On K(1)-local SU-bordism

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CHAPTER 1

Introduction and statement of results

One of the highlights in algebraic topology was the invention of generalized homology and cohomology theories by Whitehead and Brown in the 1960s. Prominent examples are real and complex K-theories first given by Atiyah and Hirzebruch and bordism theories with respect to different structure groups first given by Thom. By Brown's representability theorem every generalized cohomology theory can be represented by a spectrum and these spectra are the center of interest in modern algebraic topology.

Bordism theories with respect to some structure group G, e.g. $G = O$, SO, U, SU, Sp, and $Spin$ are defined as follows: Let M be a smooth, closed, n-dimensional manifold and $G = \{G_n\}$ be a sequence of topological groups with maps $G_n \to G_{n+1}$ compatible with their orthogonal representations $G_n \to O(n)$.

Definition 1.1. *A G-structure on M is a homotopy class of lifts $\tilde{\nu}$ of the classifying map of the stable normal bundle ν*

$$\begin{tikzcd} & BG \arrow[d] \\ M \arrow[r, "\nu"] \arrow[ur, dotted, "\tilde{\nu}"] & BO \end{tikzcd}$$

A manifold M together with a G-structure is called a G-manifold.

For each of the classical groups this gives us the G-bordism ring Ω^G_* and a Thom spectrum MG with $\Omega^G_* = MG_* = MG_*(pt) = \pi_* MG$. Further we have a homology theory $MG_*(-)$ and a cohomology theory $MG^*(-)$. Since we have inclusion maps on group level and since the Thom construction is functorial we get the following tower:

$$\begin{tikzcd} MSU \arrow[r] \arrow[d] & MSpin \arrow[d] \\ MU \arrow[r] & MSO \arrow[d] \\ & MO \end{tikzcd}$$

On the level of homotopy one knows at least rationally that the coefficient groups are polynomial rings and one asks for a decomposition on the level of spectra. In 1966 Andersen, Brown and Peterson gave an additive 2-local splitting of $MSpin$

$$MSpin_{(2)} \simeq \bigvee_{n(J)\text{ even},\, 1\notin J} ko\langle 4n(J)\rangle \vee \bigvee_{n(J)\text{ odd},\, 1\notin J} ko\langle 4n(J)+2\rangle \vee \bigvee_{i\in I} \Sigma^{d_i} H\mathbb{Z}/2$$

with $J = (i_1, ..., i_k)$ a finite sequence and $n(J) = i_1 + ... + i_k$. Bordism theories are multiplicative homology theories and their Thom spectra are ring spectra. Moreover they admit even richer structures called E_∞ structures, i.e. not only the coherent diagrams of commutativity and associativity commute up to homotopy but there are also diagrams of higher coherence. These E_∞ structures should be taken into account and therefore we are interested in a splitting in the category of E_∞ ring spectra.

Unfortunately this access raises several other difficulties. Analysing the above additive splitting of 2-local spin bordism by Anderson, Brown and Peterson, the Eilenberg-MacLane part $H\mathbb{Z}/2$ turns out to be a difficult problem. In this situation the modern viewpoint is to apply chromatic homotopy theory and to look at the chromatic tower or at certain monochromatic layers. In our case we consider localizations with respect to the first Morava K-theory $K(1)$. At $p = 2$ we have

$$L_{K(1)} \cong L_{S\mathbb{Z}/2} L_{K_{(2)}}$$

and the Eilenberg-MacLane part disappears. This is our approximation to bordism theories. Algebraically this access offers a lot of extra structure since $\pi_0 E$ of a $K(1)$-local E_∞ ring spectrum E admits a θ-algebra structure.

In [**Lau01**] Laures gives a $K(1)$-local splitting of E_∞ spectra

$$MSpin \cong T_\zeta \wedge \bigwedge_{i=1}^{\infty} TS^0$$

where T is the free functor left adjoint to the forgetful functor from E_∞ spectra to spectra and $\wedge$ is the coproduct in the category of E_∞ spectra with $\bigwedge TS^0 \cong T(\bigvee S^0)$. Such a splitting is also desireable for other bordism theories and a lot of different techniques are involved to get such a splitting.

In this work we study $K(1)$-local SU bordism. A main result is detecting an E_∞ summand T_ζ for a nontrivial element $\zeta \in \pi_{-1} L_{K(1)} S^0 \cong \mathbb{Z}_2$

$$\begin{tikzcd}
TS^{-1} \arrow[r, "\zeta"] \arrow[d, "T*"] & S^0 \arrow[d] \arrow[ddr, bend left] & \\
TD^0 = S^0 \arrow[r] \arrow[drr, bend right] & T_\zeta \arrow[dr, dotted] & \\
& & MSU
\end{tikzcd}$$

meaning that T_ζ is the resulting E_∞ spectrum when attaching a 0-cell along ζ. To this end, we construct an Artin-Schreier class $b \in KO_0 MSU$ satisfying $\psi^3 b = b + 1$ which implies that $\zeta = 0$ in $\pi_{-1} MSU$.

Another important result is the construction of spherical classes in $K_* MSU$. Although we do not have a complete splitting, comparison with the spin bordism case shows that spherical classes play an important role: They correspond to free E_∞ summands TS^0. In this work, we perform the construction of spherical classes via calculations of Adams operations on $K_*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$ whose module generators map to the algebra generators of $K_* BSU$. Later we can use Bott's theory of cannibalistic classes to lift the Adams operations to the level of Thom spectra.

Since the K-homology of $\mathbb{CP}^\infty$ is isomorphic to the ring of numerical polynomials, we are able to provide an alternative calculation of the Adams operations on $K_*\mathbb{CP}^\infty$ using Mahler series expansion in p-adic analysis.

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CHAPTER 2

Some homotopical algebra

1. Generalized cohomology theories and spectra

In this section we want to recall the basic notations of generalized cohomology theories and spectra as their representing objects. We will see the correspondence between them and have a look at their fundamental properties. The relevant homotopy category is the stable homotopy category.

Definition 2.1. *A generalized cohomology theory E consists of a sequence $\{E^n\}_{n\in\mathbb{Z}}$ of contravariant homotopy functors*

$$E^n : \mathbf{CWPairs} \to \mathbf{AbGroups}$$

together with natural transformations

$$\delta : E^n(X) \to E^{n+1}(X, A)$$

satisfying the axioms

- **Excision:** *The projection $(X, A) \to X/A$ induces an isomorphism*
$$\tilde{E}^n(X/A) \to E^n(X, A)$$
for all pairs (X, A).
- **Exactness:** *The long sequence of abelian groups*
$$... \to E^n(X, A) \to E^n(X) \to E^n(A) \xrightarrow{\delta} E^{n+1}(X, A) \to ...$$
is exact for all pairs (X, A).
- **Strong additivity:** *For every family of spaces $\{X_i\}_{i\in I}$ the natural map*
$$E^n(\coprod_{i\in I} X_i) \to \prod_{i\in I} E^n(X_i)$$
is an isomorphism.

Proposition 2.1. *Every generalized cohomology theory E enjoys the following properties:*

(1) *For a pointed topological space X there is a natural direct sum splitting*
$$E^n(X) \cong \tilde{E}^n(X) \oplus E^n(*).$$
(2) *For a family $\{X_i\}_{i\in I}$ of pointed topological spaces the map of reduced cohomology groups*
$$\tilde{E}^n(\bigvee_{i\in I} X_i) \to \prod_{i\in I} \tilde{E}^n(X_i)$$
is an isomorphism.

(3) *For a pointed topological space X we have natural isomorphisms*
$$\tilde{E}^n(X) \xrightarrow[\delta]{\cong} E^{n+1}(CX, X) \xleftarrow[excision]{\cong} \tilde{E}^{n+1}(\Sigma X)$$

(4) **Mayer-Vietoris:** *For $X = X_1 \cup X_2$ (open covering) we have the long exact sequence*

$$\dots \to E^n(X) \overset{(i_1^*,i_2^*)}{\to} E^n(X_1) \oplus E^n(X_2) \overset{j_1^*-j_2^*}{\to} E^n(X_1 \cap X_2) \overset{\delta}{\to} E^{n+1}(X) \to \dots$$

(5) **Milnor sequence:** *For a filtration $X = \operatorname{colim} X_i$ we get a short exact sequence with the derived limit*

$$0 \to \lim{}^1 E^{n-1}(X_i) \to E^n(X) \to \lim E^n(X_i) \to 0$$

which detects phantom maps.

These cohomology functors are representable by a sequence of spaces and with the suspension isomorphism we naturally get the following definition:

Definition 2.2. *A spectrum X is a sequence of pointed topological spaces $X_0, X_1, X_2, \dots$ together with structure maps*

$$\sigma_n : X_n \wedge S^1 \to X_{n+1}$$

or the adjoint map $\tilde{\sigma} : X_n \to \Omega X_{n+1}$ respectively. If $\tilde{\sigma}$ is a weak equivalence X is called an Ω-spectrum.

By Brown's representability theorem every generalized cohomology theory can be represented by an Ω-spectrum. On the other hand every spectrum defines a cohomology (and homology) theory. It is worth mentioning that every spectrum can functorially be turned into an Ω-spectrum. As an illustrative example, we define an Ω-spectrum for complex K-theory.

Example 2.1. *(K-theory) First of all we make use of Bott periodicity, i.e. there is a homotopy equivalence $\Omega^2 BU \cong \mathbb{Z} \times BU$, and we define an Ω-spectrum K by setting*

$$K_n = \begin{cases} \mathbb{Z} \times BU & \text{if } n \text{ is even,} \\ \Omega BU & \text{if } n \text{ is odd} \end{cases}$$

with structure maps adjoint to

$$(\tilde{\sigma} : K_n \to \Omega K_{n+1}) = \begin{cases} \text{the Bott equivalence} & \mathbb{Z} \times BU \xrightarrow{\cong} \Omega^2 BU & \text{if } n \text{ is even,} \\ \text{the identification} & \Omega BU \xrightarrow{\cong} \Omega(\mathbb{Z} \times BU) & \text{if } n \text{ is odd.} \end{cases}$$

This is called the complex topological K-theory spectrum. Its homotopy groups are

$$\pi_n K = \begin{cases} \pi_0(\mathbb{Z} \times BU) \cong \mathbb{Z} & \text{if } n \text{ is even} \\ \pi_1 BU = 0 & \text{if } n \text{ is odd.} \end{cases}$$

Example 2.2. *(KO-theory) Similarly, we obtain the real topological K-theory spectrum KO using real Bott periodicity, i.e.*

$$\Omega^8 BO \cong \mathbb{Z} \times BO.$$

Its homotopy groups are given in the following table:

n mod 8	0	1	2	3	4	5	6	7
$\pi_n KO$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0

In the above examples we recalled the additive homotopy groups, but as we know there is also a multiplicative structure. For example the coefficient rings of K and KO are $\pi_*K = \mathbb{Z}[u^{\pm 1}]$ with the Bott element $u \in \pi_2 K$ as invertible element and

$$\pi_*KO = \mathbb{Z}[\eta, \alpha, \beta^{\pm 1}]/(2\eta = 0, \eta^3 = 0, \eta\alpha = 0, \alpha^2 = 4\beta).$$

Definition 2.3. *A cohomology theory E^* is called multiplicative if it is equipped with a product*

$$\times : \tilde{E}^p(X) \otimes \tilde{E}^q(Y) \to \tilde{E}^{p+q}(X \wedge Y)$$

which is associative, graded commutative, unital and stable.

A multiplicative theory is realized by a ring spectrum, i.e. a spectrum $E = (E_n)_n$ together with maps

$$\mu_{mn} : E_m \wedge E_n \to E_{m+n} \quad \text{and } \eta_n : S^n \to E_n$$

such that the following diagrams representing the properties *associativity, commutativity, unit* and *stability* commute up to homotopy:

$$\begin{tikzcd}
E_k \wedge E_m \wedge E_n \arrow[r] \arrow[d] & E_{k+m} \wedge E_n \arrow[d] \\
E_k \wedge E_{m+n} \arrow[r] & E_{k+m+n}
\end{tikzcd}
\qquad
\begin{tikzcd}
E_m \wedge E_n \arrow[rr] \arrow[dr] & & E_n \wedge E_m \arrow[dl] \\
& E_{m+n} &
\end{tikzcd}$$

$$\begin{tikzcd}
S^n \wedge E_n \arrow[r] \arrow[dr] & E_m \wedge E_n \arrow[d, "\mu"] & E_m \wedge S^n \arrow[l] \arrow[dl] \\
& E_{m+n} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma S^n \arrow[r] \arrow[d] & \Sigma E_n \arrow[d] \\
S^{n+1} \arrow[r] & E_{n+1}
\end{tikzcd}$$

With these notations the product

$$\times : \tilde{E}^p(X) \otimes \tilde{E}^q(Y) \to \tilde{E}^{p+q}(X \wedge Y)$$

of the multiplicative cohomology theory E^* is given by

$$(f : X \to E_m \, , \, g : Y \to E_n) \mapsto (f \wedge g : X \wedge Y \to E_m \wedge E_n \xrightarrow{\mu_{mn}} E_{m+n}).$$

Having the above notations for associativity and commutativity of ring spectra in mind, one might think of higher coherence conditions (i.e. having a smash product of four or more spaces we want to have commutativity up to homotopy when evaluating in different order). This leads to the notion of an E_∞ ring spectrum, which comes with an E_∞ operad controlling the coherence. This is the sort of extra structure all our spectra (bordism spectra, K-theory spectra, Eilenberg-MacLane spectra) have and in this world our bordism splitting takes place. Another (in fact Quillen equivalent) model for E_∞ spectra are symmetric spectra which come up in the next paragraph.

2. Symmetric spectra over topological spaces

There are a lot of highly structured ring spectra: E_∞ spectra, $\mathbb{S}$-algebras, symmetric ring spectra and strict commutative ring spectra. They are all Quillen equivalent and their homotopy category is the stable homotopy category. Therefore it does not matter which model we use, but it gives a good feeling to have safe foundations. In this section we consider sequential spectra over pointed topological spaces and refer to [**HSS00**], [**EKMM97**] and [**Sch08**]. Let $\mathcal{T}_*$ denote the category of pointed topological spaces.

Definition 2.4. *A symmetric spectrum X consists of*

- *a sequence $X_0, X_1, ... \in \mathcal{T}_*$*
- *structure maps $\sigma : X_n \wedge S^1 \to X_{n+1}$*
- *symmetric operations $\Sigma_n \curvearrowright X_n$ such that*

$$\sigma^p : X_n \wedge S^p \xrightarrow{\sigma \wedge S^{p-1}} X_{n+1} \wedge S^{p-1} \xrightarrow{\sigma \wedge S^{p-2}} \cdots \xrightarrow{\sigma} X_{n+p}$$

are $\Sigma_n \times \Sigma_p$-equivariant.

A map $f : X \to Y$ of symmetric spectra is a family of maps $f_n : X_n \to Y_n$ of Σ_n-equivariant maps such that

$$\begin{array}{ccc} X_n \wedge S^1 & \xrightarrow{\sigma} & X_{n+1} \\ {\scriptstyle f_n \wedge S^1}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ Y_n \wedge S^1 & \xrightarrow{\sigma} & Y_{n+1} \end{array}$$

commutes. This gives us the category of symmetric spectra Sp^Σ.

Example 2.3 (Suspension spectrum)**.** *For a pointed topological space $X \in \mathcal{T}_*$ the suspension spectrum $\Sigma^\infty X$ is defined by $(\Sigma^\infty X)_n := X \wedge S^n$ and the structure map is given by the identity morphism*

$$X \wedge S^n \wedge S^1 \to X \wedge S^{n+1}.$$

Example 2.4 (Sphere spectrum)**.** *The sphere spectrum $\mathbb{S} = (S^0, S^1, S^2, ...)$ is the suspension spectrum for $K = S^0$.*

Example 2.5 (Eilenberg-MacLane spectrum $H\mathbb{Z}$)**.** *With $S^1 = \Delta^1/\partial\Delta^1$ the n-sphere is given a simplicial structure by $S^n = S^1 \wedge ... \wedge S^1$. Then let $(\mathbb{Z} \otimes S^n)_k$ be the free abelian group on the unpointed k-simplices of S^n. Define the Eilenberg-MacLane spectrum $H\mathbb{Z}$ by $(H\mathbb{Z})_n := |\mathbb{Z} \otimes S^n|$ to be the realization of the simplicial abelian group. $H\mathbb{Z}_n$ is a $K(\mathbb{Z}, n)$ since for every simplicial abelian group $\pi_n|A| = H_n A$ and here we have*

$$\pi_m|\mathbb{Z} \otimes S^n| = H_m\mathbb{Z} \otimes S^n = H_m C_. S^n = \begin{cases} \mathbb{Z} & for n = m \\ 0 & for n \neq m \end{cases}$$

with $C_. S^n$ the singular chain complex. The action $\Sigma_n \curvearrowright S^n = S^1 \wedge ... \wedge S^1$ is given by permuting the factors and the structure maps are induced by $S^n \wedge S^1 \to S^{n+1}$.

Example 2.6 (Unoriented bordism spectrum MO)**.** *The construction is given for bordism theory with respect to the orthogonal group but generalizes to other groups in the obvious way. Construct $EO(n) := |k \to O(n)^{k+1}|$ as the realization of the simplicial complex. The orthogonal group $O(n)$ acts on this space by multiplication on the right. This gives us the classifying space*

$$BO(n) := EO(n)/O(n).$$

Take the associated bundle

$$\xi_n : EO(n) \times_{O(n)} \mathbb{R}^n \to EO(n)/O(n) = BO(n)$$

and define its Thom space

$$Thom(\xi_n) = D_\xi/S_\xi = \frac{EO(n) \times_{O(n)} D^n}{EO(n) \times_{O(n)} S^{n-1}} \cong EO(n)_+ \wedge_{O(n)} (D^n/S^{n-1}).$$

Since D^n/S^{n-1} is $O(n)$-equivariantly isomorphic to $\mathbb{R} \cup \{\infty\}$ and $O(n)$ acts on $\mathbb{R} \cup \{\infty\}$ preserving $\{\infty\}$ we have an $O(n)$-action on S^n. Defining the n^{th} space of MO to be $MO_n := EO(n)_+ \wedge_{O(n)} S^n$, the Thom space of ξ_n gives a symmetric spectrum with symmetric operations $\Sigma_n \subset O(n)$ coming from coordinate permutations.

Definition 2.5 (Symmetric sequences)**.** *A symmetric sequence consists of a sequence*

$$X_0, X_1, ... \in \mathcal{T}_*$$

and symmetric operations $\Sigma_n \curvearrowright X_n$. A map $f : X \to Y$ is a family of Σ_n-equivariant maps $f_n : X_n \to Y_n$. This category is denoted by $\mathcal{T}_*^\Sigma$.

For $X, Y \in \mathcal{T}_*^\Sigma$ we can define their tensor product $X \otimes Y$ by

$$(X \otimes Y)_n := \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q$$

with $\Sigma_p \times \Sigma_q$-diagonal operation: for $(g, h) \in \Sigma_p \times \Sigma_q \subset \Sigma_n$, $\alpha \in \Sigma_n$ let $(\alpha(g, h), x, y) \sim (\alpha, gx, hy)$ be equivalent. The so defined tensor product admits a unit $U = (S^0, *, *, ...)$

$$(U \otimes X)_n = \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} U_p \wedge X_q \cong (\Sigma_n)_+ \Sigma_n X_n \cong X_n,$$

hence $U \otimes X \cong X$. Furthermore the tensor product admits a twist isomorphism $\tau : X \otimes Y \to Y \otimes X$ sending (α, x, y) to (α^s, y, x) with

$$s(1, ..., q, q + 1, ..., q + p) = (p + 1, ..., p + q, 1, ..., p)$$

the (p, q)-shuffle.

Proposition 2.2. $(\mathcal{T}_*^\Sigma, \otimes, \tau)$ *is a symmetric monoidal category.*

In the following we want to define a smash-product in the category of symmetric spectra Sp^Σ. Let $S := (S^0, S^1, ...)$ be the symmetric sphere sequence.

Proposition 2.3. *S is a commutative monoid in $\mathcal{T}_*^\Sigma$, i.e. there exist maps $\mu : S \otimes S \to S$ and $\eta : U \to S$ such that*

$$\begin{tikzcd} S \otimes S \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & S \otimes S \arrow[dl, "\mu"] \\ & S & \end{tikzcd}$$

commutes.

To get an idea where this multiplication map μ comes from, recall that

$$\mathrm{Hom}_G^*(\bigvee X_j, Y) \cong \prod \mathrm{Hom}_G^*(X_j, Y)$$

and for a subgroup $H \subset G$ we have

$$\mathrm{Hom}_G^*(G_+ \wedge_H X, Y) \cong \mathrm{Hom}_H^*(X, res_H^G Y).$$

Hence the map $\mu : S \otimes S \to S$ reduces to maps

$$\mu_n : \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} S^p \wedge S^q \to S^n$$

which restrict to $\tilde{\mu}_n : S^p \wedge S^q \to S^n$ when considering the Young-subgroups $\Sigma_p \times \Sigma_q \subset \Sigma_n$.

Definition 2.6 (Category of left S-modules)**.** *A left S-module is a symmetric sequence $X \in \mathcal{T}_*^\Sigma$ with a map $S \otimes X \to X$ such that*

$$\begin{tikzcd} (S \otimes S) \otimes X \arrow[rr] \arrow[dr, "\mu\otimes id"'] & & S \otimes (S \otimes X) \arrow[rr, "id\otimes m"] & & S \otimes X \arrow[dl, "m"] \\ & S \otimes X \arrow[rr, "m"] & & X & \end{tikzcd}$$

commutes.

This gives us an equivalence of categories:

$$\begin{array}{lcl} \text{left S-modules } X \in \mathcal{T}_*^\Sigma & \longleftrightarrow & \text{symmeric spectra } X \in Sp^\Sigma \\ (S \otimes X)_n \overset{m_n}{\to} X_n & & S^p \wedge X_q \to X_{p+q} \\ & & \Sigma_p \times \Sigma_q\text{-equivariant} \end{array}$$

Definition 2.7 (Smash-Product)**.** *Let $X, Y \in Sp^\Sigma \cong$ **left S-mod**. Then*

$$X \wedge Y := X \otimes_S Y := \text{coequalizer}(X \otimes S \otimes Y \underset{(m_X \circ \tau)\otimes 1_Y}{\overset{1_X \otimes m_Y}{\rightrightarrows}} X \otimes Y)$$

is a left S-module since $(S \otimes -)$ preserves colimits.

Proposition 2.4. *$(Sp^\Sigma, \wedge)$ is a symmetric monoidal category.*

3. Complex oriented theories and computational methods

Complex oriented theories are special generalized cohomology theories which have a big advantage: They are computable. We briefly recall some basic results. Let E be a multiplicative cohomology theory.

Definition 2.8. *E is called complex orientable if there is a class $x \in \tilde{E}^2\mathbb{CP}^\infty$ mapping to $1 \in \tilde{E}^0 S^0$ under the map*

$$\tilde{E}^2\mathbb{CP}^\infty \longrightarrow \tilde{E}^2\mathbb{CP}^1 \cong \tilde{E}^2 S^2 \overset{\Sigma^{-2}}{\longrightarrow} \tilde{E}^0 S^0$$

induced by the inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$. Any choice of x is a complex orientation of E.

Proposition 2.5. *The map $E^*[x]/(x^{n+1}) \to E^*(\mathbb{CP}^n)$ mapping x to $x_{|\mathbb{CP}^n}$ is an isomorphism.*

Proof. The above map is well defined since one can cover $\mathbb{CP}^n$ with open contractible sets $U_0, ..., U_n$ implying the existence of $x_i \in E^2(\mathbb{CP}^n, U_i)$ with $x_{i|\mathbb{CP}^n} = x_{|\mathbb{CP}^n}$. Multiplying all these

$$x_0 \cdots x_n \in E^{2(n+1)}(\mathbb{CP}^n, \bigcup_{i=0}^n U_i) = 0$$

shows that $x^{n+1}_{|\mathbb{CP}^n} = (x_0 \cdots x_n)_{|\mathbb{CP}^n} = 0$. Next we set up the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = H^p(\mathbb{CP}^n, E^q(pt)) \Rightarrow E^{p+q}(\mathbb{CP}^n);$$

with $F^{p,q} = \ker(E^{p+q}X \to E^{p+q}X_{p-1})$ and $X = \mathbb{CP}^n$ we consider

$$F^{2,0} = \ker(E^2\mathbb{CP}^n \to E^2(pt)) \ni x_{|\mathbb{CP}^n}$$

and look at the associated graded $F^{2,0}/F^{3,-1} \cong E_\infty^{2,0} \subset E_2^{2,0}$. We have the map

$$E_\infty^{2,0} = H^2(\mathbb{CP}^1, E^0(pt)) = E^0(pt) \cdot t$$

given by $x_{|\mathbb{CP}^n} \mapsto t$. By multiplicativity the spectral sequence collapses and we have the following isomorphism of graded rings

$$E^*(\mathbb{CP}^n) \cong E^*[x_{|\mathbb{CP}^n}]/(x_{|\mathbb{CP}^n}^{n+1}).$$

□

Corollary 2.1. *$E^*\mathbb{CP}^\infty \cong E^*[\![x]\!]$.*

Corollary 2.2. *$E^*(\mathbb{CP}^\infty \times ... \times \mathbb{CP}^\infty) \cong E^*[\![x_1, ..., x_n]\!]$.*

Theorem 2.1. *Let E be a complex oriented cohomology theory, X a space and $\xi \to X$ a complex vector bundle. Then there exists a unique system of cohomology classes $c_i(\xi) \in \tilde{E}^{2i}(X)$ with the following properties:*

- *(normalization) For the tautological bundle $\lambda^* \to \mathbb{CP}^\infty$ we have $c_1(\lambda^*) = x$.*
- *(naturality) For all maps $f : X \to Y$ we have $f^*c_i(\xi) = c_i(f^*\xi)$.*
- *(Cartan formula) For the total Chern class $c = 1 + c_1 + c_2 + ...$ we have $c(\xi \oplus \eta) = c(\xi)c(\eta)$.*

Lemma 2.1. *Let E be a complex oriented theory with a complex orientation $x \in E^2\mathbb{CP}^\infty$:*

1. *$E^*\mathbb{CP}^\infty \cong E^*[\![x]\!]$ the power series ring in x over E^*.*
2. *$E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) = E^*\mathbb{CP}^\infty \hat{\otimes}_{E^*} E^*\mathbb{CP}^\infty$.*
3. *$E_*\mathbb{CP}^\infty$ is a free E^* module with generators $\beta_i \in E_{2i}\mathbb{CP}^\infty$, $i \geq 0$ dual to x^i, i.e. $\langle x^i, \beta_j \rangle = \delta_{ij}$.*
4. *$E_*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) = E_*\mathbb{CP}^\infty \otimes_{E_*} E_*\mathbb{CP}^\infty$.*
5. *The diagonal $\mathbb{CP}^\infty \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ induces a coproduct ψ on $E_*\mathbb{CP}^\infty$ with $\psi(\beta_n) = \sum_{i+j=n} \beta_i \otimes \beta_j$.*

4. Bousfield localization of spectra

Bousfield localization theory is an analogue and a generalization of arithmetic localization theory in algebra. While arithmetic localization takes place for example in the category of rings and is done with respect to some prime ideals, Bousfield localization theory takes place in the stable homotopy category *SHC* and can be done with respect to any generalized homology theory E_* (represented by the spectrum E).

Definition 2.9. *A map of spectra $f : X \to Y$ is called an E-equivalence if it induces an isomorphism $f_* : E_*X \xrightarrow{\cong} E_*Y$ in E_*-homology. With this a spectrum Z is called E-local if it has the E-extension property for every E-equivalence $f : X \to Y$:*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Z \\ Y \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

Equivalently Z is E-local if the functor $[-, Z]$ takes E-equivalences to isomorphisms.

Definition 2.10. *We call $\gamma_E(X) : X \to X_E$ an E-localization if*

1. *γ_E is an E-equivalence*

(2) X_E *is* E*-local.*

Remark 2.1. *This definition is equivalent to the definition given by Bousfield* [**Bou79**] *and Ravenel* [**Rav84**]: *A spectrum* Y *is* E*-local if for each* E*-acyclic spectrum* X (*i.e.* $E_*X = 0$) *follows*: $[X, Y] = 0$. *The reason is that a map* $f : X \to Y$ *gives the cofiber sequence*

$$X \xrightarrow{f} Y \to Z = cofiber(f)$$

and a long exact sequence

$$... \to [Z, T] \to [Y, T] \to [X, T] \to ...$$

i.e. having $[X, T] = 0$ *for an* E*-acyclic* X *we get isomorphisms* $[Z, T] \xrightarrow{\cong} [Y, T]$ *and thus a lift* $Z \to T$ (*and vice versa*).

It follows directly from the definition that $\gamma_E(X)$ (if it exists) is unique up to isomorphism. The existence is given via

Theorem 2.2 (Bousfield). *With the above notation we have*

(1) $\gamma_E(X)$ *always exists*
(2) $\gamma_E(X)$ *assembles an idempotent functor*

$$L_E : SHC \to SHC_{E\text{-local objects}}$$

(3) L_E *is (up to equivalence) the categorical localization of SHC with respect to E-equivalences, i.e. given a functor* $SHC \to D$ *such that* E*-equivalences are inverted we get the commutative diagram*

$$\begin{tikzcd} SHC \arrow[r, "E\text{-equiv. inv.}"] \arrow[d] & D \\ L_E SHC \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

Collecting results from [**Bou79**], [**EKMM97**] and [**Lau01**] we formulate an overview theorem:

Theorem 2.3. *The localization functor* L_E *has the following properties:*

(1) *it is idempotent, i.e.* $L_E L_E = L_E$
(2) *if* $W \to X \to Y$ *is a cofiber sequence, so is* $L_E W \to L_E X \to L_E Y$
(3) *the homotopy inverse limit of* E*-local spectra is* E*-local*
(4) *if* E *is a ring spectrum and* X *is a* E*-module spectrum then* X *is* E*-local*
(5) *the localization functor* L_E *can be chosen to preserve* E_∞*-structures*
(6) *at* $p = 2$ *we have* $L_{K(1)} = L_{S\mathbb{Z}/2} L_{K_{(2)}}$

Next we introduce the so-called Bousfield classes which compare the localization functors.

Definition 2.11 (Bousfield classes). *For spectra* E *and* F *we say* $E \geq F$ *if* $f : X \overset{\sim_E}{\to} Y$ *implies* $f : X \overset{\sim_F}{\to} Y$. *This defines an equivalence relation by*

$$E \sim F \quad \text{if } E \geq F \text{ and } F \geq E.$$

The equivalence class $\langle E \rangle$ *is called the Bousfield class of* E.

Note that the relation $E \geq F$ gives a canonical factorization:

$$\begin{tikzcd} & L_E SHC \arrow[d, dotted, "\exists!"] \\ SHC \arrow[ur] \arrow[r] & L_F SHC \end{tikzcd}$$

Example 2.7 (p-localization)**.** *For the Moore spectrum $E = M\mathbb{Z}_{(p)}$ (i.e. E is connected $\pi_{<0}E = 0$ and the only non-vanishing homology is in degree zero $H_0(E) \cong \mathbb{Z}_{(p)}$) we have*

$$(M\mathbb{Z}_{(p)})_* X \cong \pi_* X \otimes \mathbb{Z}_{(p)},$$

and $M\mathbb{Z}_{(p)}$-localization is realized by

$$X \mapsto X \wedge M\mathbb{Z}_{(p)},$$

which we also call p-localization due to its effect in homotopy.

Example 2.8 (rationalization)**.** *Similarly we have*

$$M\mathbb{Q}_*(X) = \pi_*(X) \otimes \mathbb{Q}$$

and rationalization is given by

$$X \mapsto X \wedge M\mathbb{Q}.$$

In particular $L_{\mathbb{Q}} S = M\mathbb{Q}$.

Definition 2.12 (smashing spectrum)**.** *If $L_E X = X \wedge L_E S$, the spectrum E is called smashing.*

We have seen that the above Moore spectra are smahing and will for reasons of notation denote the rationalization of a spectrum X by $X_{\mathbb{Q}}$ and the p-localization by $X_{(p)}$. With these notations we have a local-global arithmetic square analogy.

Proposition 2.6 (arithmetic square)**.** *Let X be a finite spectrum, then*

$$\begin{tikzcd} X \arrow[r, "\prod_p L_{\mathbb{Z}_{(p)}}"] \arrow[d, "L_{\mathbb{Q}}"'] & \prod_p X_{(p)} \arrow[d, "L_{\mathbb{Q}}"] \\ X_{\mathbb{Q}} \arrow[r, "L_{\mathbb{Q}}(\prod_p L_{\mathbb{Z}_{(p)}})"'] & (\prod_p X_{(p)})_{\mathbb{Q}} \end{tikzcd}$$

is a homotopy pullback square.

4.1. K-theoretic localization of spectra. Following section 8 of [**Rav84**] we have

Theorem 2.4. *Let K and KO be the spectra representing complex and real K-theory, respectively. Then $K \wedge X = pt$ if and only if $KO \wedge X = pt$, so L_K and L_{KO} represent the same functors and we have the same Bousfield classes $\langle K \rangle = \langle KO \rangle$.*

Proof. Consider $\mathbb{CP}^2 = S^2 \cup_\eta e^4$ with the Hopf map $\eta : S^3 \to S^2$ being the attaching map of e^4. Let us denote by $S^0 \cup_\eta e^2$ the suspension spectrum with $\mathbb{CP}^2$ being the second suspension. Due to Adams we have

$$KU = KO \wedge \mathbb{CP}^2.$$

Hence having the cofiber sequence

$$S^3 \xrightarrow{\eta} S^2 \to \mathbb{CP}^2$$

we get by smashing with KO

$$\Sigma^3 KO \xrightarrow{\Sigma^2\eta^{st}} \Sigma^2 KO \to KO \wedge \mathbb{CP}^2 = \Sigma^2 KU$$

and get by applying Σ^{-2} in the stable homotopy category the famous cofiber sequence

$$\Sigma KO \xrightarrow{\eta} KO \to KU.$$

Looking at this sequence $KO \wedge X = pt$ implies $K \wedge X = pt$ and conversely if $K \wedge X = pt$ then η induces an automorphism of KO_*X. But since η is nilpotent ($\eta^4 = 0$) we have $KO_*X = 0$. $\square$

5. Algebraic manipulations of spectra

When working with ring spectra algebraically one often looks at them as algebraic rings. And as the study of rings is often simplified by passage to its quotients and localizations one also wants to transfer these techniques to ring spectra. As in [**HS98**] we want to recall the construction of quotients and localizations. Suppose that E is a ring spectrum and that $\pi_*E = R$ is commutative. Given $x \in R$, define the spectrum $E/(x)$ by the cofibration

$$\Sigma^{|x|}E \xrightarrow{x\cdot} E \to E/(x).$$

If x is a non-zero divisor then $\pi_*E/(x)$ is isomorphic to the ring $R/(x)$ and in "good" cases $E/(x)$ is a ring spectrum and the map $E \to E/(x)$ is a map of ring spectra. Such a "good" situation is given if one has a regular sequence $\{x_1, x_2, ..., x_n, ...\} \subset R$ in which one can iterate the above situation and form a ring spectrum $E/(x_1, x_2, ..., x_n, ...)$ with

$$\pi_*E/(x_1, x_2, ..., x_n, ...) \cong R/(x_1, x_2, ..., x_n, ...)$$

such that the natural map

$$E \to E/(x_1, x_2, ..., x_n, ...)$$

is a map of ring spectra. Considering the case of localizations, suppose that $S \subset R$ is a closed subset. Since $S^{-1}R$ is a flat R-module, the functor

$$S^{-1}R \otimes_R E_*(-)$$

is a homology theory denoted by $S^{-1}E$. In "good" cases it is represented by a ring spectrum, and the localization can be described by a map of ring spectra

$$E \to S^{-1}E.$$

Now we want to apply the above constructions to the Brown-Peterson spectrum BP (confer [**Rav84**]). In this case everything is "good" and all the constructions can be made. Recall that

$$BP_* \cong \mathbb{Z}_{(p)}[v_1, ..., v_n, ...] \quad \text{with} \quad |v_n| = 2p^n - 2.$$

For $0 < n < \infty$ the Morava K-theory ring spectra $K(n)$ and the Johnson-Wilson ring spectra $E(n)$ are defined by the isomorphisms

$$\begin{aligned} K(n)_* &\cong \mathbb{F}_p[v_n, v_n^{-1}] \\ E(n)_* &\cong \mathbb{Z}_{(p)}[v_1, ..., v_n, v_n^{-1}] \end{aligned}$$

with the understanding that they are constructed from BP using a combination of the above methods.

5.1. The chromatic framework. In the following paragraph we want to describe the chromatic framework which gives us the motivation for $K(1)$-localization.

Theorem 2.5 (chromatic convergence theorem)**.** *Let p be a fixed prime and denote localization with respect to the Johnson-Wilson theories $E(n)_*$ $(n \geq 0)$ by L_n. Then there are natural maps $L_nX \to L_{n-1}X$ for all spectra X, and if X is a p-local finite spectrum, then the natural map*

$$X \to \operatorname{holim} L_nX$$

is a weak equivalence.

Recall from of [**Rav84**, p. 361] that the spectra $E(n)$ and $K(0) \vee K(1) \vee ... \vee K(n)$ have the same Bousfield classes. As said above there are natural transformations $L_n \to L_{n-1}$ and compatible transformations $1 \to L_n$ giving the so-called *chromatic tower*

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & L_2S \arrow[d] \\
 & L_1S \arrow[d] \\
S \arrow[r] \arrow[ru] \arrow[ruu] \arrow[ruuu] & L_0S
\end{tikzcd}$$

Looking at the tower we are interested in the difference between L_n and L_{n-1}. On the one hand the fiber M_n of the transformation $L_n \to L_{n-1}$ is known as the monochromatic layer, and on the other hand the difference of L_n and L_{n-1} is measured by the functor $L_{K(n)}$, which is localization with respect to the n^{th} Morava K-theory. From [**HG94**] we cite that there are natural equivalences

$$L_{K(n)}M_nF \cong L_{K(n)}F \quad \text{and} \quad M_nL_{K(n)}F \cong M_nF,$$

so the homotopy types of $L_{K(n)}F$ and M_nF determine each other.

Theorem 2.6 (arithmetic square)**.** *Let $K(n)_*$ denote the n^{th} Morava K-theory. There is a natural commutative diagram*

$$\begin{tikzcd}
L_nX \arrow[r] \arrow[d] & L_{K(n)}X \arrow[d] \\
L_{n-1}X \arrow[r] & L_{n-1}L_{K(n)}X
\end{tikzcd}$$

which for any spectrum X is a homotopy pullback square.

Informally one might say that, having a p-local spectrum X, the basic building blocks for the homotopy type of X are the Morava K-theory localizations $L_{K(n)}X$. Comparing the stable homotopy category with the integers in arithmetic, it is the localization functors $L_{K(n)}$ which take over the role of the primes.

6. A resolution of the $K(1)$-local sphere

In this section we are going to introduce a very useful $K(1)$-local fiber sequence which comes up from the resolution of the $K(1)$-local sphere and which will help us constructing an Artin-Schreier class and in the construction of spherical classes. Although we only need this special fiber sequence, we will sketch quite briefly also the general setup for the resolution of the $K(n)$-local sphere, which will give us some motivation for all that stuff: It is the theory of formal group laws which is the starting point. We refer to [**Re97**]: Having the Honda formal group law of height n – which is characterized by its p-series $[p]_{\Gamma_n}(x) = x^{p^n}$ – we apply the theory of Lubin-Tate deformation theories which is Landweber exact and gives homology theories called *Morava E-theories* E_n. Considering the automorphisms $\mathrm{Aut}(\Gamma_n)$ of the Honda formal group law Γ_n (also known as the Morava stabilizer group $\mathbb{S}_n$) one considers the group

$$\mathbb{G}_n = \mathrm{Aut}(\Gamma_n) \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

which gives a group action on E_{n*}. The Hopkins-Miller theorem states that $\mathbb{S}_n$ gives an action on the spectrum E_n itself, and the Adams-Novikov spectral sequence

$$E_2^{s,t} := H^s(\mathbb{S}_n, (E_n)_t)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)} \Rightarrow \pi_{t-s} L_{K(n)} S^0$$

provides computational methods for calculating the homotopy groups of the $K(n)$-local sphere.

6.1. The case $n = 1$. For $n = 1$ the Honda formal group law Γ_1 coincides with the multiplicative formal group law $\mathbb{G}_m(x, y) = x + y + xy = (1 + x)(1 + y) - 1$ because their p-series coincide modulo p

$$[p]_{\mathbb{G}_m}(x) = (1 + x)^p - 1 \equiv x^p = [p]_{\Gamma_1}(x).$$

In our p-local setting we have $\mathrm{Aut}(\mathbb{G}_m) \cong \mathbb{Z}_p^\times$ and thus we have

$$\mathbb{G}_1 = \mathbb{S}_1 \cong \mathbb{Z}_p^\times,$$

where $\mathbb{G}_1 \cong \mathbb{Z}_p \times C_{p-1}$ for p odd and $\mathbb{G}_1 \cong \mathbb{Z}_2 \times C_2$ for $p = 2$. Following [**GHMR**] there is a short exact sequence of continuous $\mathbb{G}_1$-modules

$$0 \to \mathbb{Z}_p[\![\mathbb{G}_1/F]\!] \to \mathbb{Z}_p[\![\mathbb{G}_1/F]\!] \to \mathbb{Z}_p \to 0$$

where F is the maximal finite subgroup of $\mathbb{G}_1$. These resolutions of the trivial module are analogues of the fibrations

$$L_{K(1)}S^0 \cong E_1^{h\mathbb{G}_1} \to E_1^{hF} \to E_1^{hF}$$

with the notation meaning the homotopy fixed point spectra with respect to the given group. We note that p-adic complex K-theory $K\mathbb{Z}_p$ is a model for E_1 and the homotopy fixed point spectrum $E_1^{hC_2}$ can be identified with 2-adic real K-theory $KO\mathbb{Z}_2$. Since every p-adic unit $k \in \mathbb{Z}_p^\times$ gives an Adams operation ψ^k and vice versa, and for example 3 is a topolocal generator for $\mathbb{Z}_2^\times/\{\pm 1\}$, we can write

$$L_{K(1)}S^0 \to KO\mathbb{Z}_2 \overset{\psi^3-1}{\to} KO\mathbb{Z}_2.$$

This is a resolution of the $K(1)$-local sphere – and the $K(1)$-local sphere is the fiber of the Adams operation $\psi^3 - 1$. Hence this is our desired fiber sequence.

Remark 2.2. At this point we want to point out that the profinite group $\mathbb{G}_1/F \cong \mathbb{Z}_p$ has the pro-group ring $\mathbb{Z}_p[\![\mathbb{G}_1/F]\!]$ which is isomorphic to the power series ring $\Lambda = \mathbb{Z}_p[\![T]\!]$ called the *Iwasawa algebra*. The Iwasawa algebra plays an important role in number theory when studying $\mathbb{Z}_p$-extensions of number fields and has been studied by many people for a long time. We refer to [**Wa97**] for number theoretic properties and to [**HM07**] for the interpretation for $K(1)$-local spectra. In [**HM07**] the resolution of the $K(1)$-local sphere is stated as

$$S^0 \to KO \xrightarrow{T} KO.$$

for $p = 2$. Maybe more topology could be deduced from the analysis of the Iwasawa algebra.

7. Thom isomorphism

Having a complex vector bundle $\xi : E \to X$ with X compact Hausdorff one can construct the Thom space $\mathrm{Th}(\xi) = D(\xi)/S(\xi)$. If the bundle ξ admits a Thom class $\tau \in \tilde{K}^*(\mathrm{Th}(\xi))$ we get an isomorphism of modules

$$K^*(X) \xrightarrow{\cong} \tilde{K}^*(\mathrm{Th}(\xi))$$

called the Thom isomorphism. $\tilde{K}^*(\mathrm{Th}(\xi))$ is the free $K^*(X)$-module with single basis element the Thom class $\tau \in \tilde{K}^*(\mathrm{Th}(\xi))$.

7.1. Generalization.

Let E^* be a generalized multiplicative cohomology theory.

Definition 2.13. *A class $\tau \in E^*(D(\xi), S(\xi))$ is said to be a Thom class for ξ if for every $x \in X$ the restriction of τ to $E^*(D(\xi_x), S(\xi_x))$ is an $E^*(pt)$-module generator.*

Having a Thom class $\tau \in E^d(D(\xi), S(\xi))$ the homotopy equivalence $p : D(\xi) \to X$ inducing an isomorphism $p^* : E^*(X) \xrightarrow{\cong} E^*(D(\xi))$ leads us to the definition of the Thom isomorphism

$$E^*(X) \to E^{*+d}(D(\xi), S(\xi)) \cong \tilde{E}^{*+d}(\mathrm{Th}(\xi))$$

by applying the cup product and mapping

$$\alpha \mapsto p^*(\alpha) \cup \tau.$$

This gives an isomorphism of graded modules over $E^*(pt)$.

Remark 2.3. *For a trivial bundle of dimension 1 the Thom isomorphism reduces to the suspension isomorphism.*

To prove the Thom isomorphism for compact X one proceeds by induction over the open sets in a trivialization of ξ using the suspension isomorphism as the starting case and the Mayer-Vietoris sequence to carry out the inductive step.

Remark 2.4. *There is also a homology Thom isomorphism*

$$\tilde{E}_{*+d}(\mathrm{Th}(\xi)) \xrightarrow{\cong} E_*(X),$$

using the cap product with the Thom class

$$\cap : \tilde{E}_{*+d}(\mathrm{Th}(\xi)) \times \tilde{E}^d(\mathrm{Th}(\xi)) \to E_*(X)$$

instead of the cup product.

7.2. Strong form of the Thom isomorphism. Following along the lines of [**MR81**] one can state a strong form of the Thom isomorphism theorem. In this context we assume given a stable spherical fibration $\nu : X \to BF$ over a locally finite CW complex X, and a ring spectrum E orienting ν, i.e. a Thom class $\tau : \mathrm{Th}(\nu) \to E$ whose restriction to a fiber $S^0 \hookrightarrow \mathrm{Th}(\nu)$ is the unit of E.

Theorem 2.7 (Mahowald, Ray)**.** *There is a homotopy equivalence*

$$\alpha(\tau) : E \wedge \mathrm{Th}(\nu) \to E \wedge X_+$$

which on homotopy groups induces the traditional Thom isomorphism

$$\phi_\tau : E_*(\mathrm{Th}(\nu)) = \pi_*(E \wedge \mathrm{Th}(\nu)) \xrightarrow{\alpha(\tau)_*} \pi_*(E \wedge X_+) = E_*(X_+).$$

Proof. First suppose that X has finite dimension, so that ν lifts to $\nu_n : X \to BF_n$ for suitably large n. Let $p_n : S(\nu_n) \to X$ be the associated n-sphere fibration, so that $\mathrm{Th}(\nu_n) = X \cup_{p_n} CS(\nu_n)$. Now we define the Thom diagonal

$$\Delta : \mathrm{Th}(\nu_n) \to \mathrm{Th}(\nu_n) \wedge X_+$$

to be

$$\Delta(x) = \begin{cases} (x, p_n(x)) & \text{for } x \neq \infty \\ \infty & \text{for } x = \infty \end{cases}$$

and consider the composite

$$\alpha(\tau) : E \wedge \mathrm{Th}(\nu_n) \xrightarrow{\mathrm{id}\wedge\Delta} E \wedge \mathrm{Th}(\nu_n) \wedge X_+ \xrightarrow{\mathrm{id}\wedge\tau\wedge\mathrm{id}} E \wedge \Sigma^{n+1} E \wedge X_+ \xrightarrow{\mu\wedge\mathrm{id}} E \wedge \Sigma^{n+1} X_+$$

where μ is the product in E. On homotopy groups this map induces a homomorphism

$$\phi_\tau : E_{*+n+1}(\mathrm{Th}(\nu_n)) \to E_*(X_+)$$

which is the usual homology Thom isomorphism, i.e. cap product with τ. $\square$

The above theorem suffices for our purposes, i.e. we have $K_*MU \cong K_*BU$ and $K_*MSU \cong K_*BSU$.

Example 2.9. *To define the Thom isomorphism $\Phi : K_*MU \to K_*BU$ we use the Thom diagonal*

$$MU \xrightarrow{\Delta} BU \wedge MU,$$

and choose a Thom class $MU \xrightarrow{\tau} K$. Thus we define the Thom isomorphism: An element $\overline{f} \in K_n MU$ represented by $S^n \xrightarrow{f} MU \wedge K$ is mapped to $\Phi\overline{f}$, more explicitly

$$S^n \xrightarrow{f} MU \wedge K \xrightarrow{\Delta\wedge 1} BU \wedge MU \wedge K \xrightarrow{BU\wedge\tau\wedge 1} BU \wedge K \wedge K \xrightarrow{BU\wedge\mu} BU \wedge K.$$

CHAPTER 3

The algebraic structure of $K(1)$-local E_∞ ring spectra

The following chapter is about an algebraic extra structure which comes up as the homotopy of $K(1)$-local E_∞ ring spectra: The so-called θ-algebra structure. In the beginning we briefly recall the geometric objects: An E_∞ ring spectrum is a ring spectrum X with an E_∞ operad $E = \{E(n)\}$ acting on X. An operad is called E_∞ if $E(n) \simeq *$ for all n and $E(n)$ has an free action of the symmetric group Σ_n. The functor being left-adjoint to the forgetful functor

$$E_\infty\text{spectra} \to \text{spectra}$$

is the free algebra functor $TX = \bigvee T_n X$ with

$$T_n X = E(n) \wedge_{\Sigma_n} X^{\wedge n}$$

where Σ_n acts on $X^{\wedge n}$ by permuting the factors.

1. Operads

Historically one can think of loop spaces as monoids up to higher homotopy. A monoid structure on a set S is a family of maps $M(k) : S^k \to S$ (one map $M(k)$ for each k) with $M(1) = id$ and $\{M(k)\}$ closed under multivariable composition.

Example 3.1. *To think of a loop space as a monoid up to higher homotopy is the following: Take $S = \Omega Z$ for a based space Z and the composition of loops $M_r : (\Omega Z)^2 \to \Omega Z$ for $r \in (0, 1)$ and define*

$$\mathcal{M}(k) = \{(\Omega Z)^k \to \Omega Z\};$$

then $\mathcal{M}(k) \cong \mathcal{A}(k)$ is the set of k closed intervals in $[0, 1]$ with disjoint interiors being as a subset of $\mathbb{R}^{2n}$ contractible.

Proposition 3.1. *For a loop space $Y = \Omega Z$ there exists a sequence of subspaces $\mathcal{M}(k) \subset Map(Y^k, Y)$ such that*

(1) $\mathcal{M}(1) \ni id$
(2) $\mathcal{M} = \{\mathcal{M}(k)\}_{k\geq 0}$ *is closed under multivariable composition*
(3) *each $\mathcal{M}(k)$ is contractible*

Theorem 3.1 (Converse theorem)**.** *For any connected Y satisfying the above three conditions there exists a space Z with $Y = \Omega Z$.*

This leads us to a provisional definition:

Definition 3.1. *A non-symmetric operad $\mathcal{O}$ is a collection of subspaces $\mathcal{O}(k) \subset Map(Y^k, Y)$ such that*

(1) $\mathcal{O}(1) \ni id_Y$
(2) $\mathcal{O}$ *is closed under multivariable composition*

Example 3.2. $\mathcal{A} = \{\mathcal{A}(k)\}_{k\geq 0}$ *little intervals non-symmetric operad*

Example 3.3. *$End(Y) = \{Map(Y^k, Y)\}_{k\geq 0}$ endomorphism operad*

Definition 3.2. *An A_∞ operad is a non-symmetric operad with $\mathcal{O}(k) \simeq *$ (associativity up to higher homotopy)*

Definition 3.3. *An E_∞ operad has also a permutation action $\Sigma_k \curvearrowright \mathcal{O}(k)$ (commutativity up to higher homotopy)*

Next we want to give a definition of an operad in modern language. Let $(\mathcal{C}, \wedge, S)$ be a symmetric monoidal category.

Definition 3.4. *An operad C in $\mathcal{C}$ is a monoid in $(\mathcal{C}^\Sigma, \circ, U)$*

This gives us the category of operads in $\mathcal{C}$. Notation: $oper\mathcal{C} = \mathcal{O}$. Now we want to recall the notion of the composition product: For $X, Y \in \mathcal{C}^\Sigma$ define $X \circ Y$ by

$$(X \circ Y)_n := \bigvee_{k\geq 0} X_k \wedge_{\Sigma_k} (Y^{\otimes k})_n.$$

Then $X \in \mathcal{C}^\Sigma$ gives a functor $X : \mathcal{C} \to \mathcal{C}$ by $X(A) := \bigvee_{n\geq 0} X_n \wedge_{\Sigma_n} A^{\wedge n}$. This gives us for $X, Y \in \mathcal{C}^\Sigma$, $A \in \mathcal{C}$ the identity

$$(X \circ Y)(A) \cong X(Y(A)).$$

Remark 3.1. *For $C \in \mathcal{C}$ the functor $C : \mathcal{C} \to \mathcal{C}$, $X \mapsto \bigvee_{n\geq 0} C_n \wedge_{\Sigma_n} X^{\otimes n}$ is a triple on $\mathcal{C}$ (or monad), i.e. a monoid on $(Fun(\mathcal{C}, \mathcal{C}), \circ)$.*

Definition 3.5. *For T a triple on $\mathcal{C}$, a T-algebra X is an object X in $\mathcal{C}$ together with a map $\omega : T(X) \to X$ such that*

$$\begin{tikzcd}
T(T(X)) \arrow[r, "T\omega"] \arrow[d, "\cong"] & T(X) \arrow[dd, "\omega"] \\
(T \circ T)(X) \arrow[d, "\mu"] & \\
T(X) \arrow[r, "\omega"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\varepsilon_X"] \arrow[dr, "="'] & T(X) \arrow[d, "\omega"] \\
& X
\end{tikzcd}$$

A map $X \xrightarrow{f} Y$ is a T-algebra map if

$$\begin{tikzcd}
T(X) \arrow[r, "Tf"] \arrow[d] & T(Y) \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

This gives us the category of T-algebras Alg^T.

Example 3.4. *Let $\mathcal{C} = (\mathbf{AbGr}, \otimes)$ and R a monoid in $\mathcal{C}$ (i.e. a ring $R \in \mathbf{AbGr}$, $\mu : R\otimes_{\mathbb{Z}} R \to R$ the multiplication map and $\varepsilon : \mathbb{Z} \to R$ the unit) with $T : \mathbf{AbGr} \to \mathbf{AbGr}$, $A \mapsto R \otimes A$. Then*

$$T \circ T \to T :\Leftrightarrow \underbrace{R \otimes R \otimes A}_{=T^2(A)} \xrightarrow{\mu\otimes A} \underbrace{R \otimes A}_{T(A)}$$

$$id \to T :\Leftrightarrow A \cong \mathbb{Z} \otimes A \to R \otimes A = T(A).$$

Hence T is a triple: $T \in Fun(\mathbf{AbGr}, \mathbf{AbGr})$, $T^2 \to T$, $id \to T$. We now conclude what the T-algebras are: Take $X \in Alg^T$, $T(X) \to X$, i.e. $R \otimes X \to X$ implying

$$Alg^T \simeq R - mod.$$

Example 3.5. *Take $C \in \mathcal{O}$ in $\mathcal{T}_*$ with $C_n = S^0$. For $X \in Alg^C$, $C(X) \to X$ and*

$$\bigvee_{n \geq 0} C_n \wedge_{\Sigma_n} X^n \to X,$$

i.e.

$$S^0 \wedge_{\Sigma_n} X^n \to X \quad \overset{1:1}{\leftrightarrow} \quad X^{\wedge n}/\Sigma_n \to X$$

Hence Alg^C are commutative monoids in $(\mathcal{T}_*, \wedge)$.

Example 3.6. *$C_n = S^0 \in Sp^\Sigma$, $C \in (Sp^\Sigma)^\Sigma$ gives Alg^C = commutative ring spectra.*

2. Dyer-Lashof operations for $K(1)$-local E_∞ spectra

An E_∞ structure ξ on E determines power operations

$$P_n : E^0X \to E^0T_nX$$

by setting

$$P_n(x) : T_nX \to TX \overset{Tx}{\to} TE \overset{T\xi}{\to} E$$

for each $x \in E^0X$. For $X = S^0$ and $n = 2$ this gives a map

$$P_2(x) : T_2S^0 \simeq B\Sigma_{2+} \to E$$

for each $x \in \pi_0 E$. The classifying space $B\Sigma_{2+}$ reduces to two copies of S^0 in the $K(1)$-local world

Lemma 3.1 (Hopkins)**.** *The map*

$$B\Sigma_{2+} \overset{(\epsilon,Tr)}{\longrightarrow} S^0 \times S^0$$

is a weak equivalence in the category of E_∞ ring spectra. Here ϵ is induced by $\Sigma_2 \to \{e\}$ and $Tr : B\Sigma_{2+} \to S^0$ is the transfer map.

As in [**Hop98**] and [**Lau03**] one defines maps

$$\theta, \psi : S^0 \to B\Sigma_{2+}$$

by requiring that the compositions in $\pi_0 S^0 \cong \mathbb{Z}$ are

$$\begin{array}{cc} Tr \circ \theta = -1 & Tr \circ \psi = 0 \\ \epsilon \circ \theta = 0 & \epsilon \circ \psi = 1 \end{array}$$

The map $B\{e\} \to B\Sigma_2$ gives rise to a map

$$e : S^0 \simeq B\{e\}_+ \to B\Sigma_{2+}$$

and by the definition we have

$$\epsilon \circ e = 1 : S^0 \overset{e}{\to} B\Sigma_{2+} \overset{\epsilon}{\to} S^0$$

and by [**Hop98**] $Tr \circ e = 2$. Because of $\epsilon \circ e = e \circ \psi - 2\epsilon \circ \theta = 1$ and $Tr \circ e = Tr \circ \psi - 2Tr \circ \theta = 2$ it follows that $e = \psi - 2\theta$. With $\theta(x) = P_2(x)\theta$ and $\psi(x) = P_2(x)\psi$, the last equation gives

$$\psi(x) - 2\theta(x) = P_2(x)e = x^2.$$

We refer to [**Lau03**, pp. 993-994] and state without proof:

Proposition 3.2. *The operation θ turns $\pi_0 E$ into a θ-algebra.*

We briefly recall the properties of a θ-algebra.

Definition 3.6. *A θ-algebra over a ring R with unit is a commutative algebra A over R together with a function $\theta : A \to A$ such that:*

- $\theta(1) = 0$
- $\theta(a+b) = \theta(a) + \theta(b) - ab$
- $\theta(ab) = \theta(a)b^2 + a^2\theta(b) + 2\theta(a)\theta(b)$

A morphism of θ-algebras is an algebra homomorphism compatible with the θ-operation.

Proposition 3.3. *$\psi(x) = x^2 + 2\theta(x)$ is a ring homomorphism and commutes with θ.*

Proof.

$$\begin{aligned}
\psi(a+b) &= (a+b)^2 + 2\theta(a+b) = 2(\theta(a) + \theta(b) - ab) + (a+b)^2 \\
&= \psi(a) + \psi(b) \\
\psi(ab) &= a^2b^2 + 2\theta(ab) = a^2b^2 + 2(\theta(a)b^2 + a^2\theta(b) + 2\theta(a)\theta(b) \\
&= (a^2 + 2\theta(a))(b^2 + 2\theta(b)) = \psi(a)\psi(b) \\
\theta\psi(a) &= \theta(a^2 + 2\theta(a)) = \theta(a^2) + \theta(2\theta(a)) - 2a^2\theta(a) \\
&= 2\theta(a)a^2 + 2\theta(a)^2 - \theta(a)^2 + 2\theta^2(a) - 2a^2\theta(a) \\
&= \theta(a)^2 + 2\theta^2(a) = \psi\theta(a)
\end{aligned}$$

□

Example 3.7. *The 2-adic integers $\mathbb{Z}_2$ are a θ-algebra with $\theta(x) = \frac{x-x^2}{2}$ and $\psi(x) = x$. Similarly the ring of continuous functions on the 2-adics $\mathcal{C} = \mathcal{T}(\mathbb{Z}_2, \mathbb{Z}_2)$ is a θ-algebra with $\psi(f) = f$.*

Remark 3.2. *There is no example of a θ-algebra in characteristic 2, since for $(a,b) = (1,0)$ we get $\theta(0) = 0$ and for $(a,b) = (1,1)$ we get $\theta(0) = \theta(2) = 1$.*

3. The θ-algebra structure of $\pi_0 K \wedge MU$

Let $g(x) = \sum_{i=0}^{\infty} b_i x^i$ with $b_0 = 1$ be an invertible power series with coefficients in the polynomial ring $\pi_0 K \wedge MU \cong \mathbb{Z}_2[b_1, b_2, ...]$.

Theorem 3.2 (Laures)**.** *The θ-algebra structure of $\pi_0 K \wedge MU$ is determined by the equation*

$$\sum_{i \geq 0} \psi(b_i)x^i = \frac{g(1+\sqrt{1-x})g(1-\sqrt{1-x})}{g(2)}.$$

Equivalently one has to show the identity

$$\sum_{i \geq 0} \psi(b_i)x^i(2-x)^i = \psi(g(x)) = \frac{g(x)g(2-x)}{g(2)}.$$

Transforming the last equality yields

$$\begin{aligned}
\psi g(x) &= g(2)^{-1} \sum_{i\geq 0} \big(\sum_{k=0}^{i} b_{i-k}(-1)^k \sum_{n\geq k} b_n \binom{n}{k} 2^{n-k} \big) x^i \\
&= \left(\sum_{i\geq 0} \big(\sum_{k=0}^{i} b_{i-k}(-1)^k \sum_{n\geq k} b_n \binom{n}{k} 2^{n-k} \big) x^i \right) \left(\sum_{n\geq 0} (-1)^n \big(\sum_{i\geq 1} b_i 2^i \big)^n \right).
\end{aligned}$$

Although it is inviting to reduce modulo 2,

$$\psi(\sum_{i\geq 0} b_i x^i) \equiv \sum_{i\geq 0} \big(\sum_{k=0}^{i} b_k b_{i-k} \big) x^i \mod 2$$

because of $\psi(x) = x^2 + 2\theta(x)$ one has to reduce modulo 4 to get

Corollary 3.1 (Laures)**.** *In $\pi_0 K \wedge MU$ we have the following formula modulo 2:*

$$\theta(b_r) \equiv (1+b_1)b_r^2 + \sum_{i=0}^{r} b_i (b_{2r-i} + b_{2r-i+1}).$$

In particular, modulo 2 and decomposables we have (for $r > 0$)

$$\theta(b_r) \equiv b_{2r} + b_{2r+1}.$$

Proof. Since $\pi_0 K \wedge MU$ is torsion-free and $2\theta(x) = \psi(x) - x^2$, it is enough to compute the action of ψ on the generators. Let

$$D_k := \frac{d^k}{k! dx^k}|_{x=0}$$

be the normalized k^{th} derivative evaluated at 0. Then we get modulo 4

$$\begin{aligned}
D_{2r}\psi(g(x)) &\equiv D_{2r} \left(\sum_{i\geq 0} \big(\sum_{k=0}^{i} b_{i-k}(-1)^k (b_k + 2(k+1)b_{k+1}) \big) x^i \right) (1+2b_1) \\
&\equiv (1+2b_1) \sum_{k=0}^{2r} b_{2r-k}(-1)^k (b_k + 2(k+1)b_{k+1}) \\
&\equiv (1+2b_1) \left(\sum_{i+j=2r} (-1)^j b_i b_j + 2 \sum_{i+j-1=2r} j b_i b_j \right)
\end{aligned}$$

On the other hand we have modulo 4

$$\begin{aligned}
D_{2r}\psi(g(x)) &\equiv \sum_{i\geq 0} D_{2r}\psi(b_i)\psi(x^i) = \sum_{i\geq 0} \psi(b_i) D_{2r} x^i (2-x)^i \\
&\equiv (-1)^r \psi(b_r)
\end{aligned}$$

since

$$\begin{aligned}
D_{2r} x^i (2-x)^i &= D_{2r} x^i \sum_{j+k=i} 2^j \binom{i}{j} (-x)^k \equiv D_{2r} x^i \big((-x)^i + 2i(-x)^{i-1} \big) \\
&= \begin{cases} (-1)^r & \text{for } i = r \\ 0 & \text{else} \end{cases}
\end{aligned}$$

Hence

$$(-1)^r\psi(b_r) = D_{2r}\psi(g(x)) = (1+2b_1)\left(\sum_{i+j=2r}(-1)^j b_i b_j + 2\sum_{i+j-1=2r} j b_i b_j\right).$$

With $\psi(b_r) = 2\theta(b_r) + b_r^2$ we have modulo 4

$$(-1)^r(2\theta(b_r)+b_r^2) = (1+2b_1)\left(\sum_{i=0}^{r} 2(-1)^i b_i b_j - (-1)^r b_r^2 + 2\sum_{i+j-1=2r} j b_i b_j\right)$$

and by division by 2 and consideration modulo 2 we conclude

$$\theta(b_r) = (1+b_1)b_r^2 + \sum_{i=0}^{r} b_i(b_{2r-i}+b_{2r-i+1}).$$

□

Some further calculations give the following results:

Corollary 3.2. *In $\pi_0 K\wedge MU$ we have the formula modulo 4*

$$\begin{aligned}\theta(b_r) &= (-1+b_1+2b_2+2b_1^2)b_r^2 + (1+2b_1)\sum_{k=0}^{r}(-1)^{r-k}b_k \times \\ &\quad (b_{2r-k} + (2r+1-2k)b_{2r-k+1} + (2r+2+2k^2)b_{2r-k+2})\end{aligned}$$

In particular we have modulo 4 and decomposables $\theta(b_r) = 0$ for all r.

Corollary 3.3. *In $\pi_0 K\wedge MU$ we have the formula modulo 16*

$$\begin{aligned}(-1)^r\left[\psi(b_r) - 4\binom{r+1}{2}\psi(b_{r+1})\right] &= (1-2b_1-4b_2-8b_3+4b_1^2-8b_1^3) \\ \times\sum_{k=0}^{2r} b_{2r-k}(-1)^k\big(b_k + 2\binom{k+1}{1}b_{k+1} &+ 4\binom{k+2}{2}b_{k+2} + 8\binom{k+3}{3}b_{k+3}\big)\end{aligned}$$

This gives us the θ-algebra structure since we have modulo 16:

$$\psi(b_r) \equiv \left[\psi(b_r) - 4\binom{r+1}{2}\psi(b_{r+1})\right] + 4\binom{r+1}{2}\left[\psi(b_{r+1}) - 4\binom{r+2}{2}\psi(b_{r+2})\right]$$

and $\psi(x) = x^2 + 2\theta(x)$. The θ-algebra structure modulo 2^n for all n gives a unique integral θ-structure since $\mathbb{Z}_2 \cong \lim_n \mathbb{Z}/2^n$.

Corollary 3.4. *An easy calculation shows*

$$D_{2r}x^i(2-x)^i = (-1)^i 4^{i-r}\binom{i}{2i-2r}$$

This leads to

$$\begin{aligned} D_{2r}\psi(g(x)) &= g(2)^{-1}\sum_{k=0}^{2r} b_{2r-k}(-1)^k \sum_{n\geq k} b_n \binom{n}{k} 2^{n-k} \\ &= \sum_{i\geq 0} \psi(b_i)(-1)^i 4^{i-r}\binom{i}{2i-2r} \\ &= \sum_{i=r}^{2r} \psi(b_i)(-1)^i 4^{i-r}\binom{i}{2i-2r} \end{aligned}$$

4. The θ-algebra structure of $\pi_0 K \wedge MSU$

We state the θ-algebra structure of $\pi_0 K \wedge MSU$ as it is given in [**Lau03**, p. 1011 ff]. Let

$$f : \mathbb{CP}^\infty_+ \wedge \mathbb{CP}^\infty_+ \to BSU_+ \to K \wedge BSU_+$$

be the map which classifies $(1-L_1)(1-L_2)$ and $f(x,y) = \sum_{i,j} a_{ij}x^i y^j$ be the associated power series.

Theorem 3.3 (Laures)**.** *The θ-algebra structure of $\pi_0(K \wedge MSU_+)$ is determined by the identity*

$$\sum_{i,j} \psi(a_{ij})(x(2-x))^i(y(2-y))^j = \psi f(x,y) = \frac{f(x,y)f(2-x,y)}{f(2,y)}.$$

Proof. The decomposition

$$(1-L_1)(1-L_2) = (L_1L_2 - 1) + (1-L_1) + (1-L_2)$$

implies

$$\iota_* f(x,y) = \frac{g(x)g(y)}{g(x +_{\hat{\mathbb{G}}_m} y)}$$

with $\iota : BSU \to BU$ the inclusion and

$$g : \mathbb{CP}^\infty_+ \to K \wedge BU_+$$

the classifying map of $1-L$. Since ι_* is an injection we omit it from the notation. Using the naturality of ψ we get with the theorem of the previous section

$$\psi(g(x +_{\hat{\mathbb{G}}_m} y)) = \mu^*\psi(g(x)) = \frac{g(x +_{\hat{\mathbb{G}}_m} y)g(2-(x +_{\hat{\mathbb{G}}_m} y))}{g(2)}$$

and hence

$$\begin{aligned} \psi f(x,y) &= \frac{\psi g(x)\psi g(y)}{\psi g(x +_{\hat{\mathbb{G}}_m} y)} = \frac{g(x)g(2-x)g(y)g(2-y)}{g(2)g(x +_{\hat{\mathbb{G}}_m} y)g(2-(x +_{\hat{\mathbb{G}}_m} y))} \\ &= \frac{f(x,y)f(2-x,y)}{f(2,y)} \end{aligned}$$

□

CHAPTER 4

Splitting off an E_∞ summand T_ζ

In the $K(1)$-local world at the prime $p = 2$, we take the fiber sequence $S \to KO \overset{\psi^3-1}{\to} KO$ and look at the homotopy long exact sequence

$$\ldots \to \pi_0 S^0 \longrightarrow KO_0 \xrightarrow{\psi^3-1} KO_0 \longrightarrow \pi_{-1} S^0 \to \ldots$$

Since $KO_0 \cong \mathbb{Z}_2$ are the 2-adic integers and ψ^3 is a ring homomorphism, $\psi^3 - 1$ is the zero map on KO_0. Thus $KO_0 \to \pi_{-1}S^0$ is injective and the image of 1 is a non-trivial element $\zeta \in \pi_{-1}S^0 \cong \mathbb{Z}_2$. Now we are attaching a 0-cell along ζ and take the homotopy pushout in the category of E_∞ spectra:

$$\begin{tikzcd} S^{-1} \arrow[r, "\zeta"] \arrow[d, "*"] & S^0 \\ D^0 & \end{tikzcd} \qquad \begin{tikzcd} TS^{-1} \arrow[r, "\zeta"] \arrow[d, "T*"] & S^0 \arrow[d] \\ TD^0 = S^0 \arrow[r] & T_\zeta \end{tikzcd}$$

This E_∞ spectrum T_ζ will be an E_∞ summand in MSU. For this

$$\begin{tikzcd} TS^{-1} \arrow[r, "\zeta"] \arrow[d, "T*"] & S^0 \arrow[d] \arrow[ddr, bend left] & \\ TD^0 = S^0 \arrow[r] \arrow[drr, bend right] & T_\zeta \arrow[dr, dotted] & \\ & & MSU \end{tikzcd}$$

we have to show that $\zeta \in \pi_{-1}MSU$ vanishes. Considering the diagram

$$\begin{tikzcd} & KO_0S^0 \arrow[r] \arrow[d] & \pi_{-1}S^0 \arrow[d] \\ KO_0MSU \arrow[r, "\psi^3-1"] & KO_0MSU \arrow[r] & \pi_{-1}MSU \end{tikzcd}$$

it is sufficient to find an element $b \in KO_0MSU$ mapping to 1, because on the one hand the element $1 \in KO_0S^0$ maps to $1 \in KO_0MSU$ going to $0 \in \pi_{-1}MSU$ due to the long exact sequence, and on the other hand the element $1 \in KO_0S^0$ maps to $\zeta \in \pi_{-1}S^0$, which has to vanish in $\pi_{-1}MSU$ because the diagram commutes.

Definition 4.1. *An Artin-Schreier class is a class $b \in KO_0MSU$ with $\psi^3 b = b + 1$.*

In the following part we construct such a class rationally and then give a construction of an SU-manifold which realizes this class.

1. The image of $MSU_* \to MU_*$

In MSU_* every torsion is 2-torsion which is the kernel of $MSU_* \to MU_*$ concentrated in dimensions $8k+1$ and $8k+2$ for $k \geq 0$; in these cases $MSU_{8k+1} \cong MSU_{8k+2}$ is an $\mathbb{F}_2$ vector space whose dimension is the number of partitions of k (compare [**CF66b**]). Due to a theorem by Thom, complex bordism is rationally represented by complex projective spaces:

Theorem 4.1 (Thom).

$$MU_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^n | n \geq 1].$$

The obstruction for a U-manifold to be an SU-manifold is the first Chern class c_1 of the tangent bundle. Hence a manifold $M \in MSU_4$ is rationally a linear combination

$$M = A \cdot \mathbb{CP}^1 \times \mathbb{CP}^1 + B \cdot \mathbb{CP}^2 \quad \text{with} \quad c_1^2[M] = 0;$$

in the above notation we always mean their bordism classes and have omitted the brackets for brevity. An example of an SU-manifold is the Kummer surface

$$\mathcal{K} = K3 = \{z \in \mathbb{CP}^3 | z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$$

which is U-bordant to $K3 \sim_U 18(\mathbb{CP}^1)^2 - 16\mathbb{CP}^2$. Indeed $MSU_4 = \mathbb{Z}\langle K3 \rangle$ since the Todd-genus ($\hat{A}$-genus respectively) of an SU-manifold is even and $Td(K3) = 2$. It turns out that we cannot construct an Artin-Schreier class out of a class in MSU_4 since we need an SU-manifold with $\hat{A} = 1$. Therefore we are interested in the image of $MSU_8 \to MU_8$. Rationally this is a linear combination

$$M = A \cdot \mathbb{CP}^4 + B \cdot \mathbb{CP}^1 \times \mathbb{CP}^3 + C \cdot (\mathbb{CP}^2)^{\times 2} + D \cdot (\mathbb{CP}^1)^{\times 4} + E \cdot (\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2;$$

requiring the first Chern class to vanish implies the conditions $c_1^4[M] = c_1 c_3[M] = c_1^2 c_2[M] = 0$ in the Chern numbers. To express them as linear equations in the coefficients we first have to calculate the total Chern classes of the complex projective spaces and their products:

$$\begin{aligned}
c(T\mathbb{CP}^4) &= c(1 \oplus T\mathbb{CP}^4) = c(5L^*) = (1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 \\
c(T(\mathbb{CP}^1 \times \mathbb{CP}^3)) &= pr_1^* c(T\mathbb{CP}^1) \cdot pr_2^* c(T\mathbb{CP}^3) = (1+x_1)^2 (1+x_2)^4 \\
&= (1+2x_1)(1 + 4x_2 + 6x_2^2 + 4x_2^3) \\
&= 1 + (2x_1 + 4x_2) + (8x_1x_2 + 6x_2^2) + (12x_1x_2^2 + 4x_2^3) + 8x_1x_2^3 \\
c(T(\mathbb{CP}^2 \times \mathbb{CP}^2)) &= pr_1^* c(T\mathbb{CP}^2) \cdot pr_2^* c(T\mathbb{CP}^2) = (1+x_1)^3(1+x_2)^3 \\
&= (1 + 3x_1 + 3x_1^2)(1 + 3x_2 + 3x_2^2) \\
&= 1 + (3x_1 + 3x_2) + (3x_1^2 + 9x_1x_2 + 3x_2^2) + (9x_1^2x_2 + 9x_1x_2^2) + 9x_1^2x_2^2 \\
c(T(\mathbb{CP}^1)^{\times 4}) &= (1+x_1)^2(1+x_2)^2(1+x_3^2)(1+x_4)^2 \\
&= (1+2x_1)(1+2x_2)(1+2x_3)(1+2x_4) \\
&= 1 + 2(x_1 + x_2 + x_3 + x_4) \\
&\quad + 4(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) \\
&\quad + 8(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) + 16x_1x_2x_3x_4 \\
c(T((\mathbb{CP}^1)^2 \times \mathbb{CP}^2)) &= (1+x_1)^2(1+x_2)^2(1+x_3)^3 = (1+2x_1)(1+2x_2)(1+3x_3+3x_3^2) \\
&= 1 + (2x_1 + 2x_2 + 3x_3) + (4x_1x_2 + 6x_1x_3 + 6x_2x_3 + 3x_3^2) \\
&\quad + (6x_1x_3^2 + 6x_2x_3^2 + 12x_1x_2x_3) + 12x_1x_2x_3^2
\end{aligned}$$

Now we calculate the Chern numbers $c_1^4(TM)[M]$, $c_1c_3(TM)[M]$ and $c_1^2c_2(TM)[M]$ by evaluating them on the complex projective spaces:

$$\begin{aligned}
c_1^4(T\mathbb{CP}^4)[\mathbb{CP}^4] &= (5x)^4[\mathbb{CP}^4] = 625 \\
c_1^4[\mathbb{CP}^1 \times \mathbb{CP}^3] &= (2x_1+4x_2)^4[\mathbb{CP}^1 \times \mathbb{CP}^3] = 512x_1x_2^3[\mathbb{CP}^1 \times \mathbb{CP}^3] = 512 \\
c_1^4[\mathbb{CP}^2 \times \mathbb{CP}^2] &= 3^4(x_1+x_2)^4[\mathbb{CP}^2 \times \mathbb{CP}^2] = 486x_1^2x_2^2[\mathbb{CP}^2 \times \mathbb{CP}^2] = 486 \\
c_1^4[(\mathbb{CP}^1)^{\times 4}] &= 2^4(x_1+x_2+x_3+x_4)^4[(\mathbb{CP}^1)^{\times 4}] \\
&= 2^4 \cdot 4! \cdot x_1x_2x_3x_4[(\mathbb{CP}^1)^{\times 4}] = 384 \\
c_1^4[(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2)] &= (2x_1+2x_2+3x_3)^4[(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2)] \\
&= 432x_1x_2x_3^2[(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2)] = 432
\end{aligned}$$

gives the equation

$$c_1^4[M] = 0 = 625A + 512B + 486C + 384D + 432E,$$

evaluation of $c_1c_3(TM)[M]$

$$\begin{aligned}
c_1c_3(T\mathbb{CP}^4)[\mathbb{CP}^4] &= 5x \cdot 10x^3[\mathbb{CP}^4] = 50 \\
c_1c_3[\mathbb{CP}^1 \times \mathbb{CP}^3] &= (2x_1+4x_2)(12x_1x_2^2+4x_2^3)[\mathbb{CP}^1 \times \mathbb{CP}^3] = 56x_1x_2^3[\mathbb{CP}^1 \times \mathbb{CP}^3] = 56 \\
c_1c_3[\mathbb{CP}^2 \times \mathbb{CP}^2] &= (3x_1+3x_2)(x_1^3+9x_1^2x_2+9x_1x_2^2+x_2^3)[\mathbb{CP}^2 \times \mathbb{CP}^2] \\
&= 54x_1^2x_2^2[\mathbb{CP}^2 \times \mathbb{CP}^2] = 54 \\
c_1c_3[(\mathbb{CP}^1)^{\times 4}] &= 16(x_1+x_2+x_3+x_4)(x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4)[(\mathbb{CP}^1)^{\times 4}] \\
&= 64x_1x_2x_3x_4[(\mathbb{CP}^1)^{\times 4}] = 64 \\
c_1c_3[(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2)] &= (2x_1+2x_2+3x_3)(6x_1x_3^2+6x_2x_3^2+12x_1x_2x_3)[(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2)] \\
&= 60x_1x_2x_3^2[(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2)] = 60
\end{aligned}$$

gives the equation

$$c_1c_3[M] = 0 = 50A + 56B + 54C + 64D + 60E,$$

and evaluation of $c_1^2c_2(TM)[M]$

$$\begin{aligned}
c_1^2c_2(T\mathbb{CP}^4)[\mathbb{CP}^4] &= (5x)^2 \cdot 10x^2[\mathbb{CP}^4] = 250x^4[\mathbb{CP}^4] = 250 \\
c_1^2c_2[\mathbb{CP}^1 \times \mathbb{CP}^3] &= (2x_1+4x_2)^2(x_1^2+8x_1x_2+6x_2^2)[\mathbb{CP}^1 \times \mathbb{CP}^3] \\
&= 224x_1x_2^3[\mathbb{CP}^1 \times \mathbb{CP}^3] = 224 \\
c_1^2c_2[\mathbb{CP}^2 \times \mathbb{CP}^2] &= (3x_1+3x_2)^2(3x_1^2+9x_1x_2+3x_2^2)[\mathbb{CP}^2 \times \mathbb{CP}^2] \\
&= 216x_1^2x_2^2[\mathbb{CP}^2 \times \mathbb{CP}^2] = 216 \\
c_1^2c_2[(\mathbb{CP}^1)^{\times 4}] &= 16(x_1+x_2+x_3+x_4)^2 \times \\
&\quad (x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)[(\mathbb{CP}^1)^{\times 4}] \\
&= 192x_1x_2x_3x_4[(\mathbb{CP}^1)^{\times 4}] = 192 \\
c_1^2c_2[(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2)] &= (2x_1+2x_2+3x_3)^2 \times \\
&\quad (4x_1x_2+6x_1x_3+6x_2x_3+3x_3^2)[(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2)] \\
&= 204x_1x_2x_3^2[(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2)] = 204
\end{aligned}$$

gives the equation

$$c_1^2c_2[M] = 0 = 250A + 224B + 216C + 192D + 204E.$$

Hence we consider the system of linear equations

$$\begin{array}{rrrrrrrrrrr} c_1^4[M]= & 0= & 625\mathrm{A} & + & 512\mathrm{B} & + & 486\mathrm{C} & + & 384\mathrm{D} & + & 432\mathrm{E} \\ c_1c_3[M]= & 0= & 50\mathrm{A} & + & 56\mathrm{B} & + & 54\mathrm{C} & + & 64\mathrm{D} & + & 60\mathrm{E} \\ c_1^2c_2[M]= & 0= & 250\mathrm{A} & + & 224\mathrm{B} & + & 216\mathrm{C} & + & 192\mathrm{D} & + & 204\mathrm{E} \end{array}$$

which is integrally equivalent to the following system of homogeneous linear equations:

$$\begin{array}{rrrrrrrrrr} 0= & 25\mathrm{A} & + & 8\mathrm{B} & & & & & & \\ 0= & & + & 4\mathrm{B} & & & + & 16\mathrm{D} & + & 9\mathrm{E} \\ 0= & & & & - & 27\mathrm{C} & + & 48\mathrm{D} & + & 15\mathrm{E} \end{array}$$

The space of solutions is 2-dimensional. We know one solution $K3 \times K3$, i.e. the square of the Kummer surface, having the parameter representation

$$(A, B, C, D, E) = (0, 0, 256, 324, -576)$$

or

$$\mathcal{K}^2 = K3 \times K3 \sim_U 256\mathbb{CP}^2 \times \mathbb{CP}^2 + 324(\mathbb{CP}^1)^{\times 4} - 576(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2.$$

Another independent solution is given in parameter representation as $(A, B, C, D, E) = (8, -25, -12, -23, 52)$ or as

$$N := 8\mathbb{CP}^4 - 25\mathbb{CP}^1 \times \mathbb{CP}^3 - 12\mathbb{CP}^2 \times \mathbb{CP}^2 - 23(\mathbb{CP}^1)^{\times 4} + 52(\mathbb{CP}^1)^{\times 2} \times \mathbb{CP}^2.$$

Hence we can rationally describe bordism classes of SU-manifolds under the injection $MSU_8 \to MU_8$ via

$$M = k \cdot (K3)^2 + l \cdot N$$

with $k, l \in \mathbb{Q}$. In the next section we take the values $(k, l) = (\frac{1}{4}, 12)$ and study its K-theory class under the map $MU_* \to K_*MU$ using Miscenkos formula which gives us an Artin-Schreier class.

2. Formal group laws and Miscenkos formula

Formal group laws. In the following part we briefly recall the notions of the theory of formal group laws which we use to construct the morphism $MU_* \to K_*MU$. We restrict to commutative, one-dimensional formal group laws.

Definition 4.2. *Let R be a commutative ring with unit. A formal group law over R is a power series $F(x, y) \in R[\![x, y]\!]$ satisfying*

(1) $F(x, 0) = x = F(0, x)$
(2) $F(x, y) = F(y, x)$
(3) $F(x, F(y, z)) = F(F(x, y), z)$.

These axioms correspond to the existence of a neutral element, commutativity and associativity in the group case. Obviously we can write $F(x, y) = x + y + \sum_{i,j\geq 1} a_{ij}x^iy^j$ with $a_{ij} = a_{ji}$, and in terms of the power series it is clear that there exists an inverse, i.e. a formal power series $\iota(x) \in R[\![x]\!]$ such that $F(x, \iota(x)) = 0$. Formal group laws are naturally related to complex oriented theories in the following way: The Euler class of a tensor product of line bundles defines a formal group law

$$\widehat{G_E}(x, y) = e(L_1 \otimes L_2) \in E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \pi_*E[\![x, y]\!]$$

with $x = e(L_1)$ and $y = e(L_2)$.

Example 4.1. *The additive formal group law $\mathbb{G}_a(x, y) = x + y$ arises as an orientation of singular cohomology. The multiplicative formal group law $\mathbb{G}_m(x, y) = x + y - xy$ comes up as an orientation of complex K-theory. In the following we will encounter the universal formal group law F_u via complex cobordism (MU-theory).*

Definition 4.3. *Let F and G be formal group laws. A homomorphism $f : F \to G$ is a power series $f(x) \in R[\![x]\!]$ with constant term 0 such that $f(F(x, y)) = G(f(x), f(y))$. It is an isomorphism if it is invertible, i.e. if $f'(0)$ (the coefficient of x) is a unit in R, and a strict isomorphism if $f'(0) = 1$. A strict isomorphism from F to the additive formal group law $\mathbb{G}_a$ is called a logarithm for F, denoted $\log_F(x)$. Its inverse power series is called exponential, denoted $\exp_F(x)$.*

Example 4.2. *Over a $\mathbb{Q}$-algebra every formal group law is isomorphic to the additive formal group law. Especially the logarithm of the universal formal group law is given by*

$$\log_{MU}(x) = \sum_{n \geq 0} \frac{[\mathbb{CP}^n]}{n+1} x^{n+1}.$$

Proposition 4.1. *If x_1, x_2 are two complex orientations for $E^*(-)$, then their associated formal group laws F_1 and F_2 are isomorphic.*

In the context of formal group laws let F_{MU} denote the universal formal group law

$$F_{MU}(x, y) = x + y + \sum_{i,j \geq 1} a_{ij} x^i y^j$$

with the coefficients $a_{ij} \in L$ in the Lazard ring with degree $|a_{ij}| = 2 - 2(i + j)$. Let

$$F_K(x, y) = x + y + vxy$$

denote the multiplicative formal group law corresponding to the K-theory spectrum with v the inverse Bott element with $|v| = -2$. Now we are going to construct a morphism

$$f : MU_* \to K_*MU$$

such that the induced formal group law

$$f^* F_{MU}(x, y) := x + y + \sum_{i,j \geq 1} f(a_{ij}) x^i y^j$$

is the formal group law F_K twisted by the invertible power series

$$g(x) = \sum_{i \geq 0} b_i x^{i+1}$$

(with $b_0 = 1$) defined by

$${}^g F_K(x, y) := g(F_K(g^{-1}(x), g^{-1}(y))) = g(g^{-1}(x) + g^{-1}(y) + v g^{-1}(x) g^{-1}(y))$$

with $g^{-1}(g(x)) = x$ the inverse function.

Boardman homomorphism. The element $a_{ij} \in \pi_{2(i+j-1)}$ can be represented by a weakly almost complex manifold. To ask for the (normal) characteristic numbers of this manifold is (essentially) equivalent to asking for the image of a_{ij} under the Hurewicz homomorphism

$$\pi_* MU \to H_* MU.$$

We introduce the Boardman homomorphism, which is (slightly) more general than the Hurewicz homomorphism. Let E be a (commutative) ring spectrum, then for any (space or spectrum) Y we consider the map

$$Y \cong S^0 \wedge Y \overset{i\wedge 1}{\to} E \wedge Y.$$

Composing a map $X \to Y$ with this map induces a homomorphism

$$B : [X, Y]_* \to [X, E \wedge Y]_*$$

called the Boardman homomorphism. The Hurewicz homomorphism is recovered by setting $X = S^0$ and $E = H$ (the Eilenberg-MacLane spectrum representing singular homology).

Since $E \wedge Y$ is at least a module spectrum over the ring spectrum E, we may obtain information about $[X, E \wedge Y]_r = (E \wedge Y)^{-r}(X)$ from $E_*(X)$, for example there is a universal coefficient theorem

$$\begin{tikzcd}
{[X,Y]_*} \arrow[rr, "B"] \arrow[dr, "\alpha"'] & & {[X, E\wedge Y]_*} \arrow[dl, "p"] \\
& \mathrm{Hom}_{\pi_* E}(E_*X, E_*Y) &
\end{tikzcd}$$

where $\alpha(f) = f_* : E_*X \to E_*Y$ is the induced map in E-homology and p is defined by $(p(h))(k) = \langle h, k\rangle \in E_*Y$ using the Kronecker pairing

$$(E \wedge Y)^*(X) \otimes E_*X \to E_*Y$$

with

$$h \otimes k \mapsto \langle h, k\rangle : S \to E \wedge X \overset{1\wedge h}{\to} E \wedge E \wedge Y \overset{\mu\wedge 1}{\to} E \wedge Y.$$

Miscenkos formula. We recall that power series of the form $g(x) = x + b_1x^2 + b_2x^3 + ...$ are strict isomorphisms

$$g : F \xrightarrow{\cong} {}^gF = g(F(g^{-1}x, g^{-1}y))$$

and want to give the explicit coefficients of the inverse power series $g^{-1}(x) = \sum_{i\geq 0} c_i x^{i+1}$. We calculate the first coefficients taking everything modulo x^6 and using the identity

$$\begin{aligned}
x &\equiv g^{-1}(g(x)) = g(x) + c_1 g(x)^2 + c_2 g(x)^3 + c_3 g(x)^4 + c_4 g(x)^5 + ... \quad (\mod x^6) \\
&\equiv x + b_1x^2 + b_2x^3 + b_3x^4 + b_4x^5 \\
&\quad +c_1(x^2 + 2b_1x^3 + (2b_2 + b_1^2)x^4 + (2b_3 + 2b_1b_2)x^5) \\
&\quad +c_2(x^3 + 3b_1x^4 + (3b_2 + 3b_1^2)x^5 + c_3(x^4 + 4b_1x^5) + c_4x^5
\end{aligned}$$

Comparing coefficients gives the system of equations

$$\begin{aligned} 0 &= c_1 + b_1 \\ 0 &= c_2 + 2b_1c_1 + b_2 \\ 0 &= c_3 + 3b_1c_2 + c_1(2b_2 + b_1^2) + b_3 \\ 0 &= c_4 + 4b_1c_3 + c_2(3b_2 + 3b_1^2) + c_1(2b_3 + 2b_1b_2) + b_4 \end{aligned}$$

resulting in

$$\begin{aligned} c_1 &= -b_1 \\ c_2 &= 2b_1^2 - b_2 \\ c_3 &= -5b_1^3 + 5b_1b_2 - b_3 \\ c_4 &= 14b_1^4 - 21b_1^2b_2 + 6b_1b_3 + 3b_2^2 - b_4. \end{aligned}$$

Applying the residue theorem of complex analysis proves the following (as done in [**Ada74**, p. 65 Prop. (7.5)]):

Proposition 4.2. *Denoting the degree $2n$-part of an inhomogeneous polynomial with a lower index n we have*

$$c_n = \frac{1}{n+1}(\sum_{i\geq 0} b_i)_n^{-(n+1)} \quad \textit{and} \quad b_n = \frac{1}{n+1}(\sum_{i\geq 0} c_i)_n^{-(n+1)}.$$

Next we explicitly calculate ${}^gF_K(x,y) = g(g^{-1}x + g^{-1}y + vg^{-1}xg^{-1}y)$:

$$\begin{aligned} {}^gF_K(x,y) = \; & x + y + (v+2b_1)xy + (b_1v - 2b_1^2 + 3b_2)(x^2y + xy^2) \\ & + (2vb_2 - 2vb_1^2 + 4b_3 - 8b_1b_2 + 4b_1^3)(x^3y + xy^3) \\ & + (v^2b_1 - 3vb_1^2 + 2b_1^3 - 6b_1b_2 + 6vb_2 + 6b_3)x^2y^2 \\ & + (5vb_1^3 - 8vb_1b_2 + 25b_1^2b_2 + 3vb_3 - 10b_1^4 - 14b_1b_3 - 6b_2^2 + 5b_4) \\ & \quad \times (x^4y + xy^4) \\ & + (4vb_1^3 - 18vb_1b_2 - 4b_1^4 + 8b_1^2b_2 - 2v^2b_1^2 + 3v^2b_2 - 3b_2^2 \\ & \quad - 16b_1b_3 + 12vb_3 + 10b_4) \times (x^3y^2 + x^2y^3) \\ & + \text{higher order terms}. \end{aligned}$$

This implies:

$$\begin{aligned} a_{11} &\mapsto v + 2b_1 \\ a_{21} &\mapsto vb_1 - 2b_1^2 + 3b_2 \\ a_{31} &\mapsto 2vb_2 - 2vb_1^2 + 4b_3 - 8b_1b_2 + 4b_1^3 \\ a_{22} &\mapsto v^2b_1 - 3vb_1^2 + 2b_1^3 - 6b_1b_2 + 6vb_2 + 6b_3 \\ a_{41} &\mapsto 5vb_1^3 - 8vb_1b_2 + 25b_1^2b_2 + 3vb_3 - 10b_1^4 - 14b_1b_3 - 6b_2^2 + 5b_4 \\ a_{32} &\mapsto 4vb_1^3 - 18vb_1b_2 - 4b_1^4 + 8b_1^2b_2 - 2v^2b_1^2 + 3v^2b_2 - 3b_2^2 - 16b_1b_3 + 12vb_3 + 10b_4 \end{aligned}$$

Recall that the complex manifold $\mathbb{CP}^n$ defines an element $[\mathbb{CP}^n] \in \pi_{2n}MU$. The Hurewicz homomorphism

$$\pi_*MU \to H_*MU$$

tells us that the image of $[\mathbb{CP}^n]$ in $H_{2n}MU$ is $(n+1)c_n$ since the formula $(\sum_{i\geq 0} b_i)_n^{-(n+1)}$ gives the normal Chern numbers of $\mathbb{CP}^n$. The most important formula for us will be

$$[\mathbb{CP}^n] = (n+1)c_n = (\sum_{i\geq 0} a_{1i})_n^{-1}$$

leading to

$$\boxed{\begin{array}{l} [\mathbb{CP}^1] = -a_{11} \\ [\mathbb{CP}^2] = -a_{12} + a_{11}^2 \\ [\mathbb{CP}^3] = -a_{13} - a_{11}^3 + 2a_{11}a_{12} \\ [\mathbb{CP}^4] = -a_{14} + a_{11}^4 + a_{12}^2 + 2a_{11}a_{13} \end{array}}$$

Substituting these formulas we get

$$\begin{aligned} [N] &= -112vb_1^3 + 340vb_1b_2 + 256b_1^2b_2 - 60vb_3 - 184b_1^4 + 40b_1b_3 \\ &\quad +12b_2^2 - 40b_4 + 48v^2b_2 + 58v^2b_1^2 + 22v^3b_1 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{4}[K3^2] &= v^4 + 24v^3b_1 + 120v^2b_1^2 + 48v^2b_2 - 288vb_1^3 + 448vb_1b_2 \\ &\quad +144b_1^4 - 576b_1^2 + 576b_2^2. \end{aligned}$$

Defining

$$M := \frac{1}{4}K3^2 + 12N$$

we get

$$\begin{aligned} [M] &= v^4 + 16\cdot(18v^3b_1 + 51v^2b_1^2 + 39v^2b_2 - 102vb_1^3 + 283vb_1b_2 \\ &\quad -45vb_3 - 129b_1^4 + 30b_1b_3 + 156b_1^2b_2 + 45b_2^2 - 30b_4). \end{aligned}$$

3. Construction of an SU-manifold with $\hat{A} = 1$

To split off the spectrum T_ζ from MSU one essentially uses the existence of an Artin-Schreier class $b \in KO_0MSU$ satisfying $\psi^3 b = b + 1$. Via Miscenkos formula we have seen that such a class can be constructed with the logarithm construction if there is a Bott manifold whose associated K-theory class is congruent to v^4 modulo 16. Essentially we have to find a Bott manifold in SU bordism, i.e. an SU-manifold M with $\hat{A}([M]) = 1$ giving a periodicity element in MSU_*.

Main idea. The Hopf bundle $\sigma : S^7 \to S^4$ with fiber $S^3 \cong SU(2)$ on the one hand admits an SU structure and on the other hand generates $Im(J)_7 \cong \Omega_7^{fr} \cong \pi_7^{st} \cong \mathbb{Z}/240$. Since $Td(D(\sigma)) = 1/240$ and since $240[\sigma] = 0$ in Ω_7^{fr} implies the existence of a framed manifold R^8 with $\partial R^8 = -240\sigma$, we define

$$B := 240D(\sigma) \cup_{240\sigma} R^8$$

which serves as the desired Bott manifold, i.e. $Td(B) = \hat{A}(B) = 1$.

$Sp(1)$-principal bundles over S^4. With the identifications $Sp(1) \cong SU(2) \cong S^3$ and $Sp(2)/Sp(1) \cong S^7$ and

$$\frac{Sp(2)}{Sp(1) \times Sp(1)} \cong \mathbb{HP}^1 \cong S^4$$

we take the canonical $Sp(1)$-principal bundle over S^4

$$\begin{array}{ccc} Sp(1) \cong S^3 & \longrightarrow & S^7 \\ & & \big\downarrow \\ & & S^4 \end{array}$$

i.e. the bundle whose associated line bundle

$$E := S^7 \times_{Sp(1)} \mathbb{H}^1 \to S^4$$

satisfies $\langle c_2(E), [S^4] \rangle = 1$. We know that every G-principal bundle is given as the pullback of the universal G-principal bundle via the classifying map

$$\begin{array}{ccc} f^*EG & \longrightarrow & EG \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow{f} & BG. \end{array}$$

In other words the functor G-$Pb(-)$ is representable by BG and

$$[B, BG] \cong G\text{-}Pb(B) \text{ via } f \mapsto f^*EG.$$

In the case of $Sp(1)$-principal bundles over S^4 we get

$$[S^4, BSp(1)] = [\Sigma S^3, BSp(1)] \cong [S^3, \Omega BSp(1)] = [S^3, Sp(1)] = [S^3, S^3] \cong \mathbb{Z}.$$

The canonical $Sp(1)$-principal bundle over S^4 is associated to $1 \in \mathbb{Z}$. We see that the disk bundle $Q := D(E)$ with $\pi : Q \to S^4$ has as boundary $\partial Q = \partial D(E) = S(E)$ the original principal bundle.

Splitting of the tangent bundle TQ. In general for a smooth vector bundle $\xi : E \to M$ the total space E is again a smooth manifold. Now we are interested in the structure of the tangential bundle TE. There are two induced bundles, namely the induced tangential bundle and that of the total space:

$$\begin{array}{ccc} \xi^*TM & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \xi \\ TM & \longrightarrow & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \xi^*E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \xi \\ E & \xrightarrow{\xi} & M \end{array}$$

These already give an isomorphism

$$TE \cong \xi^*TM \oplus \xi^*E.$$

Such a splitting of a tangent bundle is geometrically called a connection. With the notation of above restricting the tangent bundle of the vector bundle to the disk bundle we get the splitting

$$TQ \cong \pi^*E \oplus \pi^*TS^4;$$

note that the second summand is stably trivial.

The Hopf bundle is an SU manifold. The Hopf bundle $\sigma : S^7 \to S^4$ with fiber $S^3 \cong SU(2)$ is not only an $SU(2)$-bundle but also an SU manifold. A manifold M has an SU structure if its stable tangent bundle TM is a complex vector bundle with a trivialization of its determinant bundle $det(TM) \cong 1_{\mathbb{C}}$.

$$\begin{array}{ccccc} D(\sigma) & \longrightarrow & \lambda_{taut} & \longrightarrow & \lambda_{taut} \\ \downarrow & & \downarrow & & \downarrow \\ S^4 & \longrightarrow & BSU & \longrightarrow & BU. \end{array}$$

From the splitting above we see the SU structure, since the 8-dimensional bundle splits into two 4-dimensional bundles and TS^4 is stably trivial and E is chosen to have vanishing c_1.

Evaluation of the Todd genus. We recall $Td = e^{c_1/2}\hat{A}$ and see that for SU manifolds the Todd-genus and the $\hat{A}$-genus coincide. From [**Hi56**] the degree 8-term of the Todd genus is given in Chern classes by:

$$T_4 = \frac{1}{720}(-c_4 + c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4).$$

For the evaluation the Chern classes c_1 and c_4 do not contribute, due to the SU structure and since Q is a homotopy 4-sphere, respectively. Next we emphasize that while for closed stably almost complex manifolds the Todd genus maps to the integers; the situation for (U, fr) manifolds is different. A (U, fr) manifold M^n is a differentiable manifold M with a given complex structure on its stable tangent bundle TM and a given compatible framing of TM restricted to the boundary ∂M. Their Chern numbers depend only on the bordism classes in $\Omega_n^{U,fr}$ and hence we have a Todd genus

$$Td : \Omega_{2n}^{U,fr} \to \mathbb{Q}.$$

Moreover there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{2n}^{U} & \longrightarrow & \Omega_{2n}^{U,fr} & \longrightarrow & \Omega_{2n-1}^{fr} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Td} & & \downarrow{\scriptstyle Td} & & \downarrow{\scriptstyle e_{\mathbb{C}}} & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

where $e_{\mathbb{C}}$ is the Adams e-invariant. This is worked out in [**CF66a**]. As done on page 95 of [**CF66a**] we can now evaluate the Todd genus

$$\begin{aligned} \langle Td(TQ), [Q, \partial Q]\rangle &= \langle \frac{1}{720} 3c_2^2(E), [Q, \partial Q]\rangle = \frac{1}{240}\langle c_2^2(E), [Q, \partial Q]\rangle \\ &= \frac{1}{240}\langle c_2(E), [S^4]\rangle = \frac{1}{240}. \end{aligned}$$

Remark on the relation to K-theory. In modern formulation the Todd genus is associated to the multiplicative formal group law and therefore to K-theory. Let $P(x)$ be a power series with 1 as constant coefficient. Its logarithm g is given by

$$g^{-1}(x) = \frac{x}{P(x)}.$$

Complex oriented cohomology theories always come with a formal group law $F(x, y)$ which can be expressed as

$$F(x, y) = g^{-1}(g(x) + g(y)).$$

For the Todd genus we have $P(x) = \frac{x}{1-e^{-x}}$ implying $y = g^{-1}(x) = 1 - e^{-x}$. This gives us $g(x) = -\ln(1 - x)$ and thus

$$\begin{aligned} F(x, y) &= 1 - \exp[-(-\ln(1 - x) - \ln(1 - y))] \\ &= 1 - \exp(\ln(1 - x) + \ln(1 - y)) \\ &= 1 - (1 - x)(1 - y) = x + y - xy, \end{aligned}$$

which is the multiplicative formal group law coming from complex K-theory.

Definition of the Bott manifold. Since $\partial Q = S^7$ is framed and $[\partial Q] \in \Omega_7^{fr} \cong \pi_7^s \cong \mathbb{Z}/240$ we have $240[\partial Q] = 0$, i.e. there exists a framed manifold R^8 with $\partial R^8 = -240\partial Q$. We define a Bott-manifold by

$$B := 240Q \cup_{240\partial Q} R^8$$

and see that indeed $\hat{A}(B) = Td(B) = 240Td(Q) + 0 = 1$.

4. Construction of an Artin-Schreier class

Having a Bott manifold with associated K-theory class congruent to v^4 modulo 16 we can use the power series of the logarithm

$$\log(1 + x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n + 1}$$

to define

$$b = -\frac{\log([M])}{\log(3^4)}.$$

Proposition 4.3. *The class b is an Artin-Schreier class.*

Proof.

$$\psi^3 b = -\frac{\log([M]/3^4)}{\log(3^4)} = -\frac{\log([M])}{\log(3^4)} + \frac{\log(3^4)}{\log(3^4)} = b + 1.$$

Here the stable Adams operation $\psi^k : K \to K$ is defined levelwise by $\frac{\Psi^k}{k^n} : K_{2n} \to K_{2n}$ with Ψ^k being the unstable Adams operation. Inverting powers of $k \in \mathbb{Z}_2^\times$ is not a problem since everything is 2-completed. $\square$

5. Construction of an E_∞ map $T_\zeta \to MSU$

The fiber sequence $X \to KO \wedge X \xrightarrow{\psi^3 - 1} KO \wedge X$ induces the unit map $\pi_0 KO \to \pi_{-1} S^0$ mapping $1 \mapsto \zeta$. Now we define T_ζ to be the homotopy pushout in the category of $K(1)$-local E_∞ ring spectra:

$$\begin{array}{ccc} TS^{-1} & \xrightarrow{T*} & T* = S^0 \\ \downarrow{\scriptstyle \zeta} & & \downarrow \\ S^0 & \longrightarrow & T_\zeta \end{array}$$

with TX the free E_∞ spectrum generated by the pointed space X. As the Hurewicz image of $\zeta \in \pi_{-1}MSU$ is zero we get a map $T_\zeta \to MSU$:

$$\begin{tikzcd}
TS^{-1} \arrow[r, "T*"] \arrow[d, "\zeta"] & T* = S^0 \arrow[d] \arrow[ddr, bend left] & \\
S^0 \arrow[r] \arrow[drr, bend right] & T_\zeta \arrow[dr, dotted] & \\
& & MSU
\end{tikzcd}$$

6. Split map - direct summand argument

To get T_ζ as a direct summand, one has to construct a split p such that the composition

$$T_\zeta \xrightarrow{i} MSU \xrightarrow{p} T_\zeta$$

is the identity. This can be done using the *Spin* splitting of Laures [**Lau03**]

$$\begin{tikzcd}
T_\zeta \arrow[d, shift left, "i"] \arrow[dr, bend left] & \\
T_\zeta \wedge \bigwedge_{i=1}^{\infty} TS^0 \arrow[u, shift left, "p"] & MSpin \arrow[l, "\simeq"']
\end{tikzcd}$$

and showing that the extended triangle commutes

$$\begin{tikzcd}
& MSU \arrow[dd, "g"] \\
T_\zeta \arrow[ur, "h"] \arrow[d, shift left, "i"] \arrow[dr, bend left] & \\
T_\zeta \wedge \bigwedge_{i=1}^{\infty} TS^0 \arrow[u, shift left, "p"] & MSpin \arrow[l, "\simeq"']
\end{tikzcd}$$

6.1. Comparison of the Artin-Schreier classes. The *SU* Artin-Schreier class constructed above is naturally also a *Spin* Artin-Schreier class. Refering to [**Lau02**] we have

Lemma 4.1. *Let b and b' be two Artin-Schreier elements of $\pi_0 KO \wedge MSpin$. Then there is an E_∞ self homotopy equivalence κ of $MSpin$ which carries b to b'.*

Proof. The short exact sequence

$$0 \to \pi_0 MSpin \to \pi_0 KO \wedge MSpin \overset{\psi^3 - 1}{\to} \pi_0 KO \wedge MSpin \to 0$$

with $(\psi^3 - 1)b = (\psi^3 - 1)b' = 1$ tells us that b and b' can only differ by a class $a \in \pi_0 MSpin$. Let κ be the E_∞ map of

$$MSpin \cong T_\zeta \wedge \bigwedge TS^0$$

which is the identity on each TS^0 and restricts to

$$\iota + a\delta : C_\zeta \to MSpin$$

on T_ζ. Then its inverse is defined in the same way with a replaced by $-a$. $\square$

With the notations T_ζ^{SU} and T_ζ^{Spin} for the E_∞ spectra we get from the different Artin-Schreier classes, we have the following diagram with E_∞ maps:

$$
\begin{tikzcd}
T_\zeta^{SU} \arrow[r] \arrow[d, "\simeq"] & MSU \arrow[l, bend left] \arrow[dd] \\
T_\zeta^{Spin} \arrow[d, shift left, "\iota"] \arrow[dr, bend left] & \\
T_\zeta \wedge \bigwedge_{i=1}^{\infty} TS^0 \arrow[u, shift left, bend left, "{(id,*,*,\ldots)}"] & MSpin \arrow[l, "\simeq"']
\end{tikzcd}
$$

CHAPTER 5

Detecting free E_∞ summands TS^0

1. Introduction to Adams operations in K-theory

1.1. Basics. In K-theory we have not only the ring structure, but also certain ring homomorphisms $\psi^k : K(X) \to K(X)$.

Theorem 5.1 (Adams). *There exist ring homomorphisms $\psi^k : K(X) \to K(X)$, defined for all compact Hausdorff spaces X and all integers $k \geq 0$, satisfying:*

(1) $\psi^k f^* = f^* \psi^k$ *for all maps* $f : X \to Y$ *(naturality)*
(2) $\psi^k(L) = L^k$ *if L is a line bundle*
(3) $\psi^k \circ \psi^l = \psi^{kl}$
(4) $\psi^p(\alpha) \equiv \alpha^p$ *mod p for p prime*

At first we consider the special case when E is a sum of line bundles L_i

$$\psi^k(L_1 \oplus ... \oplus L_n) = L_1^k + ... + L_n^k.$$

Next we are looking for a general definition of $\psi^k(E)$ which specializes to the above. We use the exterior powers $\lambda^i(E)$ to define

$$\lambda_t(E) = \sum_i \lambda^i(E) t^i \in K(X)[\![t]\!].$$

By naturality of the exterior power construction we have

$$\begin{aligned} \lambda_t(E_1 \oplus E_2) &= \sum_i \lambda^i(E_1 \oplus E_2) t^i = \sum_i \bigoplus_{k=0}^{i} (\lambda^k(E_1) \otimes \lambda^{i-k}(E_2)) t^i \\ &= (\sum_i \lambda^i(E_1) t^i) \otimes (\sum_k \lambda^k(E_2) t^k) = \lambda_t(E_1) \otimes \lambda_t(E_2). \end{aligned}$$

In the case $E = L_1 \oplus ... \oplus L_n$ we get

$$\lambda_t(E) = \prod_i \lambda_t(L_i) = \prod_i (1 + L_i t) = 1 + \sigma_1 + ... + \sigma_n;$$

thus comparing the coefficients, $\lambda^j(E) = \sigma_j(L_1, ..., L_n)$ is the j^{th} elementary symmetric polynomial. Hence we take the Newton polynomials s_k and define

$$\psi^k(E) = s_k(\lambda^1(E), ..., \lambda^k(E)),$$

which specializes to the formula above when E is a sum of line bundles. Finally we prove that the stated properties are fulfilled.

Proof. The naturality

$$f^*(\psi^k(E)) = \psi^k(f^*(E))$$

follows from $f^*(\lambda^i(E)) = \lambda^i(f^*(E))$. The additivity $\psi^k(E_1 \oplus E_2) = \psi^k(E_1) + \psi^k(E_2)$ is done via the naturality and the splitting principle. Since $p : F(E) \to X$ induces an injection $p^* : K^*(X) \to K^*(F(E))$ we get

$$\begin{aligned} p^*\psi^k(E_1 \oplus E_2) &= \psi^k p^*(E_1 \oplus E_2) = \psi^k(L_1 \oplus ... \oplus L_m \oplus L'_1 \oplus ... \oplus L'_n) \\ &= \psi^k(p^*E_1 \oplus p^*E_2) = p^*\psi^k E_1 \oplus p^*\psi^k E_2. \end{aligned}$$

Since ψ^k is additive on vector bundles, we get via the Grothendieck construction an additive operation on $K(X)$ defined by

$$\psi^k(E_1 - E_2) = \psi^k(E_1) - \psi^k(E_2).$$

Next we prove multiplicativity: If E is the sum of line bundles L_i and E' is the sum of line bundles L'_j then $E \otimes E'$ is the sum of line bundles $L_i \otimes L'_j$ implying

$$\begin{aligned} \psi^k(E \otimes E') &= \sum_{i,j} \psi^k(L_i \otimes L_j) = \sum_{i,j} (L_i \otimes L'_j)^k \\ &= \sum_{i,j} L_i^k \otimes L_j'^k = \sum_i L_i^k \sum_j L_j'^k = \psi^k(E)\psi^k(E'). \end{aligned}$$

Thus ψ^k is multiplicative on vector bundles and it follows formally that it is multiplicative on elements of $K(X)$. For $\psi^k \circ \psi^l = \psi^{kl}$ the splitting principle and additivity reduce to the case of line bundles where

$$\psi^k(\psi^l(L)) = L^{kl} = \psi^{kl}(L).$$

Likewise for $\psi^p(\alpha) \equiv \alpha^p \bmod p$, since for $E = L_1 + ... + L_n$ we have modulo p

$$\psi^p(E) = L_1^p + ... + L_n^p \equiv (L_1 + ... + L_n)^p = E^p.$$

$\square$

1.2. Adams operations on the K-theory spectrum. One recalls that Adams operations $\psi^k : K \to K$ are operations on the ring spectrum K itself and that we have naturality, i.e. for every map $f : X \to Y$ the diagram

$$\begin{tikzcd} K \wedge X \arrow[r, "\psi^k \wedge X"] \arrow[d, "K \wedge f"'] & K \wedge X \arrow[d, "K \wedge f"] \\ K \wedge Y \arrow[r, "\psi^k \wedge Y"] & K \wedge Y \end{tikzcd}$$

commutes. Also Adams operations behave well with respect to the complexification map from KO-theory to K-theory, i.e. the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & KO \wedge X \arrow[r, "\psi^k \wedge 1"] \arrow[d, "c \wedge 1"] & KO \wedge X \arrow[d, "c \wedge 1"] \\ & K \wedge X \arrow[r, "\psi^k \wedge 1"] & K \wedge X \end{tikzcd}$$

commutes. Having $u \in K_*X$ we get $\psi^k(u)$ by

$$\begin{tikzcd} S \arrow[r, "u"] \arrow[rr, bend right, "\psi^k(u)"'] & K \wedge X \arrow[r, "\psi^k \wedge X"] & K \wedge X. \end{tikzcd}$$

1.3. The Kronecker pairing.

Lemma 5.1. *We have* $\langle \psi^k a, \psi^k b\rangle = \psi^k\langle a, b\rangle$ *and* $\psi^{k^{-1}}$ *is adjoint to* ψ^k *via the Kronecker pairing* $\langle , \rangle : K^*X \otimes K_*X \to K_*$.

PROOF. With f representing $b \in K_*X$ and g representing $a \in K^*X$ we have

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[dr, "\psi^k(f)"'] & K \wedge X \arrow[r, "1\wedge g"] \arrow[d, "\psi^k\wedge 1"] & K \wedge K \arrow[r, "\mu"] \arrow[d, "\psi^k\wedge\psi^k"] & K \arrow[d, "\psi^k"] \\
 & K \wedge X \arrow[r, "1\wedge\psi^k(g)"] & K \wedge K \arrow[r, "\mu"] & K
\end{tikzcd}$$

The line above gives $\langle a, b\rangle$, and the lower line $\langle \psi^k a, \psi^k b\rangle$, giving the first statement. For the second one we know that ψ^k is a ring map, and with $b' = \psi^{k^{-1}}b$ we get

$$\langle \psi^k a, b\rangle = \langle \psi^k a, \psi^k b'\rangle = \psi^k\langle a, b'\rangle = \langle a, b'\rangle = \langle a, \psi^{k^{-1}} b\rangle.$$

□

2. Calculating operations on $K_*\mathbb{CP}^\infty$

2.1. Adams operations on $K_*\mathbb{CP}^\infty$. As for any complex oriented theory, we have $K^*\mathbb{CP}^\infty \cong \pi_*K[\![x]\!] \cong \mathbb{Z}_p[\![x]\!]$ (the last isomorphism is due to the $K(1)$-local point of view). With β_i being dual to x^i the statement in homology is $K_*\mathbb{CP}^\infty \cong \mathbb{Z}_p\langle\beta_i\rangle$. Next we want to apply the pairing

$$K^*X \otimes K_*X \to \pi_*K$$

and show its compatibility with Adams operations. Given $a \in K_*X$ (i.e. $S \to K \wedge X$) and $x \in K^*X$ (i.e. $X \to K$) we have the commuting diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[dr, dotted, "\psi^k_* a"'] & K \wedge X \arrow[d, "\psi^k\wedge 1"] & & \\
 & K \wedge X \arrow[r, "1\wedge x"] & K \wedge K \arrow[r, "\mu"] & K.
\end{tikzcd}$$

Since ψ^k is a ring homomorphism and $\psi^k(L) = L^k$ for line bundles L, we have for the generator $x = 1 - L$ (here we take the tautological line bundle over $\mathbb{CP}^\infty$

$$\psi^k(x^n) = (1-(1-x)^k)^n.$$

The Kronecker pairing gives us $\langle \psi^k x, a\rangle = \langle x, \psi^{k^{-1}} a\rangle$, the duality was $\langle x^k, \beta_k\rangle = 1$, hence basis expansion gives us

$$\psi^{k^{-1}}\beta_n = \sum_{j\geq 0}\langle x^j, \psi^{k^{-1}}\beta_n\rangle\beta_j$$

with $\langle x^j, \psi^{k^{-1}}\beta_n\rangle = \langle \psi^k x^j, \beta_n\rangle = \langle (1-(1-x)^k)^j, \beta_n\rangle$. We calculate

$$\begin{aligned}
(1-(1-x)^3)^j &= x^j(3+x(x-3))^j = x^j\sum_{s=0}^{j}\binom{j}{s}3^{j-s}x^s(x-3)^s \\
&= x^j\sum_{s=0}^{j}\binom{j}{s}3^{j-s}x^s\sum_{t=0}^{s}\binom{s}{t}x^t(-3)^{s-t}
\end{aligned}$$

Hence we have

$$\langle \psi^3 x^j, \beta_i \rangle = (-1)^{i-j} \sum_{s+t=i-j} \binom{j}{s}\binom{s}{t} 3^{j-t}.$$

Explicitly we have

$$\begin{aligned}
(3x-3x^2+x^3)^2 &= 9x^2-18x^3+15x^4-6x^5+x^6\\
(3x-3x^2+x^3)^3 &= 27x^3-81x^4+108x^5-81x^6+36x^7-9x^8+x^9\\
(3x-3x^2+x^3)^4 &= 81x^4-324x^5+594x^6-648x^7+459x^8-216x^9+66x^{10}\\
&\quad -12x^{11}+x^{12}\\
(3x-3x^2+x^3)^5 &= 243x^5-1215x^6+2835x^7-4050x^8+3915x^9-2673x^{10}\\
&\quad +1305x^{11}-450x^{12}+105x^{13}-15x^{14}+x^{15}\\
(3x-3x^2+x^3)^6 &= 729x^6-4374x^7+12393x^8-21870x^9+26730x^{10}-23814x^{11}\\
&\quad +15849x^{12}-7938x^{13}+2970x^{14}-810x^{15}+153x^{16}-18x^{17}+x^{18}\\
(3x-3x^2+x^3)^7 &= 2187x^7-15309x^8+51030x^9-107163x^{10}+158193x^{11}\\
&\quad -173502x^{12}+145719x^{13}-95175x^{14}+48573x^{15}-19278x^{16}\\
&\quad +5859x^{17}-1323x^{18}+210x^{19}-21x^{20}+x^{21}\\
(3x-3x^2+x^3)^8 &= 6561x^8-52488x^9+201204x^{10}-489888x^{11}+847098x^{12}\\
&\quad -1102248x^{13}+1115856x^{14}-896184x^{15}+576963x^{16}-298728x^{17}\\
&\quad +123984x^{18}-40824x^{19}+10458x^{20}-2016x^{21}\\
&\quad +276x^{22}-24x^{23}+x^{24}\\
(3x-3x^2+x^3)^9 &= 19683x^9-177147x^{10}+767637x^{11}-2125764x^{12}+4212162x^{13}\\
&\quad -6337926x^{14}+7501410x^{15}-7138368x^{16}+5535297x^{17}\\
&\quad -3523257x^{18}+1845099x^{19}-793152x^{20}+277830x^{21}-78246x^{22}\\
&\quad +17334x^{23}-2916x^{24}+351x^{25}-27x^{26}+x^{27}\\
(3x-3x^2+x^3)^{10} &= 59049x^{10}-590490x^{11}+2854035x^{12}-8857350x^{13}+19781415x^{14}\\
&\quad -33776028x^{15}+45730170x^{16}-50257260x^{17}+45522405x^{18}\\
&\quad -34314030x^{19}+21640365x^{20}-11438010x^{21}+5058045x^{22}\\
&\quad -1861380x^{23}+564570x^{24}-138996x^{25}+27135x^{26}\\
&\quad -4050x^{27}+435x^{28}-30x^{29}+x^{30}.
\end{aligned}$$

Ordering the terms we get

$$\begin{aligned}
\psi^{3^{-1}}\beta_1 &= 3\beta_1\\
\psi^{3^{-1}}\beta_2 &= -3\beta_1+9\beta_2\\
\psi^{3^{-1}}\beta_3 &= \beta_1-18\beta_2+27\beta_3\\
\psi^{3^{-1}}\beta_4 &= 15\beta_2-81\beta_3+81\beta_4\\
\psi^{3^{-1}}\beta_5 &= -6\beta_2+108\beta_3-324\beta_4+243\beta_5\\
\psi^{3^{-1}}\beta_6 &= \beta_2-81\beta_3+594\beta_4-1215\beta_5+729\beta_6\\
\psi^{3^{-1}}\beta_7 &= 36\beta_3-648\beta_4+2835\beta_5-4374\beta_6+2187\beta_7
\end{aligned}$$

$$\begin{aligned}
\psi^{3^{-1}}\beta_8 &= -9\beta_3 + 459\beta_4 - 4050\beta_5 + 12393\beta_6 - 15309\beta_7 + 6561\beta_8 \\
\psi^{3^{-1}}\beta_9 &= \beta_3 - 216\beta_4 + 3915\beta_5 - 21870\beta_6 + 51030\beta_7 - 52488\beta_8 + 19683\beta_9 \\
\psi^{3^{-1}}\beta_{10} &= 66\beta_4 - 2673\beta_5 + 26730\beta_6 - 107163\beta_7 + 201204\beta_8 - 177147\beta_9 + 59049\beta_{10}.
\end{aligned}$$

2.2. Adams operations on $K_*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$. Analogously we have

$$K^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \pi_* K[\![x,y]\!] \cong \mathbb{Z}_p[\![x,y]\!]$$

and with $\beta_i \otimes \beta_j$ being dual to $x^i y^j$, we have

$$K_*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \mathbb{Z}_p\langle \beta_i \otimes \beta_j \rangle.$$

Now the pairing above gives us

$$\begin{aligned}
\langle x^i y^j, \psi^{k^{-1}}(\beta_m \otimes \beta_n)\rangle &= \langle \psi^k x^i \psi^k y^j, \beta_m \otimes \beta_n\rangle \\
&= \langle (1-(1-x)^k)^i (1-(1-y)^k)^j, \beta_m \otimes \beta_n\rangle
\end{aligned}$$

and by basis expansion we have

$$\psi^{k^{-1}}(\beta_i \otimes \beta_j) = \sum_{m,n} \langle x^m y^n, \psi^{k^{-1}}(\beta_i \otimes \beta_j)\rangle \beta_m \otimes \beta_n.$$

With the map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{f} BSU$ classifying the virtual SU bundle $(1-L_1)(1-L_2)$, the module generators $\beta_i \otimes \beta_j$ are mapped by f_* to algebra generators in K_*BSU. By naturality of the Adams operations this allows us to calculate the Adams operations on K_*BSU. We can calculate the pairing factor by factor:

$$\begin{aligned}
\psi^{3^{-1}}\beta_i \otimes \beta_j &= \sum_{m,n} \langle \psi^3(x^m y^n), \beta_i \otimes \beta_j\rangle \beta_m \otimes \beta_n \\
&= \sum_{m,n} \langle \psi^3 x^m, \beta_i\rangle \langle \psi^3 y^n, \beta_j\rangle \beta_m \otimes \beta_n \\
&= \sum_m \langle \psi^3 x^m, \beta_i\rangle \beta_m \bigotimes \sum_n \langle \psi^3 y^n, \beta_j\rangle \beta_n \\
&= (\psi^{3^{-1}}\beta_i) \otimes (\psi^{3^{-1}}\beta_j)
\end{aligned}$$

The following table contains the mod 2 coefficients a_k of $\psi^{3^{-1}}\beta_i = \sum a_k \beta_k$.

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
$\psi^{3^{-1}}\beta_1 =$	1									
$\psi^{3^{-1}}\beta_2 =$	1	1								
$\psi^{3^{-1}}\beta_3 =$	1		1							
$\psi^{3^{-1}}\beta_4 =$		1	1	1						
$\psi^{3^{-1}}\beta_5 =$					1					
$\psi^{3^{-1}}\beta_6 =$		1	1		1	1				
$\psi^{3^{-1}}\beta_7 =$					1		1			
$\psi^{3^{-1}}\beta_8 =$			1	1		1	1	1		
$\psi^{3^{-1}}\beta_9 =$			1		1				1	
$\psi^{3^{-1}}\beta_{10} =$					1		1		1	1

2.3. Integral Adams operations on K_*BSU and 2-structures. First we recall from [**Lau02**] the structure of K_*BSU: Let L be the tautological line bundle over $\mathbb{CP}^\infty$ and $\beta_i \in K_{2i}\mathbb{CP}^\infty$ be dual to $c_1(L)^i$. As above let f be the map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to BSU$ that classifies the bundle $(1-L_1)(1-L_2)$ with L_i the tautological line bundle over the i^{th} factor. Now choose for each natural number k and $1 \leq i \leq k-1$ integers n_k^i such that

$$\sum_{i=1}^{k-1} n_k^i \binom{k}{i} = gcd\{\binom{k}{1}, .., \binom{k}{k-1}\}.$$

Defining elements $d_k = \sum_{i=1}^{k-1} n_k^i f_*(\beta_i \otimes \beta_{k-i}) \in K_{2k}BSU$ we get

$$K_*BSU \cong \mathbb{Z}_2[d_2, d_3, ...].$$

With the notation $a_{ij} = f_*(\beta_i \otimes \beta_j) \in \pi_{2(i+j)}K \wedge BSU$ and the fact that

$$K^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \mathbb{Z}_2[\![x, y]\!]$$

with $\beta_i \otimes \beta_j \in K_*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$ being dual to $x^i y^j$, we calculate the Adams operations by applying basis expansion:

$$\psi^k a_{ij} = \sum_{m,n} \langle x^m y^n, \psi^k a_{ij} \rangle a_{mn}.$$

Here again the Kronecker pairing gives

$$\begin{aligned} \langle x^m y^n, \psi^k f_*(\beta_i \otimes \beta_j) \rangle &= \langle \psi^{k^{-1}} f^*(x^m y^n), \beta_i \otimes \beta_j \rangle \\ &= \langle \psi^{k^{-1}}(x^m y^n), \beta_i \otimes \beta_j \rangle \\ &= \langle \psi^{k^{-1}} x^m, \beta_i \rangle \cdot \langle \psi^{k^{-1}} y^n, \beta_j \rangle. \end{aligned}$$

It is easily seen that

$$gcd\{\binom{k}{1}, ..., \binom{k}{k-1}\} = \begin{cases} p & \text{for } k = p^s \\ 1 & \text{else.} \end{cases}$$

The generators a_{ij} defined above satisfy certain relations which we want to describe in the following.

Definition 5.1. *The binomial coefficients associated to the formal group law F*

$$\binom{k}{i, j}_F \in \pi_{2(i+j-k)}E$$

are defined by the equation

$$(x +_F y)^k = \sum_{i,j} \binom{k}{i, j}_F x^i y^j.$$

Example 5.1 (K-theory and the multiplicative formal group law)**.** *For $E = K$ and $F = \hat{\mathbb{G}}_m$ with $\hat{\mathbb{G}}_m(x, y) = x + y - v^{-1}xy$ we have*

$$(x +_{\hat{\mathbb{G}}_m} y)^k = (x + y - v^{-1}xy)^k = \sum_{s=0}^{k} \sum_{t=0}^{s} \binom{k}{s}\binom{s}{t} (-v)^{s-k} x^{k-s+t} y^{k-t}$$

and hence

$$\binom{k}{i, j}_{\hat{\mathbb{G}}_m} = \binom{k}{2k-i-j}\binom{2k-i-j}{k-j} (-v)^{k-i-j}.$$

Lemma 5.2 (Laures)**.** *The following relations hold for i, j, k:*

$$\begin{aligned} a_{0,0} = 1 \quad &; \quad a_{0i} = a_{i0} = 0 \text{ for all } i \neq 0 \\ a_{ij} &= a_{ji} \\ \sum_{l,s,t} \binom{l}{s,t}_{\hat{G}_m} a_{j-s,k-t} a_{il} &= \sum_{l,s,t} \binom{l}{s,t}_{\hat{G}_m} a_{lk} a_{i-s,j-t}. \end{aligned}$$

For our calculations we choose the following non-vanishing coefficients n_k^i for our basis elements d_k:

k \ i	1	2	3	4	5	6	7	8	9	10
2	1									
3	1									
4	-1	1								
5	1									
6	1	1	-1							
7	1									
8	9			-1						
9	-9		1							
10	1	11			-2					

$$\begin{aligned} \psi^{3^{-1}} d_2 &= \psi^{3^{-1}} f_*(\beta_1 \otimes \beta_1) = f_*(\psi^{3^{-1}}(\beta_1 \otimes \beta_1)) \\ &= f_*(3\beta_1 \otimes 3\beta_1) = 9 f_*(\beta_1 \otimes \beta_1) = 9 d_2 \\ \psi^{3^{-1}} d_3 &= f_*(\psi^{3^{-1}} \beta_1 \otimes \beta_2) = f_*(3\beta_1 \otimes (-3\beta_1 + 9\beta_2)) \\ &= 27 d_3 - 9 d_2 \\ \psi^{3^{-1}} d_4 &= \psi^{3^{-1}}(-f_*(\beta_1 \otimes \beta_3) + f_*(\beta_2 \otimes \beta_2)) \\ &= -f_*(3\beta_1 \otimes (\beta_1 - 18\beta_2 + 27\beta_3)) + f_*((-3\beta_1 + 9\beta_2) \otimes (-3\beta_1 + 9\beta_2)) \\ &= -9a_{11} + 54a_{12} - 81a_{13} + 9a_{11} - 54a_{12} + 81a_{22} = 81(-a_{13} + a_{22}) = 81 d_4. \end{aligned}$$

To calculate the Adams operation on the higher d_k, one has to invest the 2-structure condition from above. An equivalent way to handle this is to see $K \wedge BSU_+$ as a complex oriented ring theory with complex orientation $x_{K\wedge BSU_*} = (1 \wedge \eta)_* x_K$. The classifying map

$$(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)_* \xrightarrow{f_+} BSU_+ \xrightarrow{\eta\wedge 1} K \wedge BSU_+$$

can be regarded as a power series

$$f(x, y) = 1 + \sum_{i,j\geq 1} b_{ij} x^i y^j \in (K \wedge BSU)^0(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$$

for some $b_{ij} \in K_{2(i+j)}BSU$. Indeed we have $b_{ij} = a_{ij}$ (compare [**Lau02**])

$$\begin{aligned} b_{ij} &= \sum_{k,l\geq 1} b_{ij} (1 \wedge \eta)_* \langle \beta_i \otimes \beta_j, x^k y^l \rangle \\ &= \langle (1 \wedge \eta)_* \beta_i \otimes (1 \wedge \eta)_* \beta_j, 1 + \sum_{k,l\geq 1} b_{kl} x^k y^l \rangle \\ &= \langle (1 \wedge \eta)_* (\beta_i \otimes \beta_j), f^*(\eta \wedge 1) \rangle \\ &= (\mu f (1 \wedge \eta))_* (\beta_i \otimes \beta_j) = a_{ij}. \end{aligned}$$

Lemma 5.3. *For the power series $f(x,y)$ above, the following is straightforward to check:*

$$\begin{aligned} f(x,0) &= f(0,y) = 1 \\ f(x,y) &= f(y,x) \\ f(x,y)f(x +_{\hat{G}_m} y) &= f(x +_{\hat{G}_m} y, z) f(y,z). \end{aligned}$$

In the sense of [**AHS01**] *such an f is called a 2-structure.*

Comparison of the coefficients of x^2yz gives the relation

$$a_{21} + 2a_{22} = a_{11}^2 + a_{31}.$$

With this we calculate

$$\begin{aligned} \psi^{3^{-1}} d_5 &= \psi^{3^{-1}} f_*(\beta_1 \otimes \beta_4) = f_*(3\beta_1 \otimes (15\beta_2 - 81\beta_3 + 81\beta_4)) \\ &= 243 d_5 - 243 a_{13} + 45 a_{12} \\ &= 243 d_5 - 243 a_{13} + 45 a_{12} + 243(2a_{22} + a_{21} - a_{11}^2 - a_{31}) \\ &= 243 d_5 + 486(-a_{13} + a_{22}) + 288 a_{12} - 243 a_{11}^2 \\ &= 243 d_5 + 486 d_4 + 288 d_3 - 243 d_2^2. \end{aligned}$$

To calculate further operations we need some additional relations:

coefficient of	relation
x^2yz	$2a_{22} - a_{31} + a_{21} + a_{11}^2$
x^3yz	$2a_{14} + a_{11}a_{12} - a_{13} - a_{23}$
x^2y^2z	$6a_{14} - 6a_{13} + 2a_{22} - a_{11}^2 - a_{11}$
x^3yz^2	$3a_{33} - 2a_{23} + a_{11}a_{13} - a_{11}a_{22} - a_{12}^2$
x^4yz	$5a_{15} - 2a_{24} - 3a_{14} + 2a_{11}a_{13} + a_{12}^2$

With this we compute:

$$\begin{aligned} \psi^{3^{-1}} d_6 &= \psi^{3^{-1}}(f_*(\beta_1 \otimes \beta_5) + f_*(\beta_2 \otimes \beta_4) - f_*(\beta_3 \otimes \beta_3)) \\ &= f_*(3\beta_1 \otimes (-6\beta_2 + 108\beta_3 - 324\beta_4 + 243\beta_5)) \\ &\quad + f_*((-3\beta_1 + 9\beta_2) \otimes (15\beta_2 - 81\beta_3 + 81\beta_4)) \\ &\quad - f_*((\beta_1 - 18\beta_2 + 27\beta_3) \otimes (\beta_1 - 18\beta_2 + 27\beta_3)) \\ &= 729 d_6 - 1215 a_{14} + 243 a_{23} + 513 a_{13} - 189 a_{22} - 27 a_{12} - a_{11} \\ &\quad [+243(2a_{14} - a_{23} + a_{11}a_{12} - a_{13})] \\ &= 729 d_6 - 729 d_5 + 270 a_{13} - 189 a_{22} + 243 a_{11}a_{12} - 27 a_{12} - a_{11} \\ &\quad [-81(-a_{13} + 2a_{22} + a_{12} + a_{11}^2)] \\ &= 729 d_6 - 729 d_5 - 351 d_4 + 243 d_2 d_3 - 108 d_3 - 81 d_2^2 - d_2. \end{aligned}$$

Analogously we can go on

$$\psi^{3^{-1}} d_7 = \psi^{3^{-1}} f_*(\beta_1 \otimes \beta_6) = 3^7 d_7 + 3a_{12} - 243 a_{13} + 1782 a_{14} - 3645 a_{15}$$

and reduce the term above to polynomials in the d_k.

3. Bott's formula and cannibalistic classes

Due to Bott [**Bot69**], one can calculate Adams operations on the Thom space by calculating them on the base space, multiplying with the cannibalistic class $\theta_k(E)$ and applying the Thom isomorphism. Here the Thom space is constructed with respect to the bundle E. Let

$$i_! : K(X) \to \tilde{K}(X^E)$$

denote the Thom isomorphism; then we have

$$\psi^k(i_! x) = i_! \theta_k(E)\psi^k(x)$$

for $x \in K(X)$.

3.1. Stable cannibalistic classes in K. In order to calculate Adams operations on K_*MSU using the formula above, one has to calculate cannibalistic classes. These classes are introduced by Bott in [**Bot69**] and are defined for complex vector bundles over compact spaces X and are characterized by the properties

- $\theta^k(L) = 1 + L^* + ... + (L^*)^{k-1}$ for all line bundles L
- $\theta^k(\xi + \xi') = \theta^k(\xi)\theta^k(\xi')$ for all complex bundles ξ, ξ'.

In particular, this implies that for the trivial bundle of rank n (simply denoted by n) we have $\theta^k(\xi+n) = k^n\theta^k(\xi)$. To define *stable operations* $\hat{\theta}^k$ we assume k to be an odd number and set

$$\hat{\theta}^k(\xi) := \frac{\theta^k(\xi)}{k^{dim_{\mathbb{C}}\xi}} \in K(X).$$

Next we calculate the cannibalistic classes for the universal SU-bundle

$$\begin{array}{ccc} (1-L_1)(1-L_2) & & \\ \downarrow & & \\ \mathbb{CP}^\infty \times \mathbb{CP}^\infty & \longrightarrow & BSU. \end{array}$$

Notice that it is indeed an SU bundle because the first Chern class vanishes:

$$\begin{aligned} c(1 + L_1L_2 - L_1 - L_2) &= \frac{1 \cdot c(L_1L_2)}{c(L_1)c(L_2)} = \frac{1 + x_1 + x_2}{(1 + x_1)(1 + x_2)} \\ &= (1 + x_1 + x_2)(1 - x_1 + x_1^2...)(1 - x_2 + x_2^2...) \\ &= 1 + (x_1 + x_2 - x_1 - x_2) + ... \end{aligned}$$

Since $\theta^k(1) = k$ and $\theta^k(L) = 1 + L^* + ... + (L^*)^{k-1} = \frac{1-(L^*)^k}{1-L^*}$ (formally) we have

$$\begin{aligned} \theta^k((1 - L_1)(1 - L_2)) &= \theta^k(1)\theta^k(L_1L_2)\theta^k(-L_1)\theta^k(-L_2) \\ &= k\frac{(1 - (L_1^*L_2^*)^k)(1 - L_1^*)(1 - L_2^*)}{(1 - L_1^*L_2^*)(1 - (L_1^*)^k)(1 - (L_2^*)^k)}. \end{aligned}$$

Remark 5.1. *Choosing $x = 1 - L_1$ and $y = 1 - L_2$ as the generators of*

$$K_*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \mathbb{Z}_2[\![x, y]\!]$$

we can change to another orientation $x' = 1 - \frac{1}{1-x} = -\sum_{k\geq 1} x^k$. Hence we get $x' = 1 - L_1^*$ and $y' = 1 - L_2^*$, respectively. We compute

$$\hat{\theta}^k((1 - L_1)(1 - L_2)) = k\frac{q_k(x' + y' - x'y')}{q_k(x')q_k(y')}$$

where $q_k(x') = \frac{1-(1-x')^k}{x'}$.

We notice that $1 - L = -L \otimes (1 - L^*)$ and with the notations $x = 1 - L_1$ and $y = 1 - L_2$ we get

$$L_1^* = (1-x)^{-1} \text{ and } L_2^* = (1-y)^{-1}.$$

Hence for $k = 3$ we can write

$$\begin{aligned}\theta^3((1-L_1)(1-L_2)) &= 3\frac{1+(1-x)^{-1}(1-y)^{-1}+(1-x)^{-2}(1-y)^{-2}}{(1+(1-x)^{-1}+(1-x)^{-2})(1+(1-y)^{-1}+(1-y)^{-2})} \\ &= 3\frac{(1-x)(1-y)+1+(1-x)^{-1}(1-y)^{-1}}{((1-x)+1+(1-x)^{-1})((1-y)+1+(1-y)^{-1})}\end{aligned}$$

and see that the cannibalistic class is invariant under $\psi^{-1} : L \mapsto L^*$.

3.2. Calculating $\theta^3((1-L_1)(1-L_2))$.

Lemma 5.4. *For the cannibalistic class θ^3 we have the description*

$$\theta^3((1-L_1)(1-L_2)) = 3\frac{1+(1-x)(1-y)+(1-x)^2(1-y)^2}{(3-3x+x^2)(3-3y+y^2)}$$

and the coefficients of the power expansion

$$\frac{1}{3-3x+x^2} = \sum_{k\geq 0} a_k x^k$$

satisfy the recurrence relation $a_0 = a_1 = \frac{1}{3}$ and $a_{n+2} = a_{n+1} - \frac{1}{3}a_n$, or more explicitly,

$$\begin{aligned} a_{6n} = a_{6n+1} &= (-1)^n\, 3^{-(3n+1)} \\ a_{6n+2} &= (-1)^n\, 2\cdot 3^{-(3n+2)} \\ a_{6n+3} &= (-1)^n\, 3^{-(3n+2)} \\ a_{6n+4} &= (-1)^n\, 3^{-(3n+3)} \\ a_{6n+5} &= 0. \end{aligned}$$

PROOF. Let $f(x) = \sum_{k\geq 0} a_k x^k$ denote the generating function of the recurrence relation $a_0 = a_1 = \frac{1}{3}$ and $a_{k+2} = a_{k+1} - \frac{1}{3}a_k$. Then we have

$$\begin{aligned} f(x) &= \sum_{k\geq 0} a_k x^k = a_0 + a_1 x + \sum_{k\geq 2}(a_{k-1} - \frac{1}{3}a_{k-2})x^k \\ &= a_0 + a_1 x + x(f(x) - a_0) - \frac{1}{3}x^2 f(x) \\ &= a_0 + f(x)(x - \frac{1}{3}x^2), \end{aligned}$$

hence

$$f(x) = \frac{1}{3-3x+x^2}.$$

□

Corollary 5.1. *The coefficients c_{mn} of the power expansion of the cannibalistic class*

$$\theta^3((1-L_1)(1-L_2)) = 3\frac{1+(1-x)(1-y)+(1-x)^2(1-y)^2}{(3-3x+x^2)(3-3y+y^2)} = \sum_{m,n\geq 0} c_{mn} x^m y^n$$

are given by

$$\begin{aligned} c_{mn} &= 9a_m a_n - 9a_{m-1}a_n - 9a_m a_{n-1} + 3a_{m-2}a_n + 15a_{m-1}a_{n-1} + 3a_m a_{n-2} \\ &\quad -6a_{m-2}a_{n-1} - 6a_{m-1}a_{n-2} + 3a_{m-2}a_{n-2}. \end{aligned}$$

The coefficents with negative indices are understood to be zero.

Corollary 5.2. *The coefficients c_{mn} are symmetric, i.e. $c_{mn} = c_{nm}$ and we have*

$$c_{0n} = \begin{cases} 1 \text{ for } n = 0 \\ 0 \text{ for } n \geq 1 \end{cases} \qquad \text{and} \qquad c_{1n} = 3a_{n+1}.$$

If both indices are ≥ 2 we have with $0 \leq i, k \leq 5$:

$$c_{6m+i,6n+k} = (-1)^{m+n} \cdot 3^{-(3m+3n+\lfloor \frac{i+k}{2} \rfloor)} b_{ik}$$

with

$$b_{ik} = \begin{cases} 2 \text{ if } i-k = 0 \\ 1 \text{ if } i-k = \pm 1, \pm 2 \\ 0 \text{ if } i-k = \pm 3 \\ -1 \text{ if } i-k = \pm 4, \pm 5. \end{cases}$$

Corollary 5.3. *We can also write this as*

$$c_{mn} = 3^{-\lfloor \frac{m+n}{2} \rfloor} \begin{cases} 2 & \text{if } m-n \equiv 0 & \mod 6 \\ 1 & \text{if } m-n \equiv \pm 1, \pm 2 & \mod 12 \\ 0 & \text{if } m-n \equiv 3 & \mod 6 \\ -1 & \text{if } m-n \equiv \pm 4, \pm 5 & \mod 12 \end{cases}$$

for positive indices, whereas

$$c_{0n} = \begin{cases} 1 \text{ for } n = 0 \\ 0 \text{ else.} \end{cases}$$

4. Spherical classes in K_*MSU

We are now able to calculate spherical classes in K_*MSU. For this purpose we collect all relevant notions: Let $a = \tau^* b \in K^0 MSU$ be an arbitrary class, f the classifying map of the virtual SU-bundle $(1 - L_1)(1 - L_2)$ and

$$d_k = \sum_{i=1}^{k-1} n_k^i f_*(\beta_i \otimes \beta_{k-i})$$

the generators of $K_*BSU \cong \pi_* K[d_2, d_3, ...]$. Writing

$$\theta^3((1 - L_1)(1 - L_2)) = \sum_{m,n \geq 0} c_{mn} x^m y^n$$

for the cannibalistic class of the SU-bundle above we get:

$$\begin{aligned}
\langle a, \psi_M^{3^{-1}} d_k \rangle &= \langle \psi_M^3(\tau^* b), d_k \rangle = \langle \theta^3 \tau \psi_B^3(b), d_k \rangle \\
&= \langle \theta^3((1-L_1)(1-L_2))\psi_B^3(b), \sum_{i=1}^{k-1} n_k^i f_*(\beta_i \otimes \beta_{k-i}) \rangle \\
&= \sum_{i=1}^{k-1} n_k^i \sum_{m,n\geq 0} c_{mn} \langle f^*(\psi_B^3(b)), \beta_{i-m} \otimes \beta_{k-i-n} \rangle \\
&= \sum_{i=1}^{k-1} n_k^i \sum_{m,n\geq 0} c_{mn} \langle b, \psi_B^{3^{-1}} f_*(\beta_{i-m} \otimes \beta_{k-i-n}) \rangle .
\end{aligned}$$

Lemma 5.5. *The Adams operations on K_*MSU are computable via the formula*

$$\psi_M^{3^{-1}} \Phi_* d_k = \Phi_* \Big(\sum_{m,n\geq 0} c_{mn} \sum_{i=1}^{k-1} n_k^i \psi_B^{3^{-1}} f_*(\beta_{i-m} \otimes \beta_{k-i-n}) \Big),$$

where Φ_* is the Thom isomorphism.

Sample calculations (dropping the Thom isomorphism from the notation and writing $\psi_M^{3^{-1}}$ for the Adams operation on the level of the Thom spectrum) give with respect to the n_k^i chosen above:

$$\begin{aligned}
\psi_M^{3^{-1}} d_2 &= 9d_2 + \frac{2}{3} \\
\psi_M^{3^{-1}} d_3 &= 27d_3 - 9d_2 + \frac{1}{3} \\
\psi_M^{3^{-1}} d_4 &= 81d_4 + 2d_2 + \frac{1}{3} \\
\psi_M^{3^{-1}} d_5 &= 243d_5 + 486d_4 + 288d_3 - 243d_2^2.
\end{aligned}$$

4.1. Construction of spherical classes. Modulo 2 and omitting the Thom isomorphism, we get:

$$\begin{aligned}
\psi_M^{3^{-1}} d_2 &= d_2 \\
\psi_M^{3^{-1}} d_3 &= d_3 + d_2 + 1 \\
\psi_M^{3^{-1}} d_4 &= d_4 + 1 \\
\psi_M^{3^{-1}} d_5 &= d_5 + d_2^2.
\end{aligned}$$

These calculations give the following spherical classes modulo 2:

- d_2 in degree 4
- $d_3^2 + d_5 + d_4 + d_2^2$ in degree 12
- $d_4^2 + d_4$ in degree 16
- $d_5^2 + d_2^2 d_5$ in degree 20.

Remark 5.2. *We observe that there is no spherical class in degree 6 and conjecture that there is no spherical class in degree $4k+2$.*

Corollary 5.4. *Since $\pi_6 T_\zeta = 0$, this gives $MSU_6 = 0$ in the $K(1)$-local world.*

4.2. Lifting mod p spherical classes. Having mod p spherical classes we are interested in getting integral spherical classes and constructing a spherical class basis for K_*MSU. We use the following algebraic lemma.

Lemma 5.6. *Assume A and B to be p-complete. Let $f : A \to B \cong \mathbb{Z}_p[b_i]$ be such that there are $a_i \in A$ with $f(a_i) \equiv b_i$ modulo p. Then $f : A \to B$ is surjective.*

We want to apply this lemma for $A = K_*MSU$ and assume that we have a basis of mod p spherical classes $A \cong \mathbb{Z}_p[a_i]$ with $\psi^3 a_i \equiv a_i$ modulo p.

Proposition 5.1. *There are elements b_i such that*

(1) $A \cong \mathbb{Z}_p[b_i]$ *and*
(2) $\psi^3 b_i = b_i$

Proof. We make use of the bootstrap method: Assume $\psi^3 a_i = a_i + pa'$ with $a' = \sum c_j a_j$. Then we have

$$\psi^3 pa' = p\psi^3 a' = p\psi^3(\sum c_i a_i) = p(\sum c_i \psi^3 a_i) + p^2 a'',$$

and go on the same way. $\square$

5. Umbral calculus

5.1. Mahler series in p-adic analysis. The binomial polynomials define continuous functions

$$\binom{\cdot}{k} : \mathbb{Z}_p \to \mathbb{Z}_p, \qquad x \mapsto \binom{x}{k}.$$

Since $\mathbb{N}$ is dense in $\mathbb{Z}_p$, we have $\left\|\binom{\cdot}{k}\right\| = \sup_{\mathbb{N}} \left|\binom{n}{k}\right| \leq 1$. Because of $\binom{k}{k} = 1$, equality holds in fact. In p-adic analysis we know that for a given sequence $(a_i)_{i\geq 0}$ in $\mathbb{C}_p$ with $|a_i| \to 0$, the series $\sum_{k\geq 0} a_k \binom{\cdot}{k}$ is a continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$. It is quite remarkable that conversely, every continuous function $\mathbb{Z}_p \to \mathbb{C}_p$ can be represented this way. This result has been obtained by Mahler.

Definition 5.2. *A Mahler series is a series $\sum_{k\geq 0} a_k \binom{\cdot}{k}$ with coefficients $|a_k| \to 0$ in $\mathbb{C}_p$.*

With the notation of the norm $\|f\| = \sup_{\mathbb{Z}_p} |f(x)|$ and the finite-difference operator ∇

$$(\nabla f)(x) = f(x+1) - f(x),$$

and its k-fold iterated version ∇^k, we have:

Theorem 5.2 (Mahler)**.** *Let $f : \mathbb{Z}_p \to \mathbb{C}_p$ be a continuous function and put $a_k = \nabla^k f(0)$. Then $|a_k| \to 0$, and the series $\sum_{k\geq 0} a_k \binom{\cdot}{k}$ converges uniformly to f. Moreover $\|f\| = \sup_{k\geq 0} |a_k|$.*

5.2. The ring of numerical polynomials. Let A denote the ring

$$A := \{f \in \mathbb{Q}[\omega] \text{ such that } f(\mathbb{Z}) \subset \mathbb{Z}\},$$

which we call the ring of numerical poynomials.

Remark 5.3. *This ring has been studied for a long time - historically Pascal considered elements $\binom{w}{i} = \frac{w(w-1)\ldots(w-i+1)}{i!}$ and Fermat studied $\frac{w^p - w}{p}$ for p a prime. In fact Newton found out that $1, w, \binom{w}{2}, \binom{w}{3}, \ldots$ are a basis for A.*

In p-adic analysis the p-completion of A, i.e.

$$\hat{A}_p = \{f \in \mathbb{Q}_p[\![\omega]\!] : f(\mathbb{Z}_p) \subset \mathbb{Z}_p\}$$

is, by a theorem of Mahler, the ring of continuous functions $f : \mathbb{Z}_p \to \mathbb{Z}_p$, and its elements can be written as Mahler series

$$f(\omega) = \sum_{i=0}^{\infty} a_i \binom{\omega}{i} \quad \text{with} \quad a_i \to 0.$$

Integrally we can identify $K_0\mathbb{CP}^\infty \cong A$, i.e. the K-homology of $\mathbb{CP}^\infty$ equals the ring of numerical polynomials. The duality

$$K^0\mathbb{CP}^\infty \cong \mathrm{Hom}(K_0\mathbb{CP}^\infty, \mathbb{Z})$$

is given as follows: The series $\sum a_i t^i \in \mathbb{Z}[\![t]\!] \cong K^0\mathbb{CP}^\infty$ maps to the homomorphism given by $\binom{w}{i} \mapsto a_i$ on basis elements.

5.3. Alternative description of the Adams operations. We have seen that $K_*\mathbb{CP}^\infty$ is the ring of continuous functions on $\mathbb{Z}_p$ which are given as Mahler series. Its module generators $\beta_i \in K_{2i}\mathbb{CP}^\infty$ represent the function $\beta_i(T) = \binom{T}{i}$. Application of a base change leads to an interesting observation: At the prime 2 we have:

$$\begin{aligned}
\binom{3T}{1} &= 3\binom{T}{1}\\
\binom{3T}{2} &= 9\binom{T}{2} + 3\binom{T}{1}\\
\binom{3T}{3} &= 27\binom{T}{3} + 18\binom{T}{2} + \binom{T}{1}\\
\binom{3T}{4} &= 81\binom{T}{4} + 81\binom{T}{3} + 15\binom{T}{2}\\
\binom{3T}{5} &= 243\binom{T}{5} + 324\binom{T}{4} + 108\binom{T}{3} + 6\binom{T}{2}\\
\binom{3T}{6} &= 729\binom{T}{6} + 1215\binom{T}{5} + 594\binom{T}{4} + 81\binom{T}{3} + \binom{T}{2}
\end{aligned}$$

and this is exactly the Adams operation $\psi^{3^{-1}}$ on β_k with respect to the generator $x = L - 1 \in K^*\mathbb{CP}^\infty$. The generator used before results in the same operation up to an alternating sign.

Lemma 5.7. *The Adams operation $\psi^{3^{-1}}$ on $K_*\mathbb{CP}^\infty$ is given by*

$$\psi^{3^{-1}}\beta_i(T) = \beta_i(3T),$$

or, equivalently as the Mahler series

$$\psi^{3^{-1}}\binom{T}{i} = \binom{3T}{i} = \sum_{j\geq 1} a_j \binom{T}{j},$$

where

$$a_j = \sum_{s+t=i-j} \binom{j}{s}\binom{s}{t} 3^{j-t}.$$

PROOF. Due to Mahler's theorem, the j^{th} coefficient satisfies

$$a_j = \nabla^j \binom{3x}{i}\Big|_{x=0},$$

i.e. it can be expressed using the j-fold iterated finite difference operator. Starting the calculations we get

$$\nabla\binom{3T}{i} = 3\binom{3T}{i-1} + 3\binom{3T}{i-2} + \binom{3T}{i-3}.$$

Comparing this to the calculation of the Adams operation in $K^*\mathbb{CP}^\infty$ with respect to the generator $x = L - 1$ we get

$$\psi^3 x = 3x + 3x^2 + x^3$$

and see that taking the j^{th} power of $3x + 3x^2 + x^3$ is exactly the same as taking the j^{th} iterated finite difference operator ∇^j. Hence the calculations coincide and the claim follows. $\square$

Corollary 5.5. *For a 2-adic unit $k \in \mathbb{Z}_2^\times$ we have*

$$\left(\psi^{k^{-1}}\beta_i\right)(T) = \beta_i(kT) = \binom{kT}{i}.$$

PROOF. Since 3 is a topological generator of $\mathbb{Z}_2^\times$, the sequence $a_n = 3^n$ contains a subsequence $(a_{i_n})_n$ converging to k, and we have

$$\begin{aligned} \psi^{k^{-1}}\beta_i(T) &= \lim_n \psi^{a_{i_n}^{-1}}\beta_i(T) = \lim_n \psi^{3^{-a_{i_n}}}\beta_i(T) \\ &= \lim_n \psi^{3^{-1}}\cdots\psi^{3^{-1}}\beta_i(T) = \lim_n \beta_i(3^{a_{i_n}}T) \\ &= \beta_i(kT). \end{aligned}$$

$\square$

CHAPTER 6

Open questions and concluding remarks

Working towards a full E_∞ splitting of the $K(1)$-local bordism spectrum MSU one has to know that the quotient $M\mathrm{SU}/T_\zeta$ is free in the $K(1)$-local stable homotopy category. This implies a basis corresponding to the spherical classes which is geometrically realized as free E_∞ summands TS^0. I conjecture that MSU splits at $p = 2$ as $M\mathrm{SU} \cong T_\zeta \wedge \bigwedge_{i=1}^{\infty} TS^0$, which looks very similar to the E_∞ splitting of $M\mathrm{Spin}$. Maybe the E_∞ comparison map $M\mathrm{SU} \to M\mathrm{Spin}$ is close to an E_∞ equivalence. An indication for this is the vanishing of MSU_6 after $K(1)$-localization. Integrally this is false, because the comparison map is neither injective nor surjective on the level of homotopy. To give examples we mention that $\mathbb{HP}^2$ is not a complex manifold, but it is spin, thus the comparison map is not surjective. While the 3-connected manifold $\mathbb{HP}^2$ represents a non-trivial spin bordism class, it does not admit a stably complex structure (since its signature is odd, cf. [**CF66a**]). On the other hand we have $M\mathrm{Spin}_6 = 0$, but by construction the projective variety

$$\mathcal{K} = \{z \in \mathbb{CP}^4 | z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0\}$$

has vanishing first Chern class and represents a non-zero class in $M\mathrm{SU}_6$. Recall from [**ABP66**] that an SU-manifold is null bordant if and only if its Chern numbers and KO-characteristic numbers vanish.

n	0	1	2	3	4	5	6	7	8
π_n^s	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/16$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
$Im(J)_n$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/16$	$\mathbb{Z}/2$
KO_n	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}_{(2)}$	0	0	0	$\mathbb{Z}_{(2)}$
$M\mathrm{SU}_n$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)}$
$M\mathrm{Spin}_n$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}_{(2)}$	0	0	0	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)}$

The 2-primary part of some relevant homotopy groups

Desiring progress in a full SU-splitting, one has to get a better understanding of the spherical classes. One approach is to apply better arithmetic techniques, another approach is to interpret the Adams operations as a precomposition of automorphisms as in the example of $K_*\mathbb{CP}^\infty$. Again another access is the study of symmetric 2-cocycles in the sense of [**Lau02**] and [**AHS01**]. This is an interesting arithmetic problem and it poses quite a challenge to calculate the corresponding spherical classes.

Another problem which is not solved yet is a $K(1)$-local additive decomposition of $M\mathrm{SU}$ in terms of K-theory. In [**Pen82**] Pengelley gives a 2-local additive splitting of MSU

$$MSU_{(2)} \cong \bigvee_i \Sigma^{d_i} BoP \vee \bigvee_j \Sigma^{d'_j} BP$$

into a wedge of suspensions of Brown-Peterson spectra BP and a wedge of suspensions of other indecomposable spectra BoP, which bear similarities to the BP spectrum and connective K-theory ko. In [**Ho97**] Marc Hovey conjectures that the spectrum BoP splits into a wedge of suspensions of K-theory spectra K and KO, so that in this case MSU splits additively like

$$MSU \cong \bigvee K \vee \bigvee KO.$$

Using results from [**Hop98**], more specifically that $\pi_* K \wedge K \cong \mathrm{Hom}_{cts}(\mathbb{Z}_2^\times, \pi_* K)$ and $\pi_* K \wedge KO \cong \mathrm{Hom}_{cts}(\mathbb{Z}_2^\times/\{\pm 1\}, \pi_* K)$, we see that

$$\begin{aligned} K_* MSU &\cong \bigoplus K_* K \oplus \bigoplus K_* KO \\ &\cong \bigoplus \mathrm{Hom}_{cts}(\mathbb{Z}_2^\times, \pi_* K) \oplus \bigoplus \mathrm{Hom}_{cts}(\mathbb{Z}_2^\times/\{\pm 1\}, \pi_* K). \end{aligned}$$

It is highly desirable to get a precise additive splitting. Such a description would offer comforting methods to calculate Adams operations.

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