Fredholm Operators and K-Theory

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Goal: sketch the proof of today's theorem

EFT's form a spectrum for KO-theory

and explain a main ingredient

Fredholm operators form such a spectrum

Plan of the Talk:

- Introduction
  real C^*-algebras, Hilbert modules, Fredholm operators
- Theorem: $KO^{-k}(X;\mathcal{A}) \cong [X, \hat{F}^k_*(H_\mathcal{A}) \times KO_k(\mathcal{A})]$
- A brief look on Bott periodicity
- Sketch of proof of the main theorem

Real C^*-Algebras and K-Theory

Definition: A (real) C^*-algebra is a (real) Banach-$*$-algebra which $*$-isometrically isomorphic to a closed subalgebra of $\mathcal{L}(H)$, the algebra of bounded linear operators of a (real) Hilbert space H.
Equivalently, it is a Banach-$*$-algebra $\mathcal{A}$ satisfying for any $x \in \mathcal{A}$: $\|x^*x\| = \|x\|^2$ and $1 + x^*x$ is invertible in $\tilde{\mathcal{A}}$.

Definition: For a unital C^*-algebra $\mathcal{A}$ and a compact Hausdorff space X, we define $KO(X; \mathcal{A})$ to be the Grothendieck group of $\mathcal{A}$-bundles over X.
For a pair of compact spaces (X, Y) we define a triple (E, F, α) to consist of two $\mathcal{A}$-bundles E, F and an $\mathcal{A}$-bundle isomorphism $\alpha: E|_Y \to F|_Y$. We call (E_1, F_1, α_1) and (E_2, F_2, α_2) equivalent if there are $\mathcal{A}$-bundle isomophisms $f: E_1 \to E_2$, $g: F_1 \to F_2$ satisfying $\alpha_2 \circ f|_Y = g|_Y \circ \alpha_1$ and denote by $KO(X, Y; \mathcal{A})$ the group of stable equivalence classes of triples.
For a locally compact space we define $KO(X; \mathcal{A}) := KO(X_+, +; \mathcal{A})$.

Definition:
Higher KO-groups are defined by

$$KO^{-n}(X, Y; \mathcal{A}) := KO((X \setminus Y) \times \mathbb{R}^n; \mathcal{A})$$

Remarks:

- As in topological K-theory, there is a long exact sequence of K-theory as well as a Meyer-Vietoris sequence
- By defining $K_n(\mathcal{A}) := KO^{-n}(*;\mathcal{A})$ we recover operator K-theory (operator K-theory: $K_n(\mathcal{A}) = \pi_{n-1}(GL(A))$)
- sometimes useful: $K_n(C_0(X;\mathcal{A})) = KO^{-n}(X;\mathcal{A})$

Hilbert Modules

Definition:

A pre-Hilbert module E over a C^*-algebra $\mathcal{A}$ is a right-$\mathcal{A}$-module with an $\mathcal{A}$-valued inner product $(\cdot,\cdot): E\times E \to \mathcal{A}$ satisfying $(x,x)\geq 0\ \forall x$, $(x,x)=0 \Leftrightarrow x=0$, $(x,y)=(y,x)^*$, $(x,ya)=(x,y)a$ $(a\in\mathcal{A})$.

On a pre-Hilbert module one can define a norm by $\|x\|_E = \|(x,x)\|^{\frac{1}{2}}$.
A pre-Hilbert module which is complete with respect to this norm is called a Hilbert module.

Example:

For $n\in\mathbb{N}\cup\{\infty\}$ let $\mathcal{A}^n := \bigoplus_{i=1}^{n}\mathcal{A}$. This can be made a Hilbert module by $(x,y)=\sum_{i=1}^{n} x_i^* y_i$. We denote $H_{\mathcal{A}} := \mathcal{A}^\infty$.

Theorem:

For any countably generated Hilbert-$\mathcal{A}$-module E, $E\oplus H_{\mathcal{A}} \cong H_{\mathcal{A}}$.

Definition:

For two Hilbert $\mathcal{A}$-modules E, F we define $\mathcal{L}(E,F)$ to be the set of all bounded linear maps $T: E \to F$ which admit an adjoint $T^*: F \to E$ satisfying $(Tx, y)_F = (x, T^*y)_E$. The C^*-algebra $\mathcal{L}(E,E)$ is denoted by $\mathcal{L}(E)$.

Definition:

Let $\mathcal{K}(E,F)$ be the norm closure of the $\mathcal{L}(E)$-$\mathcal{L}(F)$-bimodule generated by operators of the form $z \xmapsto{\theta_{x,y}} x(y,z)$ in $\mathcal{L}(E,F)$. Operators in $\mathcal{K}(E,F)$ are called generalized compact operators.

Proposition:

$$\mathcal{L}(E) \cong M(\mathcal{K}(E)), \quad \mathcal{L}(H_{\mathcal{A}}) \cong M(\mathcal{A} \otimes \mathcal{K})$$

Definition:

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and let E be a Hilbert-$\mathcal{A}$-$\mathcal{B}$-bimodule (in particular $(x,y)_{\mathcal{A}} z = x(y,z)_{\mathcal{B}}$ and $(ax,ax)_{\mathcal{B}} \le \|a\|^2 (x,x)_{\mathcal{B}}$). If $\operatorname{span}\{(x,x)_{\mathcal{A}} \mid x \in E\}$ and $\operatorname{span}\{(x,x)_{\mathcal{B}} \mid x \in E\}$ are dense in $\mathcal{A}$ and $\mathcal{B}$ respectively, then $\mathcal{A}$ and $\mathcal{B}$ are called strongly Morita equivalent.

Theorem:

Strongly Morita equivalent σ-unital C^*-algebras have the same K-theory.

Fredholm Operators

Definition:

Let $\mathcal{A}$ be a C^*-algebra and E and F be Hilbert $\mathcal{A}$-modules. An operator $T \in \mathcal{L}(E,F)$ is called an $\mathcal{A}$-Fredholm operator if it has a parametrix, i.e. an operator $S \in \mathcal{L}(F,E)$ satisfying

$$ST - id_E \in \mathcal{K}(E) \qquad TS - id_F \in \mathcal{K}(F).$$

The set of all such operators will be called $\mathrm{Fred}(E,F)$.

Theorem: (Mingo)

For any unital C^*-algebra $\mathcal{A}$ and any compact Hausdorff space X,

$$K_0(X;\mathcal{A}) \cong [X, \mathrm{Fred}(H_{\mathcal{A}})] .$$

Proposition:

For any Fredholm operator $F \in \mathcal{L}(H_{\mathcal{A}})$ there is a compact perturbation with closed range and finitely generated kernel and cokernel.

Sketch of proof:

We can find a partial isometry U satisfying $U|F| = F$ modulo compacts. Again modulo compacts, we can take the logarithm H of $|F|$. Then $G = e^H U$ is the requested operator ($\mathrm{rg}\, G = \mathrm{rg}\, V$, $\ker G = \ker V$).

Definition:

Let $F \in \mathrm{Fred}(H_{\mathcal{A}})$ and let G be a compact perturbation of F having closed range and finitely generated kernel and cokernel. We define $\mathrm{index}\, F = [\ker G] - [\ker G^*] \in K_0(\mathcal{A})$. By a result of Kasparov's this is well defined.

Remark:

Any operator in $\mathcal{L}(H_{\mathcal{A}})$ having closed range and finitely generated kernel and cokernel is Fredholm.

Properties of the index

- homomorphism w.r.t. $\circ$ (path of Fredholm's from $FG \oplus 1$ to $F \oplus G$)
- locally constant
- surjective (main ingredient: $\forall$ cp. proj. p $\exists$ isometry w : $p = 1 - ww^*$, fin. gen. proj. modules can be characterized by compact pr.)
- if $\mathrm{index}(F) = 0$, there is an invertible compact perturbation of F

Corollary:

Because the group $\mathcal{L}(H_{\mathcal{A}})^\times$ is connected, we get an isomorphism

$$[\mathrm{Fred}(H_{\mathcal{A}})] \xrightarrow[\mathrm{index}]{\cong} K_0(\mathcal{A})$$

Mingo's theorem now follows:

$$\begin{aligned}[X, \mathrm{Fred}(H_{\mathcal{A}})] &\cong [\mathrm{Fred}(H_{C(X)\otimes\mathcal{A}}) \cap C(X) \otimes M(A \otimes \mathcal{K})] \\ &\cong [\mathrm{Fred}(H_{C(X)\otimes\mathcal{A}})] \cong K_0(C(X) \otimes \mathcal{A}) \cong K_0(X; \mathcal{A})\end{aligned}$$

Theorem:

Let $\mathcal{A}$ be a unital C^*-algebra. We fix a representation of Cl_k on $H_{\mathcal{A}}$ and denote the images of the generators by $e_1, \dots, e_k$. Furthermore, we let $F_k(H_{\mathcal{A}}) = \{T \in \mathrm{Fred}(H_{\mathcal{A}}) \mid T^* = -T, \forall l < k : e_l T = -T e_l\}$ and $\tilde{F}_k(H_{\mathcal{A}})$ be the connected component of e_k in $F_k(H_{\mathcal{A}})$. Then for any compact Hausdorff space X,

$$KO^{-k}(X; \mathcal{A}) \cong [X, \tilde{F}_k(H_{\mathcal{A}}) \times KO_k(\mathcal{A})].$$

We denote by $Q(H_{\mathcal{A}})$ the algebra $\mathcal{L}(H_{\mathcal{A}}) / \mathcal{K}(H_{\mathcal{A}})$. By the long exact sequence of K-theory and $K^{-m}(X; \mathcal{L}(H_{\mathcal{A}})) = 0$ ($\mathcal{L}(H_{\mathcal{A}})^\times$ is contractible), we get $K^{1-m}(X; \mathcal{A}) \cong K^{-m}(X; Q(H_{\mathcal{A}}))$.

By a result of Karoubi's $K^r(X; \mathcal{B}) \cong [X, K_{-r}(\mathcal{B}) \times \lim \mathrm{grad}^r(M_k \mathcal{B})]$ for any C^*-algebra $\mathcal{B}$, where $\mathrm{grad}^r(M_k \mathcal{B})$ denotes the connected component of ε_1 in the space of gradings on $M_k \mathcal{B}$ which are Cl-antilinear.

The map $h: gl^r(\mathcal{B}) \to \mathrm{grad}^r(\mathcal{B})$, $g \mapsto g \varepsilon_1 g^{-1}$ induces a homeomorphism $gl^r(\mathcal{B}) / gl'^r(\mathcal{B}) \to \mathrm{grad}^r \mathcal{B}$.

$gl^r(\mathcal{B}) = \mathcal{B}$-$Cl_r$-automorphisms of $Cl_{r,1} \otimes \mathcal{B}$

$gl'^r(\mathcal{B}) =$ subgroup of $Cl_{r,1}$-automorphisms

Now let $\pi: \mathcal{L}(H_{\mathcal{A}}) \to Q(H_{\mathcal{A}})$ be the projection and let σ be a continuous cross-section. This can be used to average over the Clifford group, showing that $\mathrm{Fred}^{r,1}(H_{\mathcal{A}}) \to \pi(\mathrm{Fred}^{r,1}(H_{\mathcal{A}}))$ is a fibration with contractible fibers. Since $\mathrm{Grad}^{r,1}(Q(H_{\mathcal{A}}))$ is a deformation retract ($x \mapsto x(x^*x)^{-\frac{1}{2}}$), we have a homotopy equivalence $\mathrm{Fred}^{r,1}(H_{\mathcal{A}}) \to \mathrm{Grad}^{r,1}(H_{\mathcal{A}})$.

This respects the components, so we get $\tilde{F}^{r,1}(H_{\mathcal{A}}) \simeq \mathrm{grad}^{r,1}(Q(H_{\mathcal{A}}))$.

Since $[X, \mathrm{Fred}(H_{\mathcal{A}})] = [X, \mathrm{Fred}(H_{\mathcal{A}}^k)]$, and the fibration $\mathrm{Fred}(H_{\mathcal{A}}^k) \to Gl_k(Q(H_{\mathcal{A}}))$ has contractible fibers, we get by Mingo's theorem, that the inclusion $Gl_k(Q(H_{\mathcal{A}})) \to Gl_{k+1}(Q(H_{\mathcal{A}}))$ is a homotopy equivalence.

Because $Cl_{r+n, s+n} \cong M_{2^n}(Cl_{r,s})$, this yields that also the inclusions $gl^{r,s}(M_k(Q(H_{\mathcal{A}}))) \to gl^{r,s}(M_{k+1}(Q(H_{\mathcal{A}})))$ are homotopy equivalences and by the homotopy sequence of the fibration

$$gl^{r,s+1}(M_k(Q(H_{\mathcal{A}}))) \to gl^{r,s}(M_k(Q(H_{\mathcal{A}}))) \to \mathrm{grad}^{r,s}(M_k(Q(H_{\mathcal{A}})))$$

one sees that $\mathrm{grad}^{r,s}(M_k(Q(H_{\mathcal{A}}))) \to \mathrm{grad}^{r,s}(M_{k+1}(Q(H_{\mathcal{A}})))$ is a homotopy equivalence, too.

Karoubi's result from above together with $\mathrm{Fred}^{r,s}(H_{\mathcal{A}}) \simeq \mathrm{grad}^{r,s}(Q(H_{\mathcal{A}}))$ now yields our theorem.

A Brief Look on Bott Periodicity

Definition:

Let $\mathcal{A}$ and $\mathcal{B}$ be separable σ-unital C^*-algebras. A Kasparov $(\mathcal{A}, \mathcal{B})$-bimodule is a triple (E, ϕ, T) consisting of a countably generated graded Hilbert-$\mathcal{B}$-module E, a graded $*$-homomorphism $\phi: \mathcal{A} \to \mathcal{L}(E)$ and $T \in \mathcal{L}(E)$ of degree 1 satisfying

$$(T - T^*)\phi(a) \in \mathcal{K}(E), \quad (T^2 - 1)\phi(a) \in \mathcal{K}(E), \quad [T, \phi(a)] \in \mathcal{K}(E).$$

Two such triples (E_1, ϕ_1, T_1) and (E_2, ϕ_2, T_2) are called orthogonally equivalent, if there is an isometric isomorphism $U \in \mathcal{L}(E_1, E_2)$ satisfying $T_1 = U^* T_2 U$ and $\phi_1(a) = U^* \phi_2(a) U$. We denote the set of orthogonal equivalence classes by $\mathbb{E}(\mathcal{A}, \mathcal{B})$.

Definition:

Two Kasparov-$(\mathcal{A}, \mathcal{B})$-bimodules (E_0, ϕ_0, T_0), (E_1, ϕ_1, T_1) are called homotopic if there is $(E, \phi, T) \in \mathbb{E}(\mathcal{A}, \mathcal{B} \hat{\otimes} C([0,1]))$ such that for $i = 0, 1$ the triples $(E \hat{\otimes}_{f_i} \mathcal{B}, f_i \circ \phi, f_{i*}(T))$ and (E_i, ϕ_i, T_i) are orthogonally equivalent ($f_i : \mathcal{B} \hat{\otimes} C[0,1] \to \mathcal{B}$, $g \mapsto g(t)$ is the evaluation map).

Definition:

The abelian group of homotopy classes in $\mathbb{E}(\mathcal{A}, \mathcal{B})$ is called $KKO(\mathcal{A}, \mathcal{B})$.

Theorem:

$$KKO(\mathbb{R}, \mathcal{B}) \cong KO(\mathcal{B})$$

(The isomorphism is induced by $\mathcal{L}(H_{\mathcal{B}}) \twoheadrightarrow Q(H_{\mathcal{B}})$)

If we define $KKO_m(\mathcal{A}, \mathcal{B}) = KKO(\mathcal{A}, \mathcal{B} \otimes Cl_{0,m})$, we can state the periodicity theorem of KK-theory:

$$KKO_m(\mathcal{A} \otimes C_0(\mathbb{R}^m), \mathcal{B}) \cong KKO(\mathcal{A}, \mathcal{B}) \cong KKO_{-m}(\mathcal{A}, \mathcal{B} \otimes C_0(\mathbb{R}^m))$$

Corollary:

$$
\begin{aligned}
KO_0(\mathcal{B}) &\cong KKO(\mathbb{R}, \mathcal{B}) \\
&\cong KKO(\mathbb{R}, \mathcal{B} \otimes C_0(\mathbb{R}^n) \otimes Cl_{0,n}) \\
&\cong KO_n(\mathcal{B} \otimes Cl_{0,n}),
\end{aligned}
$$

and because $Cl_{0,n}$ and $Cl_{0,n+8}$ are Morita equivalent $\left(Cl_{0,n+8} \cong M_{16}(Cl_{0,n})\right)$, we obtain Bott periodicity.

Sketch of the main proof

We have seen, in the special case $\mathcal{A} = \mathbb{R}$, that $KO^{-t}(X) \cong [X, \tilde{F}_t]$. It remains to show $EFT_n \simeq \tilde{F}_n$. This can be done by showing

$$
\begin{aligned}
EFT_n &\simeq \mathrm{Conf}^{t_1\text{odd}}_{Cl_n}(\bar{\mathbb{R}}, \infty) \\
\tilde{F}_n &\simeq \mathrm{Conf}_n \subset \mathrm{Conf}^{t_1\text{odd}}_{Cl_n}(\tilde{\mathbb{R}}, \pm\infty)
\end{aligned}
$$

$$
\mathrm{Conf}^{t_1\text{odd}}_{Cl_n}(\tilde{\mathbb{R}}, \pm\infty) \to \mathrm{Conf}^{t_1\text{odd}}_{Cl_n}(\bar{\mathbb{R}}, \infty)
$$

is a quasi-fibration with contractible fibre.