Ninth exercise sheet Advanced Algebra II.

Problem 1 (2 points)**.** *Let R be a convex subring of a real closed field K and $\Gamma = K^\times/R^\times$ the target of the valuation defined by R. Show that Γ is a $\mathbb{Q}$-vector space!*

Problem 2 (2 points)**.** *Let R be a ring and A an integral R-algebra. Show that* $\mathrm{Sper}A$ *is R-proper!*

Problem 3 (6 points)**.** *Let A be a PID, $\mathfrak{p} \in \mathrm{Spec}A$ a real prime ideal, $\mathfrak{p} = \pi A$ with $\pi \in A$ and $\overline{r} = r \bmod \mathfrak{p}$. Fix an ordering of $A/\mathfrak{p}$ and let $\mathfrak{P}_o = \{r \in r \mid \overline{r} \geq 0\} \in \mathrm{Sper}A$. Show that there are precisely two elements $\mathfrak{P}_\pm \in \mathrm{Sper}A$ contained in $\mathfrak{p}_o$ with $\mathrm{supp}\mathfrak{P}_\pm = \{0\}$, one with $\pi \in \mathfrak{P}_+$ and one with $-\pi \in \mathfrak{P}_-$.*

Problem 4 (4 points)**.** *In the situation of the previous problem, let $\mathfrak{P}$ be one of $\mathfrak{P}_\pm$, K the quotient field of A ordered by $\mathfrak{P}$ and R the convex hull of A in K. Show that $R = A_\mathfrak{p}$, the localization of A at $\mathfrak{p}$.*

In particular, R is a DVR.

Problem 5 (2 points)**.** *In the situation of the previous two problems, let $\mathcal{R}_{A,\mathfrak{P}}$ be the convex hull of A in the real closure $\mathfrak{K}(\mathfrak{P})$ of K. Show that the valuation group of $\mathcal{R}_{A,\mathfrak{P}}$ is isomorphic to $\mathbb{Q}$!*

In particular, $\mathcal{R}_{A,\mathfrak{P}}$ has rank one and the map

$$\mathrm{Spec}_{\mathcal{R}_{A,\mathfrak{P}}} \to \overline{\{\mathfrak{P}\}} \tag{2.3.1}$$

discussed in Proposition 2.3.4 from the lecture is bijective in this case. In the situation of Problem 8, for a non-zero polynomial

$$P = \sum_{\alpha \in \mathbb{N}^m} p_\alpha X^\alpha \in A = R[X_1, \dots, X_m]$$

let $\mathrm{lse}_\trianglelefteq(P)$ (resp. $\mathrm{mse}_\trianglelefteq(P)$) be the $\trianglelefteq$-minimum (resp. maximum) of the set of $\alpha \in \mathbb{N}^m$ with $p_\alpha \neq 0$ and let

$$\mathrm{lsc}_\trianglelefteq(P) = p_{\mathrm{lse}_\trianglelefteq(P)}$$
$$\mathrm{msc}_\trianglelefteq(P) = p_{\mathrm{mse}_\trianglelefteq(P)}.$$

In the following we apply this with $R = K$ a real closed field.

Problem 6 (4 points)**.** *Show that*

$$\mathfrak{P}^{(0)}_\trianglelefteq = \{f \in A \mid f = 0 \text{ or } \mathrm{lsc}_\trianglelefteq(f) > 0\}$$

is a prime cone in A!

Problem 7 (4 points)**.** *Show that*

$$\mathfrak{P}_{\trianglelefteq}^{(\infty)} = \left\{ f \in A \;\middle|\; f = 0 \textit{ or } \operatorname{msc}_{\trianglelefteq}(f) > 0 \right\}$$

is a prime cone in A!

Four of the 24 points from this sheet are bonus points. Solutions should be submitted in the lecture Friday, June 21.