Autoequivalences of derived categories of local K3 surfaces

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1 Introduction

Let X be a smooth projective K3 surface over some field k and let $C \subseteq X$ be a (-2)-curve on X, i.e. $C \cong \mathbb{P}^1_k$ and $C.C = -2$. The sheaves $\mathcal{O}_C(i) \in \mathcal{D}^b(X), i \in \mathbb{Z}$, are then spherical objects in the (bounded) derived category $\mathcal{D}^b(X)$ of X (see 2.9). By definition, an object $E \in \mathcal{D}^b(X)$ is called spherical if

$$\operatorname{Ext}^p_X(E,E) \cong \begin{cases} k, & p = 0,2 \\ 0, & \text{otherwise} \end{cases}$$

To every spherical object $E \in \mathcal{D}^b(X)$ one can associate a spherical twist

$$T_E : \mathcal{D}^b(X) \longrightarrow \mathcal{D}^b(X),$$

which is an exact autoequivalence of $\mathcal{D}^b(X)$ (see 2.14). On some object $F \in \mathcal{D}^b(X)$ the twist $T_E(F)$ is defined via a distinguished triangle

$$\operatorname{Ext}^*_X(E,F) \otimes_k E \longrightarrow F \longrightarrow T_E(F) \longrightarrow *[1].$$

So every (-2)-curve $C \subseteq X$ gives rise to autoequivalences

$$T_{\mathcal{O}_C(i)} \in \operatorname{Aut}(\mathcal{D}^b(X)), i \in \mathbb{Z},$$

of the derived category $\mathcal{D}^b(X)$. It is one aim of this master thesis to describe the subgroup

$$\langle T_{\mathcal{O}_C(i)}, i \in \mathbb{Z} \rangle \subseteq \operatorname{Aut}(\mathcal{D}^b(X))$$

by generators and relations. More generally, we consider some family $E_i \in \mathcal{D}, i \in \Gamma$, of spherical objects in a k-linear K3 category $\mathcal{D}$ (see 2.2) and try to present the subgroup

$$\langle T_{E_i}, i \in \Gamma \rangle \subseteq \operatorname{Aut}(\mathcal{D})$$

by generators and relations. In general, this task seems too hard, so we restrict ourselves to families of spherical objects with prescribed combinatorial data. These families are called Γ-configurations (see 2.16), where Γ denotes some (possibly multi-edged) graph. Let Γ be such a graph. Then a family $(E_i \mid i \in \Gamma)$ of spherical objects in a triangulated category $\mathcal{D}$ is called a Γ-configuration if for $i, j \in \Gamma, i \neq j$, we have:

$$\dim_k \operatorname{Hom}_{\mathcal{D}}(E_i, E_j[p]) = \begin{cases} \text{number of edges in } \Gamma \text{ from i to j}, & p = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We will mainly consider the case where Γ is a Dynkin diagram of type ADE or an affine Dynkin diagram of type $\tilde{\mathrm{A}}$. In [ST01] P. Seidel and R. Thomas show that in the case of an A_n-configuration $(E_i \mid i \in \mathrm{A}_n)$ the subgroup

$$\langle T_{E_i}, i \in \mathrm{A}_n \rangle \subseteq \operatorname{Aut}(\mathcal{D})$$

is isomorphic to the braid group B_{A_n} of type A_n (see 2.19 for a definition of B_{A_n}). After giving an introduction to spherical twists in chapter 2 we present a proof of C. Brav and H. Thomas ([BT11]) for a generalization of this result in type ADE (see chapter 3, theorem 3.16):

Theorem 1.1. *Let $\mathcal{D}$ be a k-linear K3 category. If Γ is a Dynkin diagram of type* ADE *and $(E_i \in \mathcal{D} \mid i \in \Gamma)$ a Γ-configuration, then the subgroup*

$$\langle T_{E_i}, i \in \Gamma \rangle \subseteq \mathrm{Aut}(\mathcal{D})$$

is isomorphic to the braid group B_Γ of type Γ.

In chapter 4 we investigate the case of a Γ-configuration for the affine Dynkin diagram $\Gamma = \tilde{\mathrm{A}}_1$. We derive the following theorem (see chapter 4, theorem 4.16):

Theorem 1.2. *Let $\mathcal{D}$ be a k-linear K3 category. If Γ is the affine Dynkin diagram of type $\tilde{\mathrm{A}}_1$ and $(E_i \in \mathcal{D} \mid i \in \Gamma)$ a Γ-configuration, then the subgroup*

$$\langle T_{E_i}, i \in \Gamma \rangle \subseteq \mathrm{Aut}(\mathcal{D})$$

is isomorphic to the braid group B_Γ of type Γ.

Recall that the braid group $B_{\tilde{\mathrm{A}}_1}$ of type $\tilde{\mathrm{A}}_1$ is just a free group on two generators (see 2.19). But for the proof of this theorem the following presentation of $B_{\tilde{\mathrm{A}}_1}$ turns out to be more useful:

$$B_{\tilde{\mathrm{A}}_1} \cong \langle s_i, i \in \mathbb{Z} \mid s_{i-1}s_i = s_{j-1}s_j, i, j \in \mathbb{Z} \rangle$$

(see 4.3 and [McC05, example 4.8]). As a corollary, we obtain (see 2.17,4.3 and 4.16):

Corollary 1.3. *Let X be a K3 surface over k and $C \subseteq X$ a (-2)-curve on X. Then the subgroup*

$$\langle T_{\mathcal{O}_C(i)} \mid i \in \mathbb{Z} \rangle \subseteq \mathrm{Aut}(\mathcal{D}^b(X))$$

is freely generated by $T_{\mathcal{O}_C}$ and $T_{\mathcal{O}_C(1)}$.

Chapter 5 deals with another question. Theorem 4.16 from chapter 4 raises the question, whether for two $\tilde{\mathrm{A}}_1$-configurations (E_0, E_1) and (E'_0, E'_1) in k-linear K3 categories $\mathcal{D}$ and $\mathcal{D}'$ the triangulated categories $\langle E_0, E_1 \rangle$ and $\langle E'_0, E'_1 \rangle$ are actually isomorphic. (Here, $\langle E_0, E_1 \rangle, \langle E'_0, E'_1 \rangle$ denote the smallest full, triangulated subcategories in $\mathcal{D}$ and $\mathcal{D}'$ containing E_0 and E_1 resp. E'_0 and E'_1.) We answer this question in chapter 5 by showing the following theorem. To state it, we define the semi-simple ring $R := k \times k$, which admits an automorphism

$$\sigma : R \longrightarrow R$$

permuting the two idempotents $e_1 := (1, 0)$ and $e_2 := (0, 1)$. We denote by $M := R^2$ the R-R-bimodule R^2 with usual left multiplication and σ-twisted right multiplication of R. Let $A := \Lambda^*(M)$ and view this as a dg-algebra (see 5.1) over R with trivial differential (for a precise definition of A, see 5.2).

Theorem 1.4. *Let $\mathcal{D}$ be a full triangulated category of the derived category of a K3 surface. Assume that (E_0, E_1) is an $\tilde{\mathrm{A}}_1$-configuration in $\mathcal{D}$. Then there is a fully, faithful functor*

$$F : \langle E_0, E_1 \rangle \longrightarrow \mathcal{D}(A)$$

into the derived category of A sending $E_i, i = 0, 1,$ to the right module $e_{i+1}A$. Hence, we have an equivalence

$$\langle E_0, E_1 \rangle \xrightarrow{\simeq} \langle e_1A, e_2A \rangle$$

of $\langle E_0, E_1 \rangle$ with the smallest full, triangulated subcategory $\langle e_1A, e_2A \rangle$ of $\mathcal{D}(A)$ containing e_1A and e_2A.

Hence the isomorphism type of $\langle E_0, E_1 \rangle$ is indeed independent of the particular $\tilde{\mathrm{A}}_1$-configuration (E_0, E_1). The proof of this theorem uses Keller's classification of algebraic triangulated categories via derived categories of dg-categories. The main step in the proof of theorem 5.21 is that we show in theorem 5.20 that the twisted exterior algebra A is Koszul and hence intrinsically formal (theorem 5.19). We thank Prof. Catharina Stroppel for suggesting this possibility to us.

In our last chapter 6 we discuss a geometric example for the case $\Gamma = \tilde{\mathrm{A}}_2$ and derive a partial result on faithfulness for the affine braid group $B := B_{\tilde{\mathrm{A}}_2}$. As the example, we take two (-2)-curves $C, D \subseteq X$ on a K3 surface X meeting in a single point $x \in X$ and look at the Γ-configuration

$$(\mathcal{O}_C, \mathcal{O}_D, \mathcal{O}_{C \cup D}[1])$$

(see 6.1). Our partial result 6.8 can be stated as follows (for the definition of strict decreasing see 6.5):

Proposition 1.5. *Let $\alpha \in B$ be strict decreasing, $\alpha \neq 1$. Then α does not act as the identity on $\mathcal{D}^b(X)$.*

To derive this partial result we prove the following presentation of the braid group B:

$$B \cong \langle a_i, b_j, i, j \in \mathbb{Z} \mid a_j b_i a_j = b_i a_j b_i,\ a_{i-1} a_i = a_{j-1} a_j,\ b_{i-1} b_i = b_{j-1} b_j,$$

$$a_0 a_1 b_j = b_{j+1} a_0 a_1,\ b_0 b_1 a_j = a_{j+1} b_0 b_1,\ i, j \in \mathbb{Z} \rangle$$

(see 6.2).

Finally, we want to mention a paper of [Kea12] on faithfulness of braid group actions of $\tilde{\mathrm{A}}_1$-configurations on symplectic manifolds. For more informations about Homological Mirror Symmetry and the analogy between spherical twists and generalized Dehn twists we refer to [Huy06, chapter 13.2] and [ST01, chapter 1.c)].

Acknowledgements: Prof. Dr. Daniel Huybrechts deserves a special thank for offering me this well chosen topic. It was interesting because of its link to current research, i.e. Bridgeland's conjecture, but still accessible for me. Moreover, I want to thank him for his constant help in preparing this master thesis and for carefully reading a preliminary version of this thesis. For explaining Keller's theorem to me I want to thank Hanno Becker. A crucial point in chapter 5 is theorem 5.19, it was told to me by Prof. Dr. Catharina Stroppel and I am deeply grateful for her explanations.

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2 Spherical twists

In this chapter we briefly introduce the necessary background on K3 categories, spherical objects and spherical twists needed for this master thesis. We try to give several examples illuminating these notions.

2.1 Preliminaries on triangulated categories

If $\mathcal{D}$ is a triangulated category over a field k, we denote by $[n]$ the n-th power of the shift functor

$$[1] : \mathcal{D} \longrightarrow \mathcal{D}.$$

For two objects E, F in $\mathcal{D}$ and $p \in \mathbb{Z}$ we will write $[E, F]_p$ instead of $\mathrm{Hom}_{\mathcal{D}}(E, F[p])$. We also set

$$[E, F]_* := \bigoplus_{p \in \mathbb{Z}} [E, F]_p[-p],$$

which we view as a complex in the derived category $\mathcal{D}(k)$ of k.

By definition, a K3 surface X over k will mean a smooth projective surface over k, such that $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

Definition 2.1. *Let X be K3 surface over a field k. Then we define the* **derived category of X** *as*

$$\mathcal{D}^b(X) := \mathcal{D}^b(Coh(X)).$$

So $\mathcal{D}^b(X)$ is the bounded derived category of the abelian category of coherent sheaves on X.

Our examples of triangulated categories will be derived categories of K3 surface over some field k and full triangulated subcategories of such. So our triangulated categories will have special properties, which can be summarized in the abstract notion of a K3 category. Let us denote by $V^\vee$ the k-dual of a k-vector space V.

Definition 2.2. *Let $\mathcal{D}$ be a k-linear triangulated category. An autoequivalence*

$$S : \mathcal{D} \longrightarrow \mathcal{D}$$

is called a **Serre functor** *for $\mathcal{D}$ if there exists isomorphisms*

$$\Phi_{E,F} : [E, F]_0 \xrightarrow{\simeq} [F, S(E)]_0^\vee$$

for any $E, F \in \mathcal{D}$, which are natural in E and F. We call $\mathcal{D}$ a **K3 category** *if the double shift*

$$[2] : \mathcal{D} \longrightarrow \mathcal{D}$$

is a Serre functor, i.e. for $E, F \in \mathcal{D}$ there are natural isomorphisms

$$[E, F]_0 \longrightarrow [F, E]_2^\vee.$$

As the terminology suggests, derived categories of K3 surfaces provide examples of K3 categories.

Example 2.3. 1. *Let X be a K3 surface. Then $\mathcal{D}^b(X)$ is a K3 category. Indeed, [Huy06, theorem 3.12] shows that the composition*

$$\omega_X \otimes_{\mathcal{O}_X} (-) \circ (-[2]) : \mathcal{D}^b(X) \longrightarrow \mathcal{D}^b(X)$$

of $\omega_X \otimes_{\mathcal{O}_X} (-)$ and the double shift $[2]$ is a Serre functor for $\mathcal{D}^b(X)$. But $\omega_X \cong \mathcal{O}_X$, so $[2]$ is a Serre functor for $\mathcal{D}^b(X)$ and $\mathcal{D}^b(X)$ is a K3 category.

2. *If $\mathcal{D}$ is a K3 category, then every full triangulated subcategory of $\mathcal{D}$ is again a K3 category. This is immediate from the definitions. In particular, we see that there are K3 categories, which do not arise as the derived category of a K3 surface. (For example, one can use that the K-theory of a K3 surface cannot be generated by two elements.)*

3. *The derived category $\mathcal{D}^b(A)$ of an abelian surface A is also a K3 category as $\omega_A \cong \mathcal{O}_A$ is trivial. Using Orlov's result on the existence of Fourier–Mukai-kernels ([Huy06, theorem 5.12]) one can show that $\mathcal{D}^b(A)$ is not equivalent to a full subcategory of the derived category $\mathcal{D}^b(X)$ of a K3 surface X. In fact, any fully faithful Fourier–Mukai-transform $\mathcal{D}^b(A) \longrightarrow \mathcal{D}^b(X)$ would be an equivalence (see [Huy06, proposition 7.6]), which is impossible ([Huy06, corollary 10.2]).*

Every triangulated category comes with its group of autoequivalences.

Definition 2.4. *Let $\mathcal{D}$ be a k-linear triangulated category. We denote by*

$$\operatorname{Aut}(\mathcal{D}) := \{\ \Phi : \mathcal{D} \longrightarrow \mathcal{D} \mid \Phi \text{ is a } k\text{-linear, exact equivalence } \}/\sim$$

the **group of autoequivalences** *of $\mathcal{D}$. Two autoequivalences are identified in $\operatorname{Aut}(\mathcal{D})$ if they are isomorphic as exact functors.*

In 2.3 we will encounter spherical twists, which are autoequivalences of triangulated categories. In a geometric example, i.e. $\mathcal{D} = \mathcal{D}^b(X)$ for X some K3 surface, the shift functor, automorphisms of X and tensoring with line bundles on X yield a subgroup

$$\mathbb{Z} \times \operatorname{Aut}(X) \ltimes \operatorname{Pic}(X) \subseteq \operatorname{Aut}(\mathcal{D}).$$

2.2 Spherical objects

For this section fix a field k and a k-linear K3 category $\mathcal{D}$. All schemes in this chapter are assumed to be schemes over k.

Definition 2.5. *An object $E \in \mathcal{D}$ is called* **spherical** *(or better* **2-spherical***), if*

$$[E, E]_* \cong k \oplus k[-2],$$

that is $[E, E]_p \cong k$ if $p = 0, 2$ and $[E, E]_p = 0$ otherwise.

Remark 2.6. 1. *For a general definition of spherical objects in triangulated categories see [BT11, definition 3.1].*

2. *Slightly rewritten, an object $E \in \mathcal{D}$ is spherical if and only if its* Ext*-algebra $[E, E]_*$ is isomorphic (as a graded algebra) to the singular cohomology of the 2-sphere:*

$$[E, E]_* \cong H^*_{sing}(S^2, k).$$

This is a first motivation for the terminology "spherical".

3. *As the double shift in $\mathcal{D}$ is a Serre functor, a spherical object $E \in \mathcal{D}$ has the smallest possible* Ext*-algebra of non-zero objects in $\mathcal{D}$. In particular, if $\Phi \in \mathrm{Aut}(\mathcal{D})$ is an autoequivalence of $\mathcal{D}$ and $E \in \mathcal{D}$ spherical, then $\Phi(E)$ is again spherical.*

Example 2.7. *Let X be a K3 surface and let $\mathcal{L} \in \mathrm{Pic}(X)$ be a line bundle on X. Then $\mathcal{L}$ is spherical, as*

$$[\mathcal{L}, \mathcal{L}]_* \cong [\mathcal{O}_X, \mathcal{O}_X]_* \cong H^*(X, \mathcal{O}_X) \cong k \oplus k[-2].$$

Another class of spherical objects in the derived category of a K3 surface X, already mentioned in the introduction, is provided by (-2)-curves on X.

Definition 2.8. *Let X be a K3 surface. A curve $C \subsetneq X$ is called a* (-2)**-curve** *if C is isomorphic to $\mathbb{P}^1_k$ and the self.intersection $C.C$ equals -2.*

If X is a K3 surface, then an irreducible curve $C \subsetneq X$ (smooth or not) is smooth and rational if and only if the self -intersection number $C.C = \deg_C(\mathcal{O}_C(C))$ is -2 (see [Har77, exercise V.1.3]). Hence an integral curve C on X is a (-2)-curve if and only if $C \cong \mathbb{P}^1 k$ is smooth and rational.

Lemma 2.9. *Let X be a K3 surface and $C \subsetneq X$ a (-2)-curve. Then for every $j \in \mathbb{Z}$ the object $\mathcal{O}_C(j) \in \mathcal{D}^b(X)$ is spherical.*

Proof. Let $E := \mathcal{O}_C(j)$. Then clearly, $[E, E]_0 \cong k$ and hence (by Serre duality) $[E, E]_2 \cong k$. Moreover, $[E, E]_p = 0$ for $p \notin \{0, 1, 2\}$. Finally, the Riemann–Roch formula for surfaces (see [Har77, theorem 1.6]) implies that

$$\chi(E, E) := \sum_{p \in \mathbb{Z}} (-1)^p \dim_k [E, E]_p = -C.C = 2.$$

Hence $[E, E]_1 = 0$ and E is spherical. □

Remark 2.10. *The following converse to lemma 2.9 also holds: If k is algebraically closed and $C \subsetneq X$ is an integral curve, such that $\mathcal{O}_C \in \mathcal{D}^b(X)$ is spherical, then $C \cong \mathbb{P}^1$. To prove this, we look at the exact sequence*

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

and apply $[\mathcal{O}_C, -]_0$ to get an exact sequence

$$[\mathcal{O}_C, \mathcal{O}_X(-C)]_1 \xrightarrow{\alpha} [\mathcal{O}_C, \mathcal{O}_X]_1 \longrightarrow [\mathcal{O}_C, \mathcal{O}_C]_1 = 0.$$

But the map α vanishes (as it is induced by a global section $s \in \mathcal{O}_X(C)$ vanishing on C), so

$$H^1(C, \mathcal{O}_C) \cong [\mathcal{O}_C, \mathcal{O}_X]_1^\vee = 0$$

and hence $C \cong \mathbb{P}^1_k$.

A fundamental property of spherical objects is the following:

Lemma 2.11. *Let $E \in \mathcal{D}$ be spherical, then for every object $F \in \mathcal{D}$ the composition of morphism yields a perfect pairing*

$$[E,F]_1 \otimes [F,E]_1 \longrightarrow [E,E]_2 \cong k.$$

Proof. See [BT11, remark 3.1] for the case that the triangulated category is arbitrary (i.e. not a K3 category). In our case, $\mathcal{D}$ is a K3 category and thus, by definition, we have functorial isomorphisms

$$\Phi_{E,F} : [E,F]_0 \xrightarrow{\simeq} [F,E]_2^\vee$$

for $F \in \mathcal{D}$. Setting $F = E$ we get a canonical map (called "trace map"):

$$\mathrm{tr}_E := \Phi_{E,E}(\mathrm{Id}_E) : [E,E]_2 \longrightarrow k,$$

which is an isomorphism as E is spherical. By functoriality, the isomorphism $\Phi_{E,F}$ is given by

$$\Phi_{E,F}(f)(g) = \mathrm{tr}_E(gf)$$

for any $F \in \mathcal{D}$ and $f \in [E,F]_1,\ g \in [F,E]_1$. Thus inverting the trace map tr_E yields the result. □

2.3 Spherical twists

As in the previous section, we fix a field k and a k-linear K3 category $\mathcal{D}$. Instead of

$$Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow Y_1[1],$$

we will write

$$Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow *[1]$$

to denote triangles in $\mathcal{D}$.

Definition 2.12. *Let $E \in \mathcal{D}$ be spherical. We define the* **spherical twist** *T_E as a functor*

$$T_E : \mathcal{D} \longrightarrow \mathcal{D}$$

by a distinguished triangle

$$[E,F]_* \otimes E \xrightarrow{\mathrm{eval}} F \longrightarrow T_E(F) \longrightarrow *[1]. \tag{1}$$

Remark 2.13. *1. The object $[E,F]_* \otimes E$ is defined in [ST01, chapter 2.a)] and equals $\bigoplus_p [E,F]_p \otimes E[-p]$. A summand $[E,F]_p \otimes E[-p]$ is (by definition) the $\dim_k [E,F]_p$-fold direct sum of copies of $E[-p]$. The evaluation map*

$$\mathrm{eval} : [E,F]_p \otimes E[-p] \longrightarrow F$$

is then defined as follows: Let $\delta \in [E,F]_p$. Then δ defines a morphism

$$\delta : E[-p] \longrightarrow F$$

and on the summand $\delta \otimes E[-p] = E[-p]$ of $[E,F]_p \otimes E[-p]$ the evaluation map is given by δ.

2. *The above definition of the twist functor T_E is sloppy and ignores some difficulties concerning to the non-functoriality of mapping cones in triangulated categories. Also, our definition defines T_E not on morphisms and only up to non-unique isomorphisms. However, these problems can be solved by the additional assumption that $\mathcal{D}$ comes with a fixed enhancement (to get functorial cones), see [BT11, chapter 3]. In the case of a full triangulated subcategory of the derived category of a K3 surface such an enhancement can always be found. We will from now on assume that our triangulated category $\mathcal{D}$ comes with some fixed enhancement and do not pursue this topic any further - for our purposes, the above triangle is enough. Actually, we will not need to know how T_E is defined on morphisms.*

3. *If $\mathcal{D} = \mathcal{D}^b(X)$ is the derived category of a K3 surface, then spherical twists can be defined using a Fourier–Mukai-kernel. This circumvents the necessity of some enhancement (see [Huy06, definition 8.3]).*

4. *In the paper [ST01] P. Seidel and R. Thomas explain the origin of spherical twists. Under Homological Mirror Symmetry spherical twists correspond to generalized Dehn twists along Lagrangian spheres. This is the second motivation for the terminology "spherical".*

The following theorem is due to P. Seidel and R. Thomas (see [ST01, theorem 1.2]).

Theorem 2.14. *Let $E \in \mathcal{D}$ be a spherical object. Then the spherical twist T_E is an autoequivalence of $\mathcal{D}$.*

We summarize some basic properties of spherical twists.

Proposition 2.15. *Let $E \in \mathcal{D}$ be a spherical object. Then we have the following:*

1) $T_E(E) \cong E[-1]$ and $T_E(F) \cong F$ if $[E, F]_* = 0$.

2) If $\Phi : \mathcal{D} \longrightarrow \mathcal{D}$ is an exact autoequivalence, then $T_{\Phi(E)} \cong \Phi \circ T_E \circ \Phi^{-1}$.

3) $T_{E[1]} \cong T_E$.

Proof. The first assertion is elementary, but we give a full proof showing how the triangle (1) can be used to compute spherical twists. Let $0 \neq \beta \in [E, E]_2$. Then

$$[E, E]_* \otimes E = \mathrm{Id}_E \otimes E \oplus \beta \otimes E[-2] \cong E \oplus E[-2]$$

and the triangle (1) is isomorphic to the triangle:

$$E \oplus E[-2] \overset{(\mathrm{Id}_E, \beta[-2])}{\longrightarrow} E \longrightarrow T_E(E) \longrightarrow *[1],$$

which is isomorphic to

$$E \oplus E[-2] \overset{(\mathrm{Id}_E, 0)}{\longrightarrow} E \longrightarrow T_E(E) \longrightarrow *[1]$$

via the automorphism

$$\begin{pmatrix} \mathrm{Id}_E & \beta[-2] \\ 0 & \mathrm{Id}_{E[-2]} \end{pmatrix}$$

of $E \oplus E[-2]$. Hence,

$$T_E(E)[-1] \cong E[-2]$$

and thus $T_E(E) \cong E[-1]$. The second assertion can be found in [Huy06, lemma 8.21]. Finally, the last assertion follows easily from 2) as T_E is exact or directly from the triangle (1). $\square$

Our main interest will be in the interplay of several spherical twists. Namely, we will consider some special cases of the following general definition.

Definition 2.16. *Let Γ be an (undirected, possibly multi-edged) graph and let $E_i \in \mathcal{D}, i \in \Gamma$, be spherical objects labeled by the vertices of Γ. Then $(E_i \mid i \in \Gamma)$ is said to form a* **Γ-configuration** *if for $i, j \in \Gamma, i \neq j$, we have*

$$\dim_k [E_i, E_j]_p = \begin{cases} \text{number of edges in } \Gamma \text{ from } i \text{ to } j, & p = 1 \\ 0, & \text{otherwise} \end{cases}$$

We will immediately restrict ourselves to the case where Γ is a Dynkin diagram of type ADE or an affine Dynkin diagram of type $\tilde{\mathrm{A}}$.

Example 2.17. *Let X be a K3 surface and $C \subseteq X$ a (-2)-curve.*

1) *Then*

$$[\mathcal{O}_X, \mathcal{O}_C]_* \cong H^*(C, \mathcal{O}_C) \cong k,$$

so $(\mathcal{O}_X, \mathcal{O}_C[-1])$ form an A_2-configuration. Similarly, $(\mathcal{O}_X, \mathcal{O}_C(1)[-1])$ form an $\tilde{\mathrm{A}}_1$-configuration.

2) *For $j \in \mathbb{Z}$ the two spherical objects $(\mathcal{O}_C(j), \mathcal{O}_C(j+1)[-1])$ form an $\tilde{\mathrm{A}}_1$-configuration as*

$$[\mathcal{O}_C(j), \mathcal{O}_C(j+1)]_* \cong k^2,$$

which can be calculated as in 2.9. This example will occupy us in chapter 4.

3) *Let $D \subseteq X$ be another (-2)-curve such that $C \cap D$ is a 0-dimensional subscheme of X of length n. Then for $i, j \in \mathbb{Z}$*

$$[\mathcal{O}_C(i), \mathcal{O}_D(j)]_* \cong k^n[-1].$$

In fact, the local-to-global spectral sequence given by

$$E_2^{pq} = H^p(X, \mathcal{E}xt^q_X(\mathcal{O}_C(i), \mathcal{O}_D(j))) \Rightarrow \mathrm{Ext}^{p+q}_X(\mathcal{O}_C(i), \mathcal{O}_D(j))$$

shows that $\mathrm{Ext}^*_X(\mathcal{O}_C(i), \mathcal{O}_D(j)) \cong H^0(X, \mathcal{E}xt^*_X(\mathcal{O}_C(i), \mathcal{O}_D(j)))$ as $\mathcal{E}xt^*_X(\mathcal{O}_C(i), \mathcal{O}_D(j))$ is concentrated in dimension 0. A local computation shows

$$\mathcal{E}xt^*_X(\mathcal{O}_C(i), \mathcal{O}_D(j)) \cong \mathcal{O}_{C \cap D}[-1]$$

and thus

$$[\mathcal{O}_C(i), \mathcal{O}_D(j)]_* = \mathrm{Ext}^*_X(\mathcal{O}_C(i), \mathcal{O}_D(j)) \cong k^n[-1].$$

In particular, if $n = 1$ (resp. $n = 2$), we get an A_2(resp. $\tilde{\mathrm{A}}_1$)-configuration.

4) *Assume that $\pi : X \longrightarrow \mathbb{P}^1$ is an elliptic K3 surface. Then the singular fibers of π yield examples of interesting configurations. Investigating these is one motivation for this master thesis. But we can only describe the $\tilde{\mathrm{A}}_1$-case completely.*

Assume that $(E_i \mid i \in \Gamma)$ is a Γ-configuration in $\mathcal{D}$ and define Fr_Γ to be the free group on the set Γ. We will write $s_i \in \mathrm{Fr}_\Gamma, i \in \Gamma$, for the generator given by i. Then we get a homomorphism

$$T^\Gamma : \mathrm{Fr}_\Gamma \longrightarrow \mathrm{Aut}(\mathcal{D}) \tag{2}$$

by sending $s_i \in \mathrm{Fr}_\Gamma$ to the twist T_{E_i}. It is one main topic of this master thesis to determine for some graphs Γ the kernel of T^Γ in terms of Γ.

Some known relations are provided by the following lemma.

Lemma 2.18. *Let $E_1, E_2 \in \mathcal{D}$ be two spherical objects.*

1) If $[E_1, E_2]_* = 0$, then $T_{E_1}(E_2) \cong E_2$ and

$$T_{E_1}T_{E_2} \cong T_{E_2}T_{E_1}.$$

2) If $[E_1, E_2]_* \cong k[-1]$, then $T_{E_1}T_{E_2}(E_1) \cong E_2$ and

$$T_{E_1}T_{E_2}T_{E_1} \cong T_{E_2}T_{E_1}T_{E_2}.$$

Proof. In the first case proposition 2.15 implies both, $T_{E_1}(E_2) \cong E_2$ and

$$T_{E_1}T_{E_2} \cong T_{T_{E_1}(E_2)}T_{E_1} \cong T_{E_2}T_{E_1}.$$

In general, 2.15 shows that

$$T_{E_1}T_{E_2}T_{E_1} \cong T_{E_1}T_{T_{E_2}(E_1)}T_{E_2} \cong T_{T_{E_1}T_{E_2}(E_1)}T_{E_1}T_{E_2}.$$

If $[E_1, E_2]_* \cong k[-1]$ we claim that

$$T_{E_1}T_{E_2}(E_1) \cong E_2.$$

Indeed, we apply $[E_1, -]_0$ to the distinguished triangle (coming from the triangle (1))

$$E_1 \xrightarrow{f} T_{E_2}(E_1) \longrightarrow E_2 \longrightarrow *[1] \tag{3}$$

defining T_{E_2} and receive two things. First, the map

$$f : E_1 \longrightarrow T_{E_2}(E_1)$$

is non-zero and second $[E_1, T_{E_2}]_* \cong k$ (use lemma 2.11). Thus $T_{E_1}T_{E_2}(E_1)$ is the mapping cone of f, which is E_2 by (3). We can conclude

$$T_{E_1}T_{E_2}T_{E_1} \cong T_{T_{E_1}T_{E_2}(E_1)}T_{E_1}T_{E_2} \cong T_{E_2}T_{E_1}T_{E_2}.$$

□

These known relations motivate the following definition.

Definition 2.19. *Let Γ be an undirected graph. For $i, j \in \Gamma, i \neq j$, let*

$$e_{i,j} := \textit{number of edges in } \Gamma \textit{ joining } i \textit{ and } j.$$

We define the **braid group of type Γ** *as*

$$B_\Gamma := \langle s_i, i \in \Gamma \mid s_i s_j = s_j s_i \textit{ if } e_{i,j} = 0,\ s_i s_j s_i = s_j s_i s_j \textit{ if } e_{i,j} = 1,\ i, j \in \Gamma, i \neq j \rangle.$$

The relations $s_i s_j s_i = s_j s_i s_j$ are usually called braid relations. By 2.18 the homomorphism T^Γ introduced in (2) will factor through B_Γ. We will also write T^Γ to denote the induced homomorphism

$$T^\Gamma : B_\Gamma \longrightarrow \mathrm{Aut}(\mathcal{D}). \tag{4}$$

Note that $B_{\tilde{\mathrm{A}}_1}$ is just a free group on two generators.

We finish this chapter by giving some concrete examples for the action of spherical twists. For this let X be a K3 surface and $C \subseteq X$ a (-2)-curve.

The first example shows that spherical twists do not necessarily preserve the support of an object of $\mathcal{D}^b(X)$. Also, the example shows that in general spherical twists transform sheaves into genuine complexes.

Example 2.20. *Let $x \in X$ be a point and $j \in \mathbb{Z}$.*

1. *If $x \notin C$, then $T_{\mathcal{O}_C(j)}(k(x)) \cong k(x)$.*
2. *If $x \in C$, then the complex $T_{\mathcal{O}_C(j)}(k(x))$ has cohomology sheaves*

$$H^p(T_{\mathcal{O}_C(j)}(k(x))) \cong \begin{cases} \mathcal{O}_C(j-1), & p=-1 \\ \mathcal{O}_C(j), & p=0 \\ 0, & \textit{otherwise} \end{cases}$$

Proof. If $x \notin C$, then $[\mathcal{O}_C(j), k(x)]_* = 0$ and hence $T_{\mathcal{O}_C(j)}(k(x)) \cong k(x)$ by 2.15. If $x \in C$, then

$$[\mathcal{O}_C(j), k(x)]_* \cong k \oplus k[-1]$$

(for example, by the local-to-global spectral sequence for Ext). Thus $T_{\mathcal{O}_C(j)}(k(x))$ is defined by a distinguished triangle

$$\mathcal{O}_C(j) \oplus \mathcal{O}_C(j)[-1] \longrightarrow k(x) \longrightarrow T_{\mathcal{O}_C(j)}(k(x)) \longrightarrow *[1]$$

and the description of the cohomology sheaves follows easily. □

Example 2.21. *Let $j \in \mathbb{Z}$. Then*

$$T_{\mathcal{O}_C(j)}(\mathcal{O}_C(j+1)) \cong \mathcal{O}_C(j-1)[1]$$

and

$$T_{\mathcal{O}_C(j)} T_{\mathcal{O}_C(j+1)}(F) \cong \mathcal{O}_X(C) \otimes_{\mathcal{O}_X} F$$

for $F \in \mathcal{D}^b(X)$. So the composition $T_{\mathcal{O}_C(j)} \circ T_{\mathcal{O}_C(j+1)}$ is isomorphic to the functor given by tensoring with the line bundle $\mathcal{O}_X(C)$.

Proof. We show the first claim and refer to [IU05, lemma 3.15] for the second. Recall the definition of the Euler sequence on $\mathbb{P}^1_k$ (see [Har77, theorem 8.13]):

$$\mathcal{O}_C(-2) \longrightarrow \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1) \longrightarrow \mathcal{O}_C \longrightarrow *[1].$$

Tensoring with $\mathcal{O}_C(j+1)$ yields

$$\mathcal{O}_C(j-1) \longrightarrow \mathcal{O}_C(j) \oplus \mathcal{O}_C(j) \longrightarrow \mathcal{O}_C(j+1) \longrightarrow *[1],$$

so $\mathcal{O}_C(j-1)[1]$ is the mapping cone of the map

$$\mathcal{O}_C(j) \oplus \mathcal{O}_C(j) \longrightarrow \mathcal{O}_C(j+1)$$

But this mapping cone is also $T_{\mathcal{O}_C(j)}(\mathcal{O}_C(j+1))$ as $[\mathcal{O}_C(j), \mathcal{O}_C(j+1)]_* \cong k^2$ (see 2.17.2)). □

Example 2.22. *Assume that* $D \subseteq X$ *is another* (-2)*-curve and that* $C \cap D = \{x\}$ *is a single point. Then*

$$T_{\mathcal{O}_C}(\mathcal{O}_D(-x)) \cong \mathcal{O}_{C \cup D}.$$

In particular, the object $\mathcal{O}_{C\cup D} \in \mathcal{D}^b(X)$ *is spherical.*

Proof. We have a distinguished triangle

$$\mathcal{O}_C[-1] \longrightarrow \mathcal{O}_D(-x) \longrightarrow \mathcal{O}_{C\cup D} \longrightarrow *[1],$$

so $\mathcal{O}_{C\cup D}$ is the mapping cone of a non-zero morphism $\mathcal{O}_C[-1] \longrightarrow \mathcal{O}_D(-x)$. But by example 2.17

$$[\mathcal{O}_C, \mathcal{O}_D(-x)]_* \cong k[-1],$$

so the mapping cone of every non-zero morphism $\mathcal{O}_C[-1] \longrightarrow \mathcal{O}_D(-x)$ is isomorphic to $T_{\mathcal{O}_C}(\mathcal{O}_D(-x))$. Hence

$$T_{\mathcal{O}_C}(\mathcal{O}_D(-x)) \cong \mathcal{O}_{C\cup D}.$$

□

3 The case $\Gamma = \mathrm{ADE}$

We fix a K3 category $\mathcal{D}$ and a graph Γ of type ADE. In this chapter we present a proof of C. Brav and H. Thomas ([BT11, chapter 3]) showing that the homomorphism

$$T^{\Gamma} : B_{\Gamma} \longrightarrow \mathrm{Aut}(\mathcal{D})$$

coming from some Γ-configuration is always injective. For later use in chapter 6 we improve their result slightly.

3.1 A general lemma

In this section we prove a general lemma about the behavior of Hom-spaces under spherical twists. This lemma will be the starting point for the combinatorics needed in section 3.3 and chapter 4, section 4.3.

Let $E_1, E_2 \in \mathcal{D}$ be two spherical objects, we define

$$m := \max\{\ p \mid [E_2, E_1]_p \neq 0\}.$$

Lemma 3.1. *Let $Y \in \mathcal{D}$ and define*

$$l_i := \max\{\ p \mid [E_i, Y]_p \neq 0\},\ \ i = 1, 2.$$

Similarly, define

$$\lambda_i := \max\{\ p \mid [E_i, T_{E_1}(Y)]_p \neq 0\},\ \ i = 1, 2.$$

Then:

1) $\lambda_1 = l_1 + 1$

2) $\lambda_2 \leq \max\{l_2,\ l_1 + m - 1\}$

3) If $\max\{l_2,\ l_1 + m - 1\} = l_1 + m - 1$, *then* $\lambda_2 = l_1 + m - 1$.

4) If $l_2 \geq l_1 + m + 1$, *then* $\lambda_2 = l_2$.

Proof. The first statement is a trivial consequence of $T_{E_1}(E_1) \cong E_1[-1]$ and the fully faithfulness of T_{E_1}:

$$[E_1, T_{E_1}(Y)]_* \cong [T_{E_1}(E_1), T_{E_1}(Y)]_{*-1} \cong [E_1, Y]_{*-1}.$$

For the other statements consider the distinguished triangle

$$[E_1, Y]_* \otimes E_1 \longrightarrow Y \longrightarrow T_{E_1}(Y) \longrightarrow *[1]$$

giving the long exact sequence

$$.. \longrightarrow [E_2, Y]_p \longrightarrow [E_2, T_{E_1}(Y)]_p \longrightarrow \bigoplus_{i \leq m} [E_1, Y]_{p+1-i} \otimes [E_2, E_1]_i \longrightarrow ... \qquad (5)$$

Let $p > \max\{l_2,\ l_1 + m - 1\}$. Then $[E_2, Y]_p = 0$ and

$$\bigoplus_{i \leq m} [E_1, Y]_{p+1-i} \otimes [E_2, E_1]_i = 0$$

as $p+1-i > l_1+m-1+1-i \geq l_1+m-i \geq l_1$ for $i \leq m$. This shows the second claim in view of (5).

Now assume $p := \max\{l_2,\ l_1+m-1\} = l_1+m-1$. Then $[E_2, Y]_{p+1} = 0$ and

$$\bigoplus_{i \leq m} [E_1, Y]_{p+1-i} \otimes [E_2, E_1]_i = [E_1, Y]_{p+1-m} \otimes [E_2, E_1]_m \neq 0,$$

so that $[E_2, T_{E_1}(Y)]_p$ surjects onto something non-zero, therefore $[E_2, T_{E_1}(Y)]_p \neq 0$ and $\lambda_2 = l_1+m-1$. Assume finally that $l_2 \geq l_1+m+1$. Then

$$\bigoplus_{i \leq m} [E_1, Y]_{l_2-i} \otimes [E_2, E_1]_i = 0$$

and $0 \neq [E_2, Y]_{l_2}$ injects into $[E_2, T_{E_1}(Y)]_{l_2}$. We get $\lambda_2 = l_2$ and all four claims are proven. □

Now assume that $(E_i \mid i \in \Gamma)$ is a configuration of type $\Gamma = \text{ADE}$. We denote by $\mathcal{E}$ the direct sum

$$\mathcal{E} := \bigoplus_{i \in \Gamma} E_i.$$

From lemma 3.1 we conclude:

Lemma 3.2. *Let* $Y \in \mathcal{D}$ *and*

$$l_i := \max\{\ p \mid [E_i, Y]_p \neq 0\}$$

for $i \in \Gamma$. *Then*

1) $[E_i, T_{E_i}(Y)]_* \cong [E_i, Y]_{*-1}$ *for* $i \in \Gamma$, *in particular*

$$l_i + 1 = \max\{\ p \mid [E_i, T_{E_i}(Y)]_p \neq 0\}.$$

2) $[E_j, T_{E_i}(Y)]_p = 0$ *for* $i, j \in \Gamma, i \neq j$ *and* $p > \max\{l_i, l_j\}$.

3) Fix $i \in \Gamma$. *Let*

$$l := \max\{\ l_j \mid j \in \Gamma\} = \max\{\ p \mid [\mathcal{E}, Y]_p \neq 0\}$$

and

$$\lambda := \max\{\ p \mid [\mathcal{E}, T_{E_i}(Y)]_p \neq 0\}.$$

Then $l \leq \lambda \leq l+1$. *Moreover,* $\lambda = l+1$ *if and only if* $l = l_i$.

Proof. Part 1) and 2) follow directly from 3.1 and Part 3) follows from 1) and 2) with the exception of the claim that $l \leq \lambda$ if $l_i < l$. But if $l_i = l-1$, then $\lambda = l$, so we can assume that $l_i < l-1$. Then take $j \in \Gamma$ such that $l_j = l$. The lemma 3.1 can then be applied with E_i and E_j to show that $\lambda = l$. □

Lemma 3.2 is taken from [BT11, lemma 3.3].

3.2 A Garside structure for B_Γ

In this section we summarize some facts about a Garside structure for B_Γ if Γ is of type ADE. In our presentation we follow [BT11, chapter 2] closely.

Let Γ be a graph of type ADE. We will recall same basic definitions regarding the corresponding Weyl group.

Definition 3.3. *Let $H \subseteq B_\Gamma$ be the normal subgroup generated by $\{s_i^2 \mid i \in \Gamma\}$ and define the* **Weyl group of type** Γ *to be*

$$W_\Gamma := B_\Gamma / H.$$

Let σ_i, $i \in \Gamma$, be the image of s_i in W_Γ. For $\tau \in W$ we define the **length**

$$l(\tau) := \min\{\ r \mid \exists\ i_1, .., i_r \in \Gamma \text{ such that } \sigma_{i_1}..\sigma_{i_r} = \tau\}.$$

Let $\tau, \sigma \in W_\Gamma$. We call (τ, σ) a **reduced factorization** *of $\tau\sigma$ if $l(\tau\sigma) = l(\tau) + l(\sigma)$.*

An element τ is called a **left factor** *of $\alpha \in W$, if there exists a reduced factorization (τ, σ) of α with $\sigma \in W$. Similarly, an element τ is called a* **right factor** *of $\alpha \in W$, if there exists a reduced factorization (σ, τ) of α with $\sigma \in W$.*

We also recall some basic facts about W_Γ.

Lemma 3.4. 1. *The group W_Γ is finite and there exists a unique element $\delta \in W_\Gamma$ such that $l(\delta)$ is maximal.*

2. *Every element $\tau \in W_\Gamma$ is a left factor and a right factor of δ.*

Proof. See [BT11, chapter 2]. □

To relate the Weyl group W_Γ and the braid group B_Γ further one defines a set-theoretic section.

Definition 3.5. *Define a map*

$$\varphi : W_\Gamma \longrightarrow B_\Gamma$$

as follows: For $\alpha \in W_\Gamma$ choose a reduced factorization $\alpha = \sigma_{i_1}..\sigma_{i_r}$, $i_1, .., i_r \in \Gamma$, and define $\varphi(\alpha)$ as $s_{i_1}..s_{i_r} \in B_\Gamma$.

Remark 3.6. *The map φ is indeed well-defined, since different reduced expressions for α are linked via braid relations. These are relations, which also hold in B_Γ.*

Let $B_\Gamma^+ \subseteq B_\Gamma$ be the monoid generated by the s_i, $i \in \Gamma$. For $\alpha \in B_\Gamma^+$ we define (as for the Weyl group) the length of α as

$$l(\alpha) := \min\{\ r \mid \exists\ i_1, .., i_r \in \Gamma \text{ such that } s_{i_1}..s_{i_r} = \alpha\}.$$

Using this length function we can also speak of reduced factorizations, left factors and right factors for elements in B_Γ^+. In B_Γ^+ we have the element $\Delta := \varphi(\delta)$. It has the following important property: For every $\tau \in W$ the element $\varphi(\tau)$ is a left factor and a right factor of Δ in B_Γ^+.

To show the importance of this property we assume that G is a group generated by some elements $g_r \in G$, $r \in J$, and denote by G^+ the monoid generated by $g_r, r \in J$. We assume furthermore that there is an element $\Lambda \in G^+$ such that for every $g_r, r \in J$, we have $\Lambda^{-1} g_r \Lambda \in G^+$ and some element $m \in G^+$ such that $g_r m = \Lambda$, i.e. G^+ is stable under conjugation with Λ^{-1} and every $g_r, r \in J$ is a "left factor" of Λ in G^+.

Lemma 3.7. *We keep the above notations. Then:*

1) *Every element $g \in G$ can be written as $g = g^+\Lambda^j$ with $g^+ \in G^+$ and $j \in \mathbb{Z}$.*

2) *If H is an arbitrary group and $\psi : G \longrightarrow H$ a homomorphism, such that the restriction $\psi_{|G^+} : G^+ \longrightarrow H$ is injective, then ψ is injective.*

Proof. A general element $g \in G$ is a word in g_r and $g_r^{-1}, r \in J$. Now every inverse g_r^{-1} can be written as

$$g_r^{-1} = m\Lambda^{-1}$$

with $m \in G^+$, because every g_r is a left factor of Λ in G^+. So we can assume that g is a word in $g_r, r \in J$, and Λ^{-1}. In such a representation the elements Λ^{-1} can "bubble" to the right: Express $\Lambda^{-1}g_r = m\Lambda^{-1}$ with $m \in G^+$. Then Λ^{-1} has moved across g_r. So we can find a representation as in 1).

Let us consider the second claim. Assume that $g \in G$ is some element in the kernel of ψ. Write $g = mn^{-1}$ with $m, n \in G^+$. This is possible because of the first claim. Then

$$\psi(m) = \psi(n)$$

as $\psi(g) = 1$. But ψ is injective if restricted to G^+, so $m = n$ and therefore $g = 1$. This shows that ψ is injective. □

The assumption on G^+ in 3.7 is satisfied for example if every $g_r, r \in J$, is a left divisor of Λ in G^+ and the set of right divisors of Λ in G^+ is contained in the set of left divisors of Λ in G^+. We conclude:

Corollary 3.8. *Let H be a group and $\psi : B_\Gamma \longrightarrow H$ be a homomorphism, then ψ is injective if $\psi_{|B_\Gamma^+}$ is injective.*

Proof. By what we have said so far the generators $s_i \in B_\Gamma, i \in \Gamma$, and the element Δ satisfy the assumptions of 3.7. □

Remark 3.9. *For a general group G and generators $h_r, r \in \tilde{J}$, one can construct a set of generators R and a suitable Λ fulfilling the assumptions of 3.7 as follows: Choose some arbitrary element $\Lambda \in G$ and set*

$$R_1 := \{\ h_r,\ n_r := h_r^{-1}\Lambda \mid r \in \tilde{J}\},$$

$$G_1^+ := \text{ the monoid in } G \text{ generated by } R_1.$$

By construction every element h_r is a left divisor of Λ in G_1^+. Also, n_r becomes a left divisor of Λ in G_1^+ if we add $n_r^{-1}\Lambda = \Lambda^{-1}h_r\Lambda$ to R_1. But $\Lambda^{-1}h_r\Lambda$ becomes a left divisor of Λ in G_1^+ if we add $\Lambda^{-1}n_r\Lambda$. So we set

$$R := \bigcup_{n\geq 1} \Lambda^{-n}R_1\Lambda^n$$

and see that the data

$$R, \Lambda$$

$$G^+ := \text{ the monoid in } G \text{ generated by } R$$

fulfills the assumptions of 3.7. See 4.2 for an example, where this construction is applied.

Now we come to the desired Garside factorization for elements in B_Γ.

Lemma 3.10. *1) For every $\alpha \in B_\Gamma^+$ there exists a unique longest right factor $\beta \in \varphi(W_\Gamma)$.*

2) For every $\alpha \in B_\Gamma$ there exists unique $w_1, ..., w_r \in \varphi(W_\Gamma)$ such that

$$\alpha = w_1 ... w_r$$

and for $i = 1, .., r$ the element w_i is the unique longest right factor of $w_1 .. w_i$.

Proof. See [BT11, chapter 2]. □

Definition 3.11. *For an element $\alpha \in B_\Gamma^+$ the factorization $(w_1, .., w_r)$ from lemma 3.10 is called the* **Garside factorization** *of α with* **Garside factors** *$w_1, .., w_r$.*

To illustrate Garside factorizations we discuss as an example the case

$$\Gamma = \mathrm{A}_2 = (\underset{1}{\bullet} - \underset{2}{\bullet}).$$

The element δ is given by $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. Clearly, every element of

$$\varphi(W_\Gamma) = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1 = s_2s_1s_2\}$$

is a left factor and a right factor of $\Delta = s_1s_2s_1 = s_2s_1s_2$ in B_Γ. For the element $\alpha = s_1s_2s_1s_1$ the Garside factorization is given by $(s_2, s_1s_2s_1)$.

To check whether some factorization is actually a Garside factorization, one can use the following criteria.

Lemma 3.12. *1) If elements $\alpha_1, .., \alpha_r \in B_\Gamma^+$ are given, then $(\alpha_1, .., \alpha_r)$ is the Garside factorization of $\alpha_1..\alpha_r$ if and only if for every $i = 1, .., r-1$ the Garside factorization of $\alpha_i\alpha_{i+1}$ is (α_i, α_{i+1}).*

2) For $w, v \in W_\Gamma$, $(\varphi(w), \varphi(v))$ is the Garside factorization of $\varphi(w)\varphi(v)$ if and only if the following holds: For every $i \in \Gamma$ the element σ_i is a right factor of w if and only if σ_i is a left factor of v.

Proof. See [BT11, lemma 2.1 and lemma 2.2]. □

3.3 The main proposition in type ADE

In this section we proof the main proposition used in section 3.4 to show faithfulness in type ADE.

Assume that $E_i \in \mathcal{D}, i \in \Gamma$, are spherical and that $(E_i \mid i \in \Gamma)$ is a Γ-configuration (recall that Γ is assumed to be a graph of type ADE). Examples of such configurations have been given in 2.17. Define

$$\mathcal{E} := \bigoplus_{i \in \Gamma} E_i$$

to be the direct sum of all E_i.

We are aiming for a generalization of [BT11, proposition 3.1]. A first step is provided by the following lemma. We will write t_α for $T_\alpha^\Gamma \in \mathrm{Aut}(\mathcal{D})$ with $\alpha \in B^+ := B_\Gamma^+$ and t_i for $t_{s_i}, i \in \Gamma$.

Lemma 3.13. *Let $Y \in \mathcal{D}$ and $m := \sup\{\ p \mid [\mathcal{E}, Y]_p \neq 0\}$. Assume, $m \neq -\infty$. If for every $i \in \Gamma$ we have*

$$[E_i, Y]_m \neq 0,$$

then

$$[E_j, t_i Y]_{m+1} \neq 0 \Leftrightarrow j = i$$

for $i, j \in \Gamma$.

Proof. This follows from lemma 3.2: If $i = j$, then

$$[E_j, t_i Y]_{m+1} \cong [E_i, Y]_m \neq 0.$$

Conversely, if $i \neq j$, then by 3.2

$$[S_j, t_i, Y]_p = 0$$

for $p > m$. □

Now let $Y \in \mathcal{D}$ be an object fulfilling the assumptions of 3.13. The next proposition shows how an element $\alpha \in B_+$ can be reconstructed using the graded vector space $[\mathcal{E}, t_\alpha Y]_*$.

Proposition 3.14. *Let $\alpha \in B^+$, $\alpha \neq 1$, have Garside factors $(w_r, ..., w_1)$ and let $Y \in \mathcal{D}$ be with $m := \max\{\ p \mid [\mathcal{E}, Y]_p \neq 0\}$. Assume that*

$$m = \max\{\ p \mid [E_i, Y]_p \neq 0\}$$

for every $i \in \Gamma$. Then

1) $\max\{\ p \mid [\mathcal{E}, t_\alpha Y]_p \neq 0\} = r + m$

2) $[E_i, t_\alpha Y]_{k+m} \neq 0$ *if and only if s_i is a left factor of w_r.*

The proof is a long case-by-case calculation using induction. We will apply the induction hypothesis more than 20 times, so better be prepared!

Proof. We proof the proposition by induction on the length

$$l(\alpha) = \sum_{j=1}^{r} l(w_j).$$

The case $l(\alpha) = 1$ has been handled with in 3.13. Note that the second statement is wrong if $l(\alpha) = 0$.

So let $l(\alpha) \geq 2$ and $w_r = s_i u$ be a reduced expression for w_r. Define

$$\beta := w_{r-1} ... w_1.$$

Then $\beta = 1$ is possible, but $l(u\beta) \geq 1$ because $l(\alpha) \geq 2$. So the induction hypothesis may be applied to $u\beta$.

First consider the case that $u = 1$ (so $l(\beta) \geq 1$). In view of statement 2) it suffices (for proving 1)) to show

$$\max\{\ p \mid [\mathcal{E}, t_\alpha Y]_p \neq 0\} \leq r + m.$$

So let $p > r + m$. Then by induction

$$[\mathcal{E}, t_\beta Y]_{p-1} = 0$$

because β has $r - 1$ Garside factors and $p - 1 > (r - 1) + m$, therefore by lemma 3.2

$$[\mathcal{E}, t_i t_\beta Y]_p = 0.$$

To prove 2) for this case we have to show

$$[E_j, t_\alpha Y]_{k+m} \neq 0 \Leftrightarrow i = j$$

because s_i is the only left factor of w_r.

First we show that $[E_i, t_\alpha Y]_{r+m} \neq 0$. By induction we know that

$$[E_i, t_\beta Y]_{r+m-1} \neq 0$$

because β has $k - 1$ Garside factors and w_{r-1} has s_i as a left factor (see 3.12). Therefore we get

$$[E_i, t_i t_\beta Y]_{k+m} \neq 0$$

by 3.2.1).

Now let $j \in \Gamma, j \neq i$. We know by induction that

$$[E_j, t_\beta Y]_p = 0 = [E_i, t_\beta Y]_p$$

for $p \geq r + m$ because β has $r - 1$ Garside factors. So by 3.2.2) we get $[E_j, t_i t_\beta \mathcal{E}]_{r+m} = 0$.

We are finished with the case $l(w_r) = l(s_i u) = 1$, so let us consider the case that $u \neq 1$ and assume that $u = s_j v$ is a reduced expression for u with $v \in B^+$. Then $j \neq i$ (as $s_i u$ is a Garside factor of α) and we write $\gamma := v\beta$. Note that the case $\gamma = 1$ is possible (if $r = 1$ and $l(w_r) = 2$).

First let us show that $[\mathcal{E}, t_\alpha Y]_p = 0$ for $p > r + m$. Let $p > r + m$, then we have

$$[\mathcal{E}, t_\alpha Y]_p \cong [E_i, t_i t_u t_\beta Y]_p \oplus \bigoplus_{l \in \Gamma, l \neq i} [E_l, t_i t_u t_\beta Y]_p \cong [E_i, t_i t_u t_\beta Y]_p$$

because by induction $[\mathcal{E}, t_u t_\beta Y]_p = 0$ for $p > r + m$ and therefore (by 3.2.2))

$$[E_l, t_i t_u t_\beta Y]_p = 0$$

for $p > r + m$ and $l \neq i$. Finally $[E_i, t_i t_u t_\beta Y]_p \cong [E_i, t_u t_\beta Y]_{p-1} = 0$ for $p - 1 > r + m$ by induction. But (also by induction)

$$[E_i, t_u t_\beta Y]_{r+m} \neq 0 \Leftrightarrow s_i \text{ is a left factor of } u,$$

but s_i is not a left factor of u because $l(w_r) \geq 2$ and so we get $[E_i, t_u t_\beta Y]_{r+m} = 0$.

The rest of the proof shows statement 2) for the case $l(w_r) \geq 2$, namely that for $l \in \Gamma$ we have

$$[E_l, t_\alpha Y]_{r+m} \neq 0 \Leftrightarrow s_l \text{ is a left factor of } w_r.$$

First we show that $[E_i, t_i t_j t_\gamma Y]_{r+m} \neq 0$, which establishes one direction (because s_i is an arbitrary left factor of w_r), by dividing it into three cases:

Case 1 i, j not adjacent in Γ, that is $t_i t_j = t_j t_i$.

Then by induction $[E_i, t_i t_\gamma Y]_{r+m} \neq 0$ because $s_i\gamma$ has r Garside factors with last Garside factor $s_i v$ and s_i as a left factor of $s_i v$. But then

$$0 \neq [t_j E_i, t_j t_i t_\gamma Y]_{r+m} \cong [E_i, t_i t_j t_\gamma Y]_{r+m}$$

because $t_j E_i \cong E_i$ (see 2.18).

Case 2 i, j adjacent in Γ and v has s_i as a left factor.

Let $v = s_i\varepsilon$ be a reduced expression. Then $\gamma = s_i\varepsilon\beta$ and we compute (using $t_j t_i E_j = E_i$ and the braid relations):

$$[E_i, t_\alpha Y]_{r+m} = [E_i, t_i t_j t_i t_\varepsilon t_\beta Y]_{r+m} \cong [t_j t_i E_j, t_j t_i t_j t_\varepsilon t_\beta Y]_{r+m} \cong [E_j, t_j t_\varepsilon t_\beta Y]_{r+m}$$

and the last group is non-trivial (by induction applied to $s_j\varepsilon$) because $s_j\varepsilon$ is the last Garside factor of $s_j\varepsilon\beta$ and has s_j as a left factor.

So we are left with the last case:

Case 3 i, j adjacent in Γ, but v does not have s_i as a left factor.

We want to show that $[E_i, t_i t_j t_\gamma Y]_{r+m} \cong [E_i, t_j t_\gamma Y]_{r+m-1}$ is non-trivial. Consider the distinguished triangle

$$[E_j, t_\gamma Y]_* \otimes E_j \longrightarrow t_\gamma Y \longrightarrow t_j t_\gamma Y \longrightarrow *[1] \tag{6}$$

and apply $[E_i, -]_0$. This gives an exact sequence:

$$[E_i, t_\gamma Y]_{r+m-1} \longrightarrow [E_i, t_j t_\gamma Y]_{r+m-1} \longrightarrow [E_i, [E_j, t_\gamma Y]_* \otimes E_j]_{r+m} \longrightarrow [E_i, t_\gamma Y]_{r+m} \tag{7}$$

We claim that $[E_i, t_\gamma Y]_{r+m} = 0$. If $\gamma = 1$, then the claim follows because $r \geq 1$, but if $\gamma \neq 1$, then we can apply induction to γ. If γ has strictly less than r Garside factors, induction yields the claim. But if γ has r Garside factors, then $[E_i, t_\gamma \mathcal{S}]_{r+m} = 0$ because v does not admit s_i as a left factor.

By (7) (and the vanishing of the last term) it suffices to show that

$$[E_i, [E_j, t_\gamma Y]_* \otimes S_j]_{r+m} \neq 0$$

to get $[E_i, t_j t_\gamma Y]_{r+m-1} \neq 0$. We calculate (using that i, j are adjacent):

$$[E_i, [E_j, t_\gamma Y]_* \otimes E_j]_{r+m} \cong \bigoplus_p [E_j, t_\gamma Y]_{r+m-p} \otimes [E_i, E_j]_p \cong [E_j, t_\gamma Y]_{r+m-1} \otimes [E_i, E_j]_1$$

and so it suffices to show $[E_j, t_\gamma Y]_{r+m-1} \neq 0$. But $[E_j, t_\gamma Y]_{r+m-1} \cong [E_j, t_j t_\gamma]_{r+m}$ and the last group is non-trivial by induction applied to $s_j\gamma$. The case 3 is then finished.

We are still left with one statement, namely that if $[E_l, t_\alpha Y]_{r+m} \neq 0$, then s_l is a left factor of w_r. We will prove the contrapositive: If s_l is not a left factor of w_r, then $[S_l, t_\alpha Y]_{r+m} = 0$. So let $l \in \Gamma$ such that s_l is not a left factor of w_r.

First consider the case that l and i are not adjacent (remember the reduced decomposition $\alpha = s_i u\beta$ with $u\beta$ having r Garside factors but $l(u\beta) < l(\alpha)$).

We get $[E_l, t_i t_u t_\beta Y]_{r+m} \cong [t_i E_l, t_i t_u t_\beta Y]_{r+m} \cong [E_l, t_u t_\beta Y]_{r+m}$ and can apply induction to see that the last group is trivial because u (which is the last Garside factor of $u\beta$) does not have s_l as a left factor since a reduced expression $u = s_l v$ would give a reduced expression $\alpha = s_i s_l v\beta = s_l s_i v\beta$.

Thus we can assume that l, i are adjacent. It follows that $l \neq i$ because s_i is a left factor of w_r and s_l not (by assumption).

Consider the distinguished triangle

$$[E_i, t_u t_\beta Y]_* \otimes E_i \longrightarrow t_u t_\beta Y \longrightarrow t_i t_u t_\beta Y \longrightarrow *[1]$$

and apply $[E_l, -]_0$. This yields an exact sequence:

$$.. \longrightarrow [E_i, t_u t_\beta Y]_{r+m-1} \otimes [E_l, E_i]_1 \overset{\text{comp}}{\longrightarrow} [E_l, t_u t_\beta Y]_{r+m} \longrightarrow [E_l, t_i t_u t_\beta Y]_{r+m} \longrightarrow .. \tag{8}$$

Thus, if we can show that $[E_i, t_u t_\beta Y]_{r+m} = 0$ and that the composition map

$$[E_i, t_u t_\beta Y]_{r+m-1} \otimes [E_l, E_i]_1 \overset{\text{comp}}{\longrightarrow} [E_l, t_u t_\beta Y]_{r+m} \tag{9}$$

is surjective, we are done, because then (by exactness of (8)) $[E_l, t_i t_u t_\beta Y]_{r+m} = 0$. First let us show that $[E_i, t_u t_\beta Y]_{r+m} = 0$. But this is easy because we can apply induction to $u\beta$ and use that u does not admit s_i as a left factor (as $u\beta$ and $s_i u\beta = \alpha$ have r Garside factors).

Now let us show that the composition map (9) is surjective. If s_l is not a left factor of u, then (by induction applied to $u\beta$)

$$[E_l, t_u t_\beta Y]_{r+m} = 0$$

as $u\beta$ has r Garside factors. So we can assume that there is a reduced expression $u = s_l \varepsilon$. Now consider the distinguished triangle

$$[E_l, t_\varepsilon t_\beta Y]_* \otimes E_l \longrightarrow t_\varepsilon t_\beta Y \longrightarrow t_u t_\beta Y \xrightarrow{f} *[1] \tag{10}$$

and the exact sequence

$$[E_l, t_\varepsilon t_\beta Y]_{r+m} \longrightarrow [E_l, t_u t_\beta Y]_{r+m} \xrightarrow{f_*} [E_l, [E_l, t_\varepsilon t_\beta Y]_* \otimes E_l]_{r+m+1}. \tag{11}$$

If $\varepsilon\beta \neq 1$ let us apply induction to see that $[E_l, t_\varepsilon t_\beta Y]_{r+m} = 0$ because if $\varepsilon\beta$ has r Garside factors, then s_l cannot be a left factor of ε (as $s_i s_l \varepsilon\beta = \alpha$ has r Garside factors). In the case $\varepsilon = 1$ induction (if $\beta \neq 1$) yields $[E_l, t_\beta]_{r+m} = 0$ because β has $r-1$ Garside factors. If $\varepsilon\beta = 1$, then $[E_l, t_\varepsilon t_\beta Y]_{r+m} = 0$ as $r = 1$. This shows that f_* is always injective.

Now the rightmost term in (11) is given by

$$\begin{aligned}
[E_l, [E_l, t_\varepsilon t_\beta Y]_* \otimes E_l]_{r+m+1} &\cong [E_l, [E_l, t_\varepsilon t_\beta Y]_{r+m+1} \otimes E_l]_0 \oplus [E_l, [E_l, t_\varepsilon t_\beta Y]_{r+m-1} \otimes E_l]_2 \\
&\cong [E_l, [E_l, t_\varepsilon t_\beta Y]_{r+m-1} \otimes E_l]_2
\end{aligned}$$

(by induction) and thus for every map $x : E_l \longrightarrow t_u t_\beta Y[r+m]$ the composition

$$E_l \xrightarrow{x} t_u t_\beta Y[r+m] \xrightarrow{f} [E_l, t_\varepsilon t_\beta Y]_* \otimes E_l[r+m+1]$$

factors through the summand $[E_l, t_\varepsilon t_\beta Y]_{r+m-1} \otimes E_l[2]$ of $[E_l, t_\varepsilon t_\beta Y]_* \otimes E_l[r+m+1]$.

Now $[E_l, t_\varepsilon t_\beta Y]_{r+m-1} \otimes E_l[2]$ is just a direct sum of copies of $E_l[2]$ and lemma 2.11 shows that every map $E_l \longrightarrow [E_l, t_\varepsilon t_\beta Y]_* \otimes \mathcal{E}_l[2]$ factors through $E_i[1]$.

Let $x : E_l \longrightarrow t_\varepsilon t_\beta Y[r+m]$ and remember that we want to show that the composition map (9) is surjective. That is we want to factor x through some morphism

$$E_i[1] \longrightarrow t_\varepsilon t_\beta Y[r+m].$$

First of all we can find a commutative diagram (without the dotted arrow $\overline{y}$)

$$\begin{tikzcd}
E_l \arrow[r, "g"] \arrow[d, "x"] & E_i[1] \arrow[d, "y"] \arrow[dl, dotted, "\overline{y}"'] \arrow[dr, "z"] & \\
t_u t_\beta Y[r+m] \arrow[r, "f"] & {[E_l, t_\varepsilon t_\beta Y]_* \otimes E_l[r+m+1]} \arrow[r] & t_\varepsilon t_\beta Y[r+m+1]
\end{tikzcd}$$

,

where the bottom row is part of the (shifted) distinguished triangle (10), because we can factor $f \circ x$ over $E_i[1]$. We want to construct the dotted arrow $\overline{y}$ making the upper triangle in the square commutative.

We claim that $z = 0$, so that by exactness (coming from the distinguished triangle) the morphism y factors through f giving some $\overline{y}$ making the lower triangle on the left commutative. For this we will even prove that

$$[E_i[1], t_\varepsilon t_\beta Y]_{r+m+1} = [E_i, t_\varepsilon t_\beta Y]_{r+m} = 0,$$

so that z must be zero.

If $\varepsilon\beta = 1$ we are done because $r \geq 1$, so let us assume that $\varepsilon\beta \neq 1$, $[E_i, t_\varepsilon t_\beta Y]_{r+m} \neq 0$ and let us apply induction. Then we know that $\varepsilon\beta$ has r Garside factors (because $\varepsilon\beta$ cannot have more than r because α has r Garside factors) and that ε has s_i as a left factor. So we can find a reduced expression $\varepsilon = s_i\eta$ giving a reduced expression $\alpha = s_i s_l s_i \eta\beta = s_l s_i s_l \eta\beta$ which is a contradiction because by assumption s_l is not a left factor of α. Therefore we proved $[E_i, t_\varepsilon t_\beta Y]_{r+m} = 0$ and thus we have constructed the map $\overline{y}$.

Now we show that actually the upper triangle on the left must also commute, that is $x = \overline{y} \circ g$ and we finished the proof. But it was shown (see the exact sequence (11)) that composition with f is injective for maps from E_l, so we get $x = \overline{y} \circ g$ because

$$f \circ x = y \circ g = f \circ \overline{y} \circ g.$$

This settles the last piece needed for the proof. □

3.4 Proof of faithfulness in type ADE

We keep the notations from section 3.3. The main proposition 3.14 implies the following theorem.

Theorem 3.15. *Let $Y \in \mathcal{D}$ be an object such that*

$$\max\{\ p \mid [\mathcal{E}, Y]_p \neq 0\} = \max\{\ p \mid [E_i, Y]_p \neq 0\}$$

for all $i \in \Gamma$. Then for $\alpha \in B$

$$t_\alpha Y \cong Y \Leftrightarrow \alpha = 1.$$

In particular, the stabilizer of Y in B is trivial.

Proof. First let $\alpha, \beta \in B^+$ be such that $t_\alpha Y \cong t_\beta Y$. Then we know that

$$[\mathcal{E}, t_\alpha Y]_* \cong [\mathcal{E}, t_\beta Y]_*$$

and thus by 3.14 that the Garside factorizations $\alpha = w_r ... w_1$ and $\beta = v_r ... v_1$ must have the same length. In particular, if $\alpha = 1$, then $\beta = 1$. If now $\alpha \neq 1$ we can find a left factor s_i of

w_r and this left factor must also be a left factor of v_k, again by 3.14. Proceeding by induction on the length of α we get $\alpha = \beta$.

Using the same reasoning as in 3.7 we conclude that if $\alpha \in B$ is arbitrary we can write $\alpha = \beta^{-1}\gamma$ with $\beta, \gamma \in B^+$. Thus if $t_\alpha Y \cong Y$, then $t_\beta Y \cong t_\gamma Y$ and thus $\beta = \gamma$ because of the first case. But $\beta = \gamma$ implies $\alpha = 1$ and hence the theorem. $\square$

The next theorem is now an easy corollary of 3.15.

Theorem 3.16. *For every Γ-configuration $(E_i \mid i \in \Gamma)$ in $\mathcal{D}$ the group homomorphism*

$$T^\Gamma : B_\Gamma \longrightarrow \mathrm{Aut}(\mathcal{D})$$

is injective, i.e. the action of B_Γ on $\mathcal{D}$ is faithful.

Proof. By 3.15 it suffices to show the existence of some object $Y \in \mathcal{D}$ such that

$$\max\{\, p \mid [\mathcal{E}, Y]_p \neq 0\} = \max\{\, p \mid [E_i, Y]_p \neq 0\}$$

for all $i \in \Gamma$. Put $Y := \mathcal{E}$, then Y has this property. $\square$

The object $Y := \mathcal{E}$ is not the only possible choice, as the next example shows.

Example 3.17. *Let X be a K3 surface and $C, D \subseteq X$ two (-2)-curves on X meeting in a (reduced) point $x \in X$. Then*

$$(\mathcal{O}_C(i), \mathcal{O}_D(j)),\ \ i, j \in \mathbb{Z}$$

is an A_2-configuration in $\mathcal{D}^b(X)$, see example 2.17. The object $Y := k(x)$ fulfills the assumptions of theorem 3.15, see 2.20. We can conclude that the stabilizer of Y in

$$B_{\mathrm{A}_2} \cong \langle T_{\mathcal{O}_C(i)}, T_{\mathcal{O}_D(j)} \rangle$$

is trivial. In particular, the "small" object $k(x)$ already establishes faithfulness for B_{A_2} in this case.

4 The case $\Gamma = \tilde{\mathrm{A}}_1$

Fix a K3 category $\mathcal{D}$. Similarly to chapter 3 we want to establish faithfulness of braid group actions arising from Γ-configurations, but this time for the graph

$$\Gamma = \tilde{\mathrm{A}}_1 = (\underset{1}{\bullet} = \underset{2}{\bullet}).$$

So define $\Gamma := \tilde{\mathrm{A}}_1$ for the rest of this chapter. In section 4.5 we will give an application to Bridgeland's conjecture on autoequivalences of complex K3 surfaces.

4.1 A presentation of $B_{\tilde{\mathrm{A}}_1}$

In this section we aim for a presentation of the affine braid group

$$B := B_\Gamma,$$

which is much more suitable for our purposes.

Definition 4.1. *Define*

$$\tilde{B} := \langle s_i, i \in \mathbb{Z} \mid s_{i-1}s_i = s_{j-1}s_j,\ i,j \in \mathbb{Z}\rangle.$$

Let $\omega := s_0s_1 = s_1s_2 = \dots$.

By definition, we have $\omega s_i \omega^{-1} = s_{i-2}$ for $i \in \mathbb{Z}$.

Remark 4.2. *The construction of* $\tilde{B}$ *is motivated by 3.9 with* $h_0 = s_0,\ h_1 = s_1$ *and* $\Lambda = \omega = s_0s_1$. *But additionally we also added generators* $\omega^n s_i \omega^{-n}$ *with* $i = 0, 1$ *and* $n \geq 1$.

We want to prove:

Proposition 4.3. *The homomorphism*

$$B \longrightarrow \tilde{B} : s_i,\ i = 0, 1 \mapsto s_i$$

is an isomorphism.

Proof. First we show that s_0 and s_1 generate $\tilde{B}$. Let $H := \langle s_0, s_1\rangle \subseteq \tilde{B}$. As $\omega \in H$, the equation

$$\omega s_i \omega^{-1} = s_{i-2},\ i \in \mathbb{Z}$$

shows that every s_i is contained in H. Thus, $H = \tilde{B}$. But s_0, s_1 do not have any relations in $\tilde{B}$. To see this express the relation

$$s_{i-1}s_i = s_{j-1}s_j$$

with $i, j \in \mathbb{Z}$ in terms of s_0 and s_1. But $s_0s_1 = \omega = s_{i-1}s_i$ for $i \in \mathbb{Z}$. So the relation reads

$$\omega = \omega,$$

which is trivial. Hence, s_0 and s_1 do not satisfy any relations in $\tilde{B}$, so the above morphism $B \longrightarrow \tilde{B}$ is indeed an isomorphism. □

We found the presentation $\tilde{B}$ of B in [McC05, example 4.8]. From now on we will identify B with $\tilde{B}$ using the above isomorphism. We define $B^+ \subseteq B$ to be the monoid generated by $s_i, i \in \mathbb{Z}$.

Lemma 3.7 implies:

Corollary 4.4. *1) Every element $\alpha \in B$ can be written as*

$$\alpha = s_{i_1}..s_{i_r}\omega^j$$

with $i_1, .., i_r, j \in \mathbb{Z}$. If we require $i_j \neq i_{j+1} - 1$ for $j = 1, .., r-1$, this representation is unique.

2) Let H be a group and $\psi : B \longrightarrow H$ be a homomorphism. If $\psi_{|B^+}$ is injective, then ψ is injective.

Proof. By construction the assumptions of 3.7 are satisfied with $g_r = s_r, r \in \mathbb{Z}$, and $\Lambda = \omega$, so we are only left with the uniqueness statement. But the condition $i_j \neq i_{j+1}$ for $j = 1, .., r$ means that we cannot change the representation using the relations $s_i s_{i+1} = s_j s_{j+1},\ i, j \in \mathbb{Z}$ in B. So the representation is indeed unique. □

For later use we give the following definition.

Definition 4.5. *An element $\alpha \in B$ is called* ω-**free** *if in its unique representation from 4.4 we have $j = 0$. In other words, α can be written as*

$$\alpha = s_{i_1}..s_{i_r}$$

with $i_1, .., i_r \in \mathbb{Z}$ such that for every $j = 1, .., r-1$ we have

$$i_j \neq i_{j+1} - 1.$$

Remark 4.6. *If $s_i\alpha \in B^+$ with $i \in \mathbb{Z}$ is ω-free, then $s_j s_i \alpha$ is ω-free if and only if $j \neq i - 1$.*

4.2 Computation of Hom-spaces in Type $\tilde{\mathrm{A}}_1$

We fix $E_0, E_1 \in \mathcal{D}$ spherical and assume that $(E_0, E_1[-1])$ form a Γ-configuration. This just means that

$$[E_0, E_1]_* \cong k^2.$$

Let us denote the spherical twists T_{E_0} and T_{E_1} by T_0 and T_1. Recall the homomorphism (4):

$$T^\Gamma : B_\Gamma \longrightarrow \mathrm{Aut}(\mathcal{D})$$

sending $s_i,\ i = 0, 1$ to T_i. Using the identification $B = \tilde{B}$ from proposition 4.3 yields autoequivalences $T_i := T^\Gamma(s_i)$ for $i \in \mathbb{Z}$. For $\alpha \in B$ we will use the shorthand T_α instead of $T^\Gamma(\alpha)$. In particular, $T_{s_i} = T_i$ for $i \in \mathbb{Z}$. Also recall the element $\omega = s_0 s_1 \in B$. Define

$$E_{2j} := T_\omega^{-j} E_0,\ j \in \mathbb{Z}$$

and

$$E_{2j+1} := T_\omega^{-j} E_1,\ j \in \mathbb{Z}.$$

These objects are again spherical objects in $\mathcal{D}$. By construction $T_\omega(E_i) \cong E_{i-2},\ i \in \mathbb{Z}$, and $T_i \cong T_{E_i},\ i \in \mathbb{Z}$. In particular, the autoequivalences T_i are again spherical twists.

To motivate this construction, we give the following example.

Example 4.7. *Let X be a K3 surface and $C \subseteq X$ a (-2)-curve. Then set*

$$E_0 := \mathcal{O}_C \in \mathcal{D}^b(X)$$

and

$$E_1 := \mathcal{O}_C(1) \in \mathcal{D}^b(X).$$

Example 2.17 shows that $[E_0, E_1]_* \cong k^2$ and example 2.21 that in this case

$$T_\omega(-) \cong \mathcal{O}_X(C) \otimes (-)$$

and thus

$$E_j \cong \mathcal{O}_C(j)$$

as $\mathcal{O}_X(C)_{|C} \cong \mathcal{O}_C(-2)$.

Our aim is the calculation of the Hom-spaces $[E_i, E_j]_*$ for $i, j \in \mathbb{Z}$, because this Hom-spaces will play an important role for the combinatorics developed in section 4.3.

Lemma 4.8. *Let $i \in \mathbb{Z}$. Then*

$$[E_{i-1}, E_i]_* \cong k^2.$$

Proof. By applying T_ω we can restrict ourselves to the calculation of

$$[E_0, E_1]_*$$

and

$$[E_{-1}, E_0]_*.$$

For the first case the claim follows by assumption and for the second case we compute

$$[E_{-1}, E_0]_* \cong [T_0 T_1(E_1), E_0]_* \cong [E_1, E_0]_{*+2} \cong [E_0, E_1]^{\vee}_{-*} \cong k^2$$

using Serre duality and $T_i(E_i) \cong E_i[-1]$ for $i = 0, 1$. □

Next, we establish an analogue to the "Euler sequence" on $\mathbb{P}^1_k$.

Lemma 4.9. *Let $i \in \mathbb{Z}$ and let $x, y \in [E_{i+1}, E_{i+2}]_0$ be a basis. Then by completing the morphism*

$$E_{i+1} \oplus E_{i+1} \xrightarrow{(x,y)} E_{i+2}$$

*we get a distinguished triangle (called "**Euler triangle for** i" in this chapter)*

$$E_i \xrightarrow{(a,b)} E_{i+1} \oplus E_{i+1} \xrightarrow{(x,y)} E_{i+2} \longrightarrow *[1] \tag{12}$$

with $a, b \in [E_i, E_{i+1}]_0$ being a basis.

Proof. By definition we have an isomorphism

$$E_i \cong T_\omega(E_{i+2}) \cong T_{i+1} T_{i+2}(E_{i+2})$$

or equivalently an isomorphism

$$E_i[1] \cong T_{i+1} E_{i+2}. \tag{13}$$

By lemma 4.8 the (x, y) form also a basis of $[E_{i+1}, E_{i+2}]_*$ and thus by using the distinguished triangle

$$[E_{i+1}, E_{i+2}]_* \otimes E_{i+1} \xrightarrow{\text{eval}} E_{i+2} \longrightarrow T_{i+1}(E_{i+2}) \longrightarrow *[1],$$

the basis (x, y) and the second isomorphism (13) we conclude that we get a distinguished triangle

$$E_i \xrightarrow{(a,b)} E_{i+1} \oplus E_{i+1} \xrightarrow{(x,y)} E_{i+2} \longrightarrow *[1] \tag{14}$$

with some $a, b \in [E_i, E_{i+1}]_0$. We have to show that they form a basis or equivalently that they are linearly independent. Assume they are not. By changing the triangle (14) by an automorphism of $E_{i+1} \oplus E_{i+1}$ we then can assume that $b = 0$. But then (because E_{i+2} is indecomposable) E_{i+1} and E_{i+2} must be isomorphic, which is a contradiction. So the proof is finished. □

For different choices of a basis $x, y \in [E_i, E_{i+1}]_0$ we get isomorphic triangles. Hence we will speak of "the" Euler sequence (or Euler triangle) for i.

In our next lemma we finally compute the Hom-spaces $[E_i, E_j]_*$ for $i, j \in \mathbb{Z}$.

Proposition 4.10. *Let $i, j \in \mathbb{Z}$. Then:*

1) If $i \leq j$, we have

  a) for $0 \neq x \in [E_j, E_{j+1}]_0$ the map

$$[E_i, E_j]_p \xrightarrow{x_*} [E_i, E_{j+1}]_p$$

  is injective for $p = 0$ or $p = 1$.

  b) for $p \in \mathbb{Z}$ and $j > i$ the Euler triangle for j (from lemma 4.9) yields an exact sequence

$$0 \longrightarrow [E_i, E_j]_p \longrightarrow [E_i, E_{j+1} \oplus E_{j+1}]_p \longrightarrow [E_i, E_{j+2}]_p \longrightarrow 0.$$

  c) for $j > i$ we have

$$[E_i, E_j]_* \cong k^{j-i+1} \oplus k[-1]^{j-i-1}.$$

2) If $j < i$, then

$$[E_i, E_j]_* \cong k[-1]^{i-j-1} \oplus k[-2]^{i-j+1}.$$

Proof. The assertion 2) follows from the assertions in 1) by Serre duality. So we can restrict our attention to the claims for $j \geq i$. First we show how we can derive statement 1.c) about $[E_i, E_j]_*$ if the other two parts are known. Let us compute $[E_i, E_{i+2}]_*$ using the Euler triangle for i:

$$E_i \longrightarrow E_{i+1} \oplus E_{i+1} \longrightarrow E_{i+2} \longrightarrow *[1].$$

Applying $[E_i, -]_0$ and using the known spaces $[E_i, E_i]_*$ and $[E_i, E_{i+1}]_*$ (see lemma 4.8) we get

$$[E_i, E_{i+2}]_* \cong k^3 \oplus k[-1].$$

Thus we can proceed by induction using the claim 1.b) to show

$$[E_i, E_j]_* \cong k^{j-i+1} \oplus k[-1]^{j-i-1}$$

for $j > i$. The statement boils down to the calculation

$$2(j + 1 - i \pm 1) - (j - i \pm 1) = j + 2 - i \pm 1.$$

So we establish the remaining claim 1.a),1.b) about the injectivity and the Euler triangle by induction on j. For $j = i$ the injectivity assertion for some $0 \neq x \in [E_i, E_{i+1}]_0$ is clear by evaluating at the identity of E_i. So we can assume that $j > i$. Note that the statement 1.b) about the Euler triangle for j follows from the injectivity statement 1.a) for j. In fact, using the Euler apply and induction on j we get a (quite) long exact sequence

$$\begin{tikzcd}
 & 0 \arrow[r] & {[E_i, E_{j+2}]_{-1}} \arrow[dll] \\
{[E_i, E_j]_0} \arrow[r, "(a,b)_*"] & {[E_i, E_{j+1} \oplus E_{j+1}]_0} \arrow[r] & {[E_i, E_{j+2}]_0} \arrow[dll] \\
{[E_i, E_j]_1} \arrow[r, "(a,b)_*"] & {[E_i, E_{j+1} \oplus E_{j+1}]_1} \arrow[r] & {[E_i, E_{j+2}]_1} \arrow[dll] \\
0 & &
\end{tikzcd}$$

.

with some $a, b \in [E_j, E_{j+1}]_0$ both non-zero. So if the injectivity assumption in 1.a) holds for $j > i$, then the assumption 1.b) about the Euler triangle holds for j. Now we claim that if the injectivity assumption of 1.a) holds for j, then the injectivity assumption also holds for $j + 1$. This would clearly finish the proof. Assume the injectivity assumption 1.a) holds for j and let

$$x : E_{j+1} \longrightarrow E_{j+2} \text{ resp. } \delta : E_0 \longrightarrow E_{j+1}[p]$$

be homomorphisms with $p \in \{0, 1\}$ and $x \circ \delta = 0$. If x is non-zero, then we can pick a basis $x, y \in [E_{j+1}, E_{j+2}]_0$ containing x and look at the corresponding Euler triangle for j:

$$E_j \xrightarrow{(a,b)} E_{j+1} \oplus E_{j+1} \xrightarrow{(x,y)} E_{j+2} \longrightarrow *[1]$$

with some basis $a, b \in [E_j, E_{j+1}]_0$. By assumption the element $(\delta, 0) \in [E_0, E_{j+1} \oplus E_{j+1}]_p$ maps under $(x, y)_*$ to zero in $[E_i, E_{j+2}]_p$ and hence there exists (by exactness) a map $z \in [E_i, E_j]_p$ such that

$$(a \circ z, b \circ z) = (\delta, 0).$$

But b cannot be zero because a, b form a basis of $[S_j, S_{j+1}]_0$ according to lemma 4.9. Thus $z = 0$ by the injectivity assumption for j. This implies $\delta = a \circ z = 0$ and the desired injectivity assumption for $j + 1$. $\square$

4.3 The main proposition in type $\tilde{\mathrm{A}}_1$

In this section we develop the combinatorics for proving the main theorem on faithfulness in type $\tilde{\mathrm{A}}_1$ (see section 4.4). We keep the notations from section 4.1 and section 4.2.

On B^+ we can define a length l, namely

$$l(\alpha) := \min\{\ r \mid \exists\ i_1, .., i_r \in \mathbb{Z} \text{ such that } s_{i_1} .. s_{i_r} = \alpha\}.$$

The next lemma gives a description how Hom-spaces change under a twist T_i. It is a special case of 3.1 and can be seen as an analogue of 3.2.

Lemma 4.11. *Let $Y \in \mathcal{D}$ and for $j \in \mathbb{Z}$ let*

$$l_j := \max\{\ p \mid [E_j, Y]_p \neq 0\}.$$

Fix $i \in \mathbb{Z}$ and let $\lambda_j := \max\{\ p \mid [E_j, T_i(Y)]_p \neq 0\}$. Then we have the following:

1) *$\lambda_i = l_i + 1$.*

2) *Let $j = i - 1$. Then $\lambda_j \leq \max\{l_j, l_i\}$ and if $l_j > l_i$, then $\lambda_j = l_j$. If $l_i > l_j$, then $\lambda_j = l_i - 1$.*

3) *Let $j < i - 1$. Then $\lambda_j \leq \max\{l_j, l_i\}$ and if $l_j > l_i + 1$, then $\lambda_j = l_j$. If $l_i \geq l_j$, then $\lambda_j = l_i$.*

4) *Let $j > i$. Then $\lambda_j \leq \max\{l_j, l_i + 1\}$ and if $l_j \leq l_i + 1$, then $\lambda_j = l_i + 1$.*

Proof. All statements follow from 3.1 and 4.10. □

We come to our main proposition of this chapter.

Proposition 4.12. *Let $Y \in \mathcal{D}$ and let*

$$l_j := \max\{\ p \mid [E_j, Y]_p \neq 0\},\ j \in \mathbb{Z}.$$

Assume that for every $i, j \in \mathbb{Z}$ we have $l_i = l_j (\neq -\infty)$. Let $\alpha \in B^+$ be ω-free and $i \in \mathbb{Z}$ such that $s_i\alpha$ is again ω-free. Write

$$\lambda_j := \max\{\ p \mid [E_j, T_iT_\alpha(Y)]_p \neq 0\}.$$

Then $\lambda_j = \lambda_i - 1$ for $j < i - 1$ while $\lambda_{i-1} = \lambda_i - 2$ and $\lambda_j = \lambda_i$ for $j \geq i$.

In particular, we can determine i out of the λ_j as the least i such that

$$\lambda_i = \max\{\ \lambda_j \mid j \in \mathbb{Z}\}.$$

Proof. We use induction on the length of the ω-free element $s_i\alpha$ and heavily our lemma 4.11. First if $\alpha = 1$, then we have to compute

$$\lambda_j = \max\{\ p \mid [E_j, T_i(Y)]_p \neq 0\}.$$

We have $\lambda_i = l_i + 1$ and $l_j = l_i$ for every $j \in \mathbb{Z}$. By lemma 4.11 this implies $\lambda_{i-1} \leq l_i = \lambda_i$ and that $\lambda_j = l_i = \lambda_i - 1$ for $j < i - 1$ and $\lambda_j = l_i + 1 = \lambda_i$ for $j \geq i$. So we have to show only $\lambda_{i-1} = \lambda_i - 2$. But

$$[E_{i-1}, T_i(Y)]_* \cong [E_{i-1}, T_{i-1}T_i(Y)]_{*-1} \cong [E_{i+1}, Y]_{*-1} \tag{15}$$

and so $\lambda_{i-1} = l_{i+1} - 1 = \lambda_i - 2$.

We assume that the length of α is greater or equal to 1. Then write $\alpha = s_\gamma\beta$ with $\gamma \in \mathbb{Z}$. By our assumption that $s_i\alpha = s_is_\gamma\beta$ is ω-free we can conclude that $i \geq \gamma$ or $i < \gamma - 1$. Let

$$\mu_j := \max\{\ p \mid [E_j, T_\alpha(Y)]_p \neq 0\}.$$

By induction we know that $\mu_j = \mu_\gamma - 1$ for $j < \gamma - 1$ and $\mu_{\gamma-1} = \mu_\gamma - 2$ and $\mu_j = \mu_\gamma$ for $j \geq \gamma$. First consider the case that $i \geq \gamma$. Let $j < i-1$. Then $\mu_j \leq \mu_i$ and thus by the lemma 4.11 $\lambda_j = \mu_i = \lambda_i - 1$. If $j \geq i$, then

$$\mu_j = \mu_i \leq \mu_i + 1$$

(because $j \geq i \geq \gamma$) and therefore by the lemma 4.11

$$\lambda_j = \mu_i + 1 = \lambda_i.$$

This finishes the case that $i \geq \gamma$, because we can argue for λ_{i-1} as in (15) to show $\lambda_{i-1} = \lambda_i - 2$.

Now assume $i < \gamma - 1$. Then $\mu_i = \mu_\gamma - 1$ and thus for $j \geq i$ we have

$$\mu_j \leq \mu_i + 1 = \mu_\gamma.$$

We can conclude that $\lambda_j = \mu_i + 1 = \lambda_i$ by the lemma 4.11. The argument in (15) shows $\lambda_{i-1} = \lambda_i - 2$. We are left with our last case, namely that $j < i - 1$. Then $\mu_j = \mu_i$ and therefore by lemma 4.11 $\lambda_j = \mu_i = \lambda_i - 1$. This finishes the proof. □

4.4 Proof of faithfulness in type $\tilde{\mathrm{A}}_1$

We keep the notations from the previous sections and prove the following analog of theorem 3.15.

Theorem 4.13. *Let $Y \in D$ and for $i \in \mathbb{Z}$ let $l_i := \max\{\ p \mid [E_i, Y]_p \neq 0\}$. Assume that $l_i = l_j (\neq -\infty)$ for $i, j \in \mathbb{Z}$. Then*

$$B_Y := \{\alpha \in B \mid T_\alpha(Y) \cong Y\} \subseteq \langle\omega\rangle.$$

Proof. Take $\alpha \in B_Y$. Then according to 4.4 we can write $\alpha = \beta\omega^r$ with $r \in \mathbb{Z}$ and β ω-free. As $\alpha(Y) \cong Y$, we conclude

$$\beta(\tilde{Y}) = Y$$

if we set $\tilde{Y} := \omega^r(Y)$. In particular, we see that

$$\max\{\ p \mid [E_j, \beta(\tilde{Y})]_p \neq 0\} = \max\{\ p \mid [E_j, \beta(\tilde{Y}) \neq 0\}$$

for all $i, j \in \mathbb{Z}$. But then $\beta = 1$ as we can apply 4.12 with β and $\tilde{Y}$. In fact, for $i \in \mathbb{Z}$,

$$[E_i, \tilde{Y}]_* \cong [E_i, T_\omega^r(Y)]_* \cong [E_{i+2r}, Y]_*$$

as $\omega^{-r}(E_i) \cong E_{i+2r}$. Hence $\beta = 1$ and $\alpha \in \langle\omega\rangle$. □

Clearly, the action of $\langle\omega\rangle$ on $\mathcal{D}$ is faithful as the objects $E_i, i \in \mathbb{Z}$, are pairwise non-isomorphic (see 4.10), so to derive our main theorem on faithfulness in type $\tilde{\mathrm{A}}_1$ we have to ensure the existence of Y as in 4.13. For the construction of a suitable Y we first give an example in the case $\mathcal{D} = \mathcal{D}^b(X)$ with X a K3 surface and $E_i = \mathcal{O}_C(i)$.

Example 4.14. *Let X be a K3 surface and $\mathcal{D} = \mathcal{D}^b(X)$. Consider the spherical objects $E_i := \mathcal{O}_C(i)$ for a (-2)-curve $C \subseteq X$. Take a point $x \in C$. Then by 2.20 (and its proof) we have*

$$[E_i, Y]_* \cong k \oplus k[-1]$$

and the object $Y := k(x)$ fulfills the assumption of 4.13. Note also that $k(x)$ is a mapping cone of a non-zero homomorphism $E_0 \longrightarrow E_1$.

For the general case we mimic this example.

Lemma 4.15. *Let*

$$x : E_0 \longrightarrow E_1$$

be a fixed non-trivial homomorphism and let $Y \in \mathcal{D}$ *be the mapping cone of* x*. Then for all* $i \in \mathbb{Z}$

$$[E_i, Y]_* \cong k \oplus k[-1].$$

In particular, Y *fulfills the assumption in 4.13.*

Proof. Let $i \leq 0$. Then applying $[E_i, -]_0$ to the distinguished triangle

$$E_0 \xrightarrow{x} E_1 \longrightarrow Y \longrightarrow *[1]$$

gives an exact sequence

$$\begin{tikzcd}
 & 0 \arrow[r] & {[E_i, Y]_{-1}} \arrow[dll, rounded corners] \\
{[E_i, E_0]_0} \arrow[r, "x_*"] & {[E_i, E_1]_0} \arrow[r] & {[E_i, Y]_0} \arrow[dll, rounded corners] \\
{[E_i, E_0]_1} \arrow[r, "x_*"] & {[E_i, E_1]_1} \arrow[r] & {[E_i, Y]_1} \arrow[dll, rounded corners] \\
{[E_i, E_0]_2} & 0 &
\end{tikzcd}.$$

with both maps x_* being injective by lemma 4.8. If $i = 0$, then using

$$[E_0, E_0]_* \cong k \oplus k[-2]$$

and

$$[E_0, E_1]_* \cong k^2$$

the claim follows. If $i < 0$ the injectivity of the maps x_* implies

$$\dim[E_i, Y]_p = \dim[E_i, E_1]_p - \dim[E_i, E_0]_p = 1 - i \pm 1 + i - \mp 1 = 1$$

for $p \in \{0, 1\}$. The assertion for $i > 0$ follows in the same matter if for $i > 0$ the map

$$[E_i, E_0]_p \xrightarrow{x_*} [E_i, E_1]_p$$

is surjective for $p \in \{1, 2\}$. By Serre duality the surjectivity of

$$[E_i, E_0]_p \xrightarrow{x_*} [E_i, E_1]_p, p = 1, 2$$

is equivalent to the injectivity

$$[E_0, E_i]_p \xrightarrow{x^*} [E_1, E_i]_p, p = 0, 1.$$

But this can be established in the same way as the injectivity of x_* in the proof of 4.9. □

The following theorem answers the question about faithfulness of braid group actions in type $\tilde{\mathrm{A}}_1$.

Theorem 4.16. *For every* Γ*-configuration* (E_1, E_2) *in* $\mathcal{D}$ *the homomorphism*

$$T^\Gamma : B_\Gamma \longrightarrow \mathrm{Aut}(\mathcal{D})$$

from formula (4) *is injective.*

Proof. The spherical objects $E_1, E_2[1]$ satisfy the equality

$$[E_1, E_2[1]]_* \cong k^2,$$

which we imposed in this chapter. But $T_{E_2} = T_{E_2[1]} \in \mathrm{Aut}(\mathcal{D})$ (see 2.15), hence we can assume that

$$[E_1, E_2]_* \cong k^2.$$

Then 4.15 shows that we can apply 4.13 to conclude that $\mathrm{Ker}(T^\Gamma) \subseteq \langle \omega \rangle$. But

$$(T^\Gamma_\omega)^j(E_0) \ncong E_0,\ j \in \mathbb{Z}, j \neq 0$$

by 4.10. Hence, $\mathrm{Ker}(T^\Gamma) = \{1\}$. □

As an application we derive the following (see 2.17).

Corollary 4.17. *Let* X *be a K3 surface and* $C, D \subseteq X$ *two* (-2)*-curves meeting transversally in two points or in one point with multiplicity* 2. *Then* $T_{\mathcal{O}_C}$ *and* $T_{\mathcal{O}_D}$ *do not have any relations, that is they generate a free group in* $\mathrm{Aut}(\mathcal{D}^b(X))$.

To determine the group $\langle T_{\mathcal{O}_C}, T_{\mathcal{O}_D} \rangle$ as in corollary 4.17 has been the initial task of this master thesis.

4.5 Bridgeland's conjecture

In this chapter we want to link our result 4.16 to Bridgeland's conjecture on the group of autoequivalences of a complex, projective K3 surface.

Bridgeland's conjecture aims at a complete description of the full group $\mathrm{Aut}(\mathcal{D}^b(X))$ of autoequivalences of $\mathcal{D}^b(X)$ of a smooth projective complex K3 surface X. Roughly, it describes a part of the group of autoequivalences via the fundamental group of a certain open subset

$$\mathcal{P}_0^+ \subseteq (H^0(X, \mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X, \mathbb{Z}))_\mathbb{C}.$$

More precisely, T. Bridgeland ([Bri08]) constructs a homomorphism

$$\rho : \pi_1(\mathcal{P}_0^+) \longrightarrow \mathrm{Aut}(\mathcal{D}^b(X))$$

with image consisting (conjecturally) of all autoequivalences which act trivially on the cohomology of X. T. Bridgeland conjectures that the homomorphism ρ is always injective. We will show that our result on faithfulness confirms his conjecture.

The precise definition of $\mathcal{P}_0^+$ is not relevant for use. We only mention that the full class of spherical objects participate in the construction of $\mathcal{P}_0^+$, for details see [Bri08].

Our result 4.16 on faithfulness implies

Corollary 4.18. *Let* $E_0, E_1 \in \mathcal{D}^b(X)$ *be spherical and* $f, g \in \pi_1(\mathcal{P}_0^+)$ *such that* $\rho(f) = T^2_{E_0}$ *and* $\rho(g) = T^2_{E_1}$. *If* (E_0, E_1) *is an* $\Gamma := \tilde{\mathrm{A}}_1$*-configuration, then* $\rho_{|\langle f,g \rangle}$ *is injective.*

Proof. Consider the homomorphism

$$T^{\Gamma} : B_{\Gamma} \longrightarrow \mathrm{Aut}(\mathcal{D}^b(X))$$

arising from the configuration (E_0, E_1). Recall that B_{Γ} is a free group on generators s_0, s_1. By 4.16 the homomorphism T^{Γ} is injective. The subgroup $H := \langle s_0^2, s_1^2 \rangle$ is free by the well-known theorem of Schreier–Nielsen as B_{Γ} is free. Therefore, we can factor $\rho \circ T^{\Gamma}_{|H}$ over $\pi_1(\mathcal{P}_0^+)$ by sending s_0^2 to f and s_1^2 to g. The injectivity of T^{Γ} now yields the result the injectivity of

$$\rho_{|\langle f,g \rangle} : \langle f, g \rangle \longrightarrow \mathrm{Aut}(\mathcal{D}).$$

□

A result of T. Bridgeland (see [Huy11, remark 5.10]) shows that the existence of f, g as in corollary 4.18 is always satisfied.

5 The triangulated category in type $\tilde{\mathrm{A}}_1$

In this chapter we want to determine the triangulated category generated by two spherical objects in an $\tilde{\mathrm{A}}_1$-configuration and show that its isomorphism type is independent of the given $\tilde{\mathrm{A}}_1$-configuration (see 5.21). For showing this we will give a short review of Keller's classification of algebraic triangulated categories via derived categories of dg-categories (see section 5.1). We want to add that if $G_i, i \in J$, is a class of objects in a triangulated category $\mathcal{D}$, we then write

$$\langle G_i, i \in J \rangle$$

to denote the smallest full triangulated subcategory in $\mathcal{D}$ containing the objects $G_i, i \in J$. In the case $\langle G_i, i \in J \rangle = \mathcal{D}$, we say that the objects $G_i, i \in J$, generate the triangulated category $\mathcal{D}$.

5.1 Keller's classification of algebraic triangulated categories

In this chapter we want to present a theorem of B. Keller about algebraic triangulated categories. Unfortunately, stating Keller's theorem requires many definitions. Our main reference for them will be [Kel06].

Let $\mathcal{D}$ be a triangulated category and $G \in \mathcal{D}$. Define the graded algebra

$$A := [G, G]_*.$$

Sending an object $Y \in \mathcal{D}$ to $[G, Y]_*$ yields a functor

$$F : \mathcal{D} \longrightarrow A\text{-GrMod}$$

from $\mathcal{D}$ into the category of graded right modules over A. As the category A-GrMod is abelian, this functor F is not a good candidate for being an equivalence or at least fully faithful. But if one adds "more" informations to F, then a similar construction works with the abelian category A-GrMod replaced by a derived category of a dg-category. Of course, to say something reasonable over $\mathcal{D}$ one has to assume furthermore that G generates $\mathcal{D}$. But the construction also works with more than one chosen object in $\mathcal{D}$.

Definition 5.1. *Let k be a field and* Com(k) *be the category of (co)complexes over k. Then a* **differential-graded category** *or* **dg-category** *$\mathcal{A}$ over k is a small category enriched over* Com(k). *This means that for every objects $B, C \in \mathcal{A}$ the morphisms* $\mathrm{Hom}_{\mathcal{A}}(B, C)$ *are complexes in* Com(k) *such that the composition*

$$\mathrm{Hom}_{\mathcal{A}}(C, D) \otimes \mathrm{Hom}_{\mathcal{A}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(B, D)$$

is a morphism of complexes. The **cohomology** *$H^*(\mathcal{A})$ of a dg-category $\mathcal{A}$ is the graded category (i.e. category enriched over the category of graded k-modules) having objects as $\mathcal{A}$ and as morphisms the cohomology*

$$\mathrm{Hom}_{H^*(\mathcal{A})}(B, C) := H^*(\mathrm{Hom}_{\mathcal{A}}(B, C)), \ B, C \in H^*(\mathcal{A}).$$

A dg-category A with one object is also called a **dg-algebra**. It is just a graded algebra

$$A := \bigoplus_{p \in \mathbb{Z}} A^p$$

together with a differential $d : A \longrightarrow A[1]$ (that is $dd = 0$) such that for $f \in A^p, g \in A$ the Leibniz rule

$$d(fg) = d(f)g + (-1)^p f d(g)$$

holds. The cohomology of the dg-category A is the cohomology $H^*(A)$ of the complex A, which inherits a multiplication due to the Leibniz rule. (See [Kel06, chapter 2.2] for more details).

Remark 5.2. *A typical example of a dg-category can be obtained as follows: Let $M_i^\bullet, i \in J$, be complexes in some abelian category. Define a category $\mathcal{A}$ with objects $M_i^\bullet, i \in J$, and for $i, j \in J$ set*

$$\mathrm{Hom}^p(M_i^\bullet, N_j^\bullet) := \prod_{q\in\mathbb{Z}} \mathrm{Hom}(M_i^q, N_j^{q+p}),\ p \in \mathbb{Z}$$

and

$$\mathrm{Hom}_{\mathcal{A}}(M_i^\bullet, M_j^\bullet) := \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}^p(M_i^\bullet, N_j^\bullet)$$

with the differential

$$d(f) = d \circ f - (-1)^p f \circ d.$$

for $f \in \mathrm{Hom}^p(M^\bullet, N^\bullet)$. The category $\mathcal{A}$ is then a dg-category. Taking as the objects $M_i^\bullet$ all complexes of k-vector spaces we get a (big) dg-category, which we call $\mathcal{C}_{dg}(k)$ see [Kel06, chapter 2.2].

As in the case of ordinary algebras one can consider modules over a dg-category. Recall that to give an abelian group M the structure of a right module over a ring R is the same as to give a ring homomorphism

$$R^{op} \longrightarrow \mathrm{End}(M).$$

Definition 5.3. *Let $\mathcal{A}$ be a dg-category over a field k. Then a* **(right) dg-module** *over $\mathcal{A}$ is dg-functor*

$$\mathcal{M} : \mathcal{A}^{op} \longrightarrow \mathcal{C}_{dg}(k)$$

(*a dg-functor is just a functor of categories enriched over* Com(k)) *and a morphism of right dg-modules $\mathcal{M}, \mathcal{N}$ is a natural transformation $f : \mathcal{M} \longrightarrow \mathcal{N}$ such that for every $B \in \mathcal{A}$ the morphism $f_B : \mathcal{M}(B) \longrightarrow \mathcal{N}(B)$ is a morphism of complexes.*

Example 5.4. *Every object $B \in \mathcal{A}$ gives rise to a dg-module via its* Hom*-functor*

$$\mathrm{Hom}_{\mathcal{A}}(-, B) : \mathcal{A}^{op} \longrightarrow \mathcal{C}_{dg}(k).$$

As in the case of a dg-category, one defines the cohomology $H^*(\mathcal{M})$ of a dg-module $\mathcal{M}$. It is then a graded right module over the graded category $H^*(\mathcal{A})$. Using this one obtains the notion of a quasi-isomorphism of dg-modules as a morphism $f : \mathcal{M} \longrightarrow \mathcal{N}$ of dg-modules inducing an isomorphism in cohomology. We finally obtain the definition of the derived category for a dg-category.

Definition 5.5. *Let $\mathcal{A}$ be a dg-category and let $\mathcal{A}$-dgMod be the category of dg-modules. Define the* **derived category** *$\mathcal{D}(A)$ of A to be the localization of $\mathcal{A}$-dgMod with respect to the class of quasi-isomorphisms of dg-modules.*

Remark 5.6. *The derived category $\mathcal{D}(A)$ carries the structure of a triangulated category (see [Kel06, chapter 3.4]).*

Again, if A is a dg-algebra, then a dg-module over the dg-category A is the same thing as a graded module over A together with a differential

$$d : M \longrightarrow M[1]$$

such that the Leibniz rule

$$d(mf) = d(m)f + (-1)^p md(f),\ m \in M^p,\ f \in \mathcal{A}$$

holds. To show that dg-categories are in some cases not far away from dg-algebras we proof the following:

Lemma 5.7. *Let $\mathcal{A}$ be a dg-category such that the class of objects in $\mathcal{A}$ is finite. We define A as the dg-algebra*

$$A := \bigoplus_{B,C \in \mathcal{A}} [B, C]_*,$$

where the composition in $\mathcal{A}$ is extended by zero to morphisms not composable in $\mathcal{A}$. Then the categories $\mathcal{A}$-dgMod and A-dgMod are equivalent via sending a dg-module $\mathcal{M}$ over $\mathcal{A}$ to the direct sum

$$M := \bigoplus_{B \in \mathcal{A}} \mathcal{M}(B).$$

Proof. We just mention the construction of the inverse. The identities $e_B := \mathrm{Id}_B$ of the objects $B \in \mathcal{A}$ yield a decomposition of the unit in A into orthogonal idempotents:

$$1 = \sum_{B \in \mathcal{A}} e_B.$$

Let M be a dg-module over A. Writing

$$M = M1 = \bigoplus_{B \in \mathcal{A}} Me_B$$

gives a decomposition of M. Then define the dg-module $\mathcal{M}$ over $\mathcal{A}$ by

$$\mathcal{M} : \mathcal{A}^{op} \longrightarrow \mathcal{C}_{dg}(k) : B \mapsto Me_B.$$

Checking that both constructions extend to morphisms and are inverse to each other will be omitted. □

The main theorem of B. Keller stated below describes algebraic triangulated categories in terms of derived categories of dg-categories. By definition, a triangulated category is called algebraic if it is isomorphic (as a triangulated category) to the stable category of some Frobenius category (see [Kel07, chapter 8.7]). Without going into the details of this definition, we want to mention that derived categories of abelian categories are always algebraic. Also full triangulated subcategories of algebraic triangulated categories stay algebraic (see [Kel06, chapter 3.6]). So for our purpose we can restrict ourselves to algebraic triangulated categories and we will do so.

We now state Keller's theorem ([Kel06, theorem 3.8]) and sketch its proof. Let $\mathcal{D}$ be an algebraic triangulated category and $\mathcal{G} \subseteq \mathcal{D}$ a class of objects. Define a graded category $\mathcal{G}_{gr}$ as a category with objects $G \in \mathcal{G}$ and morphisms

$$\mathrm{Hom}_{\mathcal{G}_{gr}}(G, G') := \bigoplus_{p \in \mathbb{Z}} [G, G']_p$$

for $G, G' \in \mathcal{G}$. Define a functor

$$\overline{F} : \mathcal{D} \longrightarrow \mathcal{G}_{gr}\text{-GrMod}$$

into the category of graded $\mathcal{G}_{gr}$-modules by sending $Y \in \mathcal{D}$ to the module

$$\mathcal{G}_{gr}^{op} \longrightarrow k\text{-GrMod} : G \longrightarrow [G, Y]_*.$$

Theorem 5.8. *With notations as above, there exists a dg-category $\mathcal{A}$ and an exact functor*

$$F : \mathcal{D} \longrightarrow \mathcal{D}(\mathcal{A})$$

such that $H^*(A) \cong \mathcal{G}_{gr}$ and $H^* \circ F \cong \overline{F}$. Moreover, if $\mathcal{D} = \langle G, G \in \mathcal{G} \rangle$ is generated by $\mathcal{G}$, then F can be chosen to induce an equivalence

$$F : \mathcal{D} \longrightarrow \langle \mathrm{Hom}_{\mathcal{A}}(-, G), G \in \mathcal{A} \rangle$$

sending G to $\mathrm{Hom}_{\mathcal{A}}(-, G)$.

Proof. The idea behind the proof is very much like the one discussed at the beginning of this chapter but one furthermore considers everything in the dg-setting. Fix a dg-enhancement $\tilde{\mathcal{D}}$ of $\mathcal{D}$, this is possible as $\mathcal{D}$ is algebraic and it means that $\mathcal{D}$ is equivalent to the homotopy category of $\tilde{\mathcal{D}}$. Consider the dg-category $\mathcal{A}$ with objects $G \in \mathcal{G}$ and morphisms

$$\mathrm{Hom}_{\mathcal{A}}(G, G') := \mathrm{Hom}_{\tilde{\mathcal{D}}}(G, G').$$

Sending an object $Y \in \mathcal{D}$ to the dg-module

$$\mathcal{M}_Y : \mathcal{A}^{op} \longrightarrow \mathcal{C}_{gr}(k) : G \longrightarrow \mathrm{Hom}_{\tilde{\mathcal{D}}}(G, Y)$$

yields a functor $\tilde{\mathcal{D}} \longrightarrow \mathcal{A}$-dgMod, which descends to a functor

$$F : \mathcal{D} \longrightarrow \mathcal{D}(\mathcal{A}).$$

This functor F then satisfies all assertions. See [Kel94, chapter 4.3] for more details. □

The theorem of B. Keller has the disadvantage that one does not have an immediate description of the dg-category $\mathcal{A}$ and apparently there is no justified hope that the graded category $\mathcal{G}_{gr}$ determines $\mathcal{A}$ in general (at least up to quasi-equivalence). But fortunately, there is a non-empty - as we will see- class of graded categories, where this is the case.

Definition 5.9. *Let A be a graded algebra. Then A is said to be* **intrinsically formal** *if for every dg-algebra $\mathcal{A}$ such that $H^*(\mathcal{A}) \cong A$ there is a quasi-isomorphism $\mathcal{A} \cong A$ in the category of dg-algebras. (see [ST01, definition 4.6]).*

We also need

Lemma 5.10. *Let $\varphi : A \longrightarrow B$ be a morphism of dg-algebras, which is a quasi-isomorphism of complexes. Then there is an equivalence*

$$\mathcal{D}(A) \longrightarrow \mathcal{D}(B)$$

sending A to B.

Proof. See [RS07, example 7.15], taking the derived tensor product with the B-A-bimodule B yields an equivalence $\mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ sending B to A. □

Example 5.11. *Let k be a field. Then the graded algebra $A := [E, E]_*$ of a spherical object E in a k-linear K3 category $\mathcal{D}$ is intrinsically formal, see [KYZ09, theorem 2.1]. In particular, there is an equivalence*

$$\langle E \rangle \longrightarrow \langle A \rangle \subseteq \mathcal{D}(A)$$

sending E to A if A is considered as a dg-algebra with trivial differential.

5.2 The graded algebra in type $\tilde{\mathrm{A}}_1$

Let k be a field and $\mathcal{D}$ be k-linear K3 category equivalent to a full triangulated subcategory of the derived category of a K3 surface. Assume that $E_0, E_1 \in \mathcal{D}$ are spherical. Fix the graph $\Gamma := \tilde{\mathrm{A}}_1$. Assume that (E_0, E_1) is a Γ-configuration and set $\mathcal{E} := E_0 \oplus E_1$. In this section we want to determine the graded algebra

$$A := \bigoplus_p [\mathcal{E}, \mathcal{E}]_p.$$

Let $R := A_0 = \langle \mathrm{Id}_{S_0}, \mathrm{Id}_{S_1} \rangle \cong k \times k$, which is a semisimple k-algebra admitting the automorphism (as a k-algebra)

$$\sigma : R \longrightarrow R$$

permuting the two factors. Denote by $e_1 := \mathrm{Id}_{E_0} (= (1, 0))$ and $e_2 := \mathrm{Id}_{E_1} (= (0, 1))$ the two (non-trivial) idempotents of R.

Let $M := Rb_1 \oplus Rb_2$ be the free R-left module on b_1 and b_2 and give M the R-R-bimodule structure with right multiplication twisted by σ.

Proposition 5.12.

$$A_1 \cong M$$

as R-R-bimodules.

Proof. Let $x, y \in [E_0, E_1]_1$ and $\delta, \varepsilon \in [E_1, E_0]_1$ be two k-bases. Then $x, y, \delta, \varepsilon$ is a k-basis for $A_1 \cong [E_0, E_1]_1 \oplus [E_1, E_0]_1$. We claim that α defined (as a morphism of R-left-modules) by

$$\alpha : M \longrightarrow A_1 : b_1 \mapsto x + \epsilon,\ b_2 \mapsto y + \delta$$

defines an isomorphism of R-R-bimodules. For this we compute

$$\alpha(b_1 \sigma(e_1)) = \alpha(e_2 b_1) = e_2(x + \epsilon) = x = xe_1 = (x + \epsilon)e_1$$

and similarly the other cases. □

We denote by $TM = \bigoplus_{p \geq 0} M^{\otimes p}$ the tensor algebra of M considered as a bimodule over R. The last proposition implies that we can extend every bimodule homomorphism

$$\alpha : M \longrightarrow A_1$$

to a homomorphism

$$\alpha : TM \longrightarrow A$$

of graded R-algebras. Note that we speak of R-algebras, although R is not contained in the center of A or TM.

We need the following result about Serre duality. Recall that the Serre functor

$$[2] : \mathcal{D} \longrightarrow \mathcal{D}$$

yields natural isomorphisms

$$\Phi_{F_1,F_2} : [F_1, F_2]_0 \longrightarrow [F_2, F_1]_2^\vee$$

for $F_1, F_2 \in \mathcal{D}$. In particular, we can define natural trace maps

$$\mathrm{tr}_F : [F, F]_2 \longrightarrow k$$

as $\Phi_{F,F}(\mathrm{Id}_F)$ for every $F \in \mathcal{D}$.

Lemma 5.13. *Let $F_1, F_2 \in \mathcal{D}$ be two objects, and $x \in [F_1, F_2]_1$ and $\delta \in [F_2, F_1]_1$ two morphisms. Then*

$$\mathrm{tr}_{F_1}(\delta x) = -\mathrm{tr}_{F_2}(x\delta).$$

Proof. We can assume that $\mathcal{D} = \mathcal{D}^b(X)$ is the derived category of a K3 surface X and that the isomorphism

$$\Phi_{F_1,F_2} : [F_1, F_2]_0 \longrightarrow [F_2, F_1]_2^\vee$$

are obtained by the usual trace maps for locally free sheaves (see [Huy06, lemma 3.12]). Then we can apply [HL10, lemma 10.1.3] and derive the claim. □

Lemma 5.13 is the reason why we assumed that $\mathcal{D}$ is equivalent to a full triangulated subcategory of the derived category of a K3 surface. We do not know whether 5.13 holds for an arbitrary k-linear K3 category $\mathcal{D}$.

Now we can determine the algebra A.

Proposition 5.14. *We have*

$$A \cong \Lambda^*(M) := TM/\langle b_1 \otimes b_1, b_2 \otimes b_2, b_1 \otimes b_2 + b_2 \otimes b_1 \rangle$$

as R-algebras.

Proof. The pairing

$$[E_0, E_1]_1 \otimes [E_1, E_0]_1 \longrightarrow [E_i, E_i]_2, \ i = 0, 1,$$

given by composition is non-degenerate (see 2.11), so the R-algebra A is generated by A_1. Let x, y be a basis of $[E_0, E_1]_1$ and let $\delta, \epsilon \in [E_1, E_0]_1$ be the dual basis with respect to the pairing $[E_0, E_1]_1 \otimes [E_1, E_0]_1 \longrightarrow [E_0, E_0]_2$. We then have

$$\mathrm{tr}_{E_0}(\delta \circ x) = 1, \ \mathrm{tr}_{E_0}(\delta \circ y) = 0, \ \mathrm{tr}_{E_0}(\varepsilon \circ x) = 0, \ \mathrm{tr}_{E_0}(\varepsilon \circ y) = 1.$$

Define

$$\alpha : M \longrightarrow A_1$$

by sending $b_1 \mapsto x - \varepsilon,\ b_2 \mapsto y + \delta$ and extend it to TM. It is easily calculated that

$$\alpha(b_1 \otimes b_1) = (x - \varepsilon)(x - \varepsilon) = xx - x\varepsilon - \varepsilon x + \varepsilon\varepsilon = 0.$$

Similarly, $\alpha(b_2 \otimes b_2) = 0$. Moreover, we see

$$\alpha(b_1 \otimes b_2 + b_2 \otimes b_1) = (x - \varepsilon)(y + \delta) + (y + \delta)(x - \varepsilon)$$

$$= xy - \varepsilon y + x\delta - \varepsilon\delta + yx - y\varepsilon + \delta x - \delta\varepsilon = x\delta - y\varepsilon - \varepsilon y + \delta x.$$

We know that $\mathrm{tr}_{E_0}(-\varepsilon y + \delta x) = 0$ and hence $-\varepsilon y + \delta x = 0$. But by 5.13 we also know that $\mathrm{tr}_{E_1}(x\delta - y\varepsilon) = 0$. So we conclude

$$\alpha(b_1 \otimes b_2 + b_2 \otimes b_1) = 0$$

and hence $A \cong \Lambda^*(M)$ as both have the same dimension. □

The algebra $\Lambda^*(M)$ looks like an exterior algebra, but it is only a twisted version due to the twisted right module structure of M.

5.3 Proof of formality

In this section we want to proof that the graded algebra A from section 5.2 is intrinsically formal by showing that it is Koszul. Using the results of 5.1 we can then describe the triangulated category of an $\tilde{\mathrm{A}}_1$-configuration.

First we introduce the notion of a Koszul algebra. Let $R = k \times k$ with k a field and let

$$A = \bigoplus_{p \geq 0} A^p$$

be a graded algebra such that $A_0 = R$. For every $p \geq 0$ the abelian group is an R-R-bimodule via left and right multiplication in A. For a R-R-bimodule V we denote by V^* the R-R-bimodule $\mathrm{Hom}_R(V, R)$ of homomorphisms of left R-modules.

Definition 5.15. *We call A a* **quadratic algebra** *over R if the natural homomorphism of graded algebras*

$$TA_1 \longrightarrow A$$

is surjective and its kernel I is generated by the homogeneous part I_2 of degree 2. If

$$A \cong TA_1/I$$

is a quadratic algebra, then we define the **(left) quadratic dual** *of A to be the algebra*

$$A^! := TA_1^*/I^\perp,$$

where $I^\perp$ is the ideal generated by

$$\{\ \psi \in A_1^* \otimes_R A_1^* \mid \psi(I) = 0\}.$$

See [BGS96, chapter 2] for more details.

Example 5.16. *Consider the twisted exterior algebra*

$$A := \Lambda^* M = TM/\langle b_1 \otimes b_1, b_2 \otimes b_2, b_1 \otimes b_2 + b_2 \otimes b_1 \rangle$$

from section 5.2. In particular, we see that it is quadratic. The quadratic dual $A^!$ of A is given by

$$S^*(M^*) := TM^*/\langle \beta_1 \otimes \beta_2 - \beta_2 \otimes \beta_1 \rangle,$$

if β_1, β_2 is the dual basis of b_1, b_2. Note that the algebra $A^!$ is not commutative, a consequence of the twisted module structure of M.

Koszul algebras are special quadratic algebras.

Definition 5.17. *Let A be a quadratic R-algebra. Then A is called a* **Koszul algebra** *if the A-left module $A_0 = R$ admits a linear projective resolution*

$$.. \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow R \longrightarrow 0,$$

i.e. for every $i \geq 0$ the module P_i is a direct summand of a sum of the shifted free module $A[-i]$.

Remark 5.18. *1) If*

$$.. \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow R \longrightarrow 0$$

is a linear free resolution, i.e. every P_i is a direct sum of $A[-i]$, then the differentials have coefficients in A_1. This explains the name "linear".

2) For various characterizations, including one using the Koszul complex, see [BGS96, chapter 2].

3) Examples of Koszul algebras are symmetric algebras and exterior algebras, see for example [BGS96].

Now assume that A is a quadratic algebra such that $A_p,\ p \in \mathbb{Z}$, is a finitely generated R-left module. We then have the following theorem.

Theorem 5.19. *1) The algebra A is Koszul if and only if $A^!$ is Koszul.*

2) If A is Koszul, then A is intrinsically formal.

Proof. Part 1) can be found in [BGS96, proposition 2.9.1]. For part 2) we sketch a proof using A_∞-algebras (see [Kel99] for definitions). Let $\mathcal{A}$ be a dg-algebra having A as cohomology. By a theorem of Kadeishvili (see [Kel99, chapter 3.3], which can be applied in this situation as every A_p is R-projective, see [Sag10]) there is a structure of an A_∞-algebra on A such that A and $\mathcal{A}$ become quasi-isomorphic as A_∞-algebras. But the A_∞-structure of $A \cong \operatorname{Ext}^*_{A^!}(R, R)$ ([BGS96, theorem 2.10.1]) can be chosen in such a way that the internal degrees are preserved. Therefore it has to be formal by the same argument as in [Con11, corollary V.0.6] because the algebra $A^!$ is Koszul by 1). Hence A and $\mathcal{A}$ are quasi-isomorphic as A_∞-algebras. Therefore A and $\mathcal{A}$ are also quasi-isomorphic as dg-algebras and A is intrinsically formal. $\square$

For hinting 2) to us we want to thank Prof. Catharina Stroppel. Now we tie together the results from section 5.1 and 5.2. Recall the R-R-bimodule M from 5.12 and the algebra $A := \Lambda^* M := TM/\langle b_1 \otimes b_1, b_2 \otimes b_2, b_1 \otimes b_2 + b_2 \otimes b_1 \rangle$ from 5.14, which is isomorphic to the graded algebra for an $\tilde{\mathrm{A}}_1$-configuration by 5.14.

Theorem 5.20. *The algebra*

$$A = \Lambda^*(M)$$

is Koszul. In particular, A is intrinsically formal by Theorem 5.19.

Proof. By Theorem 5.19 it suffices to check that $A^!$ is Koszul. But 5.16 shows that

$$A^! \cong T(M^*)/\langle \beta_1 \otimes \beta_2 - \beta_2 \otimes \beta_1 \rangle$$

with $\beta_1, \beta_2 \in M^*$ the dual basis of b_1, b_2. Consider the "Koszul complex"

$$0 \longrightarrow A^![-2] \overset{(\beta_2,\beta_1)}{\longrightarrow} A^![-1] \oplus A^![-1] \overset{(\beta_1,\beta_2)}{\longrightarrow} A^! \longrightarrow R \longrightarrow 0, \tag{16}$$

which is obviously a linear projective resolution provided it is exact. (Here "β" denotes the right multiplication with β.) But as a sequence of R-left modules the sequence is isomorphic to the usual Koszul complex for the symmetric algebra $S^*(R^2)$, which is known to be exact. Hence, (16) is exact and therefore $A^!$ is Koszul. Applying 5.19 we get that A is Koszul. ☐

We are now in the position to determine the triangulated category generated by two spherical objects in an $\tilde{\mathrm{A}}_1$-configuration.

Theorem 5.21. *Let $\mathcal{D}$ be a full triangulated category of the derived category of a K3 surface, $E_0, E_1 \in \mathcal{D}$ spherical and assume that (E_0, E_1) form an $\tilde{\mathrm{A}}_1$-configuration. Then we get an exact, fully faithful functor*

$$\langle E_0, E_1 \rangle \longrightarrow \mathcal{D}(A)$$

sending E_0 to $e_1 A$ and E_1 to $e_2 A$. Here we consider A as a dg-algebra with a trivial differential. In particular, we get an equivalence

$$\langle E_0, E_1 \rangle \cong \langle e_1 A, e_2 A \rangle.$$

Proof. By 5.8 and 5.7 we know that there exists a dg-algebra $\mathcal{A}$ with $H^*(\mathcal{A}) \cong A$ and a fully faithful functor $\langle E_0, E_1 \rangle \longrightarrow \mathcal{D}(\mathcal{A})$. But as A is intrinsically formal (see 5.20), we get a quasi-isomorphism $\mathcal{A} \cong A$. Hence, we can identify $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(A)$ using 5.10. Following the proofs of 5.8, 5.7 and 5.10 we see that we can arrange things in such a way that under the resulting fully faithful functor

$$\langle E_0, E_1 \rangle \longrightarrow \mathcal{D}(A)$$

the objects E_i are sent to $e_{i+1} A$. ☐

With this theorem at hand can give the following amusing application.

Corollary 5.22. *With assumptions as in 5.21 we get an autoequivalence of order two of $\langle E_0, E_1 \rangle$ sending E_0 to E_1.*

Proof. The automorphism $\sigma : R \longrightarrow R$ permuting both factors can be extended to $\Lambda^*(M)$ by applying it on coefficients. Denote this extension again by σ. Then σ induces an autoequivalence of $\mathcal{D}(A)$ permuting the two modules of $e_1 A$ and $e_2 A$ of $\Lambda^*(M)$. By theorem 5.21 we can conjugate σ and get a desired automorphism of $\langle E_0, E_1 \rangle$ having the required properties. ☐

Remark 5.23. *1) Using derived McKay-correspondence an automorphism as in the corollary could also have been obtained (in the case $k \cong \mathbb{C}$) as follows. By theorem 5.21 we can assume that $E_0 = \mathcal{O}_D, E_1 = \mathcal{O}_D(-1)[1]$, where D is the exceptional divisor of a minimal resolution X of the A_1-singularity of $\mathbb{C}^2/G$ with $\mathbb{Z}/2\mathbb{Z} \cong G \subseteq SL_2(\mathbb{C})$. Derived McKay-correspondence yields an equivalence*

$$\mathcal{D}^b(X) \cong \mathcal{D}^b(Coh^G(\mathbb{C}^2))$$

sending E_0 and E_1 to the two irreducible representations V_0, V_1 of G (considered as G-equivariant sheaves supported in 0). It is then clear that tensoring with the non-trivial character of G yields a desired autoequivalence (see [KV00] for more details about the derived McKay-correspondence).

2) Theorem 5.21 yields a different way of proving theorem 4.16. Indeed, according to theorem 5.21 it suffices to take $E_0 = \mathcal{O}_C$ and $E_1 = \mathcal{O}_C(1)$ for some (-2)-curve $C \subseteq X$ on a K3 surface (such a curve exists). Then one can avoid the lengthy calculations in section 4.2 and directly apply 4.13 because calculating the Hom*-spaces $[E_i, E_j]_*$, $i, j \in \mathbb{Z}$, is much easier in this case.*

6 An example in the case $\Gamma = \tilde{\mathrm{A}}_2$

After having discussed the cases ADE or $\tilde{\mathrm{A}}_1$ we now consider an example for $\Gamma := \tilde{\mathrm{A}}_2$ and obtain a partial result on faithfulness. For this let X be a K3 surface and $C, D \subseteq X$ two (-2)-curves meeting in a single point. We define $\mathcal{D} := \mathcal{D}^b(X)$ and the spherical objects

$$A_j := \mathcal{O}_C(j),\ j \in \mathbb{Z},$$

$$B_j := \mathcal{O}_D(j),\ j \in \mathbb{Z},$$

in $\mathcal{D}$.

6.1 An example for $\Gamma = \tilde{\mathrm{A}}_2$

We want to understand the group

$$H := \langle T_{A_j}, T_{B_j} \mid j \in \mathbb{Z} \rangle \subseteq \mathrm{Aut}(\mathcal{D}).$$

Recall that

$$T_{A_0}(B_{-1}) = T_{\mathcal{O}_C}(\mathcal{O}_D(-1)) \cong \mathcal{O}_{C\cup D}$$

by 2.22.

Lemma 6.1. *We have*

$$H = \langle T_{\mathcal{O}_C}, T_{\mathcal{O}_D}, T_{\mathcal{O}_{C\cup D}} \rangle$$

as subgroups of $\mathrm{Aut}(\mathcal{D})$*. Moreover,* $(\mathcal{O}_C, \mathcal{O}_D, \mathcal{O}_{C\cup D}[1])$ *is a* Γ*-configuration.*

Proof. By 2.15 and 2.22 we see

$$T_{\mathcal{O}_{C\cup D}} \cong T_{T_{\mathcal{O}_C}(\mathcal{O}_D(-1))} \cong T_{\mathcal{O}_C} T_{\mathcal{O}_D(-1)} T_{\mathcal{O}_C}^{-1},$$

so

$$\langle T_{\mathcal{O}_C}, T_{\mathcal{O}_D}, T_{\mathcal{O}_{C\cup D}} \rangle \subseteq H.$$

On the other hand, it follows as in 2.22 or from 2.18 that

$$T_{\mathcal{O}_{C\cup D}}(\mathcal{O}_C) \cong \mathcal{O}_D(-1)[1]$$

and

$$T_{\mathcal{O}_{C\cup D}}(\mathcal{O}_D) \cong \mathcal{O}_C(-1)[1].$$

So $H \subseteq \langle T_{\mathcal{O}_C}, T_{\mathcal{O}_D}, T_{\mathcal{O}_{C\cup D}} \rangle$ by the discussion at the beginning of 4.2. That

$$(\mathcal{O}_C, \mathcal{O}_D, \mathcal{O}_{C\cup D}[1])$$

is a Γ-configuration follows from the proof of 2.22. □

6.2 A presentation of $B_{\tilde{A}_2}$

Let denote the vertices of $\Gamma = \tilde{A}_2$ by $0, 1$ and 2. Recall the definition of $B := B_\Gamma$:

$$B = \langle s_0, s_1, s_2 \mid s_i s_j s_i = s_j s_i s_j, \text{ for } i, j \in \{0, 1, 2\},\ i \neq j\rangle.$$

In view of 6.1 and 4.1 we will give a presentation of B which seems much more natural for the example of 6.1. We guess that our presentation is (at least in principle) known to people working on braid groups, but we did not find it in the literature.

Define a group $\tilde{B}$ as follows: Let $\tilde{B}$ be the group given by generators a_j, b_i with $i, j \in \mathbb{Z}$, and relations

$$a_j b_i a_j = b_i a_j b_i$$

$$\omega_a := a_{i-1} a_i = a_{j-1} a_j$$

$$\omega_b := b_{i-1} b_i = b_{j-1} b_j$$

$$\omega_a b_j = b_{j+1} \omega_a$$

$$\omega_b a_j = a_{j+1} \omega_b,\ i, j \in \mathbb{Z}.$$

Note that the relations imply that $\omega_a \omega_b = \omega_b \omega_a$.

With the notation from the previous section 6.1 we get, by construction, a homomorphism

$$\tilde{T} : \tilde{B} \longrightarrow \langle T_{A_j}, T_{B_i} \mid i, j \in \mathbb{Z}\rangle \mathrm{Aut}(\mathcal{D}), \tag{17}$$

because we added all relations which are known to be satisfied by the T_{A_j} and T_{B_i}. At least for the first three relations this has been discussed in 2.18 and chapter 4. For the last two relations one computes

$$T_{\omega_a} T_{B_j} T_{\omega_a}^{-1} \cong T_{\mathcal{O}_X(C) \otimes_{\mathcal{O}_X} \mathcal{O}_D(j)} \cong T_{\mathcal{O}_D(j+1)}$$

and

$$T_{\omega_b} T_{A_j} T_{\omega_b}^{-1} \cong T_{\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_C(j)} \cong T_{\mathcal{O}_C(j+1)}$$

because C and D meet in a single point. Note that indeed the elements ω_a and ω_b are mapped to $\mathcal{O}_X(C) \otimes_{\mathcal{O}_X} (-)$ resp. $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} (-)$ by 2.21.

Proposition 6.2. *We get an isomorphism*

$$\varphi : B \longrightarrow \tilde{B}$$

by setting $\varphi(s_0) := a_0,\ \varphi(s_1) := b_0$ *and* $\varphi(s_2) := a_0 b_{-1} a_0^{-1}$.

Proof. By proposition 4.3 we know that $\tilde{B}$ is generated by

$$a := a_0,\ b := a_1,\ c := b_{-1} \text{ and } d := b_0.$$

We throw away all other generators, express the relations in terms of a, b, c, d and see what is left. Write $b_j = \omega_a^j d \omega_a^{-j}$ for $j \neq -1$. The relation

$$\omega_a b_j = b_{j+1} \omega_a$$

then reads as

$$\omega_a^{j+1} d \omega_a^{-j} = \omega_a^{j+1} d \omega^{-j},$$

which is a trivial relation and can be thrown away. But we have to keep the relation

$$\omega_a c = d\omega_a.$$

Similarly, we can proceed with the relations $\omega_b a_j = a_{j+1}\omega_b,\ j \in \mathbb{Z}$. To handle the relation

$$a_j b_i a_j = b_i a_j b_i,$$

we argue as follows: Let us denote by $R_{i,j}$ the relation

$$a_j b_i a_j b_i^{-1} a_j^{-1} b_i^{-1}.$$

Note that

$$\omega_a R_{i,j} \omega_a^{-1} = R_{i-2,j+1}$$

and

$$\omega_b R_{i,j} \omega_b^{-1} = R_{i+1,j-2}.$$

If any two relations are conjugate in $\tilde{B}$, then we can throw one of them away. So we are, in particular, interested in the $\langle \omega_a, \omega_b \rangle$-orbits on $\{R_{i,j} \mid i,j \in \mathbb{Z}\}$. These can be investigated as follows: The elements ω_a and ω_b commute, so we look at the $\mathbb{Z}^2$-action on $\mathbb{Z}^2$ generated by addition with $(1,-2)$ and $(-2,1)$. But the orbits for that action are in bijection with the set

$$\mathbb{Z}^2/\langle (1,-2),(-2,1)\rangle,$$

which consist of three elements with representatives $(0,0),(1,0)$ and $(0,-1)$. So we can throw away all relations $R_{i,j}$, only keeping $R_{0,0}, R_{1,0}$ and $R_{0,-1}$. We therefore get an isomorphism

$$\tilde{B} \cong \langle a,b,c,d \mid ada = dad,\ bdb = dbd,\ aca = cac,\ abc = dab,\ cda = bcd\rangle.$$

Writing $c = a^{-1}ea$ yields

$$\tilde{B} \cong \langle a,b,e,d \mid ada = dad,\ bdb = dbd,\ aa^{-1}eaa = a^{-1}eaaa^{-1}ea,\ aba^{-1}ea = dab,$$

$$a^{-1}eada = ba^{-1}ead\rangle,$$

which simplifies to

$$\tilde{B} \cong \langle a,b,e,d \mid ada = dad,\ bdb = dbd,\ aea = eae,\ aba^{-1}ea = dab,\ a^{-1}ed = ba^{-1}e\rangle.$$

Finally, we delete b using $b = a^{-1}ede^{-1}a$:

$$\tilde{B} \cong \langle a,e,d \mid ada = dad,\ a^{-1}ede^{-1}ada^{-1}ede^{-1}a = da^{-1}ede^{-1}ad,\ aea = eae,$$

$$aa^{-1}ede^{-1}aa^{-1}ea = daa^{-1}ede^{-1}a\rangle.$$

This is just an awkward presentation for:

$$\tilde{B} \cong \langle a,e,d \mid ada = dad,\ a^{-1}ede^{-1}ada^{-1}ede^{-1}a = da^{-1}ede^{-1}ad,\ aea = eae,$$

$$ede = ded\rangle.$$

We multiply the relation $a^{-1}ede^{-1}ada^{-1}ede^{-1}a = da^{-1}ede^{-1}ad$ with da and get

$$dede^{-1}ada^{-1}ede^{-1}a = dada^{-1}ede^{-1}ad$$

$$\begin{aligned}\Leftrightarrow edee^{-1}ada^{-1}ede^{-1}a &= adaa^{-1}ede^{-1}ad \\ \Leftrightarrow edada^{-1}ede^{-1}a &= adede^{-1}ad \\ \Leftrightarrow eadaa^{-1}ede^{-1}a &= aedee^{-1}ad \\ \Leftrightarrow eadede^{-1}a &= aedad \\ \Leftrightarrow eae &= aea\end{aligned}$$

and so $B \cong \tilde{B}$ (via φ). □

We identify B and $\tilde{B}$ from now on using the map φ.

Definition 6.3. *Define B^+ to be the monoid in B generated by the elements $a_j, b_i,\ i, j \in \mathbb{Z}$.*

Lemma 6.4. *The monoid B^+ satisfies the assumption of 3.7 with $\Lambda = \omega_a\omega_b$.*

Proof. This follows by construction. □

For later use we make the following definition.

Definition 6.5. *Let $\alpha \in B^+$. Then we call α strict decreasing, if there is a representation*

$$\alpha = r_1..r_l$$

with $r_i \in \{a_i, b_j \mid i, j \in \mathbb{Z}\}, i = 1, .., l$, such that whenever $r_i = a_i, r_j = a_j$ with $i < j$ we have $i < j - 1$, and similarly for $r_i = b_i, r_j = b_j$.

If $\alpha \in B^+$ is strict decreasing, then looking at the relations in B we see that its representation as in 6.5 is unique. The element $\alpha = a_{-1}b_0a_0$ is strictly decreasing, while $\alpha = a_1b_0a_0$ is not.

6.3 Partial results on faithfulness in type $\tilde{\mathrm{A}}_2$

We keep the notation from the previous section and also keep identifying B and $\tilde{B}$. Let

$$R := \{a_j, b_i \mid i, j \in \mathbb{Z}\} \subseteq B^+.$$

We also recall the homomorphism (17):

$$\tilde{T} : \tilde{B} \longrightarrow \mathrm{Aut}(\mathcal{D})$$

defined by sending a_j to T_{A_j} and b_j to T_{B_j}. Composing $\tilde{T}$ with the isomorphism φ from 6.2 yields the homomorphism T^Γ (defined in 2.3, formula (4)) for the Γ-configuration

$$(A_0, B_0, T_{A_0}(B_{-1})[1]) = (\mathcal{O}_C, \mathcal{O}_D, \mathcal{O}_{C\cup D}[1]).$$

Now put $Y := k(x)$. For $\alpha \in B_+$ and $i \in \mathbb{Z}$ we define

$$l_{a_i,\alpha} := \max\{\ p \mid [A_i, T_\alpha(Y)]_p \neq 0\}$$

resp.

$$l_{b_i,\alpha} := \max\{\ p \mid [B_i, T_\alpha(Y)]_p \neq 0\}.$$

Then $l_{b_j,1} = l_{a_j,1} = 1$ for every $j \in \mathbb{Z}$ (see 2.20). More generally, we define analogously numbers $l^Z_{a_i,\alpha}$ and $l^Z_{b_i,\alpha}$ for arbitrary $Z \in \mathcal{D}$.

For describing the combinatorics in proposition 6.7 we give the following definitions.

Definition 6.6. *Let $Z \in \mathcal{D}$ and $\alpha \in B^+$.*

1) *If*

$$l^Z_{a_j,\alpha} = \begin{cases} 1, & j < i-1 \\ 0, & j = i-1 \\ 2, & j \geq i \end{cases}$$

*for some $i \in \mathbb{Z}$ we say that we have an $(OA)_i$-**picture on** $T_\alpha(Z)$. Similarly, we speak of an $(OB)_i$-**picture**.*

2) *If*

$$l^Z_{a_j,\alpha} = \begin{cases} 1, & j \leq i-1 \\ 2, & j \geq i \end{cases}$$

*for some $i \in \mathbb{Z}$ we say that we have an $(JA)_i$-**picture on** $T_\alpha(Z)$. Similarly, we speak of an $(JB)_i$-**picture**.*

3) *If $l^Z_{a_j,\alpha} = 1$, $i \in \mathbb{Z}$, we say that we have an (CA)-**picture on** $T_\alpha(Z)$. Similarly, we speak of an (CB)-**picture**.*

We will write $(OA)_i + (JB)_r$ for the situation that we have an $(OA)_i$- and an $(JB)_r$-picture on $T_\alpha(Z)$. We proceed similarly with other combinations of pictures.

Proposition 6.7. *Let $\alpha \in B_+$ be strict decreasing. Let $x \in R$ and assume that $x\alpha$ is also strict decreasing. Then*

1) *$x = a_i$ if and only if we have an $(OA)_i$-picture on $T_{x\alpha}(Y)$.*

2) *$x = b_i$ if and only if we have an $(OB)_i$-picture on $T_{x\alpha}(Y)$.*

3) *We can only have the possible pictures $(OA)_i + (JB)_r$, $r \in \mathbb{Z}$, and $(OA)_i + (CB)$ on $T_{x\alpha}(Y)$ in the case $x = a_i$. Similarly, for $x = b_i$ we only have the possible pictures $(JA)_r + (OB)_i$, $r \in \mathbb{Z}$, and $(CA) + (OB)_i$ on $T_{x\alpha}(Y)$.*

4) *If we have a constant picture (CB) (resp. (CA)) on $T_{x\alpha}(Y)$, then $x\alpha$ is contained in $\langle a_j \mid j \in \mathbb{Z} \rangle$ (resp. $\langle b_j \mid j \in \mathbb{Z} \rangle$).*

5) *If we have the picture $(OA)_i + (JB)_r$, $i, r \in \mathbb{Z}$, on $T_{x\alpha}(Y)$ and if y is an element in R such that $yx\alpha$ is again strict decreasing, then $x = a_i$ and furthermore $y = a_j$ with $j < i-1$ or $y = b_j$ with $j < r-1$. The similar assertion also holds for the picture $(JA)_i + (OB)_r$, $i, r \in \mathbb{Z}$.*

6) *If we have the picture $(OA)_i + (CB), i \in \mathbb{Z}$ on $T_{x\alpha}(Y)$, then y is an element in R such that $yx\alpha$ is strict decreasing if and only if $y = a_j$ with $j < i-1$ or $y = b_j$, $j \in \mathbb{Z}$.*

Proof. We prove the assertions using induction on the length of α in terms of the set of generators R. If $\alpha = 1$, so the length of α is zero, then the statements are easy and follow from 3.1 as in 3.14 and 4.12. So assume that α admits an expression $\alpha = z\beta$ with $z \in R$ and $\beta \in B^+$ strict decreasing. As the statements are symmetric in a and b, we can assume that $x = a_i$.

Case 1: Assume $z = a_j$, $j \in \mathbb{Z}$, so that (by induction) we have an $(OA)_j$-picture on $T_\alpha(Y)$. As $x = a_i$, the induction hypothesis together with the assumption that $x\alpha$ is strict decreasing yields $i < j-1$ and on $T_{x\alpha}$ we get a picture $(OA)_i$ (see the proof of 4.12). Furthermore, by induction, we have $(JB)_r$ or (CB) as pictures on $T_\alpha(Y)$. This implies (by applying 3.2 with A_i and B_l for every $l \in \mathbb{Z}$) an $(JB)_{r+1}, (JB)_r$ or (CB)-picture on $T_{x\alpha}(Y)$. So we are left with statement 5). Take $y \in R$ an element such that $yx\alpha$ is strict decreasing. If $y = a_h$, then as $x = a_i$ we get $h < i-1$. If $y = b_j$, then also $yz\beta$ is strict decreasing, so that $j < r-1$ (by induction) in the case of an $(JB)_r$-picture on $T_{z\beta}(Y)$. In particular,

$$j < r = r+1-1$$

showing the claim. In the case of an (CB)-picture on $T_\alpha(Y)$ (and hence also a (CB)-picture on $T_{x\alpha}(Y)$), we have that $yx\alpha$ is strict decreasing if $y = a_j$ with $j < i-1$ or $y = b_j$ with $j \in \mathbb{Z}$ because $xz\alpha$ does not contain any $b_r,\ r \in \mathbb{Z}$ (by induction). This finishes the first case.

Case 2: Assume $z = b_r$ with $r \in \mathbb{Z}$. We then consider the following two cases given by induction. First the case of an $(OB)_r + (JA)_h$-picture on $T_\alpha(Y)$, then that of an $(OB)_r + (CA)^1$-picture. So assume an $(OB)_r + (JA)_h$-picture on $T_\alpha(Y)$. By induction we know that $i < h-1$ as $x\alpha$ is assumed to be strict decreasing. Then we get an $(OA)_i + (JB)_{r+1}$-picture on $T_{x\alpha}(Y)$ by the proof of 4.12 and 3.2. To show statement 5) take $y \in R$ such that $yx\alpha$ is strict decreasing. If $y = a_j$, then $j < i-1$ as $x = a_i$. If $y = b_l$, then $l < r$ as $z = b_r$. In particular, $l < r+1-1$ showing the claim. We come to our final case, that of an $(OB)_r + (CA)$-picture and $x = a_i$. On $T_{x\alpha}(Y)$ we definitely get an $(OA)_i$-picture by 4.11. For the picture on $T_{x\alpha}(Y)$ we only get an $(OB)_{r+1}$-picture or an $(JB)_{r+1}$-picture by 3.2. We want to exclude the case of an $(OB)_{r+1}$-picture. As we have an (CA)-picture on $T_\alpha(Y)$ we know (by induction) that β is given as

$$\beta = b_{h_1} \cdot .. \cdot b_{h_t}$$

with

$$r < h_1 - 1 < h_2 - 2 < ... < h_t - t$$

as α is strict decreasing. We claim that $l^Y_{a_i,\beta} = l^Y_{b_r,\beta} = 1$. This is clear if $t = 0$. Otherwise we can apply induction to see that we have an $(CA) + (OB)_{h_1}$-picture on $T_\beta(Y)$. As α is strict decreasing, we know $r < h_1 - 1$, so in particular $l^Y_{a_i,\beta} = l^Y_{b_r,\beta} = 1$. Now we can apply the result 3.14 of C. Brav and H. Thomas presented in 3.3 to the A_2-configuration A_i, B_r and $T_\beta(Y)$. The result implies that

$$l^{T_\beta(Y)}_{a_i,b_ra_ib_r} = 2 = l^{T_\beta(Y)}_{b_r,b_ra_ib_r}$$

as a_i and b_r are left-factors of $a_ib_ra_i = b_ra_ib_r$. In particular, we get

$$l^Y_{b_r,x\alpha} = l^Y_{b_r,a_ib_r\beta} = l^Y_{b_r,b_ra_ib_r\beta} - 1 = l^{\beta(Y)}_{b_r,b_ra_ib_r} - 1 = 1.$$

This excludes the case of an $(OB)_{r+1}$-picture on $T_{x\alpha}(Y)$. Arguments as in the previous cases show that also statement 5) is true, thus we are finished with the proof. □

We obtain our partial result.

Proposition 6.8. *Let $\alpha, \beta \in B^+$ be strict decreasing. Then*

$$T_\alpha \cong T_\beta \Leftrightarrow \alpha = \beta,$$

so the homomorphism

$$T : B \longrightarrow \mathrm{Aut}(\mathcal{D})$$

is injective on the subset of strict decreasing elements.

Proof. As in the proof of 3.15 and 4.13 one can reconstruct the last factor of a strict-decreasing element $\alpha \in B^+$ using 6.7. Proceeding by induction on the length of α the claim follows. $\square$

We give an example showing what prevents us from generalizing 6.7 to more general elements in B^+.

Example 6.9. *Consider an element $\alpha = b_r a_i a_i \in B^+$, $i, r \in \mathbb{Z}$. Then using 3.1 (or its instances 3.2 and 4.11), 3.14 and that $a_i\alpha = b_r a_i b_r a_i$ one computes*

$$l_{a_j,\alpha} = \begin{cases} 2, & j < i \\ 1, & j = i \\ 3, & j \geq i+1 \end{cases}$$

and

$$l_{b_j,\alpha} = \begin{cases} 2, & j < r-1 \\ 1, & j = r-1 \\ 3, & j \geq r \end{cases},$$

so that (as a_{i+1} is not a left factor of α) one cannot reconstruct the last factor of α just by looking at the pictures on $T_\alpha(Y)$. A similar problem also occurs for $b_r a_i a_j \in B^+$ with $j \leq i$. The difficulty finally arises from the critical number $i-1$ in an $(OA)_i$- or $(OB)_i$-picture.

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