HOMOTOPY & COHOMOLOGY

GK 1150

GRK 1150, Mathematisches Institut, Universität Bonn, 53115 Bonn

Winter School

"From Field Theories to Elliptic Objects"

February, 28th till March, 4th 2006
Schloss Mickeln, Düsseldorf

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Talk No. 15

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Theorem The partition function of a supersymmetric C_n-linear conformal field theory E, n even, is a weak integral modular form of weight $n/2$.

Modular form

Def: A modular fct. $f : \mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$
$f : \mathbb{H} \to \mathbb{C}$ of weight k is a hol. fct
$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl_2 \mathbb{Z}$ $\qquad f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$

$f(\tau + 1) = f(\tau)$ $\qquad \Rightarrow$ f only depends on
$q = e^{2\pi i \tau}$ $\qquad q \in \mathbb{E}^\times$

$$f(q) = \sum_{n \in \mathbb{Z}} a_n q^n$$

- f is a weak mod. form $\Leftrightarrow$ for almost all $n < 0$ $a_n = 0$
- f is a mod. form : for all $n < 0$ $a_n = 0$
- f is integral $\Leftrightarrow$ all $a_n \in \mathbb{Z}$

Ex $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ weight 12

$c_4 = 1 + 240\sum_{k>0}\left(\sum_{d|k} d^3\right)q^k$ weight 4

$c_6 = 1 - 504\sum_{k>0}\left(\sum_{d|k} d^5\right)q^k$ weight 6

Ring of integral modulus form

$\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$

MF_* ring of weak int. mod forms

$\cong \mathbb{Z}[c_4, c_6, \Delta, \Delta^{-1}]/(c_4^3 - c_6^2 - 1728\Delta)$.

CFT's: $Ob(cB_n^2)$ = 1-dim closed spin mflds

$Mor(Y_1, Y_2)$ type I: (f, c) $f: Y_1 \to Y_2$ spin diff. $c \in c(Y_1)^{-n}$

type (Σ, ψ) Σ conf. spin bordism

$\psi \in T_{\partial g}(\Sigma)^{-n}$

$E: cB_n^2 \to Hilb$

$$\left.\begin{aligned} C(Y_1)^{-n} &\longrightarrow Hilb(E(Y_1), E(Y_2)) \\ c &\longmapsto E(f, c) \end{aligned}\right\} \text{linear}$$

$$\mathcal{F}_{\mathrm{alg}}(\Sigma)^{-n} \to \mathrm{Hilb}(E(Y_1), E(Y_2))$$
$$\Psi \mapsto E(\Sigma, \Psi)$$

Closed surfaces

$$\mathcal{F}_{\mathrm{alg}}(\Sigma) = \Lambda^{\mathrm{top}}(\ker D^+)^* =: \mathrm{Pf}(\Sigma) \quad \text{Pfaffian line}$$

$$Z_E(\Sigma)(\Psi) := E(\Sigma, \Psi) \in \mathrm{Hilb}(E(\emptyset), E(\emptyset)) = \mathbb{C}$$

$$\rightsquigarrow Z_E(\Sigma) \in (\mathrm{Pf}^{-n})^* = \mathrm{Pf}^{n}(\Sigma)$$

<u>the partition function.</u> Σ <u>Riemann surface</u>

a small dictionary

Spin geometry	Classical complex analysis
Spin structure on Σ	Square-roots S of K_Σ (the canonical line bdl)
D^+	$\bar\partial_S : \Gamma(S) \to \Gamma(K^* \otimes S) = \Gamma(S^{-1})$

From now on: Σ torus, we have

4 spin structures

K_Σ is the trivial hol. line bundle

$\Rightarrow$ the trivial line bdl. is a spin structure S_0

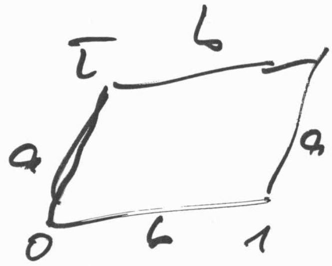
$\dim \ker \bar\partial_{S_0} = 1$

S_0 : the canonical spin structure

<u>Teichmüller space of tori</u>

$$\mathbb{H} \longrightarrow \mathcal{T}$$

$$\tau \longmapsto \underbrace{\mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}}_{=:\Sigma_\tau}$$



- any complex torus is of this form

two tori $\Sigma_\tau, \Sigma_{\tau'}$ are equivalent $\Leftrightarrow$

$\exists \underset{g}{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \in Sl_2\mathbb{Z}$ $\quad \tau' = \frac{a\tau+b}{c\tau+d}$

an iso is given by $\qquad \Sigma_\tau \xrightarrow{f_g} \Sigma_{g\tau}$

$$z \longmapsto (c\tau+d)^{-1}\cdot z$$

The Pfaffian line for the canonical spin structure defines a holomorphic line bundle on $\mathcal{T}$.

$$\begin{array}{c} Pf \to J \\ \rotatebox{90}{$\in$} \\ [\Sigma] \end{array} \qquad Pf[\Sigma] := \ker\left(\bar\partial_{(\Sigma, S_{\sigma_\Sigma})}\right)^*$$

<u>n even</u> (from now on)

$Sl_2\mathbb{Z}$ act on Pf^{-2} $\qquad g \in Sl_2\mathbb{Z}$

$\cancel{[\Sigma] \to [g\Sigma]}$ $\qquad \Sigma \xrightarrow{\phi_g} g\Sigma$

$$Pf^{-2}(\Sigma) = H^0(\Sigma, \mathcal{O}_{K_\Sigma})$$

$$\phi_g : Pf^{-2}(\Sigma) \xrightarrow{\cong} Pf^{-2}(g\Sigma)$$

A trivialization of $Pf^{-2} \to J$

$$\begin{tikzcd} \mathbb{H} \times \mathbb{C} \arrow[r, "\alpha"] \arrow[d] & Pf^{-2} \arrow[d] \\ \mathbb{H} \arrow[r, "\cong"] & J_1 \end{tikzcd} \qquad \alpha(\tau, x) := x \cdot dz_\tau$$

1-form on Σ_τ

The $Sl_2\mathbb{Z}$-action on Pf^{-2} translates into

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau, x) = \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)x\right).$$

⇒ **Prop**: An $SL_2\mathbb{Z}$-equivariant section of $Pf^{\otimes 2k}$ has the transformation behaviour of a modular function of weight k.

Corollary: The partition fct has the transf. behaviour of a mod. fct.

$2k = n$

Proof:

$$T_1 \longrightarrow Pf^{\otimes 2k}$$

$$(g\Sigma, g_* \varphi) = (\Sigma, \varphi) \quad \text{in } Mor_{\mathcal{EB}_n^2}(\emptyset, \emptyset)$$

Proof of holomorphicity of Z_E

$$A_\tau := \{0 \le \operatorname{im} z \le \operatorname{im} \tau\} / \mathbb{Z}_\tau$$

a conformal spin bordism from $S^{per,1}$ to S^{per}

$H := E(S^{per})$

on S^{per}, ∃ spin diff which interchanges the two sheets of the spin structure

$\mathbb{Z}/2$-graded

$$C_1 \cong \mathbb{C} \le C(S^{per})$$

$\Rightarrow$ H is a graded $\underset{\substack{\shortparallel \\ C_n}}{C_1^{\otimes n}}$ - module

Lemma:

$$Z_E(\tau) = \mathrm{str}_{C_n}\left(E(A_\tau, \Sigma^{-n})\right)$$

idea of proof

Consider

$$\begin{tikzcd} \rho : \mathbb{H} \arrow[r] \arrow[dr] & \mathrm{Hilb}(H, H) \\ & HS^{ev}_{C_n}(H, H) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\rho(\tau) := E(A_\tau, \Sigma^{-n})$

· semigroup

$\rho(\tau)$ is C_n-linear, even

$\rho(\tau)$ is Hilbert-Schmidt

The continuity of E should imply

$\rho(\tau)$ is ~~weakly~~ strongly differentiable; i.e. of $\mathcal{O}$

$\forall v \in H$

$$f_v(\tau) := \begin{cases} v & \tau = 0 \\ g(\tau)\cdot v & \tau \in H \end{cases}$$ is differentiable!

$$L_0 := \frac{1}{2\pi i} \frac{\partial}{\partial \tau}\Big|_{\tau=0} g(\tau) \qquad \frac{\partial}{\partial \tau} := \frac{\partial}{\partial(\mathrm{Re}\,\tau)} \pm i \frac{\partial}{\partial(\mathrm{Im}\,\tau)}$$

$$\bar{L}_0 := -\frac{1}{2\pi i} \frac{\partial}{\partial \bar\tau}\Big|_{\tau=0} g(\tau).$$

$$g(\tau) = e^{2\pi i \tau \cdot L_0} e^{-2\pi i \bar\tau \bar{L}_0}$$

$L_0, \bar{L}_0$ even, $\mathbb{C}$-linear (unbounded)

$[L_0, \bar{L}_0] = 0$

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① $g(\tau+1) = g(\tau)$ $\qquad (A_\tau = A_{\tau+1})$

② $\underline{g(-\bar\tau) = g(\tau)^*}$ $\qquad (\overline{A_\tau} = A_{-\bar\tau})$

$$\Rightarrow \left(e^{2\pi i \tau L_0} \cdot e^{-2\pi i \bar\tau \bar{L}_0}\right)^* = e^{-2\pi i \bar\tau L_0^*} e^{2\pi i \tau \bar{L}_0^*}$$

$$\| $$

$$e^{-2\pi i \bar\tau L_0} \cdot e^{2\pi i \tau \bar{L}_0} \Rightarrow \begin{array}{l} L_0 = L_0^* \\ \bar{L}_0 = \bar{L}_0^* \end{array}$$

$$\rho(1) = \mathrm{id}$$
$$\|$$
$$\exp(2\pi i (L_0 - \bar{L}_0))$$

$$\Rightarrow \mathrm{spec}(L_0 - \bar{L}_0) \subset \mathbb{Z}.$$

<u>Supersymmetry:</u>

<u>Definition:</u> E is <u>supersymmetric</u> if $\bar{L}_0$ is the square of an odd operator $\bar{G}_0$

$$Z_E(\tau) = \mathrm{str}_{C_n}(q^{L_0} \bar{q}^{\bar{L}_0}) =$$

$$\mathrm{str}_{C_n}\left(q^{L_0} \bar{q}^{\bar{L}_0}\big|_{(\ker \bar{L}_0)}\right) = \mathrm{str}_{C_n}\left(q^{L_0}\big|_{\ker \bar{L}_0}\right)$$

~~Because $\forall E$~~

$$= \sum_{k \in \mathbb{Z}} \underbrace{\mathrm{sdim}_{C_n}\left(\mathrm{Eig}(L_0, k) \cap \ker \bar{L}_0\right)}_{\in \mathbb{Z}} q^k$$

If $\exists \infty$ many $k < 0$ with $\neq 0 \Rightarrow$

$\rho(it)$ isn't Hilbert-Schmidt.

$\blacksquare$