Making All Cardinals Almost Ramsey *†‡

Arthur W. Apter§
Department of Mathematics
Baruch College of CUNY
New York, New York 10010 USA
and
The CUNY Graduate Center, Mathematics
365 Fifth Avenue
New York, New York 10016 USA
http://faculty.baruch.cuny.edu/apter
awapter@alum.mit.edu

Peter Koepke
Mathematisches Institut
Rheinische Friedrich-Wilhelms-Universität
Beringstraße 1
D-53115 Bonn, Germany
http://www.math.uni-bonn.de/people/logic/People/Koepke.html
koepke@math.uni-bonn.de

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Abstract

We examine combinatorial aspects and consistency strength properties of almost Ramsey cardinals. Without the Axiom of Choice, successor cardinals may be almost Ramsey. From fairly mild supercompactness assumptions, we construct a model of ZF + $\neg\mathrm{AC}_\omega$ in which *every* infinite cardinal is almost Ramsey. Core model arguments show that strong assumptions are necessary. Without successors of singular cardinals, we can weaken this to an *equiconsistency* of the following theories: "ZFC + There is a proper class of regular almost Ramsey cardinals", and "ZF + DC + All infinite cardinals except possibly successors of singular cardinals are almost Ramsey".

1 Introduction

Erdös and Ramsey cardinals are defined by *partition properties*. A set $X \subseteq \delta$ is *homogeneous for a partition* $F : [\delta]^{<\omega} \to 2$ iff $\forall n(|F''[X]^n| = 1)$; the *partition property* $\delta \to (\alpha)^{<\omega}_2$ is defined as

$$(\forall F : [\delta]^{<\omega} \to 2)(\exists X \subseteq \delta)(\mathrm{otp}(X) \geq \alpha \wedge X \text{ is homogeneous for } F).$$

An infinite cardinal κ is *α-Erdös* iff $\kappa \to (\alpha)^{<\omega}_2$, and it is *Ramsey* iff $\kappa \to (\kappa)^{<\omega}_2$. This suggests a natural large cardinal notion between Erdös and Ramsey cardinals.

Definition 1 *An infinite cardinal* κ *is* almost Ramsey *iff* $\forall\alpha < \kappa(\kappa \to (\alpha)^{<\omega}_2)$.

Almost Ramsey cardinals were considered before in unpublished work of

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H. Friedman and by J. Vickers and P. D. Welch in [19]. These cardinals can be viewed as "diagonal limits" in the hierarchy of Erdös cardinals.

For any uncountable almost Ramsey cardinal κ, it can be shown in ZF that the following substructure property holds: if $\lambda, \kappa', \lambda'$ are infinite cardinals satisfying $\lambda < \kappa$, $\lambda' \leq \kappa' < \kappa$, and $\lambda' \leq \lambda$, then $(\kappa, \lambda) \Rightarrow (\kappa', \lambda')$. This means that every first-order structure $(\kappa, \lambda, \dots)$ in a countable language has an elementary substructure $X \prec (\kappa, \lambda, \dots)$ with $|X| = \kappa'$ and $|X \cap \lambda| = \lambda'$. This can be viewed as a Chang's Conjecture-like two-cardinal version of the standard downward Löwenheim-Skolem theorem.

It is easy to see that many instances of $(\kappa, \lambda) \Rightarrow (\kappa', \lambda')$, in particular for successor cardinals κ, are incompatible with the Axiom of Choice. The main result of this paper yields a *choiceless* model of ZF in which *every* infinite cardinal[1] is almost Ramsey and in which the generalized downward Löwenheim-Skolem theorem holds universally. Specifically, we have the following.

Theorem 1 *Con(ZFC + There exist cardinals $\kappa < \lambda$ such that κ is 2^λ supercompact where λ is the least regular almost Ramsey cardinal greater than κ) $\implies$ Con(ZF + $\neg AC_\omega$ + Every successor cardinal is regular + Every infinite cardinal is almost Ramsey).*

In the construction, certain large cardinals are collapsed generically and become the well-ordered cardinals of a symmetric model of ZF. Due to

[1]For the purposes of this paper, in a choiceless model of ZF, infinite cardinals will always be taken as being well-ordered, i.e., as being the alephs.

the strong indestructibility of almost Ramsey cardinals, the large cardinal hypotheses in the ground model can be taken considerably weaker than in a similar construction found in [5]. (We will discuss this in greater detail at the end of Section 4.) Conversely, we use core model techniques to show that considerable large cardinal strength is necessary for the symmetric model constructions of this paper.

Theorem 2 *Assume ZF and that every infinite cardinal is almost Ramsey. Then there exists an inner model with a proper class of strong cardinals.*

Omitting successors of singular limit cardinals, consistency strengths go down to what we shall show is below the existence of Ramsey cardinals.

Theorem 3 *The following theories are equiconsistent:*

a) *ZFC + There is a proper class of regular almost Ramsey cardinals;*

b) *ZF + DC + All successor cardinals are regular + All infinite cardinals except possibly successors of singular limit cardinals are almost Ramsey.*

Before proving our theorems, we shall show some combinatorial facts about almost Ramsey cardinals and consider some obvious consistency strength questions.

2 Combinatorial aspects

Almost Ramseyness is closely connected to Erdös-type partition cardinals.

Definition 2 *For $\alpha \in Ord$, let $\kappa(\alpha)$ be the least κ such that $\kappa \to (\alpha)_2^{<\omega}$, if such a κ exists.*

The following characterization of almost Ramsey cardinals as diagonal limits of the Erdös hierarchy can be verified easily and does not involve the Axiom of Choice.

Proposition 1 *(ZF) An infinite cardinal κ is almost Ramsey iff $\kappa(\alpha)$ is defined for all $\alpha < \kappa$ and $\kappa = \bigcup_{\alpha<\kappa} \kappa(\alpha)$.*

By the classical Ramsey theorem, $\kappa(\alpha)$ is finite for $\alpha < \omega$. By Proposition 1, this immediately implies that ω is an almost Ramsey cardinal. Under the Axiom of Choice, for infinite α, $\kappa(\alpha)$ is strongly inaccessible (if α is also a limit ordinal) and $\kappa(\alpha+1) > \kappa(\alpha)$ (see [15, Propositions 7.14 and 7.15]).

Proposition 2 *(ZFC) Assume κ is almost Ramsey. Then*

a) $\forall \alpha < \kappa(\kappa(\alpha) < \kappa)$*;*

b) *κ is a strong limit cardinal.*

Proof a) Let $\alpha < \kappa$, $\alpha \geq \omega$. Then $\kappa(\alpha) < \kappa(\alpha+1) \leq \kappa$. b) is an immediate consequence of a), keeping in mind that for every limit ordinal α, $\kappa(\alpha)$ is strongly inaccessible. $\square$

Propositions 1 and 2 yield an *indestructibility property* for almost Ramsey cardinals κ under forcing which does not add bounded subsets of κ.

Proposition 3 *Let M be a transitive model of "ZFC + κ is almost Ramsey". Let $N \supseteq M$ be a transitive model of ZF such that $\forall \delta < \kappa (\mathcal{P}(\delta) \cap M = \mathcal{P}(\delta) \cap N)$. Then κ is almost Ramsey in N.*

Proof Let $\alpha < \kappa$. By Proposition 2, $(\kappa(\alpha))^M < \kappa$. $\mathcal{P}((\kappa(\alpha))^M) \cap M = \mathcal{P}((\kappa(\alpha))^M) \cap N$ implies that $(\kappa(\alpha))^N = (\kappa(\alpha))^M$. Hence $\kappa = \bigcup_{\alpha<\kappa} (\kappa(\alpha))^N$, so by Proposition 1, κ is almost Ramsey in N. $\square$

We may also infer that almost Ramsey cardinals are preserved by small forcing.

Proposition 4 *Suppose $V \vDash$ "ZFC + κ is almost Ramsey + $\mathbb{P}$ is a partial ordering such that $|\mathbb{P}| < \kappa$". Then $V^{\mathbb{P}} \vDash$ "κ is almost Ramsey".*

Proof By Propositions 1 and 2, write $\kappa = \bigcup_{\alpha \in [\alpha_0, \kappa)} \kappa(\alpha)$, where α_0 is a limit ordinal with the additional property that $|\mathbb{P}| < \kappa(\alpha_0)$. By [15, Proposition 7.15 and Exercise 10.16], for any limit ordinal $\alpha \in [\alpha_0, \kappa)$ and $\delta(\alpha) = (\kappa(\alpha))^V$, $V^{\mathbb{P}} \vDash$ "$\delta(\alpha) \to (\alpha)_2^{<\omega}$". Since κ therefore remains in $V^{\mathbb{P}}$ a limit of cardinals satisfying suitable partition properties, $V^{\mathbb{P}} \vDash$ "κ is almost Ramsey". $\square$

In ZFC, a regular almost Ramsey cardinal is strongly inaccessible. Singular almost Ramsey cardinals are much weaker. Since $\{\kappa \mid \kappa = \bigcup_{\alpha<\kappa} \kappa(\alpha)\}$ is a closed class of ordinals, we get the following.

Proposition 5 *(ZFC)*

a) Assume κ is an uncountable regular *almost Ramsey cardinal. Then the class of almost Ramsey cardinals is closed unbounded below κ.*

b) *Assume κ is an almost Ramsey cardinal which is Mahlo. Then the class of regular almost Ramsey cardinals is stationary below κ.*

c) *Assume κ is the smallest uncountable regular almost Ramsey cardinal. Then κ is not Mahlo.*

d) *Assume κ is a Ramsey cardinal. Then the class of almost Ramsey cardinals is closed unbounded below κ and the class of regular almost Ramsey cardinals is stationary below κ.*

As a corollary, we get some information on consistency strengths.

Proposition 6

a) *ZFC + There exists an uncountable regular almost Ramsey cardinal $\vdash$ $Con(ZFC$ + There exists a proper class of (singular) almost Ramsey cardinals).*

b) *$Con(ZFC$ + There exists an uncountable regular almost Ramsey cardinal) $\leftrightarrow$ $Con(ZFC$ + There exists an uncountable regular almost Ramsey cardinal which is not Mahlo).*

Let us now work without the Axiom of Choice. The following theorem of J. Silver (see [15, Theorem 9.3]) shows that homogeneous sets for partitions are basically equivalent to sets of (order) indiscernibles for first-order structures.

Proposition 7 *(ZF) For infinite ordinals α, the partition property $\kappa \to (\alpha)^{<\omega}_2$ is equivalent to the following: for any first-order structure $\mathcal{M} = (M, \dots)$ in a countable language S with $\kappa \subseteq M$, there is a set $X \subseteq \kappa$,* $\mathrm{otp}(X) \geq \alpha$ *of* indiscernibles*, i.e., for all S-formulas $\varphi(v_0, \dots, v_{n-1})$, $x_0, \dots, x_{n-1} \in X$, $x_0 < \dots < x_{n-1}$, $y_0, \dots, y_{n-1} \in X$, $y_0 < \dots < y_{n-1}$,*

$$\mathcal{M} \models \varphi(x_0, \dots, x_{n-1}) \textit{ iff } \mathcal{M} \models \varphi(y_0, \dots, y_{n-1}).$$

For limit α, indiscernibility can be strengthened to *good* indiscernibility.

Proposition 8 *(ZF) Assume $\kappa \to (\alpha)^{<\omega}_2$, where α is a limit ordinal. Then for any first-order structure $\mathcal{M} = (M, \dots)$ in a countable language S with $\kappa \subseteq M$, there is a set $X \subseteq \kappa$,* $\mathrm{otp}(X) \geq \alpha$ *of* good indiscernibles*, i.e., for all S-formulas $\varphi(v_0, \dots, v_{m-1}, w_0, \dots, w_{n-1})$, $x_0, \dots, x_{n-1} \in X$, $x_0 < \dots < x_{n-1}$, $y_0, \dots, y_{n-1} \in X$, $y_0 < \dots < y_{n-1}$, and $a_0 < \dots < a_{m-1} < \min(x_0, y_0)$,*

$$\mathcal{M} \models \varphi(a_0, \dots, a_{m-1}, x_0, \dots, x_{n-1}) \textit{ iff } \mathcal{M} \models \varphi(a_0, \dots, a_{m-1}, y_0, \dots, y_{n-1}).$$

Proof We may assume that the structure $\mathcal{M}$ contains a unary predicate Ord for the ordinals in M (this includes all ordinals less than κ) and a collection of Skolem functions for ordinal-valued existential statements, i.e., for every S-formula $\varphi(v, \vec{w})$, there is a function f of $\mathcal{M}$ such that

$$\mathcal{M} \models \forall \vec{w} (\exists v (\mathrm{Ord}(v) \wedge \varphi(v, \vec{w})) \to \varphi(f(\vec{w}), \vec{w})).$$

Choose a set $X \subseteq \kappa$, $\mathrm{otp}(X) = \alpha$ of indiscernibles for $\mathcal{M}$ such that its minimum, $\min(X)$, is minimal for all such sets of indiscernibles. Assume towards a contradiction that X is not good. Then there is an S-formula $\varphi(v_0, \dots, v_{m-1}, w_0, \dots, w_{n-1})$, $x_0, \dots, x_{n-1} \in X$, $x_0 < \cdots < x_{n-1}$, $y_0, \dots, y_{n-1} \in X$, $y_0 < \cdots < y_{n-1}$, and $a_0 < \cdots < a_{m-1} < \min(x_0, y_0)$ such that

$$\mathcal{M} \vDash \varphi(a_0, \dots, a_{m-1}, x_0, \dots, x_{n-1}) \text{ and } \mathcal{M} \vDash \neg\varphi(a_0, \dots, a_{m-1}, y_0, \dots, y_{n-1}).$$

Since α is a limit ordinal, we may take $z_0, \dots, z_{n-1} \in X$, $z_0 < \cdots < z_{n-1}$ such that $x_{n-1} < z_0$ and $y_{n-1} < z_0$.

In case

$$\mathcal{M} \vDash \varphi(a_0, \dots, a_{m-1}, z_0, \dots, z_{n-1}),$$

one has

$$\mathcal{M} \vDash \neg\varphi(a_0, \dots, a_{m-1}, y_0, \dots, y_{n-1}) \text{ and } \mathcal{M} \vDash \varphi(a_0, \dots, a_{m-1}, z_0, \dots, z_{n-1}),$$

where $y_0 < \cdots < y_{n-1} < z_0 < \cdots < z_{n-1}$. In case

$$\mathcal{M} \vDash \neg\varphi(a_0, \dots, a_{m-1}, z_0, \dots, z_{n-1}),$$

one has

$$\mathcal{M} \vDash \varphi(a_0, \dots, a_{m-1}, x_0, \dots, x_{n-1}) \text{ and } \mathcal{M} \vDash \neg\varphi(a_0, \dots, a_{m-1}, z_0, \dots, z_{n-1}),$$

where $x_0 < \cdots < x_{n-1} < z_0 < \cdots < z_{n-1}$. So in both cases, we have an ascending $2n$-tuble of indiscernibles such that the first half behaves differently from the second half with respect to the formula φ and the parameters $a_0, \dots, a_{m-1}$. So without loss of generality, we may assume that

$x_0 < \cdots < x_{n-1} < y_0 < \cdots < y_{n-1}$ and

$\mathcal{M} \models \varphi(a_0, \ldots, a_{m-1}, x_0, \ldots, x_{n-1})$ and $\mathcal{M} \models \neg\varphi(a_0, \ldots, a_{m-1}, y_0, \ldots, y_{n-1})$.

Write $\vec{x} = x_0, \ldots, x_{n-1}$ and $\vec{y} = y_0, \ldots, y_{n-1}$. Since $\mathcal{M}$ contains Skolem functions, there are functions $f_0, \ldots, f_{m-1}$ of $\mathcal{M}$ which compute parameters like $a_0, \ldots, a_{m-1}$:

$\mathcal{M} \models (\exists v_1 < x_0)(\exists v_2 < x_0) \cdots (\exists v_{m-1} < x_0)(f_0(\vec{x}, \vec{y}) < x_0 \wedge \varphi(f_0(\vec{x}, \vec{y}), v_1, \ldots, v_{m-1}, \vec{x}) \wedge \neg\varphi(f_0(\vec{x}, \vec{y}), v_1, \ldots, v_{m-1}, \vec{y}))$.

$\mathcal{M} \models (\exists v_2 < x_0) \cdots (\exists v_{m-1} < x_0)(f_0(\vec{x}, \vec{y}) < x_0 \wedge f_1(\vec{x}, \vec{y}) < x_0 \wedge \varphi(f_0(\vec{x}, \vec{y}), f_1(\vec{x}, \vec{y}), \ldots, v_{m-1}, \vec{x}) \wedge \neg\varphi(f_0(\vec{x}, \vec{y}), f_1(\vec{x}, \vec{y}), \ldots, v_{m-1}, \vec{y}))$.

$\vdots$

$\mathcal{M} \models f_0(\vec{x}, \vec{y}) < x_0 \wedge \cdots \wedge f_{m-1}(\vec{x}, \vec{y}) < x_0 \wedge \varphi(f_0(\vec{x}, \vec{y}), f_1(\vec{x}, \vec{y}), \ldots, f_{m-1}(\vec{x}, \vec{y}), \vec{x}) \wedge \neg\varphi(f_0(\vec{x}, \vec{y}), f_1(\vec{x}, \vec{y}), \ldots, f_{m-1}(\vec{x}, \vec{y}), \vec{y})$.

Now consider $\vec{z} = z_0, \ldots, z_{n-1} \in X$, $z_0 < \cdots < z_{n-1}$ such that $y_{n-1} < z_0$.

(1) There is $k < m$ such that $f_k(\vec{x}, \vec{y}) \neq f_k(\vec{y}, \vec{z})$.

Proof Assume not. Set $\xi_0 = f_0(\vec{x}, \vec{y}), \dots, \xi_{m-1} = f_{m-1}(\vec{x}, \vec{y})$. Since $\vec{x}, \vec{y}$ and $\vec{y}, \vec{z}$ are ascending indiscernible sequences of the same length, we have

$$\mathcal{M} \models \varphi(\xi_0, \xi_1, \dots, \xi_{m-1}, \vec{x}) \wedge \neg\varphi(\xi_0, \xi_1, \dots, \xi_{m-1}, \vec{y})$$

and

$$\mathcal{M} \models \varphi(\xi_0, \xi_1, \dots, \xi_{m-1}, \vec{y}) \wedge \neg\varphi(\xi_0, \xi_1, \dots, \xi_{m-1}, \vec{z}).$$

In particular,

$$\mathcal{M} \models \varphi(\xi_0, \xi_1, \dots, \xi_{m-1}, \vec{y}) \wedge \neg\varphi(\xi_0, \xi_1, \dots, \xi_{m-1}, \vec{y}),$$

which is a contradiction. $\square$(1)

So take $k < m$ such that

(2) $f_k(\vec{x}, \vec{y}) \neq f_k(\vec{y}, \vec{z})$.

Let $(\nu_i \mid i < \alpha)$ be a strictly increasing enumeration of the set X of indiscernibles, and let $(\vec{x}^{(i)} \mid i < \alpha)$ be a partition of X into ascending sequences of length n, with

$$\vec{x}^{(i)} = \nu_{n \cdot i}, \nu_{n \cdot i+1}, \dots, \nu_{n \cdot i+n-1}.$$

We then claim that

(3) $f_k(\vec{x}^{(0)}, \vec{x}^{(1)}) < f_k(\vec{x}^{(1)}, \vec{x}^{(2)})$.

Proof By indiscernibility, (2) implies that $f_k(\vec{x}^{(0)}, \vec{x}^{(1)}) \neq f_k(\vec{x}^{(1)}, \vec{x}^{(2)})$. Assume towards a contradiction that $f_k(\vec{x}^{(0)}, \vec{x}^{(1)}) > f_k(\vec{x}^{(1)}, \vec{x}^{(2)})$. Then again by indiscernibility, we would obtain a *decreasing* $\in$-sequence

$$f_k(\vec{x}^{(0)}, \vec{x}^{(1)}) > f_k(\vec{x}^{(1)}, \vec{x}^{(2)}) > f_k(\vec{x}^{(2)}, \vec{x}^{(3)}) > \cdots,$$

which is a contradiction. $\square$(3)

The above now tells us that

$$f_k(\vec{x}^{(0)}, \vec{x}^{(1)}) < f_k(\vec{x}^{(2)}, \vec{x}^{(3)}) < f_k(\vec{x}^{(4)}, \vec{x}^{(5)}) < \cdots$$

is an *ascending* α-sequence of indiscernibles for $\mathcal{M}$ with smallest element $f_k(\vec{x}^{(0)}, \vec{x}^{(1)}) < \nu_0$. This contradicts the minimal choice of $\min(X)$. $\square$

Recall that HOD is Gödel's model of *hereditarily ordinal definable sets*. Let $<_{\mathrm{HOD}}$ be the canonical well-ordering of HOD, defined in the universe V (see [14, Lemma 13.25]).

Lemma 1 *(ZF) Let κ^+ be almost Ramsey. Then $(\kappa^+)^{\mathrm{HOD}} < \kappa^+$.*

Proof Assume towards a contradiction that $(\kappa^+)^{\mathrm{HOD}} = \kappa^+$. For $\gamma \in [\kappa, \kappa^+)$, choose the $<_{\mathrm{HOD}}$-least bijection $f_\gamma : \gamma \leftrightarrow \kappa$. Define $F : [\kappa^+ \setminus \kappa]^3 \to 2$ by

$$F(\{\alpha, \beta, \gamma\}) = \begin{cases} 0 \text{ iff } f_\gamma(\alpha) < f_\gamma(\beta) \\ 1 \text{ iff } f_\gamma(\alpha) > f_\gamma(\beta) \end{cases}, \text{ for } \alpha < \beta < \gamma.$$

Take $X \subseteq \kappa^+$ homogeneous for F, with $\mathrm{otp}(X) = \kappa + 2$. Let $\gamma = \max(X)$. Then define $h : \kappa + 1 \to \kappa$ by $h(\xi) = f_\gamma(\alpha_\xi)$, where α_ξ is the ξ-th element of X.

Case 1: $\forall x \in [X]^3(F(x) = 0)$. Then for $\xi < \zeta < \kappa + 1$, we have that $\alpha_\xi < \alpha_\zeta < \gamma$ and $\{\alpha_\xi, \alpha_\zeta, \gamma\} \in [X]^3$. Since $F(\{\alpha_\xi, \alpha_\zeta, \gamma\}) = 0$, it follows that

$$h(\xi) = f_\gamma(\alpha_\xi) < f_\gamma(\alpha_\zeta) = h(\zeta).$$

Thus $h : \kappa + 1 \to \kappa$ is order preserving, which is impossible.

Case 2: $\forall x \in [X]^3(F(x) = 1)$. Then for $\xi < \zeta < \kappa + 1$, we have that $\alpha_\xi < \alpha_\zeta < \gamma$ and $\{\alpha_\xi, \alpha_\zeta, \gamma\} \in [X]^3$. Since $F(\{\alpha_\xi, \alpha_\zeta, \gamma\}) = 1$, it follows that

$$h(\xi) = f_\gamma(\alpha_\xi) > f_\gamma(\alpha_\zeta) = h(\zeta).$$

Thus $h : \kappa+1 \to \kappa$ is a strictly descending $\kappa+1$ chain in the ordinals, which is a contradiction. $\square$

3 Almost Ramsey cardinals and the Dodd-Jensen core model

We shall show that almost Ramsey cardinals are almost Ramsey in appropriate *core models*, in particular in the Dodd-Jensen core model K^{DJ} which is presented in [8] and [9]. Since core model theory usually assumes the Axiom of Choice, we shall also use the inner model HOD or extensions HOD$[a]$ of HOD by sets $a \subseteq$ HOD. The following proposition is found in [6].

Proposition 9 *(ZF) Let* $a \subseteq \mathrm{HOD}$ *be a set. Then*

a) HOD$[a]$ *is a set-generic extension of* HOD*, so* HOD$[a] \vDash$ ZFC.

b) $(K^{\mathrm{DJ}})^{\mathrm{HOD}} = (K^{\mathrm{DJ}})^{\mathrm{HOD}[a]}$*; moreover this equality holds for every level of the hierarchy, i.e.,* $(K_\alpha^{\mathrm{DJ}})^{\mathrm{HOD}} = (K_\alpha^{\mathrm{DJ}})^{\mathrm{HOD}[a]}$ *for every* $\alpha \in \mathrm{Ord}$.

By Proposition 9, we may define $K^{\mathrm{DJ}} = (K^{\mathrm{DJ}})^{\mathrm{HOD}}$ in models without choice. The following *indiscernibles lemma* by T. Dodd and R. Jensen (see [19], as well as [8] and [9]) is used to find homogeneous sets inside K^{DJ}.

Proposition 10 *Let* κ *be an infinite cardinal. Suppose* $A \in K^{\mathrm{DJ}} \cap \mathcal{P}(K_\kappa^{\mathrm{DJ}})$ *and that* I*, an infinite set of good indiscernibles for* $\mathcal{A} = (K_\kappa^{\mathrm{DJ}}, A)$*, is such that* $\mathrm{cof}(\mathrm{otp}(I)) > \omega$*. Then there is* $I' \in K^{\mathrm{DJ}}$*,* $I' \supseteq I$ *a set of good indiscernibles for* $\mathcal{A}$*.*

Lemma 2 *(ZF) Let* $\kappa > \aleph_1$ *be almost Ramsey. Then* κ *is almost Ramsey in* K^{DJ}*.*

Proof Let $F : [\kappa]^{<\omega} \to 2$, $F \in K^{\mathrm{DJ}}$ be a partition. Let $\alpha < \kappa$. Then $\alpha + \aleph_1 < \kappa$. By Proposition 8, take a set $X \subseteq \kappa$ of *good* indiscernibles for the structure $\mathcal{M} = (K_\kappa^{\mathrm{DJ}}, F)$, with $\mathrm{otp}(X) \geq \alpha + \aleph_1$. Let X' be the initial segment of X of order type $(\alpha + \aleph_1)^{\mathrm{HOD}[X]}$. In the model $\mathrm{HOD}[X]$, X' is a set of good indiscernibles for $\mathcal{M}$ such that $\mathrm{cof}(\mathrm{otp}(X')) > \omega$. By Proposition 10 applied inside $\mathrm{HOD}[X]$, there is a set $Y \supseteq X'$, $Y \in K^{\mathrm{DJ}}$ which is a set of good indiscernibles for $\mathcal{M}$. Then Y is also homogeneous for the partition F of order type greater than or equal to α. $\square$

We are now able to prove the inner model direction of Theorem 3.

Lemma 3 Con(*ZF + All infinite cardinals except possibly successors of singular limit cardinals are almost Ramsey*) $\implies$ Con(*ZFC + There is a proper class of regular almost Ramsey cardinals*).

Proof Assume Con(ZF + All infinite cardinals except possibly successors of singular limit cardinals are almost Ramsey). If there is a proper class of *regular* almost Ramsey cardinals, then by Lemma 2, we are done. So assume that this is not the case, and let the cardinal θ be an upper bound for the set of regular almost Ramsey cardinals. Then θ^{++} and θ^{+++} are not successors of limit cardinals. By assumption, θ^{++} and θ^{+++} are almost Ramsey. By the definition of θ, θ^{++} and θ^{+++} must be singular. By [18], this implies consistency strength far above measurable cardinals, and hence the consistency of a proper class of regular almost Ramsey cardinals. $\square$

We briefly outline now how to prove the forcing direction of Theorem 3. Specifically, we wish to show that Con(ZFC + There is a proper class of regular almost Ramsey cardinals) $\implies$ Con(ZF + DC + All successor cardinals are regular + All infinite cardinals except possibly successors of singular limit cardinals are almost Ramsey). To do this, we construct the model N of Theorem 1 of [4], using the class of regular almost Ramsey cardinals in place of the class of supercompact cardinals. (We refer readers of this paper to [4] for the exact definition of N.) The proofs of Lemmas 1.1–1.7 of [4] then show that $N \models$ "ZF + DC + All successor cardinals are regular + All successor cardinals except possibly successors of singular limit cardinals are almost Ramsey", assuming that each regular almost Ramsey cardinal is indestructible under forcing with arbitrary Lévy collapses, and each regular almost Ramsey cardinal is preserved by small forcing. However, these facts just follow from Propositions 3 and 4. Finally, since by Proposition 1, ZF $\vdash$ "Any limit of almost Ramsey cardinals is an almost Ramsey cardinal", $N \models$ "All infinite cardinals except possibly successors of singular limit cardinals are almost Ramsey". This completes our discussion of Theorem 3. $\square$

In the following, we apply the core model below a proper class of strong cardinals, denoted by the class term K (see [17]). As with the Dodd-Jensen core model, we get the following.

Proposition 11 *(ZF) Let $a \subseteq \mathrm{HOD}$ be a set. Then $K^{\mathrm{HOD}} = K^{\mathrm{HOD}[a]}$.*

If there is no inner model with a proper class of strong cardinals and the Axiom of Choice holds, then the core model K satisfies certain covering properties. In particular, Theorem 8.18 of [17] tells us that if $\kappa \geq \omega_2$, then $\mathrm{cof}^V((\kappa^+)^K) \geq \mathrm{Card}^V(\kappa)$.

Lemma 4 *(ZF) Let κ^+ be almost Ramsey, where κ is a singular cardinal. Then there is an inner model with a proper class of strong cardinals.*

Proof Assume that there is no inner model with a proper class of strong cardinals. By Lemma 1, $(\kappa^+)^{\mathrm{HOD}} < \kappa^+$. Since $K \subseteq \mathrm{HOD}$, $(\kappa^+)^K < \kappa^+$. Choose a bijection $f : \kappa \leftrightarrow (\kappa^+)^K$ and a cofinal subset $Z \subseteq \kappa$ such that $\mathrm{otp}(Z) < \kappa$. The class $\mathrm{HOD}[f, Z]$ is a model of ZFC which satisfies that κ is a singular cardinal such that $(\kappa^+)^K < \kappa^+$. Then inside the model $\mathrm{HOD}[f, Z]$, $\mathrm{cof}((\kappa^+)^K) < \kappa$. This contradicts the above mentioned covering property of the core model. $\square$

We are now able to prove Theorem 2.

Proof By assumption, $\aleph_{\omega+1}$ is almost Ramsey and the successor of the singular cardinal $\aleph_\omega$. Lemma 4 now implies the desired conclusions. $\square$

We conclude Section 3 by remarking that Schindler has pointed out to us that the conclusions of Theorem 2 can be strengthened. In particular, we have the following theorem.

Theorem 4 *Suppose Φ is a large cardinal concept such that $\Phi(\kappa^+) \implies$ There is no inner model of ZFC in which κ^+ is a successor cardinal. Suppose further that there is a proper class of cardinals κ such that κ is singular and $\Phi(\kappa^+)$ is true. Then for every $n < \omega$ and every set of ordinals x, $M_n^\sharp(x)$ exists.*

Thus, Theorem 4 implies that for every $n < \omega$, there is actually an inner model with n Woodin cardinals.

4 A model in which every infinite cardinal is almost Ramsey

From certain fairly mild supercompactness assumptions, we construct a model in which all infinite cardinals are almost Ramsey. We have already stated the relevant theorem as Theorem 1. We restate it here in the form in which it will be proven.

Theorem 5 *Let $V \vDash$ "ZFC + $\kappa < \lambda$ are such that κ is 2^λ supercompact where λ is the least regular almost Ramsey cardinal greater than κ". There is then a model N of height κ such that $N \vDash$ "ZF + $\neg AC_\omega$ + Every successor cardinal is regular + Every infinite cardinal is almost Ramsey".*

Proof Our proof uses Gitik's techniques of [13]. As in [1], we differ from [13], [5], and [2] only in the length of the Radin sequence of measures used. Our presentation follows the ones given in [5], [2], and [1] (all of which are based on [13]), suitably modified to our current context of almost Ramseyness. As the necessary facts about Radin forcing are distributed throughout the literature, our bibliographical citations will reflect this.

Let $j : V \to M$ be an elementary embedding witnessing the 2^λ supercompactness of κ. Our first step is to define a Radin sequence of measures $\mu_{<\rho} = \langle \mu_\alpha \mid \alpha < \rho \rangle$ over $P_\kappa(\lambda)$. Specifically, if $\alpha = 0$, μ_α is defined by $X \in \mu_\alpha$ iff $\langle j(\beta) \mid \beta < \lambda \rangle \in j(X)$, and if $\alpha > 0$, μ_α is defined by $X \in \mu_\alpha$ iff $\langle \mu_\beta \mid \beta < \alpha \rangle =_{\text{df}} \mu_{<\alpha} \in j(X)$. ρ is then defined as the first ordinal such that $A \in \mu_\rho$ implies that for some $\alpha < \rho$, $A \in \mu_\alpha$, i.e., ρ is the first *repeat point*. By cardinality considerations and the fact that $M^{2^\lambda} \subseteq M$ (see [7], [16], [13], [5], [1], [2], or [12]), this definition makes sense, and ρ exists.

Next, using $\mu_{<\rho}$, we let $\mathbb{R}_{<\rho}$ be supercompact Radin forcing defined over $V_\kappa \times P_\kappa(\lambda)$. The particulars of the definition are virtually identical to the ones found in [5], [2], and [1], but for clarity, we repeat them here. $\mathbb{R}_{<\rho}$ is composed of all finite sequences of the form $\langle\langle p_0, u_0, C_0, \rangle, \ldots, \langle p_n, u_n, C_n\rangle, \langle \mu_{<\rho}, C\rangle\rangle$ such that the following conditions hold:

1. For $0 \le i < j \le n$, $p_i \precsim p_j$, where for $p, q \in P_\kappa(\lambda)$, $p \precsim q$ means $p \subseteq q$ and $\mathrm{otp}(p) < q \cap \kappa$.

2. For $0 \le i \le n$, $p_i \cap \kappa$ is a measurable cardinal.

3. $\mathrm{otp}(p_i)$ is the least cardinal greater than $p_i \cap \kappa$ which is a regular almost Ramsey cardinal. In analogy to the notation of [13], [5], [2], and [1], we write $\mathrm{otp}(p_i) = (p_i \cap \kappa)^*$.

4. For $0 \le i \le n$, u_i is a Radin sequence of measures over $V_{p_i \cap \kappa} \times P_{p_i \cap \kappa}(\mathrm{otp}(p_i))$ with $(u_i)_0$, the 0th coordinate of u_i, a supercompact measure over $P_{p_i \cap \kappa}(\mathrm{otp}(p_i))$.

5. C_i is a sequence of measure 1 sets for u_i.

6. C is a sequence of measure 1 sets for $\mu_{<\rho}$.

7. For each $p \in (C)_0$, where $(C)_0$ is the coordinate of C such that $(C)_0 \in \mu_0$, $\bigcup_{i \in \{0, \ldots, n\}} p_i \precsim p$.

8. For each $p \in (C)_0$, $\mathrm{otp}(p) = (p \cap \kappa)^*$ and $p \cap \kappa$ is a measurable cardinal.

Conditions (5) and (6) are both standard to any definition of Radin forcing. Conditions (1), (2), (4), and (7) are all standard to any definition of *supercompact* Radin forcing. Conditions (3) and (8) are used because of our ultimate aim of constructing a model in which all infinite cardinals are almost Ramsey. That they may be included and have the Radin forcing attain its desired goals follows by the fact that $V \vDash$ "κ is 2^λ supercompact where λ is the least regular almost Ramsey cardinal above κ". Thus, by closure, $M \vDash$ "κ is measurable and λ is the least regular almost Ramsey cardinal above κ". This means that by reflection, $\{p \in P_\kappa(\lambda) \mid p \cap \kappa$ is a measurable cardinal and $\text{otp}(p)$ is the least regular almost Ramsey cardinal greater than $p \cap \kappa\} \in \mu_0$. This will ensure that the Radin sequence of cardinals eventually produced can be used in our final symmetric inner model N.

For completeness of exposition, we recall now the definition of the ordering on $\mathbb{R}_{<\rho}$. If $\pi_0 = \langle \langle p_0, u_0, C_0 \rangle, \ldots, \langle p_n, u_n, C_n \rangle, \langle \mu_{<\rho}, C \rangle \rangle$ and $\pi_1 = \langle \langle q_0, v_0, D_0 \rangle, \ldots, \langle q_m, v_m, D_m \rangle, \langle \mu_{<\rho}, D \rangle \rangle$, then π_1 extends π_0 iff the following conditions hold.

1. For each $\langle p_j, u_j, C_j \rangle$ which appears in π_0, there is a $\langle q_i, v_i, D_i \rangle$ which appears in π_1 such that $\langle q_i, v_i \rangle = \langle p_j, u_j \rangle$ and $D_i \subseteq C_j$, i.e., for each coordinate $(D_i)_\alpha$ and $(C_j)_\alpha$, $(D_i)_\alpha \subseteq (C_j)_\alpha$.

2. $D \subseteq C$.

3. $n \leq m$.

4. If $\langle q_i, v_i, D_i \rangle$ does not appear in π_0, let $\langle p_j, u_j, C_j \rangle$ (or $\langle \mu_{<\rho}, C \rangle$) be the first element of π_0 such that $p_j \cap \kappa > q_i \cap \kappa$. Then

(a) q_i is order isomorphic to some $q \in (C_j)_0$.

(b) There exists an $\alpha < \alpha_0$, where α_0 is the length of u_j, such that v_i is isomorphic "in a natural way" to an ultrafilter sequence $v \in (C_j)_\alpha$.

(c) For β_0 the length of v_i, there is a function $f : \beta_0 \to \alpha_0$ such that for $\beta < \beta_0$, $(D_i)_\beta$ is a set of ultrafilter sequences such that for some subset $(D_i)'_\beta$ of $(C_j)_{f(\beta)}$, each ultrafilter sequence in $(D_i)_\beta$ is isomorphic "in a natural way" to an ultrafilter sequence in $(D_i)'_\beta$.

For further information on the definition of the ordering on $\mathbb{R}_{<\rho}$ (including the meaning of "in a natural way") and more facts about Radin forcing in general, readers are referred to [5], [2], [1], [7], [10], [13], [12], and [16].

We are now ready to define the partial ordering $\mathbb{P}$ used in the proof of Theorem 5. It is given by the finite support product ordered componentwise

$$\prod_{\{\langle\alpha,\beta\rangle \mid \alpha<\beta<\kappa \text{ are regular cardinals}\}} \mathrm{Coll}(\alpha, {<}\beta) \times \mathbb{R}_{<\rho},$$

where $\mathrm{Coll}(\alpha, {<}\beta)$ is the Lévy collapse of all cardinals of size less than β to α.

Let G be V-generic over $\mathbb{P}$, and let G_0 be the projection of G onto $\mathbb{R}_{<\rho}$. For any condition $\pi \in \mathbb{R}_{<\rho}$, call $\langle p_0, \ldots p_n \rangle$ *the p-part of* π. Let $R = \{p \mid \exists \pi \in G_0[p \in \mathrm{p-part}(\pi)]\}$, and let $R_\ell = \{p \mid p \in R$ and p is a limit point of $R\}$. Define three sets E_0, E_1, and E_2 by $E_0 = \{\alpha \mid$ For some $\pi \in G_0$ and some $p \in \mathrm{p-part}(\pi)$, $p \cap \kappa = \alpha\}$, $E_1 = \{\alpha \mid \alpha$ is a limit point of $E_0\}$, and $E_2 = E_1 \cup \{\omega\} \cup \{\beta \mid \exists \alpha \in E_1[\beta = \alpha^*]\}$. E_2 will be the set of cardinals in our symmetric inner model N. Let $\langle \alpha_\nu \mid \nu < \kappa \rangle$ be the continuous, increasing enumeration of E_2, and let $\nu = \nu' + n$ for some $n \in \omega$, where ν' is either a limit ordinal or 0. For $\beta \in [\alpha_\nu, \alpha_{\nu+1})$, define sets $C_i(\alpha_\nu, \beta)$ for $i = 1, 2$ according to specific conditions on ν and ν' in the following manner:

1. $\nu = \nu' \neq 0$ and $n = 0$, i.e., ν is a limit ordinal. Let $p(\alpha_\nu)$ be the element p of R such that $p \cap \kappa = \alpha_\nu$, and let $h_{p(\alpha_\nu)} : p(\alpha_\nu) \to \mathrm{otp}(p(\alpha_\nu))$ be the order isomorphism between $p(\alpha_\nu)$ and $\mathrm{otp}(p(\alpha_\nu))$. Then $C_1(\alpha_\nu, \beta) = \{h_{p(\alpha_\nu)}{}'' p \cap \beta \mid p \in R_\ell,\ p \subseteq p(\alpha_\nu), \text{ and } h^{-1}_{p(\alpha_\nu)}(\beta) \in p\}$.

2. ($\nu = \nu' + n$ and $n \neq 0$) or ($\nu = \nu' = 0$), i.e., (ν is a successor ordinal) or ($\nu = 0$). Let $H(\alpha_\nu, \alpha_{\nu+1})$ be the projection of G onto $\mathrm{Coll}(\alpha_\nu, {<}\alpha_{\nu+1})$. Then $C_2(\alpha_\nu, \beta) = H(\alpha_\nu, \alpha_{\nu+1}) \restriction \beta$, i.e., $C_2(\alpha_\nu, \beta) = \{p \in H(\alpha_\nu, \alpha_{\nu+1}) \mid \mathrm{dom}(p) \subseteq \alpha_\nu \times \beta\}$.

$C_1(\alpha_\nu, \beta)$ is used to collapse β to α_ν when ν is a limit ordinal, and is also used to generate the closed, cofinal sequence $\langle \alpha_\beta \mid \beta < \nu \rangle$. $C_2(\alpha_\nu, \beta)$ is used to collapse β to α_ν when ν is a successor ordinal or $\nu = 0$. Intuitively, the symmetric inner model $N \subseteq V[G]$ witnessing the conclusions of Theorem 5 is V_κ of the least model of ZF extending V which contains, for $\beta \in [\alpha_\nu, \alpha_{\nu+1})$, $C_1(\alpha_\nu, \beta)$ if ν is a limit ordinal, and $C_2(\alpha_\nu, \beta)$ if ν is a successor ordinal or $\nu = 0$.

To define N more precisely, it is necessary to define canonical names $\underline{\alpha_\nu}$ for the α_ν's and canonical names $\underline{C_i(\nu, \beta)}$ for the two sets just described. Recall that it is possible to decide $p(\alpha_\nu)$ (and hence $\mathrm{otp}(p(\alpha_\nu))$) by writing

$\omega \cdot \nu = \omega^{\sigma_0} \cdot n_0 + \omega^{\sigma_1} \cdot n_1 + \cdots + \omega^{\sigma_m} \cdot n_m$ (where $\sigma_0 > \sigma_1 > \cdots > \sigma_m$ are ordinals, $n_0, \ldots, n_m > 0$ are integers, and $+$, $\cdot$, and exponentiation are the ordinal arithmetical operations), letting $\pi = \langle \langle p_{ij_i}, u_{ij_i}, C_{ij_i} \rangle_{i \leq m, 1 \leq j_i \leq n_i}, \langle \mu_{<\rho}, C \rangle \rangle$ be such that $\min(p_{i1} \cap \kappa, \omega^{\text{length}(u_{i1})}) = \sigma_i$ and $\text{length}(u_{ij_i}) = \min(p_{i1} \cap \kappa, \text{length}(u_{i1}))$ for $1 \leq j_i \leq n_i$, and letting $p(\alpha_\nu)$ be p_{mn_m}. Further, $D_\nu = \{r \in \mathbb{P} \mid r \restriction \mathbb{R}_{<\rho}$ extends a condition π of the above form$\}$ is a dense open subset of $\mathbb{P}$. $\underline{\alpha_\nu}$ is the name of the α_ν determined by any element of $D_\nu \cap G$; in the notation of [13], [5], [2], and [1], $\underline{\alpha_\nu} = \{\langle r, \check{\alpha}_\nu(r) \rangle \mid r \in D_\nu\}$, where $\alpha_\nu(r)$ is the α_ν determined by the condition r.

The canonical names $\underline{C_i(\nu, \beta)}$ are defined in a manner so as to be invariant under the appropriate group of automorphisms. Specifically, there are two cases to consider. We again write $\nu = \nu' + n$, where $n \in \omega$ and ν' is either a limit ordinal or 0, and let β be as before. We also assume without loss of generality that as in [13], [5], [2], and [1], $\alpha_{\nu+1}$ is determined by D_ν. Further, we adopt throughout each of the two cases the notation of [13], [5], [2], and [1].

1. $\nu' = \nu \neq 0$ and $n = 0$. $\underline{C_1(\nu, \beta)} = \{\langle r, (\check{r} \restriction \mathbb{R}_{<\rho}) \restriction (\alpha_\nu(r), \beta) \rangle \mid r \in D_\nu\}$, where for $r \in \mathbb{P}$, $\pi = r \restriction \mathbb{R}_{<\rho}$, $\pi \restriction (\alpha_\nu(r), \beta) = \{h_{p(\alpha_\nu)(r)}{}'' p \cap \beta \mid p \in$ p-part(π), $p \subseteq p(\alpha_\nu)(r)$, $p \in R_\ell \restriction \pi$, and $h^{-1}_{p(\alpha_\nu)(r)}(\beta) \in p\}$.

2. $(\nu = \nu' + n$ and $n \neq 0)$ or $(\nu = \nu' = 0)$. $\underline{C_2(\nu, \beta)} = \{\langle r, (\check{r} \restriction \text{Coll}(\alpha_\nu(r), {<}\alpha_{\nu+1}(r))) \restriction \beta \rangle \mid r \in D_\nu\}$.

As in [13], [5], [2], and [1], since for any $r, r' \in D_\nu \cap G$, $p(\alpha_\nu)(r) = p(\alpha_\nu)(r')$, both of the definitions just given are unambiguous.

Let $\mathcal{G}$ be the group of automorphisms of [13], and let $\underline{C(G)} = \bigcup_{i=1,2}\{\psi(\underline{C_i(\nu,\beta)}) \mid \psi \in \mathcal{G},\ 0 \le \nu < \kappa, \text{ and } \beta \in [\nu,\kappa) \text{ is a cardinal}\}$. $C(G) = \bigcup_{i=1,2}\{i_G(\psi(\underline{C_i(\nu,\beta)})) \mid \psi \in \mathcal{G},\ 0 \le \nu < \kappa, \text{ and } \beta \in [\nu,\kappa) \text{ is a cardinal}\} = i_G(\underline{C(G)})$. N is then the set of all sets of rank less than κ of the model consisting of all sets which are hereditarily V definable from $C(G)$, i.e., $N = V_\kappa^{\mathrm{HVD}(C(G))}$.

The arguments of [13] allow us to conclude that $N \models$ "ZF + $\neg\mathrm{AC}_\omega$ + $\langle \aleph_\nu \mid \nu \in \mathrm{Ord}\rangle = \langle \alpha_\nu \mid \nu \in \mathrm{Ord}\rangle$ + Every successor cardinal is regular". In addition, we know that for any ordinal λ and any set $x \subseteq \lambda$, $x \in N$, $x = \{\alpha < \lambda \mid V[G] \models \varphi(\alpha, i_G(\psi_1(\underline{C_{i_1}(\nu_1,\beta_1)})), \ldots, i_G(\psi_n(\underline{C_{i_n}(\nu_n,\beta_n)})), C(G))\}$, where i_j is an integer, $1 \le j \le n$, $1 \le i_j \le 2$, each $\psi_i \in \mathcal{G}$, each β_i is an appropriate ordinal for ν_i, and $\varphi(x_0, \ldots, x_{n+1})$ is a formula which may also contain some parameters from V which we shall suppress.

Let

$$\overline{\mathbb{P}} = \prod_{i_j=2} \mathrm{Coll}(\alpha_{\nu_j}, <\beta_j) \times \mathbb{R}_{<\rho}.$$

For $\pi \in \mathbb{R}_{<\rho}$, let $\pi \restriction \lambda = \{\langle q, u, C\rangle \in \pi \mid q \cap \kappa \le \lambda\}$. For $p \in \overline{\mathbb{P}}$, $p = \langle p_1, \ldots, p_m, \pi\rangle$, $m \le n$, $\pi \in \mathbb{R}_{<\rho}$, let $p \restriction \lambda = \langle q_1, \ldots, q_m, \pi \restriction \lambda\rangle$, where $q_j = p_j$ if $\alpha_{\nu_j} \le \lambda$ and $q_j = \emptyset$ otherwise. In other words, $p \restriction \lambda$ is the part of p below or at λ. Without loss of generality, we ignore the empty coordinates and let $\overline{\mathbb{P}} \restriction \lambda = \{p \restriction \lambda \mid p \in \overline{\mathbb{P}}\}$. Let $G \restriction \lambda$ be the projection of G onto

$\overline{\mathbb{P}} \restriction \lambda$. An analogous fact to Theorem 3.2.11 of [13] holds, using the same proof as in [13], namely $x \in V[G \restriction \lambda]$. In addition, the elements of $\overline{\mathbb{P}} \restriction \lambda$ can be partitioned into equivalence classes (the "almost similar" equivalence classes of [13]) with respect to $\underline{C_{i_1}(\nu_1, \beta_1)}, \ldots, \underline{C_{i_n}(\nu_n, \beta_n)}$ via an equivalence relation to be called $\sim$ such that if $\sigma < \lambda$, τ is a suitable term for x, and $p \Vdash$ "$\sigma \in \tau$", then for any $q \sim p$, $q \Vdash$ "$\sigma \in \tau$". It thus follows as an immediate corollary of the work of [13] that if we define $\widetilde{G \restriction \lambda} = \{[p]_\sim \mid p \in G \restriction \lambda\}$, then $x \in V[\widetilde{G \restriction \lambda}]$ and $V[\widetilde{G \restriction \lambda}] \subseteq N$. Further, if $\lambda = \alpha_\nu$ and either ν is a successor ordinal or $\nu = 0$, then the work of [13] also tells us that $\widetilde{G \restriction \lambda} = G_0 \times G_1$ is V-generic over a partial ordering of the form $\mathbb{P}_0 \times \mathbb{P}_1$, where $\mathbb{P}_1 = \mathrm{Coll}(\lambda, {<}\beta)$ for some β, and $\mathbb{P}_0$ is forcing equivalent to a partial ordering $\mathbb{P}^*$ such that $|\mathbb{P}^*| < \lambda$. In what follows, we will slightly abuse notation and denote $V[G_0][G_1]$ and $V[G_1]$ by $V^{\mathbb{P}_0 \times \mathbb{P}_1}$ and $V^{\mathbb{P}_1}$ respectively.

The discussion of the proof of Theorem 5 will now be completed by the following lemma.

Lemma 5 $N \models$ "*Every α_ν is an almost Ramsey cardinal*".

Proof Fix $\nu < \kappa$ and $\lambda = \alpha_\nu$. Since by Proposition 1, ZF $\vdash$ "Any limit of almost Ramsey cardinals is an almost Ramsey cardinal", without loss of generality, we may assume that ν is a successor ordinal. Further, by the properties of the Radin forcing $\mathbb{R}_{<\rho}$, we may also assume without loss of generality that in V, λ is a regular almost Ramsey cardinal.

Suppose $N \models$ "$f : [\lambda]^{<\omega} \to 2$ is a partition". Note that f may be coded by a subset of λ. Therefore, as in our discussion above, $f \in V^{\mathbb{P}_0 \times \mathbb{P}_1}$, where $\mathbb{P}_0$ is forcing equivalent to a partial ordering $\mathbb{P}^*$ such that $|\mathbb{P}^*| < \lambda$, and $\mathbb{P}_1$ is $\mathrm{Coll}(\lambda, {<}\beta)$ for some β. Since by Proposition 3, any regular almost Ramsey cardinal δ is automatically indestructible under $\mathrm{Coll}(\delta, {<}\beta)$ (and much more), $V^{\mathbb{P}_1} \models$ "λ is almost Ramsey". Since $|\mathbb{P}^*| < \lambda$ in both V and $V^{\mathbb{P}_1}$, by Proposition 4, λ is almost Ramsey in $V^{\mathbb{P}_1 \times \mathbb{P}_0} = V^{\mathbb{P}_0 \times \mathbb{P}_1}$. Thus, for every $\alpha < \lambda$, there is $X \in [\lambda]^{\alpha}$, $X \in V^{\mathbb{P}_0 \times \mathbb{P}_1}$ which is homogeneous for f. Since $V^{\mathbb{P}_0 \times \mathbb{P}_1} \subseteq N$, $X \in N$ as well. This completes the proof of Lemma 5. $\square$

Lemma 5 completes the proof of Theorem 5. $\square$

We note that the properties of Radin forcing, together with Gitik's methods, allow us to infer that since the Radin forcing $\mathbb{R}_{<\rho}$ is defined using a long enough sequence of measures $\mu_{<\rho}$, N will contain regular limit cardinals. Also, the arguments of [5] suitably modified tell us that $N \models$ "Every singular limit cardinal is a Jonsson cardinal".

We conclude by remarking that in the models of [13] and [5], it is the case that all infinite cardinals are almost Ramsey. The methods of proof are similar to those found in this paper. The constructions use an almost huge cardinal, but the consistency strength of the assumptions employed was reduced in [3] to something in consistency strength strictly in between a supercompact limit of supercompact cardinals and an almost huge cardinal.

Our hypotheses employed for Theorem 5, of course, are considerably weaker than those of [13], [5], or [3]. There are two main reasons for this.

One is that we do not have any singular successor cardinals in our desired model N. The other is that, roughly speaking, as Proposition 3 shows, almost Ramsey cardinals λ are automatically indestructible under forcing not adding any bounded subsets of λ, meaning that no additional preparation is required prior to the construction of N. We conjecture that in Gitik's model of [11] in which all uncountable cardinals are singular, built using a proper class of strongly compact cardinals, all infinite cardinals are also almost Ramsey.

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