Algebraic Topology I Quick Questions

Algebraic Topology I Tutors

December 20, 2023

Below is a list of True/False-questions which the tutors of AlgTop 1 prepared. We recommend you try to answer them first without looking at your notes, in order to check your understanding of the notions discussed in the lecture.

There will be a second such practice sheet closer to the exam.

Problem 0.1. The equivalence $K(A;n) \simeq \Omega K(A;n+1)$ is adjoint to an equivalence $\Sigma K(A;n) \simeq K(A;n+1)$.

☐ True

☐ False

Problem 0.2. The Steenrod algebra $\mathscr{A}$ of stable cohomology operations in $\mathbb{F}_2$-homology is a polynomial algebra over $\mathbb{F}_2$.

☐ True

☐ False

Problem 0.3. If α is an element of degree i in the Steenrod algebra $\mathscr{A}$, $x \in \mathrm{H}^n(X;\mathbb{F}_2)$ is a cohomology class of degree n for a space X and $i > n$, then $\alpha(x) = 0$

☐ True

☐ False

Problem 0.4. Let V and W be vector bundles over a space X. Assume that V is isomorphic to the trivial bundle ε^n and W is isomorphic to the trivial bundle ε^m, then $V \oplus W$ is isomorphic to the trivial bundle ε^{n+m}. Here, 'isomorphic' is meant in the sense that the two bundles represent the same element in $\mathrm{Vect}^*_{\mathbb{R}}(X)$, with $*$ the respective dimension.

☐ True

☐ False

Problem 0.5. A subbundle of a trivial vector bundle is always trivializable.

☐ True

☐ False

Problem 0.6. Let $(F \to Y \to X, h)$ be a fibre sequence with Y contractible. Then the associated Serre spectral sequence with integral coefficients collapses on the E_2-page, i.e., all differentials d_r with $r \geq 2$ are trivial.

☐ True

☐ False

Problem 0.7. Let X be a simply-connected, finite CW-complex (i.e., with finitely many cells) such that $\mathrm{H}_i(X;\mathbb{Q}) = 0$ for all $i > 0$. Then $\pi_i(X,*)$ is finite, for all $i \in \mathbb{N}$ and every basepoint $*$.

☐ True

☐ False

Problem 0.8. The homotopy groups $\pi_m(S^n, *)$ are finite for all n and $m > n$.

☐ True

☐ False

Problem 0.9. Let $(F \to Y \xrightarrow{p} X, h)$ be a fibre sequence with F path-connected, and assume that $p^*\colon \mathrm{H}^i(X, \mathbb{F}_2) \to \mathrm{H}^i(Y, \mathbb{F}_2)$ is the zero map in all degrees $i > 0$. Then no class $x \in E_2^{i,0}$ with $i > 0$ on the associated Serre spectral sequence is a permanent cycle, i.e., every such x supports a non-trivial differential on some E_r-page.

☐ True

☐ False

Problem 0.10. Let $(E_r^{p,q})_{r\in\mathbb{N}}$ be a multiplicative spectral sequence with product $\cdot$, and let x, y be elements on the E_2-page. If $d_2(x \cdot y) = 0$, then $d_2(x) = 0$ or $d_2(y) = 0$.

☐ True

☐ False

Problem 0.11. The group $\pi_{2024}(S^{2022}, *)$ is generated by $\Sigma^{2020}(\eta) \circ \Sigma^{2021}(\eta)$.

☐ True

☐ False

Problem 0.12. Let G be a group. Then the classifying space $\mathrm{B}G = K(G, 1)$ is simple if and only if G is abelian.

☐ True

☐ False

Problem 0.13. If X is a connected CW-complex which only admits trivial real vector bundles, then X is contractible.

☐ True

☐ False

Problem 0.14. Let B be a CW-complex, p a prime, and $E \to B$ an n-dimensional real vector bundle over B. Then the Thom isomorphism theorem gives an isomorphism

$$\mathrm{H}^*(B; \mathbb{F}_p) \xrightarrow{\cong} \mathrm{H}^{*+n}(E, E_0; \mathbb{F}_p)$$

☐ True

☐ False

Problem 0.15. Stiefel–Whitney classes are multiplicative in the sense that if ξ, η are vector bundles over a base space B, then for all i

$$w_i(\xi \oplus \eta) = w_i(\xi) \cdot w_i(\eta)$$

☐ True

☐ False

Problem 0.16. If $x \in H^2(K(\mathbb{F}_2, 2), \mathbb{F}_2)$ is the generator, then $H^6(K(\mathbb{F}_2, 2), \mathbb{F}_2)$ is of rank two generated by x^3 and $\mathrm{Sq}^2(x^2)$.

☐ True

☐ False

Problem 0.17. Let $\xi\colon E \to B$ be a vector bundle of rank $n > 0$ over a paracompact base B. If there is a nowhere-vanishing section to ξ, then $w_n(\xi) = 0$.

☐ True

☐ False

Problem 0.18. The class of all finitely generated abelian groups in which every element is 3-torsion forms a Serre class.

☐ True

☐ False

Problem 0.19. Let $(F \to E \to B, h)$ be a fibre sequence with F and B path-connected. If $x \in H^{n-1}(F;\mathbb{F}_2)$ transgresses to $y \in H^n(B;\mathbb{F}_2)$, then $Sq^i x$ survives to the E_{n+i}-page and $d_{n+i}(\mathrm{Sq}^i x)$ is represented by $\mathrm{Sq}^i y$ under the quotient map $H^{n+i}(B,\mathbb{F}_2) \cong E_2^{n+i,0} \to E_{n+i}^{n+i,0}$.

☐ True

☐ False

Problem 0.20. Let $(F \to Y \to X, h)$ be a fibre sequence with associated Serre spectral sequence and A an abelian group. Then the E_2-page is isomorphic in degree (p,q) to $H^p(X;A) \otimes H^q(F;A)$.

☐ True

☐ False