TALK FF IV: THE ALGEBRAIC FARGUES-FONTAINE CURVE

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1. Collection of previous results

These notes are a detailed exposition of a talk I have given at a workshop in Neckarbischofsheim about the Galois group of $\mathbb{Q}_p$ as a geometric fundamental group `https://www.mathi.uni-heidelberg.de/~gqpaspi1geom/`.

We will, building on the work of previous talks, introduce the *algebraic Fargues-Fontaine curve* $X_{E,F}$. For its construction we have to choose two fields E and F. We fix $E/\mathbb{Q}_p$ a finite extension with residue field $\mathbb{F}_q$ and an algebraically closed non-archimedean extension $F/\mathbb{F}_q$. In particular, F is perfectoid. We also fix a uniformizer $\pi \in E$.

Let

$$Y^{\mathrm{ad}} := Y^{\mathrm{ad}}_{E,F} := \varinjlim_{I \subseteq]0,1[} \mathrm{Spa}(B_I)$$

be the adic space associated with E and F, which was constructed in talk FF II, also see [Far, Definition 2.5.].

Fact 1.1 (talk FF II)**.** *Y^{ad} has global sections $H^0(Y^{\mathrm{ad}}, \mathcal{O}_{Y^{\mathrm{ad}}}) = B$ and B is an integral domain.* [Far, Definition 2.5.]

The Frobenius $\varphi : F \to F : x \mapsto x^q$ induces an automorphism

$$\varphi : Y^{\mathrm{ad}} \to Y^{\mathrm{ad}}$$

such that $\varphi^{\mathbb{Z}}$ acts properly discontinously on Y^{ad}. In fact, for $\varpi \in F^{\times}$ with $|\varpi|_F < 1$ there exists a continous map

$$\delta : Y^{\mathrm{ad}} \to [0, \infty] : y \mapsto \frac{\log|\pi(\widetilde{y})|}{\log|\varpi(\widetilde{y})|}$$

satisfying $\delta(\varphi(y)) = \delta(y)^{1/q}$ for $y \in Y^{\mathrm{ad}}$, where $\widetilde{y}$ denotes the maximal generalization of the point y in Y^{ad} (compare with [Wei, Proposition 3.3.5.]). We can conclude that the quotient space

$$X^{\mathrm{ad}} := X^{\mathrm{ad}}_{E,F} := Y^{\mathrm{ad}}/\varphi^{\mathbb{Z}}$$

is naturally provided with a structure sheaf making X^{ad} an adic space, the so-called *adic Fargues-Fontaine curve* $X^{\mathrm{ad}} = X^{\mathrm{ad}}_{E,F}$. We denote by

$$\mathrm{pr} : Y^{\mathrm{ad}} \to X^{\mathrm{ad}}$$

the natural morphism of adic spaces.

It is a formal consequence of the properly discontinous action of $\varphi^{\mathbb{Z}}$ on Y^{ad} that the pullback pr^* induces an equivalence of the category of $\mathcal{O}_{X^{\mathrm{ad}}}$-modules with the category of φ-modules over $\mathcal{O}_{Y^{\mathrm{ad}}}$, i.e. $\mathcal{O}_{Y^{\mathrm{ad}}}$-modules carrying a $\varphi^{\mathbb{Z}}$-equivariant

Date: 06.05.2015.

action. For example, the structure sheaf $\mathcal{O}_{X^{\mathrm{ad}}}$ corresponds to the φ-module $\mathcal{O}_{Y^{\mathrm{ad}}}$ with its canonical isomorphism $\varphi_{\mathcal{O}_{Y^{\mathrm{ad}}}}: \varphi^*\mathcal{O}_{Y^{\mathrm{ad}}} \cong \mathcal{O}_{Y^{\mathrm{ad}}}$. More generally, for $d \in \mathbb{Z}$ we denote by $\mathcal{O}_{X^{\mathrm{ad}}}(d)$ or just $\mathcal{O}(d)$ the line bundle on X^{ad} corresponding to the φ-module $\mathcal{O}_{Y^{\mathrm{ad}}}(d)$ consisting of the sheaf $\mathcal{O}_{Y^{\mathrm{ad}}}$ with the twisted φ-action

$$\varphi_{\mathcal{O}_{Y^{\mathrm{ad}}}(d)}(f) := \pi^{-d}\varphi_{\mathcal{O}_{Y^{\mathrm{ad}}}}(f)$$

for $f \in \mathcal{O}_{Y^{\mathrm{ad}}}$. The global sections $P_d := H^0(X^{\mathrm{ad}}, \mathcal{O}_{X^{\mathrm{ad}}}(d))$ are thus given by

$$P_d = B^{\varphi_{\mathcal{O}_{Y^{\mathrm{ad}}}(d)}=1} = B^{\varphi=\pi^d}.$$

For example, $P_0 = E$ and $P_d = 0$ for $d < 0$ ([FFb, Corollary 1.15]).
Elements in $P_1 = B^{\varphi=\pi}$ can be constructed explicitly. Namely, let $\mathcal{G}$ be the formal group over $\mathcal{O}_E$ associated to a Lubin-Tate law $\mathcal{LT}$ over $\mathcal{O}_E$. Then $\mathcal{G}$ comes equipped with a logarithm $\log_{\mathcal{LT}}(-) \in T \cdot E[[T]]$ and a twisted Teichmüller lift

$$\begin{array}{rcl} [-]_Q : \mathcal{G}(\mathcal{O}_F) & \to & \mathcal{G}(W_{\mathcal{O}_E}(\mathcal{O}_F)) \\ \varepsilon & \mapsto & \lim\limits_{n\to\infty} [\pi^n]_{\mathcal{LT}}([\varepsilon^{q^{-n}}]), \end{array}$$

([FFb, Proposition 2.11]) where $[\pi]_{\mathcal{LT}}(-)$ denotes multiplication with respect to the Lubin-Tate law.

Fact 1.2. *The map*

$$\begin{array}{rcl} \mathcal{G}(\mathcal{O}_F) = (\mathfrak{m}_F, +_{\mathcal{LT}}) & \to & P_1 = B^{\varphi=\pi} \\ \varepsilon & \mapsto & \log_{\mathcal{LT}}([\varepsilon]_Q) \end{array}$$

is an isomorphism of E-vector spaces ([FFb, Theorem 4.6.]*).*

We will however just use the existence of the map $\mathcal{G}(\mathcal{O}_F) \to B^{\varphi=\pi}$. Up to convergence issues (see [FFb, Remark 4.8]) its well-definedness can be deduced as follows

$$\varphi(\log_{\mathcal{LT}}([\varepsilon]_Q)) = \log_{\mathcal{LT}}([\varepsilon^q]_Q) = \log_{\mathcal{LT}}([\pi]_{\mathcal{LT}}([\varepsilon]_Q)) = \pi \log_{\mathcal{LT}}([\varepsilon]_Q).$$

By definition, a point $y \in Y^{\mathrm{ad}}$ is called classical, if its support

$$\mathrm{supp}(y) := \{f \in B \mid f(y) = 0\} \subseteq B$$

is a maximal ideal. Similarly, define classical points in the open sets $\mathrm{Spa}(B_I) \subseteq Y^{\mathrm{ad}}$, $I \subseteq]0,1[$ with extremities in $|F^\times|_F \subseteq \mathbb{R}_{>0}$, as the points whose support is a maximal ideal. Let $Y^{\mathrm{ad}}_{\mathrm{cl}} \subseteq Y^{\mathrm{ad}}$ be the subset of classical points of Y^{ad}. By [FFb, Theorem 3.9.] $Y^{\mathrm{ad}}_{\mathrm{cl}} = \lim\limits_{I\subseteq]0,1[} \mathrm{Spa}(B_I)_{\mathrm{cl}}$. We want to point out, that for a classical point $y \in Y^{\mathrm{ad}}_{\mathrm{cl}}$ the valuation on $k(y)$ is of rank one, i.e. y is the only point in Y^{ad} with support $\mathrm{supp}(y)$. In fact, by [FFb, Theorem 4.3.] and [FFb, Corollary 3.11] each closed maximal ideal of B is generated by a primitive element of degree 1. Then by [FFb, Theorem 2.4.] the image of $W_{\mathcal{O}_E}(\mathcal{O}_F) \subseteq H^0(Y^{\mathrm{ad}}, {\mathcal{O}_{Y^{\mathrm{ad}}}}^+)$ in $k(y)$ is already a valuation ring of rank one, and hence $\mathrm{Spa}(k(y), k(y)^+) = \{y\}$. In particular, we obtain a bijection

$$Y^{\mathrm{ad}}_{\mathrm{cl}} \xrightarrow{1:1} \{\mathfrak{m} \subseteq B \text{ closed maximal ideal}\}.$$

Fact 1.3 (talks FF I, FF III)**.** *If $y \in Y^{\mathrm{ad}}_{\mathrm{cl}}$ is classical, then the residue field $k(y)$ is perfectoid with a canonical identification $k(y)^\flat \cong F$ of its tilt with the field F (*[FFb, Theorem 2.4.]*). In particular, $k(y)$ is algebraically closed. Moreover, the local ring $\mathcal{O}_{Y^{\mathrm{ad}},y}$ is a discrete valuation ring whose $\mathfrak{m}_{Y^{\mathrm{ad}},y}$-adic completion is Fontaine's ring $B^+_{\mathrm{dR},y}$ associated to the perfectoid field $k(y)$. (*[FFb, Theorem 3.9.] *and* [FFb, Definition 3.1]*)*

Let $\mathrm{Div}(Y^{\mathrm{ad}})$ be the group of divisors on Y^{ad}, i.e. locally finite sums of classical points in Y^{ad}.

Fact 1.4 (talk FF III)**.** *The map*

$$\begin{array}{ccc} \{\mathfrak{a} \subseteq B \text{ non-zero closed ideal}\} & \to & \mathrm{Div}^+(Y^{\mathrm{ad}}) \\ \mathfrak{a} & \mapsto & V(\mathfrak{a}) \end{array}$$

is an isomorphism ([FFb, Theorem 3.8.]*).*

The fact 1.4 was used to analyse the multiplicative structure of the graded E-algebra

$$P := P_{E,\pi} := \bigoplus_{d=0}^{\infty} P_d = \bigoplus_{d=0}^{\infty} B^{\varphi=\pi^d}.$$

Define the set of classical points in X^{ad} as $X^{\mathrm{ad}}_{\mathrm{cl}} := \mathrm{pr}(Y^{\mathrm{ad}}_{\mathrm{cl}}) \subseteq X^{\mathrm{ad}}$ and let $\mathrm{Div}(X^{\mathrm{ad}})$ be the group of divisors on X^{ad}, i.e. locally finite sums of classical points on X^{ad}. As X^{ad} is quasi-compact, being the image of the quasi-compact set $\mathrm{Spa}(B_I)$ for some compact interval $I \subseteq]0,1[$, divisors on X^{ad} are actually *finite* sums of classical points on X^{ad}. By definition, divisors on X^{ad} are in bijection with φ-invariant divisors on Y^{ad}

$$\mathrm{Div}(X^{\mathrm{ad}}) \cong \mathrm{Div}(Y^{\mathrm{ad}})^{\varphi=1}$$

as $\mathrm{pr}^{-1}(X^{\mathrm{ad}}_{\mathrm{cl}}) = Y^{\mathrm{ad}}_{\mathrm{cl}}$.

Fact 1.5 (talk FF III)**.** *The algebra P is* graded factorial *with irreducible elements of degree 1, i.e. every non-zero homogenous element can be written uniquely (up to the units $E^\times = P_0^\times$) as the product of homogenous elements of degree 1. More precisely, the divisor map*

$$\begin{array}{ccc} \mathrm{div} : (\bigcup_{d\geq 0} P_d \setminus \{0\})/E^\times & \to & \mathrm{Div}^+(X^{\mathrm{ad}}) \\ f & \mapsto & \mathrm{div}(f) \end{array}$$

is an isomorphism ([FFb, Theorem 4.3]*). In particular, there is a bijection*

$$\mathrm{div} : (P_1 \setminus \{0\})/E^\times \xrightarrow{1:1} X^{\mathrm{ad}}_{\mathrm{cl}}.$$

2. The algebraic Fargues-Fontaine curve

We now define the algebraic Fargues-Fontaine curve.

Definition 2.1. The *algebraic Fargues-Fontaine curve* (for given E, F and π) is defined as the E-scheme

$$X := X_{E,F} = \mathrm{Proj}(P),$$

with $P = P_{E,F,\pi} = \bigoplus_{d=0}^{\infty} B^{\varphi=\pi^d}$. Note, the ring B depends on E and F, but not on π.

The curve $X_{E,F}$ is independent of π in the sense that the choice of another uniformizer π' yields a curve X' canonically isomorphic to X as the following lemma shows. (see also [FFa, Section 7.1.4.])

Lemma 2.2. *Let $\pi_1, \pi_2 \in E$ be uniformizers with corresponding algebras*

$$P_{\pi_i} = \bigoplus_{d\geq 0} B^{\varphi=\pi_i^d}$$

for $i = 1, 2$. Then

$$\mathrm{Proj}(P_{\pi_1}) \cong \mathrm{Proj}(P_{\pi_2}),$$

canonically and $P_{\pi_1} \cong P_{\pi_2}$ non-canonically.

Proof. The field F is algebraically closed, hence the closure $L := \overline{\mathbb{F}}_q \subseteq \mathcal{O}_F$ lies in F. As the ring $W_{\mathcal{O}_E}(L)$ is henselian with algebraically closed residue field there exists $u \in W_{\mathcal{O}_E}(L)^\times$ with

$$\frac{\varphi(u)}{u} = \frac{\pi_1}{\pi_2}.$$

Note that $W_{\mathcal{O}_E}(L) \subseteq B$. In particular, the multiplications

$$\begin{array}{ccc} B^{\varphi=\pi_2^d} & \to & B^{\varphi=\pi_1^d} \\ f & \mapsto & u^d f \end{array}$$

for $d \in \mathbb{Z}$ combine to an isomorphism $\alpha_u : P_{\pi_2} \to P_{\pi_1}$. The element u is unique up to invertible elements $v \in W_{\mathcal{O}_E}(L)^{\varphi=1} = \mathcal{O}_E$. For $v \in \mathcal{O}_E^\times$ the isomorphisms $\alpha := \alpha_u$ and $\beta := \alpha_{vu}$ satisfy

$$v^d \alpha(f) = \beta(f)$$

for $f \in P_{\pi_2,d}$ homogenous of degree d. It is easy to see that two morphisms

$$\alpha, \beta : A \to A'$$

between non-negatively graded algebras, satisfying the above equation for some unit $v \in A_0'^\times$ and every $d \geq 0$ induce the same morphism on Proj. This proves the lemma. □

We will see that X is indeed a "curve", i.e. one-dimensional. In some respect, X behaves like the curve $\mathbb{P}^1_E$ over the field E although X is *not* of finite type over E. As X is defined via the Proj construction there are natural line bundles on X obtained by the shifted graded P-modules $P[d]$ for $d \in \mathbb{Z}$. Let

$$\mathcal{O}(d) := \mathcal{O}_X(d) := \widetilde{P[d]}.$$

Then the $\mathcal{O}(d)$ are line bundles on X as P is generated by P_1. The global sections of $\mathcal{O}(d)$ can be computed, using that P is graded factorial 1.5, as

$$P_d = H^0(X, \mathcal{O}_X(d)).$$

In fact, P_d injects into $H^0(X, \mathcal{O}_X(d))$ as P is an integral domain. Let conversly, $a \in H^0(X, \mathcal{O}_X(d))$ be a global section. For $t \in P_1$ there exists $d_t \geq 0$ and $g_t \in P_d$ with $a_{|D^+(t)} = \frac{g_t}{t^{d_t}}$. We may assume that g_t is not divisible by t as P is graded factorial. Choose some $t' \notin E^\times t$[1]. Then restricting to the intersection $D^+(t) \cap D^+(t') = D^+(t \cdot t')$ yields $\frac{g_t}{t^{d_t}} = \frac{g_{t'}}{t'^{d_{t'}}}$ as P is an integral domain. As P is graded factorial and t, t' are relatively prime, we can conclude $d_t = d_{t'} = 0$ and hence $g := g_t = g_{t'}$ so that a is induced by the section $g \in P_d$ as t was arbitrary.
For completeness we introduce a proof of the following lemma. To proof it we will use the adjunction

$$\mathrm{Hom}(Z, \mathrm{Spec}(A)) \cong \mathrm{Hom}(A, \Gamma(Z, \mathcal{O}_Z))$$

for a ring A and an *arbitrary* locally ringed space Z ([GD71, Proposition 1.6.3])-

[1]If such a t' does not exists, the claim is trivial, as then $P = E[t]$. But actually such a t' exists: by 3.1 the E-vector space P_1 is infinite dimensional.

Lemma 2.3. *Let* $S = \mathrm{Spec}(R)$ *be an affine scheme and*

$$A = \bigoplus_{d \geq 0} A_d$$

be a graded R*-algebra, generated by* A_1*. Let* $h : \mathrm{Proj}(A) \to S$ *be the canonical morphism. Then for* any *locally ringed space* $g : Z \to S$ *the map*

$$\begin{array}{rcl} \eta : \mathrm{Hom}_S(Z, \mathrm{Proj}(A)) & \to & \{(\mathcal{L} \in \mathrm{Pic}(Z), \gamma : g^*\widetilde{A} \to \bigoplus_{d\geq 0} \mathcal{L}^{\otimes d} \text{ surjective}\}/\cong \\ f & \mapsto & (f^*\mathcal{O}(1), f^*(\gamma_{\mathrm{can}})) \end{array}$$

is a bijection, where $\mathcal{O}(1) \in \mathrm{Proj}(\mathcal{A})$ *denotes the canonical line bundle* $\mathcal{O}(1) = \widetilde{A[1]}$ *and* $\gamma_{\mathrm{can}} : h^*(\widetilde{A}) \to \bigoplus_{d\geq 0} \mathcal{O}(d)$ *the canonical surjection.*

Proof. We first proof that the morphism γ_{can}, which is induced by the canonical morphism

$$A \to H^0(\mathrm{Proj}(A), \bigoplus_{d\geq 0} \mathcal{O}(d)),$$

is indeed surjective. As the open sets $D^+(t)$ for $t \in A_1$ cover $\mathrm{Proj}(A)$ and the question is local, we may restrict to $D^+(t)$ for some $t \in A_1$. Then the morphism γ_{can} is given by the multiplication

$$A[1/t]_0 \otimes_R A \to \bigoplus_{d\geq 0} A[1/t]_d,$$

which is easily seen to be surjective. We denote by $F(Z)$ the target of η. Then F is a sheaf with respect to local isomorphisms. We define for $t \in A_1 \setminus \{0\}$ the subfunctor

$$F_t(Z) := \{(\mathcal{L}, \gamma) \in F(Z) \mid \gamma(t) \text{ generates } \mathcal{L}\}$$

of F. The inclusion $F_t \to F$ is represented by open immersions. Indeed, for a morphism $(\mathcal{L}, \gamma) : Z \to F$ the fiber product $Z \times_F F_t$ is represented by the open subset

$$D(\gamma(t)) := \{z \in Z \mid \gamma(t) \text{ generates } \mathcal{L}_z\}.$$

We claim that F_t is represented by the scheme $\mathrm{Spec}(A[1/t]_0)$ by sending a morphism $f : Z \to \mathrm{Spec}(A[1/t]_0)$ corresponding to the morphism $f : A[1/t]_0 \to \Gamma(Z, \mathcal{O}_Z)$ to the pair

$$(\mathcal{O}_Z, \gamma : \widetilde{A}_{|Z} \to \bigoplus_{d\geq 0} \mathcal{O}_Z)$$

where γ maps a local section represented by $a \in A_d$ to $f(\frac{a}{t^d}) \in \mathcal{O}_Z$. As $\gamma(t^d) = 1$ for $d \geq 0$ the morphism γ is surjective. Let conversely, $(\mathcal{L}, \gamma) \in F_t(Z)$ be given. Define $f(a/t^d) \in \Gamma(Z, \mathcal{O}_Z)$ for $a \in A_d$ by the formula

$$\gamma(a) = f(a/t^d)\gamma(t)^d \in \mathcal{L}^{\otimes d}(Z).$$

Then $f : A[1/t]_0 \to \Gamma(Z, \mathcal{O}_Z)$ is well-defined and a homomorphism of rings. It can be checked that these morphisms $\mathrm{Spec}(A[1/t]_0) \to F_t$ and $F_t \to \mathrm{Spec}(A[1/t]_0)$ are mutually inverse. Moreover, the F_t for $t \in A_1$ cover F as A is generated by A_1 and $\gamma : g^*\widetilde{A_1} \to \mathcal{L}$ surjective. We can conclude that η is an isomorphism of functors as for every $t \in A_1$ the pullback

$$\mathrm{Spec}(A[1/t]_0) = D^+(t) = \mathrm{Proj}(A) \times_F F_t \to F_t$$

is an isomorphism. □

As $H^0(X^{\mathrm{ad}}, \bigoplus_{d\geq 0} \mathcal{O}(d)) = P$ we obtain by 2.3 a morphism

$$\alpha : X^{\mathrm{ad}} \to X$$

of locally ringed spaces satisfying $\alpha^*(\mathcal{O}_X(d)) \cong \mathcal{O}_{X^{\mathrm{ad}}}(d)$. More precisely, it has to be checked, that the open sets

$$D(t) := \{x \in X^{\mathrm{ad}} | \ t \text{ generates } \mathcal{O}_{X^{\mathrm{ad}}}(1)\}$$

for $t \in P_1$ cover X^{ad}. We first show that for $t \in P_1 \setminus \{0\}$ the vanishing locus

$$V(t) := \{x \in X^{\mathrm{ad}} \mid t(x) = 0\}$$

consists of classical points. This property can be checked on Y^{ad} and because $Y^{\mathrm{ad}} = \varinjlim_{I\subseteq]0,1[} \mathrm{Spa}(B_I)$, we may restrict to $U := \mathrm{Spa}(B_I) \subseteq Y^{\mathrm{ad}}$ for some interval $I \subseteq]0,1[$ whose extremities lie in $|F^\times|$. By [FFb, Theorem 3.9.] the ring B_I is a principal ideal domain. Assume $y \in V(t)$ for $t \in P_1 \subseteq B_I$. If $t \neq 0$, then t does not vanish at the generic point of U, and hence $V(t)$ consists of points, whose support is maximal. In other words, $V(t) \subseteq X^{\mathrm{ad}}$ consists of classical points. By 1.5 there is the bijection

$$\mathrm{div} : (P_1 \setminus \{0\})/E^\times \xrightarrow{1:1} X^{\mathrm{ad}}_{\mathrm{cl}}.$$

For $t, t' \in P_1 \setminus \{0\}$ with $t' \notin E^\times t$ (such t, t' exist as P_1 is infinite-dimensional over E, see 3.1) we therefore get

$$V(t) \cap V(t') = \emptyset,$$

which was our claim.

3. The fundamental exact sequence

In order to understand X we need the fundamental exact sequence. Fix an effective divisor

$$D = \sum_{i=1}^{n} a_i y_i \in \mathrm{Div}^+(Y^{\mathrm{ad}})$$

of degree $d := \sum_{i=1}^{n} a_i$. Assume that $y_i \notin \{y_j\}^{\varphi^{\mathbb{Z}}}$ for $i \neq j$ and let $x_i := \mathrm{pr}(y_i) \in X^{\mathrm{ad}}_{\mathrm{cl}}$. By 1.5 we know that $\{x_i\} = V(t_i)$ for some $t_i \in P_1 \setminus \{0\} = H^0(X^{\mathrm{ad}}, \mathcal{O}_{X^{\mathrm{ad}}}(1))$, which is unique up to multiplication by $E^\times = P_0^\times$. Let $t := \prod_{i=1}^{n} t_i^{a_i}$. Then the divisor of $t \in H^0(X^{\mathrm{ad}}, \mathcal{O}_{X^{\mathrm{ad}}}(d))$ is precisely $\sum_{i=1}^{n} a_i x_i$.

Theorem 3.1 (Fundamental exact sequence)**.** *For $r \geq 0$ the sequence*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(X^{\mathrm{ad}}, \mathcal{O}(r)) & \xrightarrow{t} & H^0(X^{\mathrm{ad}}, \mathcal{O}(d+r)) & \longrightarrow & \prod_{i=1}^{n} \mathcal{O}_{X^{\mathrm{ad}},x_i}/\mathfrak{m}^{a_i}_{X^{\mathrm{ad}},x_i} & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & P_r & \xrightarrow{t} & P_{d+r} & \xrightarrow{u} & \prod_{i=1}^{n} B^+_{\mathrm{dR},y_i}/\mathfrak{m}^{a_i}_{Y^{\mathrm{ad}},y_i} B^+_{\mathrm{dR},y_i} & \longrightarrow & 0
\end{array}$$

is exact, where u is the canonical evaluation morphism

$$P_{d+r} \subseteq B = H^0(Y^{\mathrm{ad}}, \mathcal{O}_{Y^{\mathrm{ad}}}) \to \mathcal{O}_{Y^{\mathrm{ad}},y_i}/\mathfrak{m}^{a_i}_{Y^{\mathrm{ad}},y_i} \cong B^+_{\mathrm{dR},y_i}/\mathfrak{m}^{a_i}_{Y^{\mathrm{ad}},y_i} B^+_{\mathrm{dR},y_i}.$$

Proof. We first show $\ker(u) = tP_r$. Let $f \in P_{d+r}$ be an element with $u(f) = 0$. We consider f as a function on Y^{ad} and look at its divisor $\operatorname{div}(f) \in \operatorname{Div}^+(Y^{\mathrm{ad}})$. As $u(f) = 0$ we get

$$\operatorname{div}(f) \geq \sum_{i=1}^{n} a_i y_i.$$

But $\operatorname{div}(f)$ is φ-invariant because $\varphi(f) = \pi^d f$, and hence

$$\operatorname{div}(f) \geq \sum_{i=1}^{n} a_i \sum_{n\in\mathbb{Z}} \varphi(y_i) = \operatorname{div}(t)$$

where t is considered as a function on Y^{ad}. Hence, by fact 1.4

$$f = gt$$

for some $g \in B$. We get $\varphi(g)\pi^d t = \pi^{d+r} gt$ and thus $g \in P_r$ as B is an integral domain.

Factoring $t = t_1 \cdot t'$ and considering for $r \geq 0$ the diagram

$$\begin{array}{ccc} P_r & \xrightarrow{t_1} & P_{r+1} \\ \downarrow = & & \downarrow t' \\ P_r & \xrightarrow{t} & P_{r+d} \end{array}$$

reduces the proof for surjectivity to the case $d = 1$ and $t = t_1$. Furthermore, we may assume $r = 0$. In fact, if $a \in C := k(y_1)$ and $u(t) = a^{1/r+1}$ for some $t \in P_1$, then $u(t^{r+1}) = a$. We thus have to show that the map

$$u : B^{\varphi=\pi} \to C = k(y)$$

is surjective. By 1.3 C is perfectoid and algebraically closed with tilt F. In particular, $\mathcal{O}_C/\pi \cong \mathcal{O}_F/\pi^\flat$ for some $\pi^\flat \in F$ with $|\pi^\flat|_F = |\pi|_C$. We will use the description $\mathcal{G}(\mathcal{O}_F) \cong B^{\varphi=\pi}$ from fact 1.2. We get the sequence of maps

$$\varprojlim_{[\pi]_{\mathcal{LT}}} \mathcal{G}(\mathcal{O}_C) \to \varprojlim_{[\pi]_{\mathcal{LT}}} \mathcal{G}(\mathcal{O}_C/\pi) \cong \varprojlim_{\varphi} \mathcal{G}(\mathcal{O}_F/\pi^\flat) \cong \varprojlim_{\varphi} \mathcal{G}(\mathcal{O}_F) = \mathcal{G}(\mathcal{O}_F).$$

We used that F is perfectoid to conclude

$$\varprojlim_{\varphi} \mathcal{G}(\mathcal{O}_F/\pi^\flat) \cong \varprojlim_{\varphi} \mathcal{G}(\mathcal{O}_F) \cong \mathcal{G}(\mathcal{O}_F).$$

Putting things together we get the map

$$\begin{array}{rccc} \Psi : & \varprojlim_{[\pi]_{\mathcal{LT}}} \mathcal{G}(\mathcal{O}_C) & \to & C \\ & (z_n)_n & \mapsto & \log_{\mathcal{LT}}(z_0) \end{array}$$

More precisely, take $(z_n)_n \in \varprojlim_{[\pi]_{\mathcal{LT}}} \mathcal{G}(\mathcal{O}_C/\pi)$ with reduction $(\overline{z}_n)_n \in \varprojlim_{[\pi]_{\mathcal{LT}}} \mathcal{G}(\mathcal{O}_C/\pi)$ and $\varepsilon \in \mathcal{G}(\mathcal{O}_F)$ with $\varepsilon^{1/q^n} = \overline{z}_n \in \mathcal{O}_F/\pi^\flat = \mathcal{O}_C/\pi$ for all n. Then

$$[\varepsilon]_Q = \lim_{n\to\infty} [\pi^n]_{\mathcal{LT}}([\varepsilon^{1/q^n}]) = \lim_{n\to\infty} [\pi^n]_{\mathcal{LT}}(z_n) = z_0,$$

showing that

$$\Psi((z_n)_n) = \log_{\mathcal{LT}}([\varepsilon]_Q) = \log_{\mathcal{LT}}(z_0).$$

The map Ψ is surjective as C is algebraically closed and we can conclude. Indeed, the formula

$$\log_{\mathcal{LT}}([\pi]_{\mathcal{LT}}(x)) = \pi \log_{\mathcal{LT}}(x)$$

for $x \in \mathcal{G}(\mathcal{O}_C)$ and the surjectivity of $[\pi]_{\mathcal{LT}} : \mathfrak{m}_C \to \mathfrak{m}_C$ (C is algebraically closed) shows that the image of $\log_{\mathcal{LT}} : \mathfrak{m}_C \to C$ contains elements of arbitrary large absolute value. But then the logarithm $\log_{\mathcal{LT}}$ has to be surjective as it has the Artin-Hasse-exponential as a local inverse near 0. □

Theorem 4.1 yields the following corollary.

Corollary 3.2. *Let $t \in P_1 \setminus \{0\}$ with vanishing locus $V(t) = \{x\} \subseteq X^{\mathrm{ad}}_{\mathrm{cl}}$ and $y \in Y^{\mathrm{ad}}_{\mathrm{cl}}$ a classical point over x. Then for $C := k(y)$ the map*

$$\begin{array}{rcl} \theta : P/tP & \to & \{g \in C[T] \mid g(0) \in E\} \\ \sum\limits_{d\geq 0} f_d & \mapsto & \sum\limits_{d \geq 0} f_d(y) T^d \end{array}$$

is an isomorphism of graded algebras. In particular, $\mathrm{Proj}(P/tP) = \{(0)\}$ *has one element.*

Proof. It is clear that θ is a morphism of graded algebras. Moreover, it is an isomorphism in degrees $d \geq 1$ by 3.1 and trivially for $d = 0$. Finally, let $\mathfrak{p} \neq 0$ be an homogenous prime ideal of the right hand side $\{g \in C[T] \mid g(0) \in E\}$. Then $cT^d \in \mathfrak{p}$ for some $d \geq 1$ and $c \in C^\times$. Multiplying by $c^{-1}T$ yields $T^{d+1} \in \mathfrak{p}$ such that $\mathfrak{p} = (T)$, a contradiction. □

4. Properties of the algebraic Fargues-Fontaine curve

Now we are ready to prove the main theorem of this talk.

Theorem 4.1. *The scheme X is noetherian, integral and regular of Krull dimension one. More precisely, for $t \in P_1 \setminus \{0\}$*

- *$D^+(t) = \mathrm{Spec}(B_t)$ with $B_t := P[1/t]_0 = B[1/t]^{\varphi=1}$ a principal ideal domain.*
- *$V^+(t) = \{\infty_t\}$ with $\infty_t \in X$ the closed point given by the homogenous prime ideal generated by t, so $\infty_t = (t) \subseteq P$.*

The map

$$\begin{array}{rcl} \mathrm{div} : (P_1 \setminus \{0\})/E^\times & \to & |X| := \{x \in X \text{ closed}\} \\ t & \mapsto & \infty_t \end{array}$$

is bijective[2].

Proof. As B is an integral domain, the curve X is integral. Pick $t \in P_1 \setminus \{0\}$. Then

$$V^+(t) \cong \mathrm{Proj}(P/tP) = \{tP\}$$

by 3.2, showing one claim. The description of B_t is clear and we can conclude that B_t is factorial as P is graded factorial. Moreover, the irreducible elements in B_t are exactly the fractions t'/t with $t' \in P_1$ not lying in $E^\times t$. We now want to prove that the ideal $(t'/t) \subseteq B_t$ is maximal. For this we use the exact sequence

$$0 \to t' \cdot P_r \to P_{r+1} \xrightarrow{\theta} k(x') \to 0$$

[2]as for $\mathbb{P}^1_E$

coming from 3.1. Here, $x' \in X^{\mathrm{ad}}_{\mathrm{cl}}$ denotes the unique point on $X^{\mathrm{ad}}_{\mathrm{cl}}$ with $t'(x') = 0$ (1.5). As $\theta(t) \neq 0$, by 3.1, the morphism θ factors over

$$P_1[1/t] \to k(x')$$

showing that $B_t/(t'/t) \to k(x')$ is surjective. Assume $f \in B_t$ satisfies $\theta(f) = f(x') = 0$. Then there exists $d \geq 1$ with

$$f = \frac{g}{t^d}$$

for some $g \in P_d$ and g automatically satisfies $g(x') = 0$. Hence $g \in t'P_{d-1}$ by the fundamental exact sequence 3.1 showing

$$B_t/(t'/t) \cong k(x').$$

We can conclude that B_t is a principal ideal domain as it is factorial with every irreducible element generating a maximal ideal. Covering X by two sets of the form $D^+(t)$ with $t \in P_1$ shows that X is noetherian and regular of Krull dimension one. Because t generates the ideal $\ker(P \xrightarrow{\mathrm{eval}} k(\infty_t)[T]) \subseteq P$ by 3.1 resp. 3.2 and P has units $E^\times$ the map

$$\begin{array}{cccc} \mathrm{div} : (P_1 \setminus \{0\})/E^\times & \to & |X| := \{x \in X \text{ closed}\} \\ t & \mapsto & \infty_t \end{array}$$

is injective. But for some $t \in P_1 \setminus \{0\}$ every irreducible element in B_t is of the form t'/t for some $t' \in P_1$ and hence div is surjective as B_t is a PID. □

For $x \in |X|$ we define

$$\deg : \mathrm{Div}(X) \to \mathbb{Z} : \sum_{x \in |X|} n_x x \mapsto \sum_{x \in |X|} n_x.$$

In other words, $\deg(x) := 1$ for $x \in |X|$. Then for every $f \in k(X)^\times$ in the function field $k(X)$ of X we have

$$\deg(\mathrm{div}(f)) = 0,$$

which can be reinterpreted as the statement that the curve X is "complete". Indeed, as P is graded factorial the case for general $f \in k(X)^\times$ is reduced to the case $f = t/t'$ with $t, t' \in P_1 \setminus \{0\}$, where it follows from 4.1, namely $\mathrm{div}(f) = \infty_t - \infty_{t'}$. All in all, we can conclude, as $X \setminus \{\infty_t\} = \mathrm{Spec}(B_t)$ with B_t a principal ideal domain, that similar to the case for $\mathbb{P}^1_E$ the degree map yields an isomorphism

$$\mathrm{Pic}(X) \cong \mathrm{Cl}(X) \xrightarrow{\deg} \mathbb{Z}$$

sending the line bundle $\mathcal{O}_X(d)$ to $d \in \mathbb{Z}$.

But not everything for X is similar to the projective line $\mathbb{P}^1_E$. For example, if $x \in |X|$ is a closed point, then the sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to k(x) \to 0$$

is exact showing that the non-zero E-vector space $k(x)/E$ embeds into the space $H^1(X, \mathcal{O}(-1))$, which is therefore in particular not zero contrary to the case for $\mathbb{P}^1_E$. But still $H^1(X, \mathcal{O}_X(d)) = 0$ for $d \geq 0$ (see [FFb, Proposition 6.5.]).

We can now compare the algebraic curve X with the adic curve X^{ad}. Recall that by 2.3 the identity $P = H^0(X^{\mathrm{ad}}, \bigoplus_{d \geq 0} \mathcal{O}_{X^{\mathrm{ad}}}(d))$ corresponds to a morphism

$$\alpha : X^{\mathrm{ad}} \to X.$$

of locally ringed spaces such that $\alpha^*(\mathcal{O}_X(d)) \cong \mathcal{O}_{X^{\mathrm{ad}}}(d)$.

Theorem 4.2. *The morphism $\alpha : X^{\mathrm{ad}} \to X$ induces bijections*

$$\begin{array}{rcl} \alpha : X^{\mathrm{ad}}_{\mathrm{cl}} & \overset{\cong}{\to} & |X| \\ \alpha : \widehat{\mathcal{O}_{X,x}} & \overset{\cong}{\to} & \widehat{\mathcal{O}_{X^{\mathrm{ad}},x^{\mathrm{ad}}}} \end{array}$$

for $x^{ad} \in X^{\mathrm{ad}}_{\mathrm{cl}}$ with $x := \alpha(x^{ad}) \in X$. In particular, for $x \in |X|$ the residue field $k(x)$ is algebraically closed and perfectoid with tilt $k(x)^\flat \cong F$ canonically up to a power of the Frobenius $\varphi : F \to F$.

Proof. By 1.5 and 4.1 sendig a section $t \in P_1 = H^0(X, \mathcal{O}_X(1)) = H^0(X^{\mathrm{ad}}, \mathcal{O}_{X^{\mathrm{ad}}})$ to its vanishing set $V(t) \subseteq X$ resp. $V(t) \subseteq X^{\mathrm{ad}}_{\mathrm{cl}}$ induces bijections of $|X|$ resp. $X^{\mathrm{ad}}_{\mathrm{cl}}$ with the set $(P_1 \setminus \{0\})/E^\times$. In the proof of 4.1 we have seen that α induces an isomorphism

$$\alpha : k(x) \to k(x^{\mathrm{ad}})$$

for $x^{\mathrm{ad}} \in X^{\mathrm{ad}}_{\mathrm{cl}}$. Moreover, if $\{x\} = V(t)$ with $t \in P_1$, then t is a uniformizer in $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X^{\mathrm{ad}},x^{\mathrm{ad}}}$ showing that the completions

$$\widehat{\mathcal{O}_{X,x}} \cong \widehat{\mathcal{O}_{X^{\mathrm{ad}},x^{\mathrm{ad}}}}$$

are isomorphic. □

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