A TANNAKIAN CLASSIFICATION OF TORSORS ON THE PROJECTIVE LINE

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Abstract. In this small note we present a Tannakian proof of the theorem of Grothendieck-Harder on the classification of torsors under a reductive group on the projective line over a field.

1. Introduction

Let k be a field, let G/k be a reductive group and let $\mathbb{P}^1_k$ be the projective line over k. In this small note we present a Tannakian proof of the classification of G-torsors on $\mathbb{P}^1_k$, thereby reproving known results of A. Grothendieck [Gro57] and G. Harder [Har68, Satz 3.4.]. To state our main theorem we denote by

$$\mathrm{Hom}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m))$$

the set of isomorphism classes of exact tensor functors

$$\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Rep}_k(\mathbb{G}_m).$$

Theorem 1.1 (cf. Theorem 3.3, Corollary 3.5)**.** *There exists a canonical bijection*

$$\mathrm{Hom}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m)) \cong H^1_{\text{ét}}(\mathbb{P}^1_k, G).$$

In particular, there exists a canonical bijection

$$\mathrm{Hom}(\mathbb{G}_m, G)/G(k) \cong H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, G).$$

If $A \subseteq G$ denotes a maximal split torus, then

$$\mathrm{Hom}(\mathbb{G}_m, G)/G(k) \cong X_*(A)_+$$

is in bijection with the set of dominant cocharacters of $A \subseteq G$, which gives a very concrete description of the set $H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, G)$. Using pure inner forms of G over k one can describe similarly the whole set $H^1_{\text{ét}}(\mathbb{P}^1_k, G)$ (cf. Lemma 3.4).

Our proof of Theorem 1.1, which originated in questions about torsors over the Fargues-Fontaine curve (cf. [Ans]), is based on the Tannakian description of G-torsors (cf. Lemma 3.1), the Tannakian theory of filtered fiber functors (cf. [Zie15]), the canonicity of the Harder-Narasimhan filtration (cf. Lemma 2.2) and, most importantly, the good understanding of the category $\mathrm{Bun}_{\mathbb{P}^1_k}$ of vector bundles on $\mathbb{P}^1_k$ (cf. Theorem 2.1). In particular, we use crucially the fact that

$$H^1_{\text{ét}}(\mathbb{P}^1_k, \mathcal{E}) = 0$$

for $\mathcal{E}$ a semistable vector bundle on $\mathbb{P}^1_k$ of slope ≥ 0.

In a last section we mention applications of Theorem 1.1 to the the computation of the Brauer group of $\mathbb{P}^1_k$ (avoiding Tsen's theorem) and to the Birkhoff-Grothendieck decomposition of $G(k((t)))$.

Date: March 1, 2017.

1.2. **Acknowledgment.** We want to thank Jochen Heinloth for his interest and for answering several questions.

2. Vector bundles on $\mathbb{P}^1_k$

Let k be an arbitrary field. We recall, in a more canonical form, the classification of vector bundles on the projective line $\mathbb{P}^1_k$ due to A. Grothendieck (cf. [Gro57]). Let

$$\mathrm{Rep}_k(\mathbb{G}_m)$$

be the category of finite dimensional representations of the multiplicative group $\mathbb{G}_m$ over k. More concretely, the category $\mathrm{Rep}_k(\mathbb{G}_m)$ is equivalent to the Tannakian category of finite dimensional $\mathbb{Z}$-graded vector spaces over k.

Over $\mathbb{P}^1_k$ there is the canonical $\mathbb{G}_m$-torsor

$$\eta\colon \mathbb{A}^2_k \setminus \{0\} \to \mathbb{P}^1_k,\ (x_0, x_1) \mapsto [x_0 : x_1],$$

also called the "Hopf bundle". Given a representation $V \in \mathrm{Rep}_k(\mathbb{G}_m)$ the contracted product

$$\mathcal{E}(V) := \mathbb{A}^2_k \setminus \{0\} \times^{\mathbb{G}_m} V \to \mathbb{P}^1_k$$

defines a (geometric) vector bundle over $\mathbb{P}^1_k$. The well known classification of the category

$$\mathrm{Bun}_{\mathbb{P}^1_k}$$

of vector bundles on $\mathbb{P}^1_k$ can now be phrased in the following way.

Theorem 2.1. *The functor*

$$\mathcal{E}(-)\colon \mathrm{Rep}_k(\mathbb{G}_m) \to \mathrm{Bun}_{\mathbb{P}^1_k}$$

is an exact, faithful tensor functor inducing a bijection on isomorphism classes.

However, the functor $\mathcal{E}(-)$ is not an equivalence. For example, by semi-simplicity of the category $\mathrm{Rep}_k(\mathbb{G}_m)$ every short exact sequence of $\mathbb{G}_m$-representations splits, but this is not true for short exact sequences of vector bundles on $\mathbb{P}^1_k$.

For $V \in \mathrm{Rep}_k(\mathbb{G}_m)$ the Harder-Narasimhan filtration of the vector bundle

$$\mathcal{E}(V)$$

has a very simple description. Namely, write

$$V = \bigoplus_{i\in\mathbb{Z}} V_i$$

with $\mathbb{G}_m$ acting on V_i by the character[1]

$$\mathbb{G}_m \to \mathbb{G}_m,\ z \mapsto z^{-i}$$

and set

$$\mathrm{fil}^i(V) := \bigoplus_{j\geq i} V_j$$

for $i \in \mathbb{Z}$. Then the Harder-Narasimhan filtration of $\mathcal{E} := \mathcal{E}(V)$ is given by

$$\ldots \subseteq \mathrm{HN}^{i+1}(\mathcal{E}) \subseteq \mathrm{HN}^i(\mathcal{E}) \subseteq \ldots \subseteq \mathcal{E}.$$

where

$$\mathrm{HN}^i(\mathcal{E}) := \mathcal{E}(\mathrm{fil}^i(V)).$$

[1] The sign is explained by the fact that the standard represention $z \mapsto z$ of $\mathbb{G}_m$ is sent by $\mathcal{E}(-)$ to $\mathcal{O}_{\mathbb{P}^1_k}(-1)$ and not to $\mathcal{O}_{\mathbb{P}^1_k}(1)$.

Lemma 2.2. *Sending a vector bundle $\mathcal{E}$ to the filtered vector bundle $\mathcal{E}$ with the Harder-Narasimhan filtration $\mathrm{HN}^\bullet(\mathcal{E})$ defines a fully faithful tensor functor*

$$\mathrm{HN}\colon \mathrm{Bun}_{\mathbb{P}^1_k} \to \mathrm{FilBun}_{\mathbb{P}^1_k}$$

into the exact tensor category of filtered vector bundles (with filtration by locally direct summands) (cf. [Zie15, Chapter 4] *for a definition of* $\mathrm{FilBun}_{\mathbb{P}^1_k}$*).*

Proof. This is clear from the description of the Harder-Narasimhan filtration. □

We remark that the functor HN is *not* exact as one sees for example by looking at the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1_k}(-1) \to \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k} \to \mathcal{O}_{\mathbb{P}^1_k}(1) \to 0$$

on $\mathbb{P}^1_k$.

Sending a filtered vector bundle $(\mathcal{E}, F^\bullet)$ to the associated graded vector bundle

$$\mathrm{gr}(\mathcal{E}) := \bigoplus_{i\in\mathbb{Z}} F^i\mathcal{E}/F^{i+1}\mathcal{E}$$

defines an exact tensor functor

$$\mathrm{gr}\colon \mathrm{FilBun}_{\mathbb{P}^1_k} \to \mathrm{GrBun}_{\mathbb{P}^1_k}$$

(cf. [Zie15, Chapter 4]).

The following lemma is immediate from Theorem 2.1, Lemma 2.2 and the fact that

$$H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \cong k.$$

Lemma 2.3. *The composite functor*

$$\mathrm{Rep}_k(\mathbb{G}_m) \xrightarrow{\mathcal{E}(-)} \mathrm{Bun}_{\mathbb{P}^1_k} \xrightarrow{\mathrm{HN}} \mathrm{FilBun}_{\mathbb{P}^1_k} \xrightarrow{\mathrm{gr}} \mathrm{GrBun}_{\mathbb{P}^1_k}$$

is an equivalence of exact categories from $\mathrm{Rep}_k(\mathbb{G}_m)$ *onto its essential image which consists of graded vector bundles*

$$\mathcal{E} = \bigoplus_{i\in\mathbb{Z}} \mathcal{E}^i$$

such that each $\mathcal{E}^i$ is semistable of slope i.

3. Torsors over $\mathbb{P}^1_k$

Let G/k be an arbitrary reductive group. In this section we want to classify G-torsors on $\mathbb{P}^1_k$ for the étale topology. For this we keep the notation from the last section. In particular, there is the functor

$$\mathcal{E}(-)\colon \mathrm{Rep}_k(\mathbb{G}_m) \to \mathrm{Bun}_{\mathbb{P}^1_k}$$

from Theorem 2.1

In order to apply the formulations from the previous section we need a more bundle theoretic interpretation of G-torsors (for the étale topology). This is achieved by the Tannakian formalism (cf. [Del90])

Lemma 3.1. *Let S be a scheme over k. Sending a G-torsor $\mathcal{P}$ over S to the exact tensor functor*

$$\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_S,\ V \mapsto \mathcal{P} \times^G (V \otimes_k \mathcal{O}_S)$$

defines an equivalence from the groupoid of G-torsors to the groupoid of exact tensor functors from $\mathrm{Rep}_k(G)$ to Bun_S. The inverse equivalence sends an exact tensor functor $\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_S$ the G-torsor $\mathrm{Isom}^{\otimes}(\omega_{\mathrm{can}}, \omega)$ of isomorphisms of ω to the canonical fiber functor $\omega_{\mathrm{can}}\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_S,\ V \mapsto V \otimes_k \mathcal{O}_S$.

In fact, for a general affine group scheme over k one has to use the fpqc-topology in Lemma 3.1. However, as G is assumed to be reductive, thus in particular smooth, a theorem of Grothendieck (cf. [Gro68, Théorème 11.7]) allows to reduce to the étale topology.

Composing an exact tensor functor

$$\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_{\mathbb{P}^1_k}$$

with the Harder-Narasimhan functor

$$\mathrm{HN}\colon \mathrm{Bun}_{\mathbb{P}^1_k} \to \mathrm{FilBun}_{\mathbb{P}^1_k}$$

defines a, a priori not necessarily exact, tensor functor

$$\mathrm{HN} \circ \omega\colon \mathrm{Rep}_k(G) \to \mathrm{FilBun}_{\mathbb{P}^1_k}.$$

But using Haboush's theorem reductivity of G actually implies that the composition $\mathrm{HN} \circ \omega$ is still exact.

Lemma 3.2. *Let*

$$\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_{\mathbb{P}^1_k}$$

be an exact tensor functor. Then the composition

$$\mathrm{HN} \circ \omega\colon \mathrm{Rep}_k(G) \to \mathrm{FilBun}_{\mathbb{P}^1_k}$$

is still exact.

Proof. The crucial observation is that the functors

$$\omega,\ \mathrm{gr} \circ \mathrm{HN}$$

are compatible with duals, and exterior resp. symmetric products. This is clear for ω as ω is assumed to be exact and follows from Lemma 2.3 for the functor $\mathrm{HN} \circ \mathrm{gr}$. In fact, for a representation $V \in \mathrm{Rep}_k(\mathbb{G}_m)$ with associated vector bundle

$$\mathcal{E} := \mathcal{E}(V)$$

we can conclude

$$\Lambda^r(\mathcal{E}) \cong \mathcal{E}(\Lambda^r(V)) \text{ resp. } \mathrm{Sym}^r(\mathcal{E}) \cong \mathcal{E}(\mathrm{Sym}^r(V))$$

by exactness of the functor $\mathcal{E}(-)$. But by Lemma 2.3

$$\mathrm{gr} \circ \mathrm{HN} \circ \mathcal{E}(-)$$

is an exact tensor equivalence of $\mathrm{Rep}_k(\mathbb{G}_m)$ with a subcategory of $\mathrm{GrBun}_{\mathbb{P}^1_k}$, which implies the stated compatibility with exterior and symmetric powers. Using this the proof can proceed similarly to [DOR10, Theorem 5.3.1]. We note that for a representation V of G there is a canonical isomorphism

$$\mathrm{Sym}^r(V^\vee) \cong \mathrm{TS}_r(V)^\vee$$

from the r-th symmetric power $\mathrm{Sym}^r(V^\vee)$ of the dual of V to the dual of the module

$$\mathrm{TS}_r(V) = (V^{\otimes r})^{S_r} \subseteq V^{\otimes r}$$

of symmetric tensors. In particular, G-invariant homogenous polynomials on V define G-invariant linear forms on $\mathrm{TS}_r(V)^\vee$.

Let now $0 \to V \xrightarrow{f} V' \xrightarrow{g} V'' \to 0$ be an exact sequence in $\mathrm{Rep}_k(G)$. We have to check that the sequence

$$0 \to \tilde{\omega}(V) \xrightarrow{\tilde{\omega}(f)} \tilde{\omega}(V') \xrightarrow{\tilde{\omega}(g)} \tilde{\omega}(V'') \to 0$$

with

$$\tilde{\omega} := \mathrm{gr} \circ \mathrm{HN} \circ \omega$$

is still exact. We claim that $\tilde{\omega}(f)$ is injective. This can be checked after taking the exterior power $\Lambda^{\dim V}$ of f because $\tilde{\omega}$ commutes with exterior powers. In particular, to prove injectivity we can reduce the claim for general f to the case $\dim V = 1$. Tensoring with the dual of V reduces further to the case the V is moreover trivial. By Haboush's theorem (cf. [Hab75]) there exists an $r > 0$ and a G-invariant homogenous polynomial $f \in \mathrm{Sym}^r(V^\vee)$ such that $f_{|V} \neq 0$. Using the above isomorphism $\mathrm{Sym}^r(V^\vee) \cong \mathrm{TS}_r(V)^\vee$ this shows that there exists an $r > 0$ such that the morphism

$$V \cong \mathrm{TS}_r(V) \xrightarrow{\mathrm{TS}_r(f)} \mathrm{TS}_r(V')$$

splits. This implies that $\tilde{\omega}(\mathrm{TS}_r(f))$ splits and thus that $\tilde{\omega}(f)$ is in particular injective because $\tilde{\omega}$ commutes with the symmetric tensors TS_r as it commutes with symmetric powers and duals.

Dualizing yields that $\tilde{\omega}(g)$ is surjective at the generic point of $\mathbb{P}^1_k$. However, the sequence

$$0 \to \tilde{\omega}(V) \xrightarrow{\tilde{\omega}(f)} \tilde{\omega}(V') \xrightarrow{\tilde{\omega}(g)} \tilde{\omega}(V'') \to 0$$

lies in the essential image of the functor $\mathrm{Rep}_k(\mathbb{G}_m) \to \mathrm{GrBun}_{\mathbb{P}^1_k}$ from Lemma 2.3. In particular, we see that the cokernel of $\tilde{\omega}(g)$ cannot have torsion, i.e., that it is zero. Finally, exactness in the middle of the sequence follows because

$$\mathrm{rk}(\tilde{\omega}(V')) = \mathrm{rk}(V') = \mathrm{rk}(V) + \mathrm{rk}(V'') = \mathrm{rk}(\tilde{\omega}(V)) + \mathrm{rk}(\tilde{\omega}(V'')).$$

This finishes the proof. □

We briefly recall some results about filtered fiber functors on $\mathrm{Rep}_k G$ (cf. [Zie15] and [Cor]). By definition a filtered fiber functor for $\mathrm{Rep}_k G$ over a k-scheme S is an exact tensor functor

$$\omega \colon \mathrm{Rep}_k G \to \mathrm{FilBun}_S$$

into the exact tensor category of filtered vector bundles (with filtration by locally direct summands) on S. Associated to each filtered fiber functor ω is an exact tensor functor

$$\mathrm{gr} \circ \omega \colon \mathrm{Rep}_k G \to \mathrm{GrBun}_S,$$

i.e., a graded fiber functor, by mapping a filtered vector bundle to its associated graded. A splitting γ of a filtered fiber functor ω is a graded fiber functor

$$\gamma \colon \mathrm{Rep}_k G \to \mathrm{GrBun}_S$$

such that

$$\omega = \mathrm{fil} \circ \gamma$$

where the exact tensor functor

$$\mathrm{fil}\colon \mathrm{GrBun}_S \to \mathrm{FilBun}_S$$

sends a graded vector bundle

$$\mathcal{E} = \bigoplus_{i\in\mathbb{Z}} \mathcal{E}^i$$

to the filtered vector bundle $(\mathcal{E}, \mathrm{fil}^\bullet\mathcal{E})$ with filtration

$$\mathrm{fil}^i\mathcal{E} = \bigoplus_{j\geq i} \mathcal{E}^j.$$

For a scheme $f\colon S' \to S$ over S let $\omega_{S'}$ be the base change of the filtered fiber functor ω to S', i.e., $\omega_{S'}$ is defined as the composition

$$\mathrm{Rep}_k G \xrightarrow{\omega} \mathrm{FilBun}_S \xrightarrow{f^*} \mathrm{FilBun}_{S'},$$

which is again a filtered fiber functor. For a filtered fiber functor ω the presheaf

$$\mathrm{Spl}(\omega)(S') := \{ \text{ set of splittings of } \omega_{S'}\}$$

on the category of S-schemes is represented by an fpqc-torsor for the affine and faithfully flat group scheme

$$U(\omega) := \mathrm{Ker}(\mathrm{Aut}^{\otimes}(\omega) \to \mathrm{Aut}^{\otimes}(\mathrm{gr}\circ\omega))$$

over S (cf. [Zie15, Lemma 4.20]). In particular, every filtered fiber functor

$$\omega\colon \mathrm{Rep}_k G \to \mathrm{FilBun}_S$$

admits a splitting fpqc-locally on S. The group scheme $U(\omega)$ can be described more explicitely (cf. [Zie15, Theorem 4.40]). Namely there exists a decreasing filtration by normal subgroups

$$U(\omega) = U_1(\omega) \supseteq \ldots \supseteq U_i(\omega) \supseteq \ldots$$

for $i \geq 1$, which has the property that for $i \geq 1$ the quotient

$$\mathrm{gr}^i U(\omega) := U_i(\omega)/U_{i+1}(\omega)$$

is abelian and isomorphic to

$$\mathrm{gr}^i U(\omega) \cong \mathrm{Lie}(\mathrm{gr}^i U(\omega)) \cong \mathrm{gr}^i\omega(\mathrm{Lie}(G)),\ i \geq 1.$$

We can now give a proof of our main theorem about the classification of G-torsors on $\mathbb{P}^1_k$. We denote for a scheme S over k by

$$\underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Bun}_S)$$

the groupoid of exact tensor functors $\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_S$ and by

$$\mathrm{Hom}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Bun}_S)$$

its set of isomorphism classes. Similarly, we use the notations

$$\underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m))$$

resp.

$$\mathrm{Hom}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m))$$

for the groupoid resp. the isomorphism classes of exact tensor functors

$$\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Rep}_k(\mathbb{G}_m).$$

Theorem 3.3. *Let G be a reductive group over k. Then the composition with $\mathcal{E}(-)$ defines faithful functor*

$$\Phi\colon \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m)) \to \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Bun}_{\mathbb{P}^1_k})$$

which induces a bijection

$$\mathrm{Hom}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m)) \cong H^1_{\text{ét}}(\mathbb{P}^1_k, G).$$

on isomorphism classes.

Proof. By Lemma 2.3 the composition

$$\mathrm{Rep}_k(\mathbb{G}_m) \xrightarrow{\mathcal{E}(-)} \mathrm{Bun}_{\mathbb{P}^1_k} \xrightarrow{\mathrm{HN}} \mathrm{FilBun}_{\mathbb{P}^1_k} \xrightarrow{\mathrm{gr}} \mathrm{GrBun}_{\mathbb{P}^1_k}$$

is an equivalence onto its essential image. In particular, the functor

$$\Phi\colon \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m)) \to \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Bun}_{\mathbb{P}^1_k})$$

is faithful and induces an injection on isomorphism classes. Thus we have to prove that every exact tensor functor

$$\omega\colon \mathrm{Rep}_k(G) \to \mathrm{Bun}_{\mathbb{P}^1_k}$$

factors as

$$\omega \cong \mathcal{E}(-) \circ \omega'$$

for some exact tensor functor

$$\omega'\colon \mathrm{Rep}_k(G) \to \mathrm{Rep}_k(\mathbb{G}_m).$$

Let $\tilde{\omega} := \mathrm{HN} \circ \omega$ be the functor

$$\tilde{\omega}\colon \mathrm{Rep}_k(G) \xrightarrow{\omega} \mathrm{Bun}_{\mathbb{P}^1_k} \xrightarrow{\mathrm{HN}} \mathrm{FilBun}_{\mathbb{P}^1_k}.$$

By Theorem 3.3 the functor $\tilde{\omega}$ is still exact, i.e., a filtered fiber functor in the terminology of [Zie15], and we can use the results recalled above. We get a $U(\tilde{\omega})$-torsor

$$\mathrm{Spl}(\tilde{\omega})$$

of splittings of $\tilde{\omega}$. But for the filtration

$$U(\tilde{\omega}) \supseteq U_2(\tilde{\omega}) \supseteq \dots$$

the graded quotients

$$\mathrm{gr}^i U(\tilde{\omega}) \cong \mathrm{gr}^i \tilde{\omega}(\mathrm{Lie}(G))$$

are semistable vector bundles of slope $i \geq 1$. Hence,

$$H^1_{\text{ét}}(\mathbb{P}^1_k, \mathrm{gr}^i U(\tilde{\omega})) = 0$$

because

$$\mathrm{gr}^i U(\tilde{\omega}) \cong \mathcal{O}_{\mathbb{P}^1_k}(i)^{\oplus n}$$

by Theorem 2.1. We can conclude that

$$H^1_{\text{ét}}(\mathbb{P}^1_k, U(\tilde{\omega})) = 1,$$

hence the $U(\tilde{\omega})$-torsor

$$\mathrm{Spl}(\tilde{\omega})$$

is in fact trivial, i.e., there exists a splitting

$$\gamma\colon \mathrm{Rep}_k G \to \mathrm{GrBun}_{\mathbb{P}^1_k}$$

of $\tilde{\omega}$ already over $\mathbb{P}^1_k$. As

$$\gamma \cong \mathrm{gr} \circ \tilde{\omega}$$

the functor γ takes its image in the full subcategory

$$\{\ \mathcal{E} = \bigoplus_{i\in\mathbb{Z}} \mathcal{E}^i \in \mathrm{GrBun}_{\mathbb{P}^1} \mid \mathcal{E}^i \text{ semistable of slope } i\},$$

which by Lemma 2.3 is equivalent to the category $\mathrm{Rep}_k\mathbb{G}_m$ of representations of $\mathbb{G}_m$. Thus there exists an exact tensor functor

$$\omega' \colon \mathrm{Rep}_k G \to \mathrm{Rep}_k \mathbb{G}_m$$

such that

$$\omega \cong \mathcal{E}(-) \circ \omega',$$

by simply setting

$$\omega' := \mathcal{E}_{\mathrm{gr}}(-)^{-1} \circ \mathrm{gr} \circ \tilde{\omega}$$

where

$$\mathcal{E}_{\mathrm{gr}}(-) \colon \mathrm{Rep}_k\mathbb{G}_m \to \{\ \mathcal{E} = \bigoplus_{i\in\mathbb{Z}} \mathcal{E}^i \in \mathrm{GrBun}_{\mathbb{P}^1} \mid \mathcal{E}^i \text{ semistable of slope } i\},$$

is the the equivalence of Lemma 2.3. □

Let

$$\omega^{\mathbb{G}_m}_{\mathrm{can}} \colon \mathrm{Rep}_k(\mathbb{G}_m) \to \mathrm{Vec}_k,\ V \mapsto V$$

be the canonical fiber functor of $\mathrm{Rep}_k(\mathbb{G}_m)$ over k. Composing with $\omega^{\mathbb{G}_m}_{\mathrm{can}}$ defines a morphism

$$\Phi \colon \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m)) \to \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Vec}_k)$$

of groupoids, where the right hand side denotes the groupoid of exact tensor functors

$$\mathrm{Rep}_k(G) \to \mathrm{Vec}_k,$$

which by Lemma 3.1 identifies with the groupoid of G-torsors on $\mathrm{Spec}(k)$. Geometrically, the morphism Φ can be identified on isomorphisms classes with the map

$$i^* \colon H^1_{\text{ét}}(\mathbb{P}^1_k, G) \to H^1_{\text{ét}}(\mathrm{Spec}(k), G)$$

restricting a G-torsor over $\mathbb{P}^1_k$ to a G-torsor over $\mathrm{Spec}(k)$ along a k-rational point $x \in \mathbb{P}^1_k(k)$.

In the following lemma we analyze the fibers of this functor Φ.

Lemma 3.4. *Let $\omega \colon \mathrm{Rep}_k(G) \to \mathrm{Vec}_k$ be an exact tensor functor and let*

$$H := \mathrm{Aut}^{\otimes}(\omega)$$

be the pure inner form of G defined by ω. Then the fiber

$$\Phi^{-1}(\omega) \subseteq \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k G, \mathrm{Rep}_k \mathbb{G}_m)$$

is equivalent to the quotient groupoid

$$[\mathrm{Hom}(\mathbb{G}_m, H)/H(k)]$$

of cocharacters of H. Moreover, passing to isomorphism classes yields a bijection

$$\mathrm{Hom}(\mathbb{G}_m, H)/H(k) \cong H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, H).$$

Proof. The first statement follows from the Tannakian formalism (cf. [Del90]). Namely, ω defines an equivalence

$$\mathrm{Rep}_k(G) \cong \mathrm{Rep}_k(H),\ V \mapsto \omega(V)$$

and the groupoid of exact tensor functors

$$\mathrm{Rep}_k H \to \mathrm{Rep}_k \mathbb{G}_m$$

which commute (with a given isomorphism) with the canonical fiber functors on $\mathrm{Rep}_k H$ resp. $\mathrm{Rep}_k \mathbb{G}_m$ is equivalent to the quotient groupoid

$$[\mathrm{Hom}(\mathbb{G}_m, H)/H(k)].$$

with $H(k)$ acting by conjugation. Clearly, for every cocharacter

$$\chi\colon \mathbb{G}_m \to H$$

the push forward

$$\eta \times^{\mathbb{G}_m} H$$

is an H-torsor, which is locally trivial in the Zariski topology, because this is true for the Hopf bundle

$$\eta\colon \mathbb{A}^2_k \setminus \{0\} \to \mathbb{P}^1_k.$$

Let conversely $\mathcal{P}$ be an H-torsor over $\mathbb{P}^1_k$ which is trivial for the Zariski topology and let

$$\omega_{\mathcal{P}}\colon \mathrm{Rep}_k H \to \mathrm{Bun}_{\mathbb{P}^1_k}, V \mapsto \mathcal{P} \times^H (V \otimes_k \mathcal{O}_{\mathbb{P}^1_k})$$

be the induced fiber functor (cf. Lemma 3.1). Let $x \in \mathbb{P}^1_k(k)$ be a point a k-rational point and let $U \subseteq \mathbb{P}^1_k$ be open subset containing $x \in U$ such that

$$\mathcal{P}_{|U}$$

is trivial. Then the exact tensor functor

$$\mathrm{Rep}_k H \xrightarrow{\omega_{\mathcal{P}}} \mathrm{Bun}_{\mathbb{P}^1_k} \xrightarrow{\mathrm{res}} \mathrm{Bun}_U$$

is isomorphic to the trivial fiber functor. This holds then also true after restricting to $x \in U$. Let

$$\varphi\colon \mathrm{Rep}_k H \to \mathrm{Rep}_k \mathbb{G}_m$$

be an exact tensor functor such that

$$\mathcal{E}(-) \circ \varphi \cong \omega_{\mathcal{P}}.$$

We can conclude that φ preserves the canonical fiber functors on $\mathrm{Rep}_k H$ resp. $\mathrm{Rep}_k \mathbb{G}_m$ because the composition

$$\mathrm{Rep}_k \mathbb{G}_m \xrightarrow{\mathcal{E}(-)} \mathrm{Bun}_{\mathbb{P}^1_k} \xrightarrow{\mathrm{res}} \mathrm{Bun}_x \cong \mathrm{Vec}_k$$

is the canonical fiber functor. In particular, there exists a cocharacter

$$\chi\colon \mathbb{G}_m \to H$$

such that $\mathcal{P}$ is obtained via pushout along χ of the Hopf bundle

$$\eta\colon \mathbb{A}^2_k \setminus \{0\} \to \mathbb{P}^1_k.$$

□

Note that we have actually shown that a G-torsor $\mathcal{P}$ is already locally trivial for the Zariski topology if there exists some open $U \subseteq \mathbb{P}^1_k$ containing a k-rational point, such that $\mathcal{P}_{|U}$ is trivial. The classification results of Grothendieck and Harder on torsors on $\mathbb{P}^1_k$ (cf. [Gro57] resp. [Har68]) are most concretely stated in the collowing form.

Corollary 3.5. *Let k be a field and let G/k be a reductive group with maximal split subtorus $A \subseteq G$. Then there exist canonical bijections*

$$X_*(A)_+ \cong \mathrm{Hom}(\mathbb{G}_m, G)/G(k) \cong H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, G),$$

where $X_*(A)_+$ denotes the set of dominant cocharacters of $A \subseteq G$.

Proof. By Lemma 3.4 it suffices to show

$$X_*(A)_+ \cong \mathrm{Hom}(\mathbb{G}_m, G)/G(k).$$

But this follows from the fact that all maximal split tori in G are conjugated over k and that the set of dominant cocharacters form a system of representatives for the action of the normalizer $N_G(A)$ of A in G on the group $X_*(A)$ of cocharacters for A. □

A description of $H^1_{\text{ét}}(\mathbb{P}^1_k, G)$, similar to the one of us, can be found in [Gil02].

4. Applications

In this section we present some applications of the classification of torsors (following (cf. [Far], which discusses analogous applications to the Fargues-Fontaine curve).

The first application is the computation of the Brauer group of $\mathbb{P}^1_k$. For this we recall the theorem of Steinberg (cf. [Ser02, Chapter 3.2.3]). If k is a field of cohomological dimension $\mathrm{cd}(k) \leq 1$, then Steinberg's theorem states that

$$H^1_{\text{ét}}(\mathrm{Spec}(k), G) = 1$$

for every smooth connected affine algebraic group G/k. In particular, the Brauer group

$$\mathrm{Br}(k) = 0$$

of such fields vanishes. For example, separably closed or finite fields are of cohomological dimension ≤ 1.

Theorem 4.1. *If k is of cohomological dimension $\mathrm{cd}(k) \leq 1$, then the Brauer group*

$$\mathrm{Br}(\mathbb{P}^1_k) \cong H^2_{\text{ét}}(\mathbb{P}^1_k, \mathbb{G}_m) = 0$$

vanishes.

Proof. By [Gro95, Corollaire 2.2.] there is an isomorphism

$$\mathrm{Br}(\mathbb{P}^1_k) \cong H^2_{\text{ét}}(\mathbb{P}^1_k, \mathbb{G}_m)$$

of the Brauer group $\mathrm{Br}(\mathbb{P}^1_k)$ parametrizing equivalence classes of Azumaya algebras over $\mathcal{O}_{\mathbb{P}^1_k}$ with the cohomological Brauer group $H^2_{\text{ét}}(\mathbb{P}^1_k, \mathbb{G}_m)$. It suffices to show that for every $n \geq 0$ the canonical map

$$H^1_{\text{ét}}(\mathbb{P}^1_k, \mathrm{PGL}_n) \to H^2_{\text{ét}}(\mathbb{P}^1_k, \mathbb{G}_m)$$

arising as a boundary map of the short exact sequence

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

is trivial. Because k is of cohomological dimension ≤ 1, there exists using Steinberg's theorem in the case $G = \mathrm{GL}_n$ or $G = \mathrm{PGL}_n$ and Theorem 3.3 together with Lemma 3.4 a commutative diagram

$$\begin{array}{ccc} H^1_{\text{ét}}(\mathbb{P}^1_k, \mathrm{GL}_n) & \longrightarrow & H^1_{\text{ét}}(\mathbb{P}^1_k, \mathrm{PGL}_n) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}(\mathbb{G}_m, \mathrm{GL}_n)/\mathrm{GL}_n(k) & \longrightarrow & \mathrm{Hom}(\mathbb{G}_m, \mathrm{PGL}_n)/\mathrm{PGL}_n(k). \end{array}$$

It suffices to show that the top horizontal arrow, or equivalently the lower horizontal arrow, is surjective. But every cocharacter

$$\chi\colon \mathbb{G}_m \to \mathrm{PGL}_n$$

can be lifted to GL_n because for the standard torus $T \cong \mathbb{G}_m^n \subseteq \mathrm{GL}_n$ there is a split exact sequence

$$0 \to X_*(\mathbb{G}_m) \to X_*(T) \to X_*(T/\mathbb{G}_m) \to 0$$

on cocharacter groups where $T/\mathbb{G}_m$ is a maximal torus of PGL_n. □

For a general field k, i.e., k not necessarily of cohomological dimension ≤ 1, the Brauer group of $\mathbb{P}^1_k$ is given by

$$\mathrm{Br}(\mathrm{Spec}(k)) \cong \mathrm{Br}(\mathbb{P}^1_k)$$

as can be calculated from Theorem 4.1 using the spectral sequence

$$E_2^{pq} = H^p(\mathrm{Gal}(\bar{k}/k), H^q_{\text{ét}}(\mathbb{P}^1_{\bar{k}}, \mathbb{G}_m)) \Rightarrow H^{p+q}_{\text{ét}}(\mathbb{P}^1_k, \mathbb{G}_m)$$

where $\bar{k}$ denotes a separable closure of k.

The next application we give is to the uniformization of G-torsors.

Theorem 4.2. *Let k be a field and let G be reductive group over k. If $x \in \mathbb{P}^1_k(k)$ is k-rational point, then every G-torsor*

$$\mathcal{P} \in H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, G)$$

which is locally trivial for the Zariski topology becomes trivial on $\mathbb{P}^1_k \setminus \{x\}$.

Proof. By Corollary 3.5 we know that every such G-torsor $\mathcal{P}$ is isomorphic to the pushout

$$\mathcal{P} \cong \eta \times^{\mathbb{G}_m} G$$

along a cocharacter

$$\chi\colon \mathbb{G}_m \to G$$

of the canonical $\mathbb{G}_m$-torsor

$$\eta\colon \mathbb{A}^2_k \setminus \{0\} \to \mathbb{P}^1_k$$

corresponding to the line bundle $\mathcal{O}_{\mathbb{P}^1_k}(-1)$ on $\mathbb{P}^1_k$. But

$$\mathcal{O}_{\mathbb{P}^1_k}(-1)_{|\mathbb{P}^1_k \setminus \{x\}}$$

is trivial because $\mathbb{P}^1_k \setminus \{x\} \cong \mathbb{A}^1_k$. This shows the claim. □

Finally, we reprove the Birkhoff-Grothendieck decomposition of $G(k((t))$ for a reductive group G over k (cf. [Fal03, Lemma 4]).

Theorem 4.3. *Let $A \subseteq G$ be a maximal split torus in G. Then there exists a canonical bijection*

$$X_*(A)_+ \cong G(k[t^{-1}])\backslash G(k((t)))/G(k[[t]]),$$

where $X_*(A)_+$ denotes the set of dominant cocharacters of $A \subseteq G$.

Proof. Let $x \in \mathbb{P}^1_k(k)$ be a k-rational point. By Beauville-Laszlo [BL95] and Lemma 3.1 there is an injective map

$$\gamma \colon G(k[t^{-1}])\backslash G(k((t)))/G(k[[t]]) \to H^1_{\text{ét}}(\mathbb{P}^1_k, G)$$

by glueing the trivial G-torsor on $\mathbb{P}^1_k \setminus \{x\}$ with the trivial G-torsor on the formal completion

$$\mathrm{Spec}(\widehat{\mathcal{O}}_{\mathbb{P}^1_k,x})$$

along an isomorphism on $\mathrm{Spec}(\mathrm{Frac}(\widehat{\mathcal{O}}_{\mathbb{P}^1_k,x}))$. Note that $\widehat{\mathcal{O}}_{\mathbb{P}^1_k,x} \cong k[[t]]$. From the remark following Lemma 3.4 we can conclude that the G-torsors obtained in this way are actually locally trivial for the Zariski topology. By Theorem 4.2 we can conversely see that the image of γ contains the set $H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, G)$. Using Corollary 3.5 we can conclude that

$$G(k[t^{-1}])\backslash G(k((t)))/G(k[[t]]) \cong H^1_{\mathrm{Zar}}(\mathbb{P}^1_k, G) \cong X_*(A)_+.$$

□

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