-1)}} \setminus \mathcal{F}^{E^{(\theta-1)}}_{\alpha^{(\theta-1)}}$. But this contradicts

$$\mathcal{P}(\kappa) \cap \mathcal{F}^{E^{(\theta-1)}}_{\alpha^{(\theta-1)}} = \mathcal{P}(\kappa) \cap \mathcal{F}^{E\restriction\delta+E_\delta}_{\alpha} = \mathcal{P}(\kappa) \cap \mathcal{F}^{E\restriction\delta+E'_\delta}_{\alpha'} = \mathcal{P}(\kappa) \cap \mathcal{F}^{{E'}^{(\theta-1)}}_{{\alpha'}^{(\theta-1)}}.$$

Hence $\mathcal{F}^{E^{(\theta-1)}}_{\alpha^{(\theta-1)}} = \mathcal{F}^{{E'}^{(\theta-1)}}_{{\alpha'}^{(\theta-1)}}$ and like in KUNEN's theory this implies $\mathcal{F}^{E\restriction\delta+E_\delta}_{\alpha} = \mathcal{F}^{E\restriction\delta+E'_\delta}_{\alpha'}$. Contradiction. $\square$