Degenerations of algebraic varieties and motivic integration

Summer term 2018, Tuesday 2-4pm, 0.011

We shall discuss various topics related to degenerations of algebraic varieties and motivic integration. In the end we should be able to understand the works of Halle-Nicaise [HN1] and Stewart-Vologodsky [SV].

We will start with a reminder on the Grothendieck ring of varieties and arc spaces. We should discuss the works of Denef-Loeser [DL1], [DL2] (for an overview one can look at Looijenga's exposition [Lo]). We will discuss one recent application of these ideas to the problem of rationality of algebraic varieties – the work of Nicaise-Shinder [NS] (generalized by Kontsevich-Tschinkel [KT]).

After that we will address the Hodge-theoretic part of the story, and give some overview of Schmid [Sch] and Cattani-Kaplan-Schmid [CKS]. To bring more geometry into the discussion, we can work out in more details the construction of Kulikov models for K3 surfaces (see [Ku], [PP] and a more recent partial generalization [Fu], [KLSV]). To understand degenerations of abelian varieties we will also need to discuss Néron models [BLR].

Finally, we should be able to bring together motivic integration and Hodge theory and go through [SV] and [HN1], [HN2]. We can also discuss more recent developments of these ideas [HN4].

For further information or if you want to give a talk in the seminar, please contact huybrech@ or aosoldat@. The **first talk will be on April 17.**

17 April: Motivic integration (Speaker: Lisa Li)
The talk should give an overview of arc spaces, Grothendieck ring of varieties, motivic measures and motivic integration. One can follow the exposition in [Be], see also the first three sections of [Lo], [DL1] and the notes [Bl]. As an application one can prove that K-equivalent varieties have equal Hodge numbers, see [Be], [DL1].

24 April: Motivic nearby fibre and motivic zeta function (Speaker: Dmitry Sustretov)
The talk should follow sections 4–6 in [Lo]. We should understand the statement and proof of Theorem 5.4 in [Lo] and its comparison with ordinary monodromy (end of section 5), see also the exposition in [DL1] (Theorem 3.3.1) and [DL3]. After that, discuss motivic zeta functions following section 6 in [Lo]. If time permits, explain the analogy with Igusa's p-adic zeta function (the end of section 6 in [Lo]).

8 May: Rationality under specialization (Speaker: Tim Bülles, Thorsten Beckmann)
The goal of the talk is to explain the proof of the main result of [NS]. One can follow this paper or its generalization [KT]. The main ingredient is Proposition 3.2.1 in [NS]. It contains the formula defining the specialization morphism, which is reinterpreted in [KT].

15 May: Motivic Thom-Sebastiani theorem (Speaker: Andrea Ricolfi)
The talk should explain section 7 from [Lo]. The goal is to prove motivic Thom-Sebastiani theorem [Lo, Theorem 7.4], [DL4] and discuss its applications.

29 May: Degenerations of Hodge structures (Speaker: Andrey Soldatenkov)
We should review the classical work of Schmid [Sch]. One should state the Nilpotent Orbit Theorem 4.9 and SL_2-orbit Theorem 5.13. After that, theorem 6.16 should be explained. This theorem defines the limit mixed Hodge structure of a degeneration, which is one of the key notions for us. If time permits (which is unlikely), we can discuss the idea of the proofs. The generalization of this work is contained in [CKS]. The notes [Ro] and references therein may also be useful.

5 June: Kulikov models (Speaker: Luca Tasin)
The goal of the talk should be to explain Kulikov's work on degenerations of K3 surfaces [Ku] using the results of [Fu]. Classical references for the subject are [PP] and [Pe], an overview can be found in [KK, Chapter 5]. It would be preferable to explain the general approach using MMP, as it appears in [Fu], and to deduce Kulikov's theorem from it.

12 June: The Clemens-Schmid exact sequence (Speaker: Isabell Grosse-Brauckmann)
The talk should explain how to relate the limit mixed Hodge structure of a degeneration with the mixed Hodge structure of the special fibre. One can follow the exposition of Morrison [Mo], or Chapter 5, §§3–5 of [KK]. The construction of the Clemens-Schmid sequence with the proofs is contained in [GNPP], section IV.7.

26 June: Néron models of abelian varieties (Speaker: Pablo Magni, Fabian Koch)
The talk should introduce Néron models and prove their existence for abelian varieties. One can follow Artin's exposition in [Ar]. The standard reference for Néron models is [BLR], where more general results are obtained.

3 July: Motivic zeta functions of abelian varieties (Speaker: Gebhard Martin)
In this talk one should explain the proof of motivic monodromy conjecture from [HN1]. A detailed exposition and extension of this result is contained in [HN3]. Many notions that are used in the proof (Chai's base change conductor, Edixhoven's filtration, etc.) are also explained in [HN3]

10 July 14-16, room 0.011: Essential skeleton of a degeneration (Speaker: Emma Brakkee)
This talk should give an idea about the essential skeleton of a degeneration – a notion introduced in [KS], section 6.6. One should follow the exposition in the sections 2 and 3 of [Ni]. Section 2 of this paper contains an introduction to the theory of Berkovich spaces, for more details one can look at [Pa]. This should be a preparation for the next talk.

10 July 16-18, room 0.006: Essential skeleton and MMP (Speaker: Daniel Huybrechts)
The talk should give an exposition of [NX], where the connection of essential skeleta to the MMP is studied (see also section 4 in [Ni]).

References

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