An Error Estimate for Euler's Method Using a Gronwall Type Argument

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Abstract. A reinterpretation of standard integration schemes for ODEs allows to use Gronwall type arguments to obtain error estimates. This has two advantages. First, the derivatives of the vector field which enter into the standard error estimates are here only needed along the approximating curves, thus improved error estimates may be obtained during the integration. Second, while the standard argument gives the exponential error bounds only in the limit, the Gronwall argument gives these bounds for each finite size time step directly thus simplifying the iteration of the bounds over many time steps.

Let $X(p,t)$ be a smooth, time-dependent vector field in $\mathbf{R}^n$, with Lipschitz bound L, so $\|X(p,t) - X(q,t)\| \leq L\,\|p-q\|$. The corresponding ODE Initial Value Problem is $\dot{p} = X(p(t),t)$, $p(t_0) = p_0$. The existence and uniqueness theorem for ODE asserts that for some interval I centered at t_0 there is a unique path $p(t)$ defined in I and satisfying both the ODE and the initial condition.

The "Euler Method" approximates $p(t)$ at a discrete set of points t_n defined by $t_n := t_{n-1} + \Delta T = t_0 + n\,\Delta T$, where ΔT is a small real number, the "time-step". The approximations $e(t_n)$ are defined inductively by $e(t_0) := p_0$, and $e(t_{n+1}) := e(t_n) + \Delta T\, X(e(t_n), t_n)$. We will depart from custom here and join the discrete approximations $e(t_n)$ with an explicit curve $e(t)$ that is defined on the interval $[t_n, t_{n+1}]$ and is completely determined by the integration scheme. For example, the "Euler Curve" is $e(t) := e(t_n) + (t - t_n)\, X(e(t_n), t_n)$ for $0 \leq (t - t_n) \leq \Delta T$. One can think of this as making the time step $(t - t_n)$ a continuous variable, but $\leq \Delta t$. For emphasis we show such reinterpretations for two other schemes although we will derive the error estimates only in the Euler case, other cases being very similar. The "Inverse Euler Curve" $E(t)$ is defined on the interval $[t_n, t_{n+1}]$ by $E(t) := E(t_n) + (t - t_n) \cdot X(E(t), t)$ for $0 \leq (t - t_n) \leq \Delta T$, and the "Half Step Curve" $h(t)$ is defined on the interval $[t_n, t_{n+1}]$, using an auxiliary midpoint curve $m(t) := h(t_n) + (t - t_n)/2 \cdot X(h(t_n), t_n)$, by $h(t) := h(t_n) + (t - t_n) \cdot X(m(t), t_n + (t - t_n)/2)$ for $0 \leq (t - t_n) \leq \Delta T$. Such reinterpretations allow us to estimate the difference or "error", $\mathrm{Err}(t) := \|p(t) - e(t)\|$ between $p(t)$ and $e(t)$ (and similarly for other schemes) by using a Gronwall-like argument. Initially we shall estimate the error for a single time-step, i.e., on the interval $t_0 \leq t \leq t_0 + \Delta T$ and later iterate the obtained estimate.

We first note that from the definition of e, $\dot{e}(t) = X(p_0, t_0)$, $t_0 \leq t \leq t_0 + \Delta T$, so $\ddot{e} = 0$. It follows that $\dot{p}(t) - \dot{e}(t) = X(p(t),t) - X(p_0,t_0) = X(p(t),t) - X(e(t),t) + X(e(t),t) - X(p_0,t_0)$, and hence $\|\dot{p}(t) - \dot{e}(t)\| \leq L\,\|p(t) - e(t)\| + \|X(e(t),t) - X(p_0,t_0)\|$.

The second term on the right is not yet in a form which a Gronwall argument can use, but since it is explicit we can define $K := \max_{t_0 \leq t \leq t_0 + \Delta t} \left\| \frac{d}{dt}(X(e(t),t) - X(p_0,t_0)) \right\|$ to obtain the differential inequality $\|\dot{p}(t) - \dot{e}(t)\| \leq L\,\|\mathrm{Err}(t)\| + K(t - t_0)$.

Note that for higher order schemes we get better inequalities here. For example, the function $X(h(t),t) - \dot{h}(t)$ and its first derivative are zero at t_0. Therefore we use the maximum of the (still explicit) second derivative $K := \max \left\| \frac{d^2}{dt^2}(X(h(t),t) - \dot{h}(t)) \right\|$ along the half step curve to obtain $\left\| \dot{p}(t) - \dot{h}(t) \right\| \leq L \left\| \mathrm{Err}(t) \right\| + \frac{K}{2}(t-t_0)^2$.

We continue the Euler case. Since $\mathrm{Err}(t) = \left\| \int_{t_0}^t (\dot{p}(t) - \dot{e}(t))\,dt \right\| \leq \int_{t_0}^t \left\| \dot{p}(t) - \dot{e}(t) \right\| dt$, we see that $\left\| \mathrm{Err}(t) \right\| \leq \psi(t)$, where ψ is the **differentiable** function:

$$\psi(t) := \left\| \mathrm{Err}(t_0) \right\| + L \int_{t_0}^t \left\| \mathrm{Err}(t) \right\| dt + K \int_{t_0}^t (t - t_0)\; dt.$$

Since $\dot{\psi} = L \left\| \mathrm{Err}(t) \right\| + K(t-t_0)$ and $\left\| \mathrm{Err}(t) \right\| \leq \psi$, we have the differential inequality $\dot{\psi} \leq L\psi + K(t-t_0)$ to which we now apply a Gronwall argument.

Compute the derivative of the function $\left(\psi + \frac{K}{L^2} + \frac{K}{L}(t-t_0)\right) \cdot e^{-L\cdot(t-t_0)}$:

$$\frac{d}{dt}\left(\left(\psi + \frac{K}{L^2} + \frac{K}{L}(t-t_0)\right) \cdot e^{-L\cdot(t-t_0)}\right) = \left(\dot{\psi} + \frac{K}{L} - L\left(\psi + \frac{K}{L^2} + \frac{K}{L}(t-t_0)\right)\right) \cdot e^{-L\cdot(t-t_0)}.$$

By the differential inequality for ψ, this function has a non-positive derivative, so all of its values are less than its "initial" value at t_0, namely $\mathrm{Err}(t_0) + \frac{K}{L^2}$. (Of course, for the Euler scheme $\mathrm{Err}(t_0) = 0$, but for all further time steps we do have an initial error.) Solving for ψ we obtain the desired one step error estimate:

$$\mathrm{Err}(t) \leq \psi(t) \leq \left(\mathrm{Err}(t_0) + \frac{K}{2}(t-t_0)^2\right) e^{L\cdot(t-t_0)} \quad \text{for} \quad t_0 \leq t \leq t_0 + \Delta T.$$

To iterate this estimate we define the starting point for the second time step as $p_1 := e(t_0 + \Delta T)$, so that we have the initial error bound

$$|p(t_0 + \Delta t) - p_1| \leq \mathrm{Err}(t_1) := \left(\mathrm{Err}(t_0) + \frac{K}{2}(t-t_0)^2\right) e^{L\cdot\Delta T}.$$

For the second time step the same argument as for the first time step gives:

$$\mathrm{Err}(t) \leq \psi(t) \leq \left(\mathrm{Err}(t_1) + \frac{K}{2}(t-t_1)^2\right) e^{L\cdot(t-t_1)} \quad \text{for} \quad t_1 \leq t \leq t_1 + \Delta T.$$

To reach some fixed time T one needs N time steps of size $\Delta t := (T - t_0)/N$. Then the N-fold iteration of the error estimate can be simplified (replace $N \cdot \Delta T$ by $(T - t_0)$, recall $\mathrm{Err}(t_0) = 0$ and use the sum of the geometric series):

$$\begin{aligned} \mathrm{Err}(t) \leq & \mathrm{Err}(t_0) \cdot e^{L(T-t_0)} + \frac{K}{2}\Delta T^2 \cdot (e^{L\cdot\Delta T} + e^{L\cdot 2\Delta T} + \ldots e^{L\cdot N\Delta T}) \\ \leq & \frac{K}{2}\Delta T^2 \cdot (e^{L(T-t_0+\Delta T)} - 1)/(e^{L\Delta t} - 1) \leq \frac{K}{2L}\Delta T \cdot (e^{L(T-t_0+\Delta T)} - 1). \end{aligned}$$

This proves in particular that with $\Delta T \to 0$ the iterated Euler curves converge uniformly to the exact solution.