GRK 1150, Mathematisches Institut, Universität Bonn, 53115 Bonn

HOMOTOPY & COHOMOLOGY

GK 1150

Winter School

"From Field Theories to Elliptic Objects"

February, 28th till March, 4th 2006
Schloss Mickeln, Düsseldorf

GRK-Sprecher:
Prof. Dr. C.-F. Bödigheimer
Arbeitsgruppe Topologie
Beringstrasse 1
D- 53115 Bonn
Email: grk1150@math.uni-bonn.de

GRK-Sekretariat:
Frau S. George
Tel: 0228-73 77 88
Fax: 0228-73 64 90

Talk No. 7

Speaker: Holger Reeker

universitätbonn
Rheinische Friedrich-Wilhelms-
Universität Bonn

Fockspaces II.

Plan: 0) Clifford algebras

1) (generalized) Lagrangians and Fockspaces

2) Functorial aspects of the construction and gluing of Fockspaces

3) Clifford algebras and Fock spaces associated to spin manifolds

---

0) Clifford algebras

V real / complex Hilbert space

$\alpha : V \to V$, $v \mapsto \bar{v} = \alpha(v)$ isometric involution

($\mathbb{C}$ anti-linear in the complex case)

$\Rightarrow b(v,w) := \langle \bar{v}, w \rangle$ symmetric bilinear form

Clifford algebra $C(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n} \Big/ {\scriptstyle v\cdot v = -b(v,v)\cdot 1,\ v\in V}$

e.g. $C_n := C(\mathbb{R}^n)$ generated by vectors $v \in \mathbb{R}^n$ subject to the relations $v\cdot v = -|v|^2 \cdot 1$

$C_{-n} := C(\mathbb{R}^n)$ generated by vectors $v \in \mathbb{R}^n$ subject to relations $v \cdot v = |v|^2 \cdot 1$.

$C_{n,m} := C(\mathbb{R}^n \oplus -\mathbb{R}^m)$ generated by $v \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ subject to the relations $v \cdot v = -|v|^2 \cdot 1$, $w \cdot w = |w|^2 \cdot 1$.

- $C(V \oplus W) \cong C(V) \otimes C(W)$
- $C(-V) \cong C(V)^{op}$

A $A \otimes B$-module M can be interpreted as a bimodule over $A - B^{op}$ via

$$a \cdot m \cdot b := (-1)^{|m||b|} (a \otimes b) m$$

In particular, a left module over $C(V \oplus -W)$ may be interpreted as a left module over $C(V) \otimes C(W)^{op}$;
or equivalently, as a $C(V)$-$C(W)$-bimodule.

1) (generalized) Lagrangians and Fockspaces

Standard construction of modules over the Clifford algebra $C(V)$. Input datum: Lagrangian $L \subset V$

Def (Lagrangian $L \subset V$): • L closed

• $b(\ell_1, \ell_2) = 0 \quad \forall\, \ell_1, \ell_2 \in L$

• $V = L \oplus \bar{L}$

---

given a Lagrangian L, the exterior algebra

$$\Lambda(\bar{L}) = \Lambda^{ev}(\bar{L}) \oplus \Lambda^{odd}(\bar{L}) = \bigoplus_{p\ \text{even}} \Lambda^p(\bar{L}) \oplus \bigoplus_{p\ \text{odd}} \Lambda^p(\bar{L})$$

is a $\mathbb{Z}/2$-graded module over the Clifford algebra $C(V)$.

algebraic Fock space $\quad F_{alg}(L) := \Lambda(\bar{L})$

fermionic Fock space $\quad F(L) := \overline{F_{alg}(L)}$

completion with respect to the inner product induced by the inner product on $\bar{L} \subset V$.

A <u>generalized Lagrangian</u> is a homomorphism $L: W \to V$ with finite dimensional kernel s.t. the closure $L_W \subset V$ of the image of L is a Lagrangian

algebraic Fock space $F_{alg}(L) := \Lambda^{top}(\ker L)^* \otimes \Lambda(\bar{L}_W)$

with $\Lambda^{top}(\ker L)^* = \Lambda^{\dim(\ker L)}(\ker L)^*$

fermionic Fock space $\quad F(L) = \overline{F_{alg}(L)}$

2) Functorial aspects of the construction

$$V \longmapsto C(V)$$

$$L \longmapsto F_{alg}(L)$$

Domain category

Ob: Hilbert spaces V with involutions

Mor(V_1, V_2): Lagrangian subspaces of $V_2 \oplus -V_1$

- V_1, V_2 Hilbert spaces with involutions
  $L_1 \subset V_2 \oplus -V_1$ a Lagrangian
  $\Rightarrow F_{alg}(L_1)$ is a graded module over $C(V_2 \oplus -V_1)$
  bimodule over $C(V_2) - C(V_1)$
- Given $L_1 \subset V_2 \oplus -V_1$ and $L_2 \subset V_3 \oplus -V_2$
  $\Rightarrow$ Lagrangian $L_3 \subset V_3 \oplus -V_1$ obtained by 'symplectic' reduction from the Lagrangian
  $L := L_2 \oplus L_1 \subset V := V_3 \oplus -V_2 \oplus V_2 \oplus -V_1$,

namely $L^{red} := \dfrac{L \cap U^{\perp_b}}{L \cap U} \subset V^{red} := \dfrac{V \cap U^{\perp_b}}{U}$

$U := \{(0, v_2, v_2, 0) \mid v_2 \in V_2\} \subset V$ isotropic subspace

$U^{\perp_b}$ annihilator of U with respect to b.

- Exercise: $V^{red} = V_3 \oplus -V_1$

Range category

Ob: graded algebras

Mor(A,B): pointed, graded B-A-bimodules

Composition: (M, m_0) B-A bimodule, (N, n_0) C-B-bimodule
$\Rightarrow (N, n_0)\circ(M, m_0) := (N \otimes_B M, n_0 \otimes m_0)$ C-A-bimodule

In type I case (i.e. the von Neumann algebra generated by $C(V)$ in $B(F(L))$ is type I): Composition of Lagrangians is compatible with the tensor product of pointed bimodules.

Gluing Lemma If the van Neumann algebra generated by $C(V_2)$ has type I, there is a unique isomorphism of pointed, graded $C(V_3)$-$C(V_1)$ bimodules

$$(F_{alg}(L_2) \otimes_{C(V_2)} F_{alg}(L_1), \Omega_2 \otimes \Omega_1) \cong (F_{alg}(L_3), \Omega_3)$$

Here we assume that L_i intersect V_j trivially.

Proof: $F_{alg}(L_2) \otimes_{C(V_2)} F_{alg}(L_1)$ is the quotient of

$$F_{alg}(L_2) \otimes F_{alg}(L_1) = F_{alg}(L_2 \oplus L_1) = F_{alg}(L)$$

modulo the subspace $\bar{U} F_{alg}(L)$ where

$$\bar{U} = \{(0, -v_2, v_2, 0) \mid v_2 \in V_2\}$$

We observe that for $\bar{u} = (0, -v_2, v_2, 0) \in \bar{U}$ and $\psi_i \in F_{alg}(L_i)$ we have

$$c(\bar{u})(\psi_2 \otimes \psi_1) = (-1)^{|\psi_2|}\left(-\psi_2\, c(\bar{v}_2) \otimes \psi_1 + \psi_2 \otimes c(v_2)\,\psi_1\right).$$

- an element $\bar{u} \in \bar{U} \subset V$, which decomposes as $\bar{u} = u_1 + \bar{u}_2 \in V = L \oplus \bar{L}$ with $u_i \in L_i$ acts on $F_{alg}(L) = \Lambda(\bar{L})$ as the sum $c(u_1) + c(\bar{u}_2)$ of the 'creation' operator $c(u_1)$ and the 'annihilation' operator $c(\bar{u}_2)$.

- $$\begin{aligned} L^{red} \oplus \bar{U} &\xrightarrow{\ \cong\ } L \\ (v, \bar{u}) &\longmapsto v + u_1 \end{aligned}$$

- <u>Fact</u>: The $C(V^{red})$-linear map

$$\Lambda(\bar{L}^{red}) \longrightarrow \Lambda(\bar{L}^{red} \oplus U) \big/ C(\bar{U})\Lambda(\bar{L}^{red} \oplus U)$$

is also an isomorphism. $\boxed{}$

3) Clifford algebras and Fock spaces associated to spin manifolds

Definition (Spin structures on conformal manifolds)
Let Σ be a d-dimensional mfd equipped with a conformal structure and $L^k \to \Sigma$ be the oriented real line bundle ($k \in \mathbb{R}$) whose fiber over $x \in \Sigma$ consists of all maps $\rho : \Lambda^d(T_x\Sigma) \to \mathbb{R}$ such that $\rho(\lambda\omega) = |\lambda|^{k/d} \rho(\omega)$

$\longrightarrow$ Riemannian metric on the weightless cotangent bundle $T_0^*\Sigma := L^{-1} \otimes T^*\Sigma$.

A spin structure on a conformal d-mfd Σ is a spin structure on the Riemannian vector bundle $T_0^*\Sigma$.

Definition (Clifford algebras associated to spin manifolds)
Let Y^{d-1} be a conformal spin mfd with spinor bundle $S \to Y$. Define

$$V(Y) := L^2(Y, L^{\frac{d-1}{2}} \otimes S)$$

Note: For $d=1$: $V(Y)$ is just a graded, real Hilbert space

$d=2$: since the Clifford algebra $C_{d-1} \cong \mathbb{C}$, $V(Y)$ is a complex vector space, better: graded, complex Hilbert space

Define $C(Y) := C(V(Y))$.

The generalized Lagrangian $L(\Sigma): W(\Sigma) \to V(\partial\Sigma)$

- Let Σ^d be a conformal spin manifold
- Picking a Riemannian metric in the given conformal class determines the Levi-Civita connection on the tangent bundle of Σ

$\Rightarrow$ the Levi-Civita connection determines connections on the spinor bundle $S = S(T_0^*\Sigma)$, the line bundles L^k and hence $L^k \otimes S$ for all $k \in \mathbb{K}$

- The corresponding Dirac operator $D = D_\Sigma$ is the composition

$$D: C^\infty(\Sigma; L^k \otimes S) \xrightarrow{\nabla} C^\infty(\Sigma; T^*\Sigma \otimes L^k \otimes S)$$

$$= C^\infty(\Sigma; L^{k+1} \otimes T_0^*\Sigma \otimes S) \xrightarrow{c} C^\infty(\Sigma; L^{k+1} \otimes S)$$

- Fact: For $k = \frac{d-1}{2}$ the Dirac operator is independent of the choice of the Riemannian metric.
- Green's formula (integration by parts) yields

$$\langle D\psi, \phi\rangle - \langle \psi, D\phi\rangle = \langle c(\nu)\psi_|, \phi_|\rangle, \quad \psi, \phi \in C^\infty(\Sigma, L^{\frac{d-1}{2}} \otimes S)$$

where $\psi_|$ and $\phi_|$ are the restrictions to $\partial\Sigma$ and ν is the unit conormal vector field.

- Replacing ψ by ψe_1:

$$\langle D\psi e_1, \phi\rangle + \langle \psi, D\phi e_1\rangle = \langle c(v)\psi_1 e_1, \phi_1\rangle$$

- Let $W(\Sigma) := \ker D^+$ where D^+ has domain $C^\infty(\Sigma, L^{\frac{d-1}{2}} \otimes S^+)$ and consider the restriction to the boundary

$$L(\Sigma) : W(\Sigma) \longrightarrow L^2(\partial\Sigma, L^{\frac{d-1}{2}} \otimes S) = V(\partial\Sigma)$$

- The closure L_Σ of the image of $L(\Sigma)$ is the <u>Hardy space</u> of boundary values of harmonic sections of $L^{\frac{d-1}{2}} \otimes S^+$. The kernel of $L(\Sigma)$ is the space of <u>harmonic spinors on Σ which vanish on the boundary</u>.

- <u>FACT</u>: $L(\Sigma)$ is a generalized Lagrangian.

In the following: $d=1$ or $d=2$. In these cases C_{d-1} is commutative.

<u>Definition</u> (The $C(\partial\Sigma)$-modules $F_{alg}(\Sigma)$ and $F(\Sigma)$)

$$F_{alg}(\Sigma) := F_{alg}(L(\Sigma))$$

algebraic Fock module over $C(\partial\Sigma)$

$d=1$ real vector space

$d=2$ complex vector space

$$F_{alg}(\Sigma) = F_{alg}(L(\Sigma)) = \Lambda^{top}(\ker L(\Sigma))^* \otimes \Lambda(\overline{L}_\Sigma)$$

- $\overline{L}_\Sigma$ is equipped with a natural inner product

$\Rightarrow \Lambda(\overline{L}_\Sigma)$ also has a natural inner product
$\langle v_1 \wedge v_2 \wedge \dots v_r, w_1 \wedge \dots w_r \rangle := \det\langle v_i;$

- If $\Sigma_0 \subseteq \Sigma$ denotes the subspace of closed components

then $\ker L(\Sigma) = \ker D^+_{\Sigma_0}$

- $D^+_{\Sigma_0} : C^\infty(\Sigma_0; L^{\frac{d-1}{2}} \otimes S^+) \to C^\infty(\Sigma_0; L^{\frac{d+1}{2}} \otimes S^-)$

is <u>skew-adjoint</u> since

$$\langle D\psi e_1, \phi\rangle + \langle \psi, D\phi e_1\rangle = \langle c(\nu)\psi e_1, \phi|\rangle = 0.$$

- for $\psi \in C^\infty(\Sigma; L^{\frac{d-1}{2}} \otimes S^+)$ and $\phi \in C^\infty(\Sigma; L^{\frac{d+1}{2}} \otimes S^-)$ the pointwise inner product of ψe_1 and ϕ gives us a section of L^d.
- integrating this over Σ gives us a complex number

$$\Rightarrow L^2(\Sigma, L^{\frac{d+1}{2}} \otimes S^-) = L^2(\Sigma, L^{\frac{d-1}{2}} \otimes S^+)^*$$

- In particular $\Lambda^{top}(\ker L(\Sigma))^* = \Lambda^{top}(\ker D^+_{\Sigma_0})^*$

$=$ Pfaffian line $Pf(\Sigma)$

of the skew-adjoint operator $D^+_{\Sigma_0}$

(real line for $d=1$, complex line for $d=2$)

- $F_{alg}(\Sigma) = \Lambda^{top}(\ker D^{+}_{\Sigma_0})^* \otimes \Lambda(L_\Sigma)$

equipped with natural inner products

- $F(\Sigma) := \overline{F_{alg}(\Sigma)}$
- $F(\Sigma)$ is still a module over $C(\partial\Sigma)$
- The Fock space $F(\Sigma)$ is a generalization of the Pfaffian line $Pf(\Sigma)$, since for a closed conformal spin manifold Σ the Fock space is equal to $Pf(\Sigma)$.

<u>Remark:</u> For $d=1$ we have $F_{alg}(\Sigma) = F(\Sigma)$ since both are finite dimensional.

<u>Remark:</u> If Σ is a conformal spin bordism from Y_1 to Y_2

$\Rightarrow F(\Sigma)$ is a left module over

$$C(\partial\Sigma) = C(Y_1)^{op} \otimes C(Y_2)$$

<u>or</u> a $C(Y_2)$-$C(Y_1)$-bimodule

Gluing surfaces Σ_1, Σ_2 conformal spin surfaces

$$\partial \Sigma_1 = Y_1 \cup Y_2 \quad , \quad \partial \Sigma_2 = Y_2 \cup Y_3$$

Define $\Sigma_3 := \Sigma_1 \cup_{Y_2} \Sigma_2$

Gluing Lemma (geometric formulation)
If Y_2 is a closed 1-manifold, there are natural isomorphisms of graded $C(Y_3)-C(Y_1)$ bimodules

$$F_{alg}(\Sigma_2) \otimes_{C(Y_2)} F_{alg}(\Sigma_1) \cong F_{alg}(\Sigma_3) .$$