A finestructural refinement of the J-hierarchy for extender models

Peter Koepke, University of Bonn

We interpolate successive levels J_α^E and $J_{\alpha+1}^E$ of JENSEN's J-hierarchy for the extender model L^E by an ω-sequence of intermediate levels

$$J_\alpha^E = F_{\omega\cdot\alpha}^E \subseteq F_{\omega\cdot\alpha+1}^E \subseteq F_{\omega\cdot\alpha+2}^E \subseteq \ldots \subseteq \bigcup_{n<\omega} F_{\omega\cdot\alpha+n}^E = J_{\alpha+1}^E\,.$$

Each F_γ^E is the underlying set of a structure $\mathcal{F}_\gamma^E = (F_\gamma^E, \in, E, \ldots)$ containing a SKOLEM function and other basic constructible operations. The next level $F_{\gamma+1}^E$ consists of all subsets of F_γ^E which are definable *without quantifiers* over the structure $\mathcal{F}_\gamma^E$. The *fine hierarchy* $(\mathcal{F}_\gamma^E)_{\gamma\in\mathrm{Ord}}$ satisfies a strong condensation theorem and other finestructural laws. One can define *finestructural extensions* (ultrapowers) of $\mathcal{F}_\gamma^E$ by extenders in E. If all proper initial segments of $\mathcal{F}_\gamma^E$ are finestructurally sound then this inherits to the finestructural extension. Higher core model theory can be based on the new fine structure theory.

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Contents

Extender models

$L^E = \bigcup_{\alpha \in \mathrm{Ord}} L^E_\alpha$, where E is a sequence $E = (E_\delta)$ of extenders.

$E_\delta = \emptyset$ or $E_\delta \colon (L^E_\gamma, \in) \to (L^E_\delta, \in)$ is an *extender*, i.e.,

- $E_\delta \restriction \kappa = \mathrm{id}$ and $E_\delta(\kappa) > \kappa$ for some critical point $\kappa < \gamma$
- $L^E_\gamma = (H_{\leqslant \kappa})^{L^E_\delta} \models \mathrm{ZFC}^-$
- $E_\delta \colon (L^E_\gamma, \in) \to (L^E_\delta, \in)$ is elementary and cofinal
- (L^E_δ, E_δ) is *amenable*, i.e., $\forall x \in L^E_\delta \; x \cap E_\delta \in L^E_\delta$
- .....

JENSEN-style finestructural analysis

$L^E = \bigcup_{\alpha \in \mathrm{Ord}} L^E_\alpha = \bigcup_{\alpha \in \mathrm{Ord}} J^E_\alpha$. Consider E_δ being a *partial* extender, i.e.,

- $\mathcal{P}(\kappa) \cap J^E_\alpha \subseteq J^E_\gamma$ and $\mathcal{P}(\kappa) \cap J^E_{\alpha+1} \not\subseteq J^E_\gamma$
- $\mathcal{P}(\kappa) \cap \mathbf{\Sigma}_n(J^E_\alpha) \subseteq J^E_\gamma$ and $\mathcal{P}(\kappa) \cap \mathbf{\Sigma}_{n+1}(J^E_\alpha) \not\subseteq J^E_\gamma$
- $\mathcal{P}(\kappa) \cap (J^E_\alpha)^{n,p} \subseteq J^E_\gamma$ and $\mathcal{P}(\kappa) \cap \mathbf{\Sigma}_1((J^E_\alpha)^{n,p}) \not\subseteq J^E_\gamma$, where $(J^E_\alpha)^{n,p} = (J^E_\rho, \Sigma_n\text{-truth predicate},...)$ is an n-th *reduct* of J^E_α
- form ultrapower $\pi_{E_\delta}: (J^E_\alpha)^{n,p} \rightarrow_{\Sigma_1} \mathrm{Ult}((J^E_\alpha)^{n,p}, E_\delta)$
- iterate the ultrapower operation, taking direct limits at limits

JENSEN interpolates $J^E_\alpha \subseteq J^E_{\alpha+1}$ by

$$J^E_\alpha = (J^E_\alpha)^{0,0} \supseteq (J^E_\alpha)^{1,p} \supseteq (J^E_\alpha)^{2,p'} \supseteq ..., J^E_{\alpha+1}$$

A monotone and continuous interpolation

Interpolate $J_\alpha^E \subseteq J_{\alpha+1}^E$ by

$$J_\alpha^E = \mathcal{F}_{\omega\cdot\alpha}^E \subseteq \mathcal{F}_{\omega\cdot\alpha+1}^E \subseteq \mathcal{F}_{\omega\cdot\alpha+2}^E \subseteq \ldots \subseteq \bigcup_{n<\omega} \mathcal{F}_{\omega\cdot\alpha+n}^E = \mathcal{F}_{\omega\cdot\alpha+\omega}^E = J_{\alpha+1}^E\,.$$

- $\mathcal{F}^E$-hierarchy defined by *quantifierfree* definability
- $\mathcal{F}_\beta^E$ contain Skolem functions for *quantifierfree* formulas
- quantifierfree definability $\leftrightarrow$ boolean combinations of Σ_1-definability
- $\mathcal{P}(\kappa) \cap F_{\omega\cdot\alpha+n}^E \subseteq F_\gamma^E$ and $\mathcal{P}(\kappa) \cap F_{\omega\cdot\alpha+n+1}^E \nsubseteq F_\gamma^E$
- form a fine ultrapower $\pi_{E_\delta}\colon F_{\omega\cdot\alpha+n}^E \to \mathrm{Ult}(F_{\omega\cdot\alpha+n}^E, E_\delta)$, ...

The fine hierarchy

Define $(\mathcal{F}_\alpha^E)_{\alpha \in \mathrm{Ord}}$ recursively

$$\mathcal{F}_\alpha^E = (F_\alpha^E, \in, E, <^E, I^E, S^E, R^E, D^E, P^E).$$

- $F_0^E = \emptyset$

- Assume $\mathcal{F}_\alpha^E$ is defined. For *quantifier-free* $\varphi(v_0, \ldots, v_{n-1}, v_n)$, $\vec{p} \in F_\alpha^E$ define the *interpretation*

$$I^E(F_\alpha^E, \varphi, \vec{p}) = \{v_n \in F_\alpha^E | \mathcal{F}_\alpha^E \models \varphi(\vec{p}, v_n)\} \tag{1}$$

Let

$$F_{\alpha+1}^E = \{I^E(F_\alpha^E, \varphi, \vec{p}) | \varphi(v_0, \ldots, v_{n-1}, v_n) \text{ q.f.}, \vec{p} \in F_\alpha^E\}.$$

Define $I^E \upharpoonright F^E_{\alpha+1}$ to extend $I^E \upharpoonright F^E_\alpha$ and the assignments made in (1); in all other cases set $I^E(\vec{x}) = \bot$.

The *rank function*: $R^E \upharpoonright F^E_{\alpha+1} \supseteq R^E \upharpoonright F^E_\alpha$, and for $y \in F^E_{\alpha+1} \setminus F^E_\alpha$ set

$$R^E(y) = F^E_\alpha .$$

The *definition function*: $D^E \upharpoonright F^E_{\alpha+1} \supseteq D^E \upharpoonright F^E_\alpha$, and for $y \in F^E_{\alpha+1} \setminus F^E_\alpha$, $D^E(y)$ is the $<_{\mathcal{L}}$-least q.f. φ such that

$$y = I^E(F^E_\alpha, \varphi, \vec{p})$$

for some $\vec{p} \in F^E_\alpha$;

then let the *parameter function* $P^E(y)$ be the least such $\vec{p}$ in the lexicographical wellordering induced by $<^E \upharpoonright F^E_\alpha$.

The *constructible wellorder*: $<^E \upharpoonright F^E_{\alpha+1}$ endextends $<^E \upharpoonright F^E_\alpha$ and for $y, y' \in F^E_{\alpha+1} \setminus F^E_\alpha$

$$\begin{aligned} y <^E y' \ \text{ iff } \ & D^E(y) <_{\mathcal{L}} D^E(y'), \text{ or } D^E(y) = D^E(y') \text{ and} \\ & P^E(y) \text{ is } <^E\text{-lexicographically smaller than } P^E(y'). \end{aligned}$$

The SKOLEM *function*: $S^E \restriction F^E_{\alpha+1} \supseteq S^E \restriction F^E_\alpha$ and for $\varphi(v_0, \ldots, v_{n-1}) \in \mathcal{L}_0$ and $\vec{p} \in F^E_\alpha$

$$S^E(F^E_\alpha, \varphi, \vec{p}\,) = \begin{cases} \text{the } <^E\text{-lexicographically minimal } \vec{q} \in F^E_\alpha \text{ such that} \\ \qquad \mathcal{F}^E_\alpha \vDash \varphi(\vec{p}\,, \vec{q}\,), \text{ if this exists;} \\ \bot\,, \text{ else.} \end{cases}$$

For all other arguments $\vec{x} \in F^E_{\alpha+1} \setminus F^E_\alpha$ set $S^E(\vec{x}\,) = \bot\,$.

For limit $\lambda \leqslant \infty$ take a union of structures

$$\mathcal{F}^E_\lambda = \bigcup_{\alpha<\lambda} \mathcal{F}^E_\alpha$$

Hierarchy properties

a) $\alpha \leqslant \gamma \rightarrow F^E_\alpha \subseteq F^E_\gamma$

b) $\alpha < \gamma \rightarrow F^E_\alpha \in F^E_\gamma$

c) F^E_γ is transitive

d) $F^E_\gamma \cap \mathrm{Ord} = \gamma$

e) $\bigcup_{\alpha \in \mathrm{Ord}} F^E_\alpha = L^E$

f) $F^E_{\omega \cdot \alpha} = J^E_\alpha$

<u>$\forall_1$-axiomatization of fine levels</u>

Theorem 1. *There is a theory $T^{\mathcal{F}}$ consisting of $\forall_1$-sentences of the form $\forall \vec{x}\, \varphi$, φ quantifier-free, with the property: if $\mathcal{M} = (M, \in, E, <^M, I^M, S^M, R^M, D^M, P^M)$ is a transitive $\mathcal{L}$-structure then $\mathcal{M} \vDash T^{\mathcal{F}}$ iff $\mathcal{M} = \mathcal{F}_\alpha^E$ for some $\alpha \leqslant \infty$.*

Proof. The abbreviation $F(z)$ for $z = I(z, v_0 = v_0, \emptyset)$ expresses that z is a level of the fine hierarchy. Let $T^{\mathcal{F}}$ consist of (the universal closures of)

1. Transitivity: $x \dot{\in} y \wedge y \dot{\in} z \wedge F(z) \rightarrow x \dot{\in} z$
2. Linearity: $F(x) \wedge F(y) \rightarrow x \dot{\in} y \vee x = y \vee y \dot{\in} x$
3. $F(R(x)) \wedge \neg x \dot{\in} R(x)$
4. $R(x) \dot{\in} z \wedge F(z) \rightarrow x \dot{\in} z$
5. Interpretation: $F(x) \wedge \vec{y} \dot{\in} x \rightarrow (z \dot{\in} I(x, \varphi, \vec{y}) \leftrightarrow z \dot{\in} x \wedge \varphi(\vec{y}, z))$
6. $P(x) \dot{\in} R(x)$

7. Naming: $x = I(R(x), D(x), P(x))$

8. $F(x) \wedge F(y) \wedge x \dot{\in} y \wedge \vec{p} \dot{\in} x \rightarrow I(x, \varphi, \vec{p}) \dot{\in} y$

9. $\neg F(x) \vee \neg \vec{p} \dot{\in} x \rightarrow I(x, \varphi, \vec{p}) = \bot$

10. $\varphi <_{\mathcal{L}} D(x) \rightarrow \neg I(R(x), \varphi, \vec{p}) = x$

11. $\vec{p} \dot{<}_{\text{lex}} P(x) \rightarrow \neg I(R(x), D(x), \vec{p}) = x$, where the lexicographical $\vec{p} \dot{<}_{\text{lex}} P(x)$ can be expressed purely in terms of $\dot{<}$.

12. $u \dot{<} v \leftrightarrow R(u) \dot{\in} R(v) \vee (R(u) = R(v) \wedge D(u) <_{\mathcal{L}} D(v)) \vee (R(u) = R(v) \wedge D(u) = D(v) \wedge P(u) \dot{<}_{\text{lex}} P(v))$

13. $S(x, \varphi, \vec{p}) \neq \bot \rightarrow S(x, \varphi, \vec{p}) \dot{\in} x \wedge \varphi(S(x, \varphi, \vec{p}), \vec{p})$

14. $F(x) \wedge \vec{p} \dot{\in} x \wedge u \dot{\in} x \wedge \varphi(u, \vec{p}) \rightarrow S(x, \varphi, \vec{p}) \neq \bot \wedge S(x, \varphi, \vec{p}) \dot{\leqslant} u$

15. $\neg F(x) \vee \neg \vec{p} \dot{\in} x \rightarrow S(x, \varphi, \vec{p}) = \bot$ □

Definition 2. *$Z \subseteq L^E$ is E-*closed* if Z is closed with respect to the operations I^E, S^E, R^E, D^E and P^E. For $X \subseteq L^E$ let $\mathcal{F}^E(X)$ be the hull of X in L^E, i.e., the $\subseteq$-smallest superset of X which is E-closed.*

Theorem 3. *Let $Z \subseteq L^E$ be E-closed. Then there are unique $\alpha \in \mathrm{Ord}$, and $D \subseteq V$, and a unique fine isomorphism*

$$\sigma: \mathcal{F}_\alpha^D \cong (Z, \in, E, <^E, I^E, S^E, R^E, D^E, P^E)$$

with $D \subseteq F_\alpha^D$.

Proof. Let $\sigma: (M, \in) \cong (Z, \in)$ be the Mostowski transitivization. Since $\forall_1$-theories transfer downwards, $(M, \in, \ldots)$ is a model of $T^{\mathcal{F}}$ and hence of the form $\mathcal{F}_\alpha^D$. □

Fine ultrapowers

Let $E_\delta\colon (F^E_\gamma, \in) \to (F^E_\delta, \in)$ with critical point κ be an *extender on* $\mathcal{F}^E_\alpha$, i.e.,

$$\forall p \subseteq F^E_\alpha, p \text{ finite: } \mathrm{Tr}(\mathcal{F}^E(\kappa \cup p)) \in F^E_\gamma$$

where $\mathrm{Tr}(X)$ is the transitivization of X. Let $p \subseteq q$ range over finite subsets of F^E_α.

$$\begin{array}{ccccccc}
 & \mathcal{F}^E(\kappa\cup p) & \subseteq & \mathcal{F}^E(\kappa\cup q) & \subseteq & \bigcup_{p\subseteq_{\mathrm{fin}} F^E_\alpha} \mathcal{F}^E(\kappa\cup p) = \mathcal{F}^E_\alpha \\
 & \uparrow\sigma_p & & \uparrow\sigma_q & & \| \\
F^E_\gamma \ni & \mathrm{Tr}(\mathcal{F}^E(\kappa\cup p)) & \xrightarrow{\sigma_{pq}} & \mathrm{Tr}(\mathcal{F}^E(\kappa\cup q)) & \xrightarrow{\sigma_q} & \operatorname{dir\,lim}_p \mathrm{Tr}(\mathcal{F}^E(\kappa\cup p)) = \mathcal{F}^E_\alpha \\
 & E_\delta\downarrow & & E_\delta\downarrow & & \pi_{E_\delta}\downarrow \\
 & E_\delta(\mathrm{Tr}(\mathcal{F}^E(\kappa\cup p))) & \xrightarrow{E_\delta(\sigma_{pq})} & E_\delta(\mathrm{Tr}(\mathcal{F}^E(\kappa\cup q))) & \xrightarrow{\sigma^*_q} & \operatorname{dir\,lim}_p E_\delta(\mathrm{Tr}(\mathcal{F}^E(\kappa\cup p))) \\
 & & & & & \| \\
 & & & & & \mathrm{Ult}(\mathcal{F}^E_\alpha, E_\delta) \\
 & & & & & \|? \\
 & & & & & \mathcal{F}^{E^*}_{\alpha^*}
\end{array}$$

Fine ultrapowers

- $\pi_{E_\delta}\colon \mathcal{F}_\alpha^E \to \mathrm{Ult}(\mathcal{F}_\alpha^E, E_\delta)$ is $\forall_1$-elementary
- if $\mathcal{F}_\alpha^E$ is *extendable* by E_δ, i.e., $\mathrm{Ult}(\mathcal{F}_\alpha^E, E_\delta)$ is wellfounded, then $\mathrm{Ult}(\mathcal{F}_\alpha^E, E_\delta) = \mathcal{F}_{\alpha^*}^{E^*}$ for some $E^*, \alpha^*$and $\pi_{E_\delta}\colon \mathcal{F}_\alpha^E \to \mathcal{F}_{\alpha^*}^{E^*}$
- $\pi_{E_\delta} \supseteq E_\delta$, $E^* \restriction \delta + 1 = E \restriction \delta$

Fine iterations

A commutative system $(\mathcal{F}_{\alpha^{(i)}}^{E^{(i)}}, \pi_{ij})_{i \leqslant j < \theta}$ is a *fine iteration* of $\mathcal{F}_{\alpha}^{E}$ if

- $\mathcal{F}_{\alpha^{(0)}}^{E^{(0)}} = \mathcal{F}_{\alpha}^{E}$
- $\pi_{i,i+1} \colon \mathcal{F}_{\tau^{(i)}}^{E^{(i)}} \to \mathcal{F}_{\alpha^{(i+1)}}^{E^{(i+1)}}$ is a fine ultrapower by some $E_{\delta}^{(i)}$, where $\tau^{(i)} \leqslant \alpha^{(i)}$ is maximal such that $E_{\delta}^{(i)}$ is an extender on $\mathcal{F}_{\tau^{(i)}}^{E^{(i)}}$; if $\tau^{(i)} < \alpha^{(i)}$ we say that $\mathcal{F}_{\tau^{(i)}}^{E^{(i)}}$ is a *truncation* at i
- if $\lambda < \theta$ is a limit ordinal then $\mathcal{F}_{\alpha^{(\lambda)}}^{E^{(\lambda)}}$, $(\pi_{ij})_{i \leqslant j < \lambda}$ is the transitive directed limit of $(\mathcal{F}_{\alpha^{(i)}}^{E^{(i)}}, \pi_{ij})_{i \leqslant j < \lambda}$
- $\mathcal{F}_{\alpha}^{E}$ is *iterable* if such iterations can be freely continued
- *Coiterations*: parallel fine iterations to make one iterate an initial segment of the other

Soundness of initial segments

Truncations should be *sound*, i.e.

$$\mathcal{F}^{E^{(i)}}_{\tau^{(i)}} = \mathcal{F}^{E^{(i)}}(\rho(\mathcal{F}^{E^{(i)}}_{\tau^{(i)}}) \cup p(\mathcal{F}^{E^{(i)}}_{\tau^{(i)}}))$$

for some canonical *projectum* ρ and *standard parameter* p.

Adding further basic functions to the $\mathcal{F}$, this can be expressed by a $\forall_1$-theory. Thus it is preserved by finestructural ultrapowers: e.g.

- $F(x) \wedge \xi < \rho(x) \wedge \vec{p} \in x \rightarrow I(x, \varphi, \vec{p}) \cap \xi \in x$
- $F(x) \rightarrow I(x, \varphi(x), \overrightarrow{p(x)}) \cap \rho(x) \notin x$
- ......