G-BUNDLES ON THE ABSOLUTE FARGUES–FONTAINE CURVE

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Abstract. We prove that the category of "vector bundles on the absolute Fargues–Fontaine curve" (more precisely the category of sections over some discrete algebraically closed field of the v-stack $\mathrm{Bun}_{\mathrm{FF}}$ of vector bundles on the Fargues–Fontaine curve) is canonically equivalent to the category of isocrystals. We deduce a similar result for "G-bundles on the absolute Fargues–Fontaine curve" for some reductive group G as well as for sections of $\mathrm{Bun}_{\mathrm{FF}}$ over classifying stacks for locally profinite groups.

1. Introduction

Let p be a prime, and let E be a non-archimedean local field with residue field $\mathbb{F}_q$ of characteristic p. Let $\mathrm{Perf}_{\mathbb{F}_q}$ be the category of perfectoid spaces over $\mathbb{F}_q$, and let $\mathrm{Bun}_{\mathrm{FF}}$ be the v-stack on $\mathrm{Perf}_{\mathbb{F}_q}$ of vector bundles on the Fargues–Fontaine curve for E, cf. [4, Section II.2]. Thus, if S is any perfectoid space over $\mathbb{F}_q$, then $\mathrm{Bun}(S)$ is the category of vector bundles on the (adic) relative Fargues–Fontaine curve $X_{E,S}$, cf. [4, Definition II.1.15]. Given now any small v-stack Y over $\mathbb{F}_q$ (in the sense of [9, Definition 12.4]) one can contemplate the category

$$\mathrm{Bun}_{\mathrm{FF}}(Y)$$

of morphisms of v-stacks $Y \to \mathrm{Bun}_{\mathrm{FF}}$ over $\mathrm{Perf}_{\mathbb{F}_q}$. More concretely, if Y is the quotient of a perfectoid space X by some equivalence relation $R = X \times_Y X$, which is again represented by some perfectoid space, then $\mathrm{Bun}_{\mathrm{FF}}(Y)$ is equivalent to the category of descent data for $X \to Y$. As the pullback of vector bundles is exact, $\mathrm{Bun}_{\mathrm{FF}}(Y)$ aquires a natural structure of an exact category.

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The category $\mathrm{Bun}_{\mathrm{FF}}(Y)$ for small v-sheaves associated with adic spaces over $\mathbb{Z}_p$ occur naturally when contemplating integral models of local Shimura varieties, cf. [7]. It seems therefore to be an interesting question to understand the category $\mathrm{Bun}_{\mathrm{FF}}(Y)$ for some specific examples of small v-sheaves Y, e.g., the small v-sheaf

$$\mathrm{Spd}(R) := \mathrm{Spec}(R)^\diamond \colon \mathrm{Perf}_{\mathbb{F}_q} \to (\mathrm{Sets}),\ S \mapsto \mathrm{Hom}_{\mathbb{F}_q}(R, \mathcal{O}_S(S))$$

associated with some perfect $\mathbb{F}_q$-algebra R, cf. [10, Section 18.3].

The main result of this paper discusses the case that $Y = \mathrm{Spd}(k)$ is the small v-sheaf associated with a (discrete) algebraically closed field k over $\mathbb{F}_q$.

Theorem 1.1 (cf. Theorem 3.4)**.** *The natural functor*

$$\mathcal{E}_k(-)\colon \mathrm{Isoc}_k \to \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$$

from the category of E-isocrystals over k to $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$ *is an equivalence of categories.*

One might call objects in $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$ "vector bundles on the absolute Fargues–Fontaine curve" although we do not literally construct a relative Fargues–Fontaine curve "$X_{E,\mathrm{Spd}(k)}$" and define $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$ as vector bundles on "$X_{E,\mathrm{Spd}(k)}$".

Let us now recall the definition of E-isocrystals over k. Let

$$W_{\mathcal{O}_E}(-) := \mathcal{O}_E \widehat{\otimes}_{W(\mathbb{F}_q} W(-)$$

be the functor of $\mathcal{O}_E$-typical Witt vectors on perfect $\mathbb{F}_q$-algebras, and $\varphi\colon W_{\mathcal{O}_E}(-) \to W_{\mathcal{O}_E}(-)$ the natural lift of the q-Frobenius $x \mapsto x^q$. Then for any perfect $\mathbb{F}_q$-algebra R the category Isoc_R is the category of pairs (D, φ_D) with D a finite projective $W_{\mathcal{O}_E}(R)[1/p]$-module and $\varphi_D\colon \varphi^* D \to D$ an isomorphism and morphisms respecting the φ_D.

Given any perfectoid space S over $\mathrm{Spd}(R)$ there exists a natural functor

$$\mathrm{Isoc}_R \to \mathrm{Bun}_{\mathrm{FF}}(S)$$

constructed as follows: Let $Y_{E,S}$ be the usual $\mathbb{Z}$-covering of $X_{E,S} = Y_{E,S}/\varphi^{\mathbb{Z}}$ from [4, Definition II.1.15]. Then to $(D, \varphi_D) \in \mathrm{Isoc}_R$ can be associated the vector bundle on $X_{E,S}$ obtained by descent of the trivial bundle $D \otimes_{W_{\mathcal{O}_E}(R)[1/p]} \mathcal{O}_{Y_{E,S}}$ along the φ-semilinear automorphism $\varphi_D \otimes \varphi$.

By naturality of this construction with respect to pullback along morphisms $S' \to S$ of perfectoid spaces over $\mathrm{Spd}(R)$, we obtain the functor

$$\mathcal{E}_R(-)\colon \mathrm{Isoc}_R \to \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(R))$$

mentioned in Theorem 1.1 if $R = k$.

If G/E is a reductive group, then we will deduce from Theorem 1.1 a similar description of "G-bundles on the absolute Fargues–Fontaine curve" as alluded to in the title, cf. Theorem 5.3.

Another case that we consider is that of $Y = [\mathrm{Spd}(k)/\underline{H}]$ being the classifying stack of some locally profinite group H. In this case $\mathrm{Bun}_{\mathrm{FF}}(Y)$ is equivalent to continuous representations of H on isocrystals, cf. Theorem 4.2.

By [1, Theorem 2.1] the functor sending $S \in \mathrm{Perf}_{\mathbb{F}_q}$ to the ∞-category of perfect complexes on $X_{E,S}$ is a v-stack of ∞-categories, which we call $\mathcal{P}erf_{\mathrm{FF}}$. We make the following conjecture.

Conjecture 1.2. *For any perfect $\mathbb{F}_q$-algebra R the category $\mathcal{P}erf_{\mathrm{FF}}(\mathrm{Spd}(R))$ is equivalent to the category of perfect complexes of isocrystals over R.*[1]

We note that Conjecture 1.2 includes a comparison of cohomology, which is already non-trivial if $R = k$. Namely, it implies that $R\mathrm{Hom}_{\mathcal{P}erf_{\mathrm{FF}}(\mathrm{Spd}(k))}(\mathcal{O}, \mathcal{O}(-1)) = 0$.

Let us now go through the contents of the sections of this paper. In Section 2 we describe the pro-étale site of $\mathrm{Spd}(k)$. Although the outcome is as expected the pro-étale site of $\mathrm{Spec}(k)$, the proof uses some difficult results from [9]. In Section 3 we prove Theorem 1.1. In Section 4 we apply Theorem 1.1 to describe the category $\mathrm{Bun}_{\mathrm{FF}}([\mathrm{Spd}(k)/\underline{H}])$ if H is a locally profinite group. If G is a reductive group over E, then in Section 5 we describe $\mathrm{Bun}_G(\mathrm{Spd}(k))$ for the v-stack Bun_G of G-bundles on the Fargues–Fontaine curve.

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2. Pro-étale v-sheaves over $\mathrm{Spd}(k)$

Let k be an algebraically closed field extension of $\mathbb{F}_q$. In this section we analyze small v-sheaves Y with a quasi-pro-étale map $Y \to \mathrm{Spd}(k)$.

Let $\mathrm{Spec}(k)_{\text{proét}}$ be the pro-étale site of $\mathrm{Spec}(k)$ in the sense of [2, Definition 4.1.1], or equivalently the site of locally profinite sets with

[1] By [1, Proposition 2.7] perfect complexes of isocrystals over R identify with perfect complexes over $W_{\mathcal{O}_E}(R)[1/p]$ with a φ-semilinear isomorphism.

topology induced by surjections of profinite sets. Explicitly, if T is a locally profinite set, then the topological space

$$\underline{T} \cong T \times |\mathrm{Spec}(k)|$$

is a scheme, when equipped with the pullback of the structure sheaf from $\mathrm{Spec}(k)$.[2] We denote this scheme over $\mathrm{Spec}(k)$ by $\underline{T} \times \mathrm{Spec}(k)$. Clearly, the functor $T \to \underline{T} \times \mathrm{Spec}(k)$ commutes with limits, and defines the aforementioned equivalence of sites.

Let S be any perfectoid space over $\mathrm{Spd}(k)$, and T a locally profinite set. Then we can define the perfectoid space

$$\underline{T} \times S$$

by equipping the topological product $T \times |S|$ with the pullback of the structure sheaf on S. Clearly, the morphism $\underline{T} \times S \to S$ is quasi-pro-étale in the sense of [9, Definition 10.1].

Let $S_{\text{proét}}$ be the pro-étale site of S in the sense of [9, Definition 8.1]. Then we get a functor

$$Z \in \mathrm{Spec}(k)_{\text{proét}} \mapsto Z^{\diamond} \times_{\mathrm{Spd}(k)} S \in S_{\text{proét}}$$

with $(-)^{\diamond}$ the functor from perfect schemes to small v-sheaves introduced [10, Section 18.3]. More explicitly, $Z = \underline{T} \times \mathrm{Spec}(k)$ is sent to $\underline{T} \times S \cong$ for T a locally profinite set.

By descent we obtain a functor $\underline{T} \times \mathrm{Spec}(k) \mapsto \underline{T} \times \mathrm{Spd}(k)$ from $\mathrm{Spec}(k)_{\text{proét}}$ to the category $\mathrm{Spd}(k)_{\text{proét}}$ of small v-sheaves Y with a quasi-pro-étale morphism to $\mathrm{Spd}(k)$. Via surjections of v-sheaves, the category $\mathrm{Spd}(k)_{\text{proét}}$ is a site.

Proposition 2.1. *The functor* $\mathrm{Spec}(k)_{\text{proét}} \to \mathrm{Spd}(k)_{\text{proét}}$ *is an equivalence of sites.*

Proof. Let C/k be a non-archimedean, algebraically closed extension of k and $S := \mathrm{Spa}(C, \mathcal{O}_C)$. By [9, Proposition 9.6] each small v-sheaf Y' with a quasi-pro-étale map $Y' \to S$ is represented by a perfectoid space (note that the definition of quasi-pro-étale in [9, Definition 10.1] includes the assumption of local separatedness). By v-descent for quasi-pro-étale maps, it suffices therefore to show that $\mathrm{Spec}(k)_{\text{proét}}$ is equivalent to the category of descent data for quasi-pro-étale maps for the groupoid $S \times_{\mathrm{Spd}(k)} S \rightrightarrows S$. Now, [9, Theorem 19.5] implies that $S \times_{\mathrm{Spd}(k)} S$ is connected. This implies that for a locally profinite set T the restriction along the diagonal $S \to S \times_{\mathrm{Spd}(k)} S$ induces a bijection

$$\mathrm{Hom}_{\text{cont}}(S \times_{\mathrm{Spd}(k)} S, T) \to \mathrm{Hom}_{\text{cont}}(S, T).$$

[2] [2, Lemma 2.2.8] deals with a similar construction.

This in turn implies that on $T \times S$, there can only exist the trivial descent datum with descent $T \times \mathrm{Spd}(k)$. As $\mathrm{Spec}(k)_{\text{proét}}$ is equivalent to $S_{\text{proét}}$ this finishes the proof. □

Corollary 2.2. *Let H be any topological group and $\underline{H} := \mathrm{Hom}_{\mathrm{cont}}(-, H)$ its associated constant sheaf on the v-site $\mathrm{Perf}_{\mathbb{F}_q}$. Then any quasi-pro-étale H-torsor on $\mathrm{Spd}(k)$ is trivial.*

Proof. By Proposition 2.1 we can conclude that each quasi-pro-étale morphism $Y \to \mathrm{Spd}(k)$ with Y a small v-sheaf, has a section because the same is true for $\mathrm{Spec}(k)_{\text{proét}}$. □

3. Vector bundles on the absolute Fargues–Fontaine curve

Let k be an algebraically closed field extension of $\mathbb{F}_q$. In this section we want to prove Theorem 1.1, i.e., that the functor

$$\mathcal{E}_k(-) \colon \mathrm{Isoc}_k \to \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$$

is an equivalence.

We start with the following lemma.

Lemma 3.1. *The category $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$ is abelian.*

Proof. Clearly, $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$ is additive because $\mathrm{Bun}_{\mathrm{FF}}(S)$ is additive for any perfectoid space S over $\mathrm{Spd}(k)$. Set $S := \mathrm{Spa}(C, \mathcal{O}_C)$ with C/k a non-archimedean, algebraically closed extension with residue field k. Let $f \colon S \to \mathrm{Spd}(k)$ be the natural morphism, which induces the exact pullback functor

$$f^* \colon \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k)) \to \mathrm{Bun}_{\mathrm{FF}}(S) \cong \mathrm{Bun}(X_{E,S}) \cong \mathrm{Bun}(X_{E,S}^{\mathrm{sch}}).$$

Here we used GAGA for the curve ([4, Section II.2.3]) to identify vector bundles on $X_{E,S}$ with vector bundles on the schematic Fargues–Fontaine curve $X_{E,S}^{\mathrm{sch}}$. As $S \to \mathrm{Spd}(k)$ is a v-cover an absolute vector bundle, i.e., an object of $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$, is equivalently given by a vector bundle $\mathcal{E}$ on $X_{E,F}$ (equivalently on $X_{E,S}^{\mathrm{sch}}$) together with a descent datum

$$\sigma_{\mathcal{E}} \colon \mathrm{pr}_1^* \mathcal{E} \cong \mathrm{pr}_2^* \mathcal{E}$$

on $X_{E,S\times_{\mathrm{Spd}(k)}S}$ (note that no schematic Fargues–Fontaine curve is defined for $S \times_{\mathrm{Spd}(k)} S$ as $S \times_{\mathrm{Spd}(k)} S$ is not affinoid). Let A be the group of automorphisms of S over $\mathrm{Spd}(k)$. Pulling back the descent datum along the morphism

$$S \times A = \coprod_{a \in A} S \times \{a\} \to S \times_{\mathrm{Spd}(k)} S, \quad (x, g) \mapsto (x, gx)$$

shows that $\mathcal{E}$ is naturally equipped with a semilinear action of A. Recall now that $X^{\mathrm{sch}}_{E,S}$ is a connected noetherian Dedekind scheme ([4, Proposition II.2.9]), and hence admits a good theory of coherent sheaves. Let $g\colon (\mathcal{E},\sigma_{\mathcal{E}}) \to (\mathcal{F},\sigma_{\mathcal{F}})$ be a morphism of descent data, with $\mathcal{E},\mathcal{F}$ vector bundles on $X^{\mathrm{sch}}_{E,S}$. We claim that the kernel and cokernel of g are locally free, which then implies that they admit compatible descent data and that $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$ is abelian. To show torsion freeness of the kernel and cokernel, we may enlarge C and in particular assume that it is a field of Mal'cev-Neumann series over k, cf. [8, Section 3]. Then note that the morphism $g\colon \mathcal{E} \to \mathcal{F}$ is A-equivariant for the natural semilinear A-actions on $\mathcal{E},\mathcal{F}$. Thus, the torsion in the cokernel and the torsion of the kernel of g aquire natural semilinear A-actions. By Lemma 3.2 they are therefore trivial as they are supported at an A-equivariant, finite set of closed points of $X^{\mathrm{sch}}_{E,S}$. This finishes the proof. $\square$

Let us recall the definition of the field of Mal'cev-Neumann series $k((t^{\mathbb{R}}))$ over k, cf. [8, Section 3]. It is the ring of formal power series

$$a = \sum_{x\in\mathbb{R}} a_x t^x$$

with coefficients $a_x \in k$, such that the support

$$\mathrm{supp}(a) := \{x \in \mathbb{R} \mid a_x \neq 0\}$$

is a well-ordered subset of $\mathbb{R}$.

The multiplication

$$\Big(\sum_{x\in\mathbb{R}} a_x t^x\Big)\Big(\sum_{x\in\mathbb{R}} b_x t^x\Big) := \sum_{x\in\mathbb{R}}\Big(\sum_{y+z=x} a_y b_z\Big)t^x$$

is well-defined due to the support condition and makes $k((t^{\mathbb{R}}))$ into a field, cf. [8, Section 3]. The natural valuation

$$\nu\colon k((t^{\mathbb{R}})) \to \mathbb{R}\cup\{\infty\},\ a \mapsto \inf\{\mathrm{supp}(a)\}$$

shows that $k((t^{\mathbb{R}}))$ is naturally a non-archimedean field, which is even algebraically closed, cf. [8, Theorem 1].

The construction of $k((t^{\mathbb{R}}))$ works with $\mathbb{R}$ replaced by any ordered group, and is in fact functorial in automorphisms of such. In particular, the multiplication by some $x \in \mathbb{R}_{>0}$ induces a continuous automorphism

$$\sigma_x\colon k((t^{\mathbb{R}})) \to k((t^{\mathbb{R}})),$$

which sends t to t^x.

Lemma 3.2. *Set* $C := k((t^{\mathbb{R}}))$, $S := \mathrm{Spa}(C,\mathcal{O}_C)$ *and* $A := \mathrm{Aut}_{\mathrm{Spd}(k)}(S)$. *Then the action of the group A on the closed points of $X^{\mathrm{sch}}_{E,S}$ has no non-empty finite orbit.*

Proof. Assume that $\{x_1, \dots, x_n\} \subseteq X_{E,S}^{\mathrm{sch}}$ is a non-empty finite A-orbit. As the morphism $X_{E,S} \to X_{E,S}^{\mathrm{sch}}$ induces a bijection on classical points ([4, Proposition II.2.9]), we can view $\{x_1, \dots, x_n\}$ as a subset of $X_{E,S}$. Let

$$f \colon Y_{E,S} \to X_{E,S} = Y_{E,S}/\varphi^{\mathbb{Z}}$$

be the natural covering, and let $y_i \in Y_{E,S}$ be a classical point mapping to x_i, cf. [4, Definition/Proposition II.1.22]. Then by the same reference

$$f^{-1}(\{x_1, \dots, x_n\}) = \{\varphi^i(y_1), \dots, \varphi^i(y_n) \mid i \in \mathbb{Z}\}.$$

By [4, Proposition II.1.8] there exists non-units $a_1, \dots, a_n \in \mathcal{O}_C$ such that y_i is the vanishing locus of $\pi - [a_i]$, with $\pi \in \mathcal{O}_E$ a fixed uniformizer. Let $\nu \colon C \to \mathbb{R}$ be the t-adic valuation introduced above. The set

$$M := \{\nu(\varphi^i(a_j)) \mid i \in \mathbb{Z}, j = 1, \dots, n\} \subseteq \mathbb{R}$$

is then countable. This implies that the subgroup $\langle M \rangle \subseteq \mathbb{R}$ generated by M is countable. Hence, there exists some $x \in \mathbb{R}_{>0} \setminus \langle M \rangle$. Then $x + m \notin M$ for all $m \in M$. Consider now the automorphism $\sigma_x \in A$ sending t to t^x from above. Clearly,

$$\nu(\sigma_x(a)) = x + \nu(a)$$

for any $a \in C$. This implies that $\sigma_x(y_1) \notin f^{-1}(\{x_1, \dots, x_n\})$. Indeed, assume that $\sigma_x(y_1) = \varphi^m(y_j)$ for some $m \in \mathbb{Z}$ and $j = 1, \dots, n$, and let $\theta \colon W_{\mathcal{O}_E}(\mathcal{O}_C) \to k(\varphi^m(y_j))$ be the natural map with kernel $(\pi - [a_j^{p^m}])$. Then $\sigma_x(y_1) = \varphi^m(y_j)$ implies

$$\theta(\pi - [\sigma_x(a_1)]) = 0$$

and thus $\theta([\sigma_x(a_1)]) = \sigma_x(a_1)^\sharp = (a_j^{p^m})^\sharp = \theta([a_j^{p^m}])$ for the $\sharp$-map from the natural identification $C \cong k(\varphi^m(y_j))^\flat$. But this is impossible because $\sigma_x(a_1)$ and $a_j^{p^m}$ have different valuations. This finishes the proof. □

Let C/k be a non-archimedean, algebraically closed extension and $S := \mathrm{Spa}(C, \mathcal{O}_C)$. By [4, Section II.2.4] each vector bundle on $X_{E,S}$ has its rank and degree, and these yield a Harder-Narasimhan formalism in the sense of [3, Chapitre 5.5]. By descent, we can then define the rank and degree for the absolute vector bundles, i.e., the objects in $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$ as well. Using the identity functor as a "generic fiber functor" (as permitted by Lemma 3.1) the degree and rank function satisfy the conditions of an abstract Harder-Narasimhan formalism on $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$. In particular, each absolute vector bundle has its Harder-Narasimhan filtration with associated gradeds given by semistable objects.

We note that if $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$, then the pullback of its HN-filtration to S is the HN-filtration of the pullback $\mathcal{E}_S \in \mathrm{Bun}_{\mathrm{FF}}(S)$. Indeed, over $S \times_{\mathrm{Spd}(k)} S$ there exists at most one HN-filtration of an object in $\mathrm{Bun}_{\mathrm{FF}}(S \times_{\mathrm{Spd}(k)} S)$, and hence the pullbacks of the HN-filtration of $\mathcal{E}_S$ along the projections $S \times_{\mathrm{Spd}(k)} S \to S$ must agree. In particular, the HN-filtration descents to a filtration of $\mathcal{E}$. But the graded pieces for this filtrations are semistable (as they are after pullback to S) and hence this filtration must be the HN-filtration of $\mathcal{E}$.

We now analyze semistable objects in $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$.

Lemma 3.3. *Let $\lambda, \mu \in \mathbb{Q}$, and $\mathcal{E}, \mathcal{F} \in \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$ be semistable of slopes λ resp. μ.*

(1) *$\mathcal{E}$ is isomorphic to a direct sum of $\mathcal{O}(\lambda)$.*
(2) *If $\lambda \neq \mu$, then* $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0$.

Here, $\mathcal{O}(\lambda) = \mathcal{E}_k(D_\lambda)$ is the vector bundle associated with the isoclinic isocrystal D_λ of slope $-\lambda$, cf. [4, Section II.2.].

Proof. By [4, Theorem III.4.5] the torsor of isomorphisms of $\mathcal{E}$ to the direct sum of $\mathcal{O}(\lambda)$ (of rank $\mathrm{rk}(\mathcal{E})$) is representable by some pro-étale v-sheaf over $\mathrm{Spd}(k)$. By Proposition 2.1 this torsor splits, showing the first claim. If $\lambda > \mu$, then the second statement is a general consequence of the HN-formalism, cf. [3, Section 5.5.]. Thus, let us assume that $\lambda < \mu$. By (1) we know that $\mathcal{E}, \mathcal{F}$ are direct sums of $\mathcal{O}(\lambda)$'s resp. $\mathcal{O}(\mu)$'s. Considering $\mathcal{E}^\vee \otimes \mathcal{F}$ then reduces to the case that $\mathcal{E} \cong \mathcal{O}$ and $\mu > 0$. Considering pushforwards along $g_S\colon X_{E',S} \to X_{E,S}$ for a finite, separable extension $E'|E$, then reduces to the case that $\mu \in \mathbb{Z}$. More precisely, for any perfectoid space S' over $\mathrm{Spd}(k)$ the pullback $\mathcal{F}_{S'}$ embeds via the unit of the adjunction into $g_{S',*}g_{S'}^*\mathcal{F}_{S'}$ and this embedding commutes with base change in S'. Hence, it descents to $\mathrm{Spd}(k)$. Now,

$$\mathrm{Hom}_{X_{E,S'}}(\mathcal{O}, g_{S',*}g_{S'}^*\mathcal{F}_{S'}) \cong \mathrm{Hom}_{X_{E',S'}}(\mathcal{O}, g_{S'}^*\mathcal{F}_{S'}),$$

again compatible with base change, and this allows to replace E by E' and $\mathcal{F}$ by $g_{S'}^*\mathcal{F}_{S'}$. If $\mu \in \mathbb{Z}_{>0}$, then each $\mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))}(\mathcal{O}, \mathcal{O}(\mu)) = 0$ as each non-zero morphism must have a torsion cokernel after pullback to some algebraically closed, non-archimedean field of Mal'cev-Neumann series and thus is surjective by 3.2. This finishes the proof. □

We can now prove the desired classification of absolute vector bundles.

Theorem 3.4. *Let k be an algebraically closed field over $\mathbb{F}_q$. Then the functor*

$$\mathcal{E}_k(-)\colon \mathrm{Isoc}_k \to \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$$

is an equivalence.

Proof. Fully faithfulness follows from Lemma 3.3 and [4, Proposition II.2.5]. Indeed, by the Dieudonné-Manin classification the category Isoc_k is semisimple, and for each simple object D_λ with slope $\lambda \in \mathbb{Q}$, [4, Proposition II.2.5] implies that $\mathrm{End}(D_\lambda) \cong \mathrm{End}(\mathcal{E}_k(D_\lambda))$ (in fact, this statement holds even for $\mathrm{Spd}(k)$ replaced by $\mathrm{Spa}(C, \mathcal{O}_C)$ with C/k non-archimedean, algebraically closed). Now Lemma 3.3 implies that there are no non-zero morphisms between semistable objects for different slopes, which shows then fully faithfulness.

To show essential surjectivity we argue by induction on the rank. The case of rank 0 is trivial. Let $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$. We first assume that $\mathcal{E}$ is semistable of slope λ. In this case, we claim that each non-zero subobject $\mathcal{F} \subsetneq \mathcal{E}$ has slope λ. By the proof of 3.1 we must have $\mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$. By induction $\mathcal{F}$ can thus be assumed to lie in the essential image of $\mathcal{E}_k(-)\colon \mathrm{Isoc}_k \to \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$. In particular, $\mathcal{F}$ is a sum of $\mathcal{O}(\mu)$'s for some (different) $\mu \in \mathbb{Q}$. As $\mathcal{F} \to \mathcal{E}$ is injective, we can deduce from Lemma 3.3 that in fact $\mathcal{F}$ must be of slope λ.

Now, we claim that for each $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$ its HN-filtration is canonically split. For this we want to use the trick of considering the negative degree function from [3, p. 5.5.2.3] to get an opposite HN-filtration, which yields a canonical splitting. However, a priori the semistable objects in $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))$ for the usual HN-formalism need not be semistable for the opposite HN-formalism for $(-\mathrm{deg}, \mathrm{rk})$. However, we checked above that each semistable object of slope λ only has only non-trivial subobjects of slope λ. In particular, they remain semistable for the opposite HN-filtration. This implies that the usual HN-filtration splits the opposite one, and hence the claim.

Using induction on the rank and Lemma 3.3 the essential surjectivity follows. □

4. Vector bundles for classifying stacks

Let k be an algebraically closed extension of $\mathbb{F}_q$, and let H be a locally profinite group. We let $\underline{H} := \mathrm{Hom}_{\mathrm{cont}}(-, H)$ be the associated small v-sheaf on $\mathrm{Perf}_{\mathbb{F}_q}$. Let $[\mathrm{Spd}(k)/\underline{H}]$ be the classifying stack of $\underline{H}$ over $\mathrm{Spd}(k)$. In this section, we want to identify the category

$$\mathrm{Bun}_{\mathrm{FF}}([\mathrm{Spd}(k)/\underline{H}]).$$

By v-descent, this category identifies with descent data for the groupoid

$$\underline{H} \times \mathrm{Spd}(k) \rightrightarrows \mathrm{Spd}(k).$$

Note that $\underline{H} \times \mathrm{Spd}(k) \cong (H \times \mathrm{Spec}(k))^\diamond$ in the notation of Section 2. In particular, $\underline{H} \times \mathrm{Spd}(k)$ is the v-sheaf associated to some perfect scheme over $\mathrm{Spec}(k)$.

We need the following lemma.

Lemma 4.1. *Let $f\colon Z \to \mathrm{Spec}(k)$ be a perfect scheme over $\mathrm{Spec}(k)$, let $f^\diamond\colon Z^\diamond \to \mathrm{Spd}(k)$ be the natural map and let $D, D' \in \mathrm{Isoc}_k$. Then the map*

$$\mathrm{Hom}_{\mathrm{Isoc}_Z}(f^*D, f^*D') \to \mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k))}(f^{\diamond,*}\mathcal{E}_k(D), f^{\diamond,*}\mathcal{E}_k(D'))$$

is an isomorphism.

Here, the left hand side denotes morphisms of E-isocrystals over Z.

Proof. By the Dieudonné-Manin classification, we may assume that D, D' are simple of slope λ resp. λ'. We may even reduce to the case that $\lambda = 0$, i.e., $D = W_{\mathcal{O}_E}(k)[1/p]$ with its standard Frobenius. As the statement is local on Z we may assume that $Z = \mathrm{Spec}(R)$ is affine. If $\lambda' \neq \lambda$ we claim that both sides vanish. As R is reduced, it embeds into a product $\prod_{i\in I} k_i$ of algebraically closed fields. Then $W_{\mathcal{O}_E}(R)[1/p]$ embeds into $\prod_{i\in I}(W_{\mathcal{O}_E}(k_i)[1/p])$ and from the case of an algebraically closed field we can deduce by the Dieudonné-Manin classification that the left hand side vanishes.

Let us show the vanishing of the right hand side. If $\lambda' < 0$, then the vanishing follows from [4, Proposition II.2.5.(1)] by descent. Thus, assume that $\lambda' > 0$. Rephrased using absolute Banach-Colmez spaces, we have to show that each morphism

$$g\colon \mathrm{Spd}(R) \to \mathcal{BC}(\mathcal{O}(\lambda'))$$

over $\mathrm{Spd}(k)$ factors over the zero section if $\lambda' > 0$. This may be checked after base change along some $S := \mathrm{Spa}(C, \mathcal{O}_C) \to \mathrm{Spd}(k)$ with C non-archimedean and algebraically closed. Over S, it suffices to test this on geometric points $S' = \mathrm{Spa}(C', C'^{,+})$ over S. Now each morphism $R \to C'$ factors over some algebraic closure of some residue field of R. Hence, we may replace R by such a field, and in this case the desired vanishing follows from Lemma 3.3. To finish the proof, we have to understand the case that $\lambda' = 0$. Then the left hand side identifies with $\underline{E}(\mathrm{Spec}(R))$ by Artin-Schreier theory, while the right hand side with $\underline{E}(\mathrm{Spd}(R))$. By [10, Proposition 18.2.1] the open and closed decompositions of $\mathrm{Spec}(R)$ and $\mathrm{Spd}(R)$ agree (this follows by considering homomorphisms to $\mathrm{Spec}(\mathbb{F}_p \times \mathbb{F}_p)$ whose associated v-sheaf is $\mathrm{Spd}(\mathbb{F}_p) \coprod \mathrm{Spd}(\mathbb{F}_p))$. In particular, $\pi_0(\mathrm{Spec}(R))$ and $\pi_0(\mathrm{Spd}(R))$ are homeomorphic. This implies the claim and the proof is finished. □

We can now identify the category $\mathrm{Bun}_{\mathrm{FF}}([\mathrm{Spd}(k)/\underline{H}])$. By [5, Proposition 5.10] the fibered category $Z \mapsto \mathrm{Isoc}_Z$ is a v-stack on perfect $\mathbb{F}_q$-schemes (even an arc-stack) and hence we can talk about $\mathrm{Isoc}_{[\mathrm{Spec}(k)/\underline{H}]}$, which more concretely identifies with the category of descent data for the groupoid

$$\underline{H} \times \mathrm{Spec}(k) \rightrightarrows \mathrm{Spec}(k).$$

Theorem 4.2. *The natural functor* $\mathrm{Isoc}_{[\mathrm{Spec}(k)/\underline{H}]} \to \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k)/\underline{H}])$ *is an equivalence.*

Proof. This follows from Lemma 4.1 and Theorem 3.4. Namely, both imply that each descent datum on some object $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(k)) \cong \mathrm{Isoc}_k$ is already defined on the corresponding isocrystal. □

Assume now that H is profinite and let $C(H, k)$ be the ring of continuous functions on H with values in k. Then

$$\underline{H} \times \mathrm{Spec}(k) \cong \mathrm{Spec}(C(H, k)).$$

Moreover, the natural maps

$$C(H, \mathbb{Z}_p)\widehat{\otimes}_{\mathbb{Z}_p} W_{\mathcal{O}_E}(k) \to C(H, W_{\mathcal{O}_E}(k)) \to W_{\mathcal{O}_E}(C(H, k))$$

are isomorphisms, where the topology on $W_{\mathcal{O}_E}(k)$ is the p-adic topology and the tensor product is p-adically complete. In fact, if $\pi \in \mathcal{O}_E$ is a uniformizer the three objects are π-torsion free, π-complete lifts of the perfect k-algebra $C(H, k)$. Inverting p we get isomorphisms

$$C(H, \mathbb{Q}_p)\widehat{\otimes}_{\mathbb{Q}_p} W_{\mathcal{O}_E}(k)[1/p] \cong C(H, W_{\mathcal{O}_E}(k)[1/p]) \cong W_{\mathcal{O}_E}(C(H, k))[1/p].$$

Objects in $\mathrm{Isoc}_{[\mathrm{Spec}(k)/\underline{H}]}$ identify therefore with isocrytals $D \in \mathrm{Isoc}_k$ together with a coaction of the Hopf algebra $C(H, \mathbb{Q}_p)$. Unraveling the coaction, this yields precisely an isocrystal D with a continuous action of H on D, when D is equipped with its natural topology as a finite dimensional $W_{\mathcal{O}_E}(k)[1/p]$-vector space. Hence, objects in $\mathrm{Isoc}_{[\mathrm{Spd}(k)/\underline{H}]}$ deserve to be called “isocrystals with continuous H-action”, even if H is only locally profinite.

5. G-bundles on the absolute Fargues–Fontaine curve

Let G be a reductive group over E. In this section we want to generalize Theorem 3.4 from GL_n-bundles to arbitrary G-bundles on the absolute Fargues–Fontaine curve.

First we recall the definition of a G-bundle that we use, cf. [10, Theorem 19.5.2], [4, Definition/Proposition III.1.1].

Definition 5.1. Let $S/\mathbb{F}_q$ be a perfectoid space. A G-bundle (or G-torsor) over $X_{E,S}$ is an exact, E-linear tensor functor

$$\mathrm{Rep}_E(G) \to \mathrm{Bun}(X_{E,S})$$

from the category of representations of G on finite-dimensional E-vector spaces to the category of vector bundles on $X_{E,S}$.

For a perfectoid space $S/\mathbb{F}_q$ we denote by $\mathrm{Bun}_G(S)$ the groupoid of G-bundles on $X_{E,S}$ and by Bun_G the fibered category associating to each S the groupoid $\mathrm{Bun}_G(S)$. As in the case of vector bundles the fibered category Bun_G is a small v-stack, cf. [4, Section III.1]

Let k be an algebraically closed field over $\mathbb{F}_q$. As for the case of $\mathrm{Bun}_{\mathrm{FF}}$, we are interested in the category

$$\mathrm{Bun}_G(\mathrm{Spd}(k))$$

of "G-bundles on the absolute Fargues–Fontaine curve".

To ease notation, we set

$$L := W_{\mathcal{O}_E}(k)[1/p],$$

which carries its natural Frobenius $\varphi\colon L \to L$.

The following gadget will be shown to be equivalent to $\mathrm{Bun}_G(\mathrm{Spd}(k))$.

Definition 5.2. The Kottwitz category $\mathcal{B}(G)$ for G is defined as follows. It has as objects the elements in

$$G(L).$$

For $b, b' \in G(L)$ the set of homomorphisms from b to b' is defined to be the set

$$\{c \in G(L) \mid cb\varphi(c)^{-1} = b'\}.$$

Finally, composition is defined by multiplication in $G(L)$.

In other words, the Kottwitz category is the quotient groupoid

$$[G(L)/\varphi - \mathrm{conj.}]$$

of $G(L)$ modulo φ-conjugacy. The set of isomorphism classes of the Kottwitz category, i.e., the quotient set of $G(L)$ modulo φ-conjugation, is Kottwitz' famous set $B(G)$, cf. [6], [4, Definition III.2.1].

By Steinberg's theorem ([11, Chapitre III.2.3, Thérème 1']) the Kottwitz category is equivalent to the groupoid of exact, E-linear tensor functors

$$\mathrm{Rep}_E G \to \mathrm{Isoc}_k.$$

In particular, we can define for any perfectoid space S over $\mathrm{Spd}(k)$ a functor

$$\mathcal{T}_S\colon \mathcal{B}(G) \to \mathrm{Bun}_G(S)$$

by composing an exact, E-linear tensor functor $\mathrm{Rep}_E G \to \mathrm{Isoc}_k$ with the exact, E-linear tensor functor $\mathrm{Isoc}_k \to \mathrm{Bun}_{\mathrm{FF}}(S)$ considered in Section 1. As the different functors $\mathcal{T}_S$ are natural with respect to pullback along some morphism $S' \to S$ over $\mathrm{Spd}(k)$, we get a functor

$$\mathcal{T} \colon \mathcal{B}(G) \to \mathrm{Bun}_G(\mathrm{Spd}(k)).$$

As a consequence of Theorem 3.4 we obtain the following.

Theorem 5.3. *The functor* $\mathcal{T} \colon \mathcal{B}(G) \to \mathrm{Bun}_G(\mathrm{Spd}(k))$ *is an equivalence.*

Proof. We know that

$$\mathcal{B}(G) \cong \mathrm{Fun}_E^{\mathrm{ex},\otimes}(\mathrm{Rep}_E G, \mathrm{Isoc}_k)$$

is the category of exact, E-linear tensor functors. By Theorem 3.4 the category Isoc_k identifies with the category of descent data for the groupoid $S \to \mathrm{Spd}(k)$ for any non-empty perfectoid space S over $\mathrm{Spd}(k)$. This implies that the category $\mathrm{Fun}_E^{\mathrm{ex},\otimes}(\mathrm{Rep}_E G, \mathrm{Isoc}_k)$ identifies with the category of descent data along $S \to \mathrm{Spd}(k)$ for the v-stack $\mathrm{Fun}_E^{\mathrm{ex},\otimes}(\mathrm{Rep}_E G, \mathrm{Bun}_{\mathrm{FF}})$ of exact, E-linear tensor functors from $\mathrm{Rep}_E G$ to $\mathrm{Bun}_{\mathrm{FF}}$. Namely, exactness can be checked v-locally and also an E-linear resp. a tensor structure can be constructed v-locally. By Definition 5.1 the v-stack $\mathrm{Fun}_E^{\mathrm{ex},\otimes}(\mathrm{Rep}_E G, \mathrm{Bun}_{\mathrm{FF}})$ is equivalent to the v-stack Bun_G. This implies that $\mathcal{B}(G)$ is equivalent to $\mathrm{Bun}_G(\mathrm{Spd}(k))$, and in fact the constructed equivalence is given by $\mathcal{T}$. □

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